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CENTRAL TWISTED GROUP ALGEBRAS

BY

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ABSTRACT. A twisted group algebra $L^1(A, G; T, \alpha)$ is *central* iff T is trivial and A commutative. (Group algebras of central extension of G are such.) We show that if $H^2(G)$ is discrete any central $L^1(A, G; \alpha)$ is a direct sum of closed ideals $L^1(A_i, G; \alpha_i)$ having as duals fibre bundles over the duals of closed ideals A_i in A, with fibres projective duals of G, and principal $G^\wedge$ bundles (where $G^\wedge$ denotes the group of characters of G) satisfying the conditions which define characteristic bundles for G abelian. (If G is compact $H^2(G)$ is always discrete, the direct sum is countable, and the bundles are locally trivial.) Applications are made to the duals of central extensions of groups and in particular to duals of "central" groups. For G commutative, $H^2(G)$ discrete, and A a C^*-algebra with identity, all central twisted group algebras $L^1(A, G; \alpha)$ (and their duals) are classified in purely algebraic terms involving $H^2(G)$, the group G, and the first Čech cohomology group of the dual of A. This result allows us, in principle, to construct all the central $L^1(A, G; \alpha)$ and their duals where A is a C^*-algebra with identity and G a compact commutative group.

1. Introduction. Twisted group algebras and special cases of them have been studied by many authors under a variety of names and using superficially distinct definitions. A discussion of the equivalence of several such formulations is given in [20]. The most remarkable instance of convergence of apparently different definitions is the relationships established by Busby in [4] between twisted group algebras and the homogeneous Banach *-algebraic bundles discussed by J. M. G. Fell in [11]. We adhere in this paper to the form of definition used in [1], [4], [5], [6], [7], [25], and [26]. Accordingly, we denote a twisted group algebra by $L^1(A, G; T, \alpha)$ where A is a separable Banach *-algebra, G a locally compact second countable (lcsc) group, T an action of G as automorphisms of A, and α is a Borel 2-cocycle for the action T. The algebra consists of the Bochner-integrable A-valued functions on G with a multiplication and involution defined in terms of T and α.

If $\circ \to K \to E \to G \to \circ$ is an exact sequence of lcsc groups, the usual group algebra $L^1(E)$ can be expressed as a twisted group algebra $L^1(L^1(K), G; T, \alpha)$ for appropriate T and α (see [6] for details). The extension is split (E is a semidirect product) iff α is a coboundary. On the other hand, E

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is a central extension iff K is abelian and T is trivial. Both conditions extend naturally to general twisted group algebras. Those with α trivial have been studied and applied widely under the name "covariance algebras" (cf., e.g. **[8]**, **[9]**, **[10]**, **[19]**). The subject of this paper is twisted group algebras $L^1(A, G; T, \alpha)$ satisfying the conditions analogous to centrality (i.e. A commutative and T trivial). We call such twisted group algebras "central" and write them as $L^1(A, G; \alpha)$. We are particularly concerned with studying the dual space consisting of equivalence classes of irreducible representations of $L^1(A, G; \alpha)$ with a quotient topology induced from the weak* topology on the indecomposable positive functionals.

2. Definitions. Let A be a commutative separable Banach *-algebra with a two-sided approximate identity of unit norm, and let G be an lcsc group with identity e. Let α be a Borel 2-cocycle on G taking values in the unitary double centralizers of A, i.e. for x, y, z in G

$$\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z),$$
$$\alpha(e, x) = \alpha(x, e) = 1.$$

By the central twisted group algebra $L^1(A, G; \alpha)$ we mean the Banach space $A \otimes_\gamma L^1(G)$ of all Bochner integrable A-valued functions with the product

$$(f * h)(x) = \int_G f(y)h(y^{-1}x)\alpha(y, y^{-1}x)\, dy$$

and involution

$$f^*(x) = \alpha(x, x^{-1})^* f(x^{-1})^* \Delta(x^{-1})$$

where Δ is the modular function on G and integration is with respect to a left Haar measure. (The operations can be extended to A^-, the double centralizer algebra of A, and $L^1(A, G; \alpha)$ can be embedded in $L^1(A^-, G; \alpha)$ when convenient.) Then $L^1(A, G; \alpha)$ is a Banach *-algebra (see **[6]** for details).

By a representation of a Banach *-algebra with identity (or bounded approximate identity) we mean a cyclic symmetric weakly continuous representation on a separable Hilbert space. (Representations which are extended to double centralizers will usually not be distinguished notationally from the original representation.) The set of unitary equivalence classes of representations will be given the quotient topology induced from the weak* topology on the normalized positive functionals under the usual mapping. (The terminology here is that of **[24]** which can be consulted for details. As in that paper we are ultimately concerned with irreducible representations, so it is natural to consider only cyclic representations.) If α is a Borel 2-cocycle on G taking values in the unit circle, a weakly Borel mapping of G to the unitary operators on a separable Hilbert space is called a (projective) α-representation if it satisfies

$$U_g U_\gamma = \alpha(g, \gamma) U_{g\gamma}.$$

Such α-representations are produced by lifting ordinary representations of $L^1(C, G; \alpha)$ (C being the complex numbers) to the double centralizers, in which G is imbedded as the set of unitary elements. (See [5] for details.)

3. Representations. If m is any nontrivial multiplicative functional on A, applying it to the members of $L^1(A, G; \alpha)$ maps the algebra homomorphically to $L^1(C, G; \alpha_m)$ where $\alpha_m = m(\alpha)$. Then applying any representation of the latter algebra (induced by any projective unitary α_m-representation of G) produces a representation of the former which we denote as $m \otimes U_m$ where U_m is the projective unitary α_m-representation of G. Clearly $m \otimes U_m$ is irreducible iff U_m is and $m \otimes U_m$ is unitarily equivalent to $m \otimes V_m$ iff U_m and V_m are unitarily equivalent. On the other hand, L. G. Brown [3] has extended the results of [4] to yield the conclusion that every twisted group algebra has a bounded approximate identity. This result allows us to conclude from Proposition 3.4 of [6] that every irreducible representation of a central twisted group algebra is equivalent to some $m \otimes U_m$, where U_m is irreducible. We denote by $G^\wedge$ the lcsc abelian group of one-dimensional representations (group characters) of G. Let P_m denote the space of normalized positive functionals on $L^1(C, G; \alpha_m)$ and let P_m° denote the indecomposable members of P_m. (All of these spaces bear the relative weak* topology, of course.)

LEMMA 1. *$G^\wedge$ acts as an order-preserving topological transformation group on P_m, and hence on P_m°.*

PROOF. For χ in $G^\wedge$ and φ in P_m, define $\chi \cdot \varphi$ by

$$(\chi \cdot \varphi)(f) = \varphi(\chi \cdot f)$$

for all f in $L^1(G)$, where $\chi \cdot f$ denotes pointwise multiplication.

First we note that $\chi \cdot \varphi$ is also in P_m. We have

$$(\chi \cdot \varphi)(f^* * f) = \varphi(\chi \cdot (f^* * f)) = \varphi((\chi \cdot f)^* * (\chi \cdot f)) \geqslant 0$$

since

$$\begin{aligned}
&[(\chi \cdot f)^* * (\chi \cdot f)](g) \\
&\quad = \int_G \bar{\alpha}(\gamma, \gamma^{-1}) \bar{\chi}(\gamma^{-1}) \bar{f}(\gamma^{-1}) \Delta(\gamma^{-1}) \chi(\gamma^{-1} g) f(\gamma^{-1} g) \alpha(\gamma, \gamma^{-1} g)\, d\gamma \\
&\quad = \chi(g) \int_G \bar{\alpha}(\gamma, \gamma^{-1}) \bar{f}(\gamma^{-1}) \Delta(\gamma^{-1}) f(\gamma^{-1} g) \alpha(\gamma, \gamma^{-1} g)\, d\gamma \\
&\quad = [\chi \cdot (f^* * f)](g),
\end{aligned}$$

and the same calculation shows the action of $G^\wedge$ preserves order. Thus if φ is in P_m° so is $\chi \cdot \varphi$.

We establish joint continuity, the other properties being immediate. For the

weak* topology on P_m it suffices to consider convergence on the $L^1(G)$ functions of compact support. For such a function f, if $\chi_i \to \chi$ then $\chi_i \cdot f \to \chi \cdot f$ in $L^1(G)$ since convergence in the weak* topology on $G^\wedge$ is well known (see [23]) to imply uniform convergence on compact subsets of G. However P_m is a compact convex set (see [23]) and hence is equicontinuous [18, 18.5 and 18.6], so the relative weak* topology on P_m (the topology of pointwise convergence on $L^1(G)$) is jointly continuous (15, Chapter 7, [17]). Thus $\varphi_j \to \varphi$ and $\chi_i \to \chi$ implies $\varphi_j(\chi_i \cdot f) \to \varphi(\chi \cdot f)$, and we have joint continuity: $\chi_i \cdot \varphi_j \to \chi \cdot \varphi$.

COROLLARY 1. *$G^\wedge$ acts by multiplication as a topological transformation group on the space $\mathfrak{A}_m$ of equivalence classes of irreducible α_m representations of G.*

PROOF. The action of χ on φ is easily calculated to induce a map of U_m to χU_m; the usual product:

$$\varphi(f) = \left\langle \int_G f(g) U_m(g)\, dg\xi, \xi \right\rangle,$$

$$\varphi(\chi \cdot f) = \left\langle \int_G f(g)\chi(g) U_m(g)\, dg\xi, \xi \right\rangle.$$

Clearly χU_m and χV_m are equivalent if U_m and V_m are. Since $\mathfrak{A}_m$ carries the quotient topology induced from P_m°, the result is immediate.

4. Characteristic principal bundles. In [25] it was shown that a twisted group algebra is commutative iff, in addition to being central as defined above, G is abelian and α is symmetric; $\alpha(x, y) = \alpha(y, x)$ a.e. The dual spaces (maximal ideal spaces with weak* topology) of such algebras were found to be a class of (not necessarily locally trivial) principal $G^\wedge$ bundles over $A^\wedge$, where $G^\wedge$ is the Pontryagin dual of G and $A^\wedge$ is the dual of A. These bundles were called characteristic principal bundles in [25] and they were characterized in more topologically natural terms and classified in [26].

The following construction of characteristic principal bundles is the one needed presently. (Details can be found in [25].) Let G be an abelian lcsc group. Let $C^1(G)$ be the circle-valued Borel measurable functions on G, normalized at the identity, with the topology of local convergence in measure and let $C^2(G)$ be the similarly defined class of functions on $G \times G$. The coboundary map δ: $C^1 \to C^2$ defined by

$$(\delta\psi)(g, \gamma) = \psi(g)\psi(\gamma)/\psi(g\gamma)$$

has as its kernel $G^\wedge$. If we call the image $B^2(G)$, then the short exact sequence $1 \to G^\wedge \to C^1(G) \to B^2(G) \to 1$ gives C^1 the structure of a principal $G^\wedge$ bundle over $B^2(G)$. For any lcsc Hausdorff space X, the characteristic principal $G^\wedge$

bundles over X are those induced by continuous mappings of the Čech compactification of X into $B^2(G)$.

We now wish to extend the concept of characteristic bundle to cover the case of G nonabelian. The proofs of Lemmas 6, 8, and 9 of [25] can be carried through with only obvious minor modifications to show that

$$1 \to G^\wedge \to C^1(G) \to B^2(G) \to 1$$

is still a short exact sequence of topological groups when G is not assumed to be abelian. Thus in this case we can still define a characteristic $G^\wedge$ principal bundle over X to be a bundle induced from $C^1(G)$ by a continuous map of the Čech compactification of X into $B^2(G)$. From this definition it follows that the original characterization of these bundles as those principal $G^\wedge$ bundles Y over X which support circle-valued functions F: $G \times Y \to T$ measurable on G and continuous on Y such that $F(e, y) = 1$ and $F(g, \chi y) = \chi(g)F(g, y)$ for all χ in $G^\wedge$ again applies to this generalization. However, the results of [26], which provide a more appealing characterization and which enable us to classify all the characteristic principal $G^\wedge$ bundles over X in the commutative case, do not generalize in an immediate or obvious way to this more general case. The lack of such a classification will restrict the scope of our results in §7.

5. The projective representation bundle of G. The normalized, circle-valued 2-cocycles form a closed subgroup $Z^2(G)$ in $C^2(G)$ and contain in turn $B^2(G)$ as a closed subgroup. As usual, $H^2(G) = Z^2/B^2$. The arguments given in [25] establishing that

$$1 \to G^\wedge \to C^1(G) \to B^2(G) \to 1$$

is a short exact sequence of topological groups, and hence that $C^1(G)$ is a principal $G^\wedge$ bundle over $B^2(G)$, remain valid without the assumption of commutativity of G which was made there. Choice of any coset representative in Z^2 for a member of H^2 induces a homeomorphism of that coset with B^2 and hence induces over the coset a principal $G^\wedge$ bundle isomorphic with $C^1(G)$. Choice of different representatives produces isomorphic bundles, so we have a principal $G^\wedge$ bundle isomorphic with $C^1(G)$ associated with each member of $H^2(G)$. (See [16] for details on bundle morphisms, induced bundles, etc.)

It often happens that $B^2(G)$ is open in $Z^2(G)$. In particular, C. C. Moore has shown this to be the case when $H^2(G)$ is countable [22, Proposition 6], and thus, in particular, when G is compact [21, Corollary 1]. In that case each coset is open and the local homeomorphisms of the cosets form a continuous map of Z^2 to B^2 inducing over Z^2 the disjoint union of the bundles over the cosets which we described previously. From here on we will be primarily concerned with the case of $B^2(G)$ open in $Z^2(G)$ (i.e. $H^2(G)$ is discrete) so we

will assume this condition to hold unless the contrary is explicitly stated.

REMARK 1. If α_m and α_n are cohomologous members of $Z^2(G)$,

$$\alpha_n(g, \gamma) = \frac{\psi(g)\psi(\gamma)}{\psi(g\gamma)}\,\alpha_m(g, \gamma),$$

then the respective spaces P_m°, $\mathfrak{U}_m$ and P_n°, $\mathfrak{U}_n$ are clearly homeomorphic under the mappings induced from

$$U_n(g) = \psi(g)U_m(g)$$

acting on the projective representations.

The action of $G^\wedge$ on P_m° and $\mathfrak{U}_m$ as a transformation group allows us to construct, over the coset of each α_m, fibre bundles with these spaces as fibres from the principal bundles isomorphic to $C^1(G)$ described above. We will refer to the disjoint union (over $H^2(G)$) of these bundles as the P° bundle of G and the projective representation bundle of G respectively.

6. Dual spaces of central twisted group algebras.

REMARK 2. The space of indecomposable positive functionals and the space of equivalent classes of irreducible representations of a central twisted group algebra $L^1(A, G; \alpha)$ (the dual of $L^1(A, G; \alpha)$) form bundles over $A^\wedge$.

PROOF. We need only exhibit continuous mappings of the spaces to $A^\wedge$. We know from §3 that every irreducible representation has the form $m \otimes U_m$, and every indecomposable positive functional arises from one of these. If φ is such a positive functional, we can find some $a \otimes f$ in $L^1(A, G; \alpha)$ with a in A, f in $L^1(G)$ and $\varphi(a \otimes f) \neq 0$. But

$$\varphi(a \otimes f) = m(a)\left\langle \int f(g)U_m(g)\,dg\xi, \xi \right\rangle \quad \text{for some } \xi.$$

We define $\pi(\varphi)$ in $A^\wedge$ by

$$[\pi(\varphi)](b) = \varphi(ba \otimes f)/\varphi(a \otimes f) = m(b)$$

which is clearly independent of the choice of $a \otimes f$.

Then if $\varphi_i \to \varphi$ pointwise on $L^1(A, G; \alpha)$ we have $\pi(\varphi_i) \to \pi(\varphi)$ pointwise on A. Thus π is a continuous map of the indecomposable positive functionals to $A^\wedge$. Moreover, the map is obviously independent of the choice of the cyclic vector ξ and thus invariant under unitary equivalence and independent of the particular positive functional φ corresponding to U_m. Thus it also induces a map of the equivalence classes of irreducible representations.

REMARK 3. The mapping α_-: $A^\wedge \to Z^2(G)$, defining by $m \to \alpha_m$, is continuous.

PROOF. For fixed g, γ, $\alpha(g, \gamma)$ has as its Gelfand representation the continuous function α_m on $A^\wedge$. Thus $n \to m$ implies $\alpha_n \to \alpha_m$ pointwise on $G \times G$, hence locally in measure.

We denote by $\alpha_\sim$: $A^\wedge \to H^2(G)$, the continuous map induced by composing

α_- with the natural projection of Z^2 to H^2. By Remark 1, all the spaces $\mathfrak{U}_m$ are homeomorphic for m in the pre-image of a point of H^2.

THEOREM 1. *If $H^2(G)$ is discrete, the dual of any central twisted group algebra $L^1(A, G; \alpha)$ is the disjoint union (over the image $a_\sim(A^\wedge)$ in $H^2(G)$) of fibre bundles. Each of these fibre bundles has as its principal bundle a characteristic principal $G^\wedge$ bundle, as its base space the pre-image of a point of H^2, $\alpha_\sim^{-1}(\alpha_\sim(m))$, and as its fibre the corresponding space $\mathfrak{U}_m$ of equivalence classes of projective representations. The dual of $L^1(A, G; \alpha)$ is, in fact, the bundle induced over $A^\wedge$ from the projective representation bundle of G by the mapping $\alpha_-: A^\wedge \to Z^2(G)$. The indecomposable positive functionals on $L^1(A, G; \alpha)$ are similarly characterized as the bundle induced over $A^\wedge$ from the P° bundle of G.*

PROOF. For each point of $\alpha_\sim(A^\wedge)$, choose some m in its inverse image. For any n in $\alpha_\sim^{-1}(\alpha_\sim(m))$, by Remark 1, some ψ_n in $C^1(G)$ induces a homeomorphism which we denote by $\psi_n \mathfrak{U}_m = \mathfrak{U}_n$. Construct the fibre bundle $C^1(G)[\mathfrak{U}_m] = (C^1(G) \times \mathfrak{U}_m)/G^\wedge$ which is isomorphic with the restriction of the projective representation bundle of G to the coset $\alpha_m B^2$.

There is a well-defined mapping which carries (the unitary equivalence class of) $n \otimes U_n$ to the equivalence class $(\psi_n, U_m)^\sim$ of (ψ_n, U_m) in this bundle, where $\psi_n U_m = U_n$. (The choice of ψ_n is arbitrary to within a factor $\chi \in G^\wedge$, but the point in the bundle is the equivalence class $(\psi_n, U_m)^\sim$ consisting of all $(\chi\psi_n, \chi^{-1}U_m)$.) To show this mapping to be continuous, consider $n \otimes U_n \to q \otimes U_q$ (i.e. convergence of unitary equivalence classes in the quotient weak* topology). We must show that $(\psi_n, \psi_n^{-1}U_n)^\sim$ converges to $(\psi_q, \psi_q^{-1}U_q)^\sim$. By Remark 2 above we know $n \to q$ and hence, by Remark 3, $\alpha_n \to \alpha_q$ almost everywhere. By Lemma 8 of [25] (the proof of which can be carried through without assuming G commutative) there is χ_n in $G^\wedge$ for each n such that $\chi_n\psi_n \to \psi_q$ almost everywhere. Let f be any function of compact support in $L^1(G)$ and consider, for all unit vectors ξ, η in the appropriate Hilbert spaces,

$$\begin{aligned}
&\inf_{\xi,\eta}\left\{\left\langle \int_G f(g)\chi_n^{-1}(g)\psi_n^{-1}(g)U_n(G)\,dg\xi, \xi\right\rangle \right.\\
&\qquad\qquad \left. - \left\langle \int_G f(g)\psi_q^{-1}(g)U_q(g)\,dg\eta, \eta\right\rangle\right\}\\
&= \inf_{\xi,\eta}\left\{\left\langle \int_G f(g)\left[\chi_n^{-1}(g)\psi_n^{-1}(g) - \psi_q^{-1}(g)\right]U_n(g)\,dg\xi, \xi\right\rangle \right.\\
&\qquad + \left\langle \int_G f(g)\psi_q^{-1}(g)U_n(g)\,dg\xi, \xi\right\rangle\\
&\qquad\qquad \left. - \left\langle \int_G f(g)\psi_q^{-1}(g)U_q(g)\,dg\eta, \eta\right\rangle\right\}.
\end{aligned}$$

Since the integrand in the first term converges to zero a.e. on its compact support, the quadratic form converges to zero. But $n \otimes U_n \to q \otimes U_q$ in the quotient topology derived from the weak* topology on the positive functionals implies that the remaining difference of two terms can be made arbitrarily small, hence the entire expression becomes arbitrarily small, i.e. $\chi_n^{-1}\psi_n^{-1}U_n \to \psi_q^{-1}U_q$. Thus $(\chi_n\psi_n, \chi_n^{-1}\psi_n^{-1}U_n) \to (\psi_q, \psi_q^{-1}U_q)$ in $C^1(G) \times \mathfrak{U}_m$ and, in the fibre bundle $(C^1(G) \times \mathfrak{U}_m)/G^\wedge$, $(\psi_n, \psi_n^{-1}U_n)^\sim$ converges to $(\psi_q, \psi_q^{-1}U_q)^\sim$.

Since projection in the dual bundle carries (the unitary equivalence class of) $n \otimes U_n$ to n and projection in $C^1(G)[\mathfrak{U}_m]$ carries $(\psi_n, \psi_n^{-1}U_n)^\sim$ to α_n, the above map (together with $\alpha_\sim$ on the base space) is a bundle morphism. It factors naturally through the induced bundle via the map carrying (the unitary equivalence class of) $n \otimes U_n$ to $(n,(\psi_n, \psi_n^{-1}U_n)^\sim)$ in $\alpha_\sim^{-1}(\alpha_\sim(m)) \times C^1(G)[\mathfrak{U}_m]$. On the other hand the inverse map carrying $(n,(\psi_n, U_m)^\sim)$ to (the unitary equivalence class of) $n \otimes \psi_n U_m$ is obviously a well-defined and continuous morphism on the induced bundle.

Thus the restriction of the dual bundle to $\alpha_\sim^{-1}(\alpha_\sim(m))$ is isomorphic with the bundle induced by the mapping $\alpha_\sim$ on this open and closed subset of $A^\wedge$ from the fibre bundle (isomorphic to $C^1(G)[\mathfrak{U}_m]$) obtained by restricting the projective representation bundle of G to $\alpha_m B^2$. It is therefore a fibre bundle of the type asserted and the dual of $L^1(A, G; \alpha)$ is the disjoint union of these open and closed subbundles. The proof for the space of indecomposable positive functionals is exactly parallel.

COROLLARY 2. *If $A^\wedge$ is connected and $H^2(G)$ is discrete, the dual of $L^1(A, G; \alpha)$ is a fibre bundle.*

COROLLARY 3. *If G is compact the disjoint union of fibre bundles specified in the theorem is countable and the bundles are locally trivial.*

PROOF. We observed previously that for G compact C. C. Moore has shown $H^2(G)$ to be countable and hence discrete. On the other hand, $G^\wedge$ is discrete, so by Remark 1 of [26] the principal bundles are locally trivial.

These results can be interpreted in terms of the duals of central extensions of groups:

COROLLARY 4. *If K is a central subgroup of an lcsc group E such that $H^2(E/K)$ is discrete, then the dual of E is a disjoint union of fibre bundles having characteristic principal bundles with the group of characters of E/K as the group of the bundles, open and closed subsets of $K^\wedge$ as bases and, in each case, an appropriate projective dual of E/K as fibre.*

COROLLARY 5. *If E/K above is compact then $H^2(E/K)$ is necessarily discrete, the fibre bundles are locally trivial, and there are countably many of them.*

COROLLARY 6. *If $K^\wedge$ above is connected then the dual of E is a single fibre bundle.*

Grosser and Moskowitz [12]–[15] have studied a class of locally compact groups they term "central", having the property that the quotient of the group modulo its center is compact. Such groups are precisely the central extensions with E/K compact and $(E/K)^\wedge$ trivial, so our theorem gives a particularly simple characterization of the duals of such groups in the case of E second countable:

COROLLARY 7. *If E is an lcsc "central" (in the sense of* [12]) *group with center K, then the dual of E is a countable disjoint union of open and closed subsets, each subset being the direct product of an open and closed subset of $K^\wedge$ with a space of (unitary equivalence classes of) projective representations of E/K corresponding to a distinct cohomology class of circle-valued cocycles on E/K.*

By the Shilov idempotent theorem, when $H^2(G)$ is discrete the mapping $\alpha_\sim: A^\wedge \to H^2(G)$ produces a decomposition of A into a direct sum of closed ideals, since the sets of $\alpha_\sim^{-1}(\alpha_\sim(m))$ partition $A^\wedge$ into open and closed subsets. Moreover, projection of $L^1(A, G; \alpha)$ on these ideals decomposes it into a direct sum of closed ideals.

COROLLARY 8. *If $H^2(G)$ is discrete, any central twisted group algebra $L^1(A, G; \alpha)$ is a direct sum of closed ideals of the form $L^1(A_i, G; \alpha_i)$ where A_i is a closed ideal in A and α_i takes values in the double centralizers of A_i. Each of these ideals has as dual a fibre bundle with characteristic principal bundle and with fibre a space of (equivalence classes of) projective representations with fixed cocycle. If G is compact the condition is fulfilled, the direct sum is countable, and the bundles are locally trivial.*

7. Classification of certain central twisted group algebras. The mapping $\alpha_\sim$: $A^\wedge \to H^2(G)$, with $H^2(G)$ discrete, defines a decomposition of both A and $L^1(A, G; \alpha)$ as a direct sum of closed ideals each of which is in its own right a Banach *-algebra or a central twisted group algebra, respectively. The central twisted group algebras arising as such ideals are characterized as those central twisted group algebras for which $\alpha_\sim$ is constant, and for these algebras the duals are fibre bundles.

The preceding observation, together with the results of [26], allows us to classify to within isomorphism all central twisted group algebras which can arise from a fixed C^* algebra with identity and a fixed lcsc *abelian* group for which $H^2(G)$ is discrete–in particular for G compact. If $H^2(G)$ is trivial, these algebras are commutative and the classification was given in [25]. A complete discussion of lcsc abelian groups with nontrivial H^2 is given by L. G. Brown

in [2]. We are prevented from obtaining a classification for nonabelian G by lack of a satisfactory generalization of the results on characteristic bundles obtained in [26] for G abelian.

Let $\mathcal{H}^1(B)$ denote the first Čech cohomology group of the compact space B, with the discrete topology, and let $\operatorname{Ext}(G, \mathcal{H}^1(B))$ be the group of measurable $\mathcal{H}^1(B)$-valued 2-cocycles on G modulo the automorphism induced by homeomorphisms of B and automorphisms of G. (These correspond, as usual, to equivalence classes of extensions, hence the notation.)

THEOREM 2. *If $H^2(G)$ is discrete, G is abelian, and $A = C(B)$, the continuous functions on a compact separable Hausdorff space B, then the isomorphism classes of all central twisted group algebras $L^1(A, G;\ \alpha)$ for which $\alpha_\sim$ is constant are in one-to-one correspondence with the set*

$$H^2(G) \times \operatorname{Ext}(G, \mathcal{H}^1(B)).$$

PROOF. The constant value of $\alpha_\sim$ can be any member of $H^2(G)$. By applying m, a point of $B = A^\wedge$, to $L^1(A, G;\ \alpha)$ one obtains a twisted group algebra $L^1(C, G;\ \alpha_m)$ where C is the complex numbers. By Theorem 2.7 of [6] these are all isomorphic since these α_m are all cohomologous–all belong to the selected member of H^2. On the other hand, noncohomologous α_m would produce nonisomorphic $L^1(C, G;\ \alpha_m)$ since it is well known that they produce nonisomorphic group extensions [27]. Since Busby has shown [5] that any isomorphism of a twisted group algebra $L^1(C(B), G;\ \alpha)$ is "proper" in the sense of preserving the role of A and G, it follows that different constant values of $\alpha_\sim$ must produce nonisomorphic algebras $L^1(A, G;\ \alpha)$. This accounts for the factor $H^2(G)$ in the classifying set.

Two $C(B)$-valued cocycles on G may fail to be cohomologous even though the C-valued cocycles produced by evaluation at every point of B are all cohomologous (see [25] for an example). It was shown in [26] that when the C-valued cocycles produced by evaluation are all trivial and G is commutative then the cohomology classes of the corresponding set of $C(B)$-valued cocycles are in one-to-one correspondence with the characteristic principal $G^\wedge$-bundles over B.

Again, we know by 2.7 of [6] that $L^1(A, G;\ \alpha)$ and $L^1(A, G;\ \alpha^1)$ are isomorphic if α and α^1 are cohomologous and clearly they are not properly isomorphic if their dual spaces are not isomorphic as bundles. By the results of [25] and [26] we that if $\alpha_\sim = \alpha^1_\sim$ then α and α^1 are cohomologous if and only if the corresponding principal bundles–and hence the respective duals–are isomorphic. By Theorem 3 of [26], however, the isomorphism classes of characteristic principal $G^\wedge$ bundles are in one-to-one correspondence with $\operatorname{Ext}(G, \mathcal{H}^1(B))$ and the result follows.

If B is connected, then $\alpha_\sim$ is always constant and the above theorem

classifies all the possible central twisted group algebras $L^1(C(B), G; \alpha)$ with G commutative. Otherwise the algebras are decomposed into ideals as noted previously. We then have:

COROLLARY 9. *Let $\mathfrak{Q}$ denote all partitions of B into open and closed subsets: $\{B_1, B_2, \ldots B_{n(Q)}\} = Q \in \mathfrak{Q}$. The isomorphism classes of all central twisted group algebras $L^1(C(B), G; \alpha)$ with $H^2(G)$ discrete and G abelian are in one-to-one correspondence with the set*

$$\prod_{Q \in \mathfrak{Q}} \prod_{B_i \in Q} \{H^2(G) \times \operatorname{Ext}(G, \mathfrak{H}^1(B_i))\}.$$

PROOF. Any α defines a partition Q and a corresponding decomposition of $L^1(C(B), G; \alpha)$ into ideals of the form $L^1(C(B_i), G; \alpha_i)$ where $(\alpha_i)_\sim$ is constant. Contrariwise, given $B_i \in Q \in \mathfrak{Q}$ and any element of $H^2(G)$ there is, for each element of $\operatorname{Ext}(G, \mathfrak{H}^1(B_i))$, a cocycle α_i taking values in $C(B_i)$ such that $(\alpha_i)_\sim$ is the given member of $H^2(G)$ and thus defining a distinct algebra $L^1(C(B_i), G; \alpha_i)$. The direct sum of these is then the desired algebra.

This theorem, in conjunction with results of [2], enables us to construct all the central twisted group algebras on a compact commutative group taking values in a C^*-algebra $C(X)$, for X triangulable, just as the corresponding result in [26] enables us to construct all the commutative twisted group algebras in such a case.

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