ISOTOPY GROUPS

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Abstract. For any mapping \( f: V \to M \) (not necessarily an embedding), where \( V \) and \( M \) are differentiable manifolds without boundary of dimensions \( k \) and \( n \), respectively, \( V \) compact, let \( [V \subset M]_f = \pi_1(M^V, E, f) \), i.e., the set of isotopy classes of embeddings with a specific homotopy to \( f \) \((E = \text{space of embeddings})\). The purpose of this paper is to enumerate \( [V \subset M]_f \). For example, if \( k > 3 \), \( n = 2k \), and \( M \) is simply connected, \( [S^k \subset M]_f \) corresponds to \( \pi_2M \) or \( \pi_2M \otimes \mathbb{Z}_2 \), depending on whether \( k \) is odd or even. In the metastable range, i.e., \( 3(k+1) > 2n \), a natural Abelian affine structure on \( [V \subset M]_f \) is defined: if, further, \( f \) is an embedding \( [V \subset M]_f \) is then an Abelian group. The set of isotopy classes of embeddings homotopic to \( f \) is the set of orbits of the obvious left action of \( \pi_1(M^V, f) \) on \( [V \subset M]_f \).

A spectral sequence is constructed converging to a theory \( H^*(f) \). If \( 3(k+1) < 2n \), \( H^0(f) = [V \subset M]_f \) provided the latter is nonempty. A single obstruction \( \Gamma(f) \in H^1(f) \) is also defined, which must be zero if \( f \) is homotopic to an embedding; this condition is also sufficient if \( 3(k+1) < 2n \). The \( E_2 \) terms are cohomology groups of the reduced deleted product of \( V \) with coefficients in sheaves which are not even locally trivial. \( [S^k \subset M]_f \) is specifically computed in terms of generators and relations if \( n = 2k \), \( k > 3 \) (Theorem 6.0.2).

1. Introduction. In this paper, in some respects a sequel to [7], we attack the general problem of classifying, up to isotopy, embeddings of a compact \( k \)-manifold \( V \) in an \( n \)-manifold \( M \) in the metastable range, i.e., where \( 3(k+1) < 2n \).

Differentiable shall mean infinitely differentiable, manifold shall mean differentiable manifold without boundary (either compact or open), with a countable base; embedding shall mean differentiable embedding, and isotopy shall mean homotopy of embeddings.

Let \( [V \subset M] \) denote the set of isotopy classes of embeddings of \( V \) in \( M \). Computation of \( [V \subset M] \) is the ultimate goal. Unfortunately, \( [V \subset M] \) has no convenient algebraic structure. Thus we introduce a new object, \( [V \subset M]_f \), where \( f: V \to M \) is a specific map. A homotopy \( f_t: V \to M \) is a specific map. A homotopy \( f_t: V \to M \), \( 0 \leq t \leq 1 \), is
called an embedding-homotopy (or \(\epsilon\)-homotopy) of \(f\) if \(f_0 = f\) and \(f_1\) is an embedding. If \(f_{u,t} : V \to M\), for \(0 \leq t, u \leq 1\) is a two-parameter homotopy such that, for all \(u\), \(f_{u,0} = f\) and \(f_{u,1}\) is an embedding, we say that the \(\epsilon\)-homotopies \(\{f_{0,t}\}\) and \(\{f_{1,t}\}\) are isotopic. \([V \subset M]_f\) is then defined to be the set of isotopy classes of \(\epsilon\)-homotopies of \(f\). If \(\{f_j\}\) is an \(\epsilon\)-homotopy of \(f\), let \([f_j] \in [V \subset M]_f\) be its isotopy class. It shall be shown that \([V \subset M]_f\) is an Abelian affine group in the metastable range, and an actual Abelian group if \(f\) is an embedding.

If \(2k - n < 0\), the group \([Sk \subset M^n]_f\) is, in this paper, expressed in terms of generators and relations, involving only \(\pi_1 M\) if \(n = 2k + 1\), and involving both \(\pi_1 M\) and \(\pi_2 M\) if \(n = 2k\). For \(n > 2k + 1\), \([V^k \subset M^n]_f\) = 0.

Generally (although not shown in this paper) the affine structure of \([V^k \subset M^n]_f\) in the metastable range depends only on the homotopy of \(M\) through dimension \(2k - n + 2\), as well as on \(V\).

The case \(n = 2k + 1\) was done in [7], but the result was incorrectly stated. See Theorem 6.0.1 for the correct version. (Another error in [7], an invalid proof of Theorems 3.3.1 and 3.3.2, is corrected here in §8. The error was pointed out by the referee of this paper.) The case \(n = 2k\) is computed for the first time in this paper, and involves evaluation of one nonzero differential and one nontrivial extension in a spectral sequence whose \(E_2\) terms are cohomology groups of the reduced deleted product of \(S^k\) with coefficients in sheaves which do not, in general, have a local product structure. See Theorem 6.0.2 for the general result. Some specific cases are as follows (where \(f\) is any embedding):

**Theorem 1.0.1.** If \(M = M^{2k}\) is simply connected, \(k \geq 3\), then:

\[
[S^k \subset M]_f = \begin{cases} 
\pi_2 M \otimes Z_2 & \text{if } k \text{ is even}, \\
\pi_2 M & \text{if } k \text{ is odd}.
\end{cases}
\]

**Theorem 1.0.2.** If \(k \geq 3\):

\[
[S^k \subset P^2 \times R^{2k-2}]_f = \begin{cases} 
Z_2 + Z_2 & \text{if } k \text{ is even}, \\
Z + Z + Z_2 & \text{if } k \text{ is odd}.
\end{cases}
\]

**Theorem 1.0.3.** If \(k \geq 3\):

\[
[S^k \subset P^{2k}]_f = \begin{cases} 
0 & \text{if } k \equiv 0 \mod 4, \\
Z_4 & \text{if } k \equiv 1 \mod 4, \\
Z_2 & \text{if } k \equiv 2 \mod 4, \\
Z_2 + Z_2 & \text{if } k \equiv 3 \mod 4.
\end{cases}
\]

Let \(\pi = \pi_1(M^V, f)\). We may define a left action:

\[\mu : \pi \times [V \subset M]_f \to [V \subset M]_f\]
as follows: if \( \{f_t\} \) is an \( e \)-homotopy of \( f \) and if \( \{f'_t\} \) is a self-homotopy of \( f \) representing \( a \in \pi \), then \( \mu(a, [f_t]) = [f'_t] \) where \( f''_t = f_{2t} \) or \( f''_{2t-1} \), depending on the value of \( t \). The action \( \mu \) respects the affine structure of \( [V \subset M]_f \), i.e., for each \( a \in \pi \), \( \mu(a, \cdot) \) is an affine automorphism (cf. Theorem 2.7.1).

Now let \( \phi: [V \subset M]_f \rightarrow [V \subset M] \) be the forgetful function, i.e., if \( \{f_t\} \) is an \( e \)-homotopy of \( f \), \( \phi(f_t) = [f_t] \), the isotopy class containing \( f_t \). Trivially, the reader can convince himself of the following:

(i) Every element of \( [V \subset M] \) lies in the image of \( \phi \) for some choice of \( f \).
(ii) If \( x, y \in [V \subset M]_f \), then \( \phi x = \phi y \) if and only if \( \mu(a, x) = y \) for some \( a \in \pi \).

Classification, up to isotopy, of embeddings of \( V \) in \( M \) in the stable range may thus be reduced to the following two problems:

I. Compute the affine structure of \( [V \subset M]_f \), for each \( f: V \rightarrow M \).

II. Determine the action \( \mu \) of \( \pi_1(M^V, f) \) on \( [V \subset M]_f \).

It is to the first of these problems that this paper is addressed.

In §2, we use Haefliger's results [4] to define the affine structure on \( [V \subset M]_f \). An alternative, equivalent definition of the affine structure is given in §4; this definition is much more geometric and easier to comprehend, but it is the Haefliger definition that yields to computation, by homotopy methods. In §3, a spectral sequence is defined, in §5, the sheaves we need are computed, and in §6 the spectral sequence is fully worked in the special cases.

Using somewhat different methods, Dax [2] and Salomonsen [12] have obtained results on the same and similar problems.

2. The affine structure of \( [V \subset M]_f \).

2.1. Pair fibrations and weak pair fibrations. Following [11], we say that a map \( p: (E, E_0) \rightarrow (X, X_0) \) is a pair fibration if for any given map \( h: (Y, Y_0) \rightarrow (E, E_0) \) and every homotopy \( f: (Y, Y_0) \rightarrow (X, X_0) \), \( 0 < t < 1 \), such that \( f_0 = ph \), there exists a homotopy \( h': (Y, Y_0) \rightarrow (E, E_0) \) such that \( h_0 = h \) and \( ph' = f \) for all \( t \).

A section of a pair fibration is a map \( g: (X, X_0) \rightarrow (E, E_0) \) such that \( pg = 1 \) (the identity). A lifting of \( f: (Y, Y_0) \rightarrow (X, X_0) \) to \( p \) is a map \( h: (Y, Y_0) \rightarrow (E, E_0) \) such that \( ph = f \). Sections and liftings are said to be homotopic if they are homotopic as sections or liftings (i.e., fiber homotopic).

All the theorems in §§2 and 3 which deal with sections of pair fibrations are equivalent to corresponding theorems which deal with liftings, since a lifting of \( f \) to \( p \) corresponds to a section of \( f^{-1}p \) (where \( f^{-1} \) is the usual pullback construction).

If \( (X, X_0) \) is a topological pair, we say that \( (A, A_0) \) is a sub-pair of \( (X, X_0) \) if \( A \subset X \) and \( A_0 = A \cap X_0 \). Then \( ((X, X_0), (A, A_0)) \) is a pair of pairs. If \( X \) is a C.W. complex, and \( X_0 \) and \( A \) are subcomplexes, it is a C.W. pair of pairs.

For the following definitions, let (as needed) \( f: ((Y, Y_0), (B, B_0)) \rightarrow \)
((X, X₀), (A, A₀)) be a map of pairs of pairs, let \( z: (A, A₀) \to (E, E₀) \) be a partial section of \( p \), let \( w: (B, B₀) \to (E, E₀) \) be a partial lifting of \( f \) to \( p \).

**Definition 2.1.1.** Let \([(X, X₀); p]\) be the set of homotopy classes of sections of \( p \).

**Definition 2.1.2.** Let \([(Y, Y₀); p]\) be the set of homotopy classes of liftings of \( f \) to \( p \).

**Definition 2.1.3.** Let \([(X, X₀), (A, A₀); p]\) be the set of rel \((A, A₀)\) homotopy classes of sections of \( p \) which extend \( z \). If \( p \) has a standard section, say \( s: (X, X₀) \to (E, E₀) \), then we shall always presume (unless otherwise stated) that \( z = s|\{(A, A₀)\} \) and suppress \( z \) in the notation.

**Definition 2.1.4.** Let \([(Y, Y₀), (B, B₀); p]\) be the set of rel \((B, B₀)\) homotopy classes of liftings of \( f \) to \( p \) which extend \( w \). If \( p \) has a standard section, \( s \), presume that \( w = sf|\{(B, B₀)\} \) and suppress \( w \) in the notation.

The following lemma will be important later.

**Lemma 2.1.5.** If \(((X, X₀), (A, A₀))\) is a C.W. pair of pairs, \( p \) is a pair fibration over \((X, X₀)\), \( \{z,\} \) is a homotopy of partial sections of \( p \) over \((A, A₀)\), there is a one-to-one correspondence \([(X, X₀), (A, A₀); p]\) such that, whenever \( \{g,\} \) is a homotopy of sections of \( p \) which extends \( \{z,\} \), \([g₀]\) (the homotopy class containing \( g₀ \)) corresponds to \([g]\).

**Proof.** By induction on the skeleta of \( X \). We omit the details.

2.2. Fiberwise suspensions. Choose standard spheres and balls \( S^{N-1} \subset B^N \) for all \( N > 0 \) (where \( S^{-1} \) is empty) such that \( S^0 \ast S^{N-1} = S^N \) and \( S^0 \ast B^N = B^{N+1} \) (where \( \ast \) = join). Let each sphere have a South pole and a North pole, preserved under inclusion \( S^N \subset S^{N+1} \). The South pole will be considered to be the base point of each sphere.

If \( p: E \to X \) is a fibration, let \( S^N p: S^N E \to X \), where \( S^N E = S^N \ast E \), the fiberwise join of \( p: E \to X \) with the trivial fibration \( X \times S^N \to X \). We give \( S^N \ast E \) the strong topology, thus insuring that \( S^N p \) is a fibration [5]. Note that \( S^0 p = p \), and that for \( N > 1 \), \( S^N p \) has two standard sections, the South polar and the North polar, denoted \( s₀ \) and \( s₁ \). If only one section is needed, \( s₀ \) shall be used. Note also that \( S p = S^1 p \) is the fiberwise two point suspension of \( p \).

Now, for any \( N > 0 \), let \( S^N p: P^N p \to X \) be defined as follows. For each \( x \in X \), let \((P^NS^N p)^{-1}\{x\}\) be the set of all maps \( s: B^N \to (S^N p)^{-1}\{x\}\) such that \( s|S^{N-1} \) is simply the identity (recall that each fiber of \( S^N p \) contains a copy of \( S^{N-1} \)). Let \( P^N S^N E \) then have the topology of a subspace of \( (S^N E)^θ_x \) with the compact-open topology. We can define a fiber-preserving inclusion \( P^N S^N E \subset P^{N+1} S^{N+1} E \) by identifying each \( s \in P^N S^N E \) with \( S^0 \ast s \).

Now if \( M > 0, N > 1 \), let \( Ω^M S^N p: Ω^M S^N E \to X \) be given by
(Ω^{M}S^{N}p)^{-1}(x) = Ω^{M}(S^{N}p)^{-1}(x), the M-fold loop space, which we take to be the set of maps (B^{M}, S^{M-1}) → ((S^{N}p)^{-1}(x), s_{0}x), for any x ∈ X. We use the compact open topology for Ω^{M}S^{M}E, and again there is an inclusion Ω^{M}S^{M}E ⊂ Ω^{M+1}S^{N+1}E.

The above constructions generalize naturally to pair fibrations. If p: (E, E_{0}) → (X, X_{0}) is a pair fibration, we define:

\[ S^{N}p: (S^{N}_{X}E, S^{N}_{X_{0}}E_{0}) → (X, X_{0}), \]
\[ P^{N}S^{N}p: (P^{N}_{X}S^{N}_{X}E, P^{N}_{X}S^{N}_{X_{0}}E_{0}) → (X, X_{0}), \]
\[ Ω^{M}S^{N}p: (Ω^{M}_{X}S^{N}_{X}E, Ω^{M}_{X}S^{N}_{X_{0}}E_{0}) → (X, X_{0}). \]

2.3 Groups and affine groups of sections and liftings. Henceforth in this section, let ((X, X_{0}), (A, A_{0})) be a C.W. pair of pairs, and p: (E, E_{0}) → (X, X_{0}) a pair fibration.

**Theorem 2.3.1.** If N > 1, M > 1, then \([X, X_{0}, (A, A_{0}); Ω^{M}S^{N}p]\) is a group. Furthermore, it is Abelian if M ≥ 2.

We omit the proof of 2.3.1: simply follow, fiberwise, the usual proof that \([X, A; Ω^{M}Y]\) is a group, Abelian if M ≥ 2, if Y is any pointed space.

**The affine structure.** Let N > 1, and fix a partial section of \(P^{N}S^{N}p\) over \((A, A_{0})\). We proceed to define a ternary operation, \(τ\), on \([X, X_{0}, (A, A_{0}); P^{N}S^{N}p]\).

Let U and L be the upper and lower hemispheres, respectively, of ∂ B^{N}. Let B_{1}^{N}, B_{2}^{N}, and B_{3}^{N} be copies of B^{N}, with upper and lower hemispheres U_{1}, L_{1}, U_{2}, etc. Now choose a homeomorphism \(φ: B^{N} → W = (B_{1}^{N} ∪ B_{2}^{N} ∪ B_{3}^{N})/∽\), where “∽” identifies U_{1} with U_{2} and L_{2} with L_{3}, and \(φ: L = L_{1}, φ: U = U_{3}\). (Note that \(φ\) has degree 1 in homology onto B_{1}^{N} and B_{3}^{N}, and −1 onto B_{2}^{N}.) Now every section of \(P^{N}S^{N}p\) may be thought of as a map \(X × B^{N} → S^{N}_{X}E\) satisfying the appropriate conditions. If \(g_{1}, g_{2}, \) and \(g_{3}\) are sections of \(P^{N}S^{N}p\), we define a new section of \(P^{N}S^{N}p\) called \(g_{1} ∩ g_{2}^{-1} ∩ g_{3}\), by commutativity of the following diagram for each i = 1, 2, 3:

\[
\begin{array}{ccc}
X × B^{N} & \xrightarrow{φ} & S^{N}_{X}E \\
\downarrow_{i} & & \uparrow_{g_{i}} \\
X × W & ⊂ X × B_{i}^{N} & = X × B^{N}
\end{array}
\]

Now if \(g\) is any section, \(g ∩ g^{-1} ∩ g\) is canonically homotopic to \(g\) itself. (To see this, let \(χ: W → B^{N}\) be the “triple folding” map obtained by identifying each \(B_{i}^{N}\) with \(B^{N}\). Now choose a rel \(S^{N-1}\) homotopy of \(χ ∘ φ: B^{N} → B^{N}\) with the identity; this gives a homotopy of \(g\) with \(g ∩ g^{-1} ∩ g = 1_{X} × (χ ∘ φ)\). This homotopy is functorial with respect to maps of pair fibrations.) Thus, since \(z ∩ z^{-1} ∩ z\) is homotopic to \(z\), we obtain, by Lemma
2.1.5, a one-to-one correspondence

\[ \xi: [(X, X_0), (A, A_0); P^{N S^N p}]^{z \wedge z^{-1}} \simeq [(X, X_0), (A, A_0); P^{N S^N p}]^{z}. \]

Now define \( \tau([g_1], [g_2], [g_3]) = [g_1][g_2]^{-1}[g_3] = \xi(g_1 \wedge g_2^{-1} \wedge g_3). \)

**Theorem 2.3.2.** (I) Under \( \tau, [(X, X_0), (A, A_0); P^{N S^N p}]^{z} \) is an affine group. If \( N > 2 \), it is an Abelian affine group. (II) If \( [(X, X_0), (A, A_0); P^{N S^N p}]^{z} \) is nonempty, its left action group is \( [(X, X_0), (A, A_0); P^{N S^N p}]^{z} \).

**Proof.** It is a routine exercise to show that \( \tau \) is associative and satisfies cancellation, and thus is an affine structure. To obtain (II), consider the map \( \eta: B^N \to Y = (B_1^N \cup B_2^N)/\sim = S^N \) where \( \sim \) identifies \( S_1^{N-1} \) with \( S_2^{N-1} \), and \( \eta \) maps all of \( S^{N-1} \) onto the common South pole, and is of degree 1 onto \( B_1^N \), degree \(-1\) onto \( B_2^N \). Then let \( [g_1][g_2]^{-1} = [g_1 \wedge g_2^{-1}], \) where, for \( i = 1 \) or 2, the following diagram commutes:

\[
\begin{array}{ccc}
X \times B^N & \xrightarrow{g_1 \wedge g_2^{-1}} & S^N \\
\downarrow \mathrm{id} \times \eta & & \uparrow \\
X \times Y & \supset X \times B_i^N & = X \times B^N
\end{array}
\]

Clearly \( g_1 \wedge g_2^{-1} \) is a section of \( \Omega^{N S^N p} \), and verification of the remaining details is routine. (Hint: \( z \wedge z^{-1} \) is canonically homotopic to a trivial section.)

The commutativity of \( \tau \) in the case \( N > 2 \) now follows from 2.3.1, and we are done with the proof of 2.3.2.

In certain cases which turn out to be important later there is a geometric definition of the affine structure. Suppose that \( U_1 \) and \( U_2 \) are open subsets of \( X \) such that \( U_1 \cup U_2 = X \), and suppose that \( z \) is a partial section of \( P^{N S^N p} \) over \( (A, A_0) \).

**Proposition 2.3.3.** If \( g_1 \) and \( g_2 \) are sections of \( \Omega^{M S^N p} \), for \( M > 1, N > 1 \), such that \( g_1 \) and \( g_2 \) are both trivial on \( (A, A_0) \), and such that \( g_1|U_2 \) is trivial and \( g_2|U_1 \) is trivial, let \( g_3 \) be the section which agrees with \( g_1 \) on \( U_1 \), with \( g_2 \) on \( U_2 \), and is trivial elsewhere. Then \( [g_1][g_2] = [g_3] \).

**Proof.** It is clear that \( g_3 \) is homotopic to \( g_1 \wedge g_2 \) on \( X - U_1 \) and also on \( X - U_2 \); they are both trivial on \( U_1 \cap U_2 \). Homotopies on the two closed sets may easily be chosen, and do not interfere.

**Proposition 2.3.4.** If \( g_1, g_2, \) and \( g_3 \) are sections of \( P^{N S^N p} \), for \( N > 1 \), such that all agree with \( z \) on \( (A, A_0) \), and such that \( g_1 \) agrees with \( g_2 \) outside of \( U_2 \), and \( g_3 \) agrees with \( g_2 \) outside of \( U_3 \), let \( g_4 \) be the section where, for all \( x \in X \),
h_4(x) = \begin{cases} 
    h_1(x) & \text{if } x \not\in U_2, \\
    h_2(x) & \text{if } x \in U_1 \cap U_2, \\
    h_3(x) & \text{if } x \not\in U_1.
\end{cases}

Then \([h_1][h_2]^{-1}[h_3] = [h_4]\).

**Proof.** Note that \(h_1 \wedge h_2^{-1} \wedge h_3\) and \(h_4\) are homotopic, albeit via different standard homotopies, on each of the two sets \(U_1\) and \(U_2\). By choosing a Urysohn function \(X \to I\) which is 0 on the complement of \(U_2\) and 1 on the complement of \(U_1\), these homotopies can be made to blend smoothly into one another over \(U_1 \cap U_2\), since on that region \(h_1, h_2, h_3,\) and \(h_4\) all agree.

2.4. Equivalence of classes of sections. Let \(((X, X_0), (A, A_0))\) be a C.W. pair of pairs, and let \(p: (E, E_0) \to (X, X_0)\) and \(p': (E', E'_0) \to (X, X_0)\) be pair fibrations. Let \(\gamma: (E, E_0) \to (E', E'_0)\) be a fiber map, and \(z: (A, A_0) \to (E, E_0)\) a partial section of \(p\). Finally, define \(z' = \gamma \cdot z\), a partial section of \(p'\) over \((A, A_0)\). We have an obvious function, induced by composition with \(\gamma\):

\([X, X_0], (A, A_0); p] \to [X, X_0], (A, A_0); p']^{z'}\).

Under certain conditions on the homotopy of the fibers, \(\gamma\) is one-to-one or onto.

Let \(E_x = p^{-1}(x)\), for any \(x \in X\), and let \((E_0)_x = E_x \cap E_0\). Similarly, let \(E'_x = (p')^{-1}(x)\) and \((E'_0)_x = E'_x \cap E'_0\).

**Theorem 2.4.1.** Suppose that \(X / A\) is finite dimensional. Let \(n = \dim(X / (X_0 \cup A))\), and \(n_0 = \dim(X_0 / A_0)\). (I) Suppose that, for all \(x \in (X - (X_0 \cup A))\), \(\gamma_*: \pi_k(E_x) \to \pi_k(E'_x)\) is one-to-one and onto for all \(0 < k < n\) and onto for \(k = n\); and that, for all \(x \in (X_0 - A)\), \(\gamma_*: \pi_k(E_0)_x \to \pi_k(E'_0)_x\) is one-to-one and onto for all \(0 < k < n_0\), and onto for \(k = n_0\). Then \(\gamma_*\) is onto. (II) Suppose that, for all \(x \in (X - (X_0 \cup A))\), \(\gamma_*: \pi_k(E_x) \to \pi_k(E'_x)\) is one-to-one and onto for all \(0 < k < n\) and onto for \(k = n + 1\); and that \(\gamma_*: \pi_k(E_0)_x \to \pi_k(E'_0)_x\) is one-to-one and onto for all \(0 < k < n_0\) and onto for \(k = n_0 + 1\) for all \(x \in (X_0 - A)\). Then \(\gamma_*\) is one-to-one.

**Proof.** This theorem is simply a pair-fibration version of a well-known result in fibrations. (See, for example, Lemma 2.2 of [8].) The proof is unaltered by the pair nature, since it is done inductively one cell at a time.

**Definition 2.4.2.** We say that \(p\) is \(n\)-connected if, for all \(k < n\), \(\pi_k(E)_x = 0\) for all \(x \in X\) and \(\pi_k(E_0)_x = 0\) for all \(x \in X_0\).

**Corollary 2.4.3.** If \(p\) is \(n\)-connected and \(\dim(X / A) < 2n\), then \([X, X_0], (A, A_0); p]\) is an Abelian affine group.

**Proof.** The inclusion \(i: p \subset P^NS^n p\) induces isomorphism on the homotopy of the fibers up through dimension \(2n\), and epimorphism in dimension \(2n + 1\).
(since each fiber of $P^NS^np$ is of the same homotopy type as the $N$-fold loop space of the $N$-fold suspension of the corresponding fiber of $p$). Thus, by 2.4.1, $\{(X, X_0), (A, A_0); p^n\} \simeq \{(X, X_0), (A, A_0); P^NS^np^n\}$ for all $N$. Apply 2.3.2, and we are done.

We remark that the Abelian affine structure on $\{(X, X_0), (A, A_0); p^n\}$, under the hypotheses of 2.4.3, can be alternatively defined in manner described in 2.3.4. Verification is trivial.

2.5. The obstruction theory. Let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, $p: (E, E_0) \to (X, X_0)$ a pair fibration, and $z$ a partial section of $p$ over $(A, A_0)$.

**Definition 2.5.1.** For any integer $i$, let

$$H^i((X, X_0), (A, A_0); p) = \lim_{N \to \infty} \left[ (X, X_0), (A, A_0); \Omega^{N-1}S^p \right],$$

the direct limit. Write $H^i((X, X_0); p)$ if $A$ is empty.

We remark in passing that $H^*( ; p)$ is a cohomology theory in a certain sense; and satisfies a version of the Eilenberg-Steenrod axioms. In particular, if $f: ((Y, Y_0), (B, B_0)) \to ((X, X_0), (A, A_0))$ is a map of C.W. pairs of pairs, there is an induced homomorphism

$$f^*: H^i((X, X_0), (A, A_0); p) \to H^i((Y, Y_0), (B, B_0); f^{-1}p),$$

and if $q: (F, F_0) \to (X, X_0)$ is another pair fibration and $\gamma: (E, E_0) \to (F, F_0)$ a fiber preserving map, there is an induced homomorphism

$$\gamma_*: H^i((X, X_0), (A, A_0); p) \to H^i((X, X_0), (A, A_0); q).$$

**Remark 2.5.2.** If $p$ is $n$-connected and $\dim(X/A) < 2n$, then $\{(X, X_0), (A, A_0); p^n\}$ (an Abelian affine group, by 2.4.3), if nonempty, has $H^0((X, X_0), (A, A_0); p)$ as its action group.

Thus, classification of sections of a pair fibration, under suitable dimensional restrictions, reduces to algebraic computation of a cohomology group. In the next section, we shall show how $H^*((X, X_0), (A, A_0); p)$ can be attacked by a familiar spectral sequence technique.

The single obstruction. We now consider the question of whether $z$ can be extended to a section of $p$. Recall that $Sp$ has two sections, $s_0$ and $s_1$, the South and North polar sections. Now $S_{x}E$ may be obtained from $E \times I$ by collapsing the ends in the appropriate manner (although with the strong, not the quotient topology). Let $u_x, x = [zx, t] \in S_{x}E$ for all $x \in A$. By 2.1.5, $\{u_x\}$ gives a one-to-one correspondence:

$$\sigma: \left[ (X, X_0), (A, A_0); Sp \right] \simeq \left[ (X, X_0), (A, A_0); Sp \right].$$

**Definition 2.5.3.** Let $\gamma(p) = \sigma(s_1) \in [(X, X_0), (A, A_0); Sp]$, the primitive single obstruction to section of $p$ extending $z$.

**Definition 2.5.4.** Let $\Gamma(p) \in H^1((X, X_0), (A, A_0); p)$ be the image of $\gamma(p)$ in the direct limit, the single obstruction to section of $p$ extending $z$. 


**Theorem 2.5.5.** (I) There exists a section of $p$ extending $z \Rightarrow \gamma(p) = 0 \Rightarrow \Gamma(p) = 0$. (II) Suppose that $p$ is $n$-connected and $\dim(X/A) < 2n + 1$. Then $\Gamma(p) = 0 \Rightarrow \gamma(p) = 0 \Rightarrow$ there exists a section of $p$ extending $z$.

**Proof.** (I) If $g$ is an extension of $z$, the homotopy $\{u_t\}$ can obviously be extended to a homotopy of $s_0$ with $s_1$, hence $\sigma(s_1) = [s_0] = 0$. The second implication is obvious. (II) If $\Gamma(p) = 0$, then $\gamma(p) = 0$ by 2.4.1. Thus, there is a homotopy between $s_0$ and $s_1$, extending $\{u_t\}$, hence a section of $PSp$ extending $iz$. By 2.4.1, $i_*: [(X, X_0), (A, A_0); p]^p \to [(X, X_0), (A, A_0); PSp]^p$ is onto, and we are done.

The single difference. Suppose now that $\{z_t\}$ is a homotopy of partial sections of $p$ over $(A, A_0)$. Suppose that $g_0$ and $g_1$ are sections of $p$ extending $z_0$ and $z_1$, respectively. Recalling the construction in the proof of 2.3.2, we have $(ig_1) \land (ig_0)^{-1}$, a section of $\Omega Pp$. Now that section extends $(iz_1) \land (iz_0)^{-1}$ which is homotopic (by continuously varying the first index from 1 to 0) to $(iz_0) \land (iz_0)^{-1}$, which in turn is canonically null-homotopic. Thus, using 2.1.5, we can define:

**Definition 2.5.6.** Let $\delta(g_0, g_1; \{z_t\}) \in [(X, X_0), (A, A_0); \Omega Pp]$, the primitive single difference be the element represented by $(ig_1) \land (ig_0)^{-1}$. If $z_t = z$ for all $t$, write $\delta(g_0, g_1; z)$.

**Definition 2.5.7.** Let $\Delta(g_0, g_1; \{z_t\}) \in H^0([(X, X_0), (A, A_0); p])$, the single difference class be the image of $\delta(g_0, g_1; \{z_t\})$ in the direct limit. Write $\Delta(g_0, g_1; z)$ if $z_t = z$ for all $t$.

**Theorem 2.5.8** (I) The homotopy $\{z_t\}$ can be extended to a homotopy of $g_0$ with $g_1 \Rightarrow \delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow \Delta(g_0, g_1; \{z_t\}) = 0$. (II) Suppose that $p$ is $n$-connected for some $n$, and $\dim(X/A) < 2n + 1$. Then $\Delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow \delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow$ there exists a homotopy of $g_0$ with $g_1$ which extends $\{z_t\}$.

**Proof.** Similar to that of 2.5.5.

2.6. Obstructions to embedding and isotopy. Much of the following material is from [7]. If $M$ is any manifold, of dimension $n$, let $PM$ be the projective bundle associated with the tangent bundle of $M$, let $R^*_M = (M^2 - DM)/T$, where $T(x, y) = (y, x)$, and let $R^*M = R^*_M \cup PM$, a manifold with boundary $PM$. Let $(\pm R^*_M, ^nPM) = (\pm R(M \times R^N), P(M \times R^N))$, and let $(\pm R^*_M, ^\infty PM)$ be the obvious union with the weak topology. The inclusion $(\pm R^*_M, PM) \subset (\pm R^*_M, ^\infty PM)$ we replace by a fibration of pairs in a standard manner: Let $YM$ be the space of all paths $\sigma: I \to \pm R^*_M$ such that $\sigma(1) \in R^*_M$ and let $Z_M$ be the space of all paths $\sigma: I \to ^\infty PM$ such that $\sigma(1) \in PM$, and let $\xi_M: (Y_M, Z_M) \to (\pm R^*_M, ^\infty PM)$ be evaluation at 0, an $(n - 2)$-connected pair fibration.

Now let $V$ be a compact manifold of dimension $k$, and let $f: V \to M$ be a differentiable map. Choose, once and for all, an embedding $i: V \subset R^\infty$. Let
$F = \mathbb{R}^*(f, i) \colon (\mathbb{R}^* V, PV) \to (\mathbb{R}^* M, \mathbb{R}^* PM)$.

We now have a diagram (basically diagram (3.2-1) of [7], with slightly changed notation):

$$
\begin{array}{c}
\begin{array}{c}
Y_M, Z_M
\end{array}
\end{array}
\downarrow_{\mathbb{I}_M}
\begin{array}{c}
\begin{array}{c}
(\mathbb{R}^* V, PV)
\end{array}
\end{array}
\xrightarrow{F}
\begin{array}{c}
\begin{array}{c}
(\mathbb{R}^* M, \mathbb{R}^* PM)
\end{array}
\end{array}
\end{array}

(2.6-1)

Now define a function $\phi \colon [V \subset M] \to [(\mathbb{R}^* V, PV); \mathbb{I}_M]_F$ as follows. If $\{f_i\}$ is an embedding homotopy of $f$, i.e., $f_0 = f$ and $f_1$ is an embedding, let $\Phi(f_i) : (\mathbb{R}^* V, PV) \to (Y_M, Z_M)$ be defined as follows. For any $0 < u < 1$ and any $r \in \mathbb{R}^* V$, $\Phi(f_i)(r)(u) = \mathbb{R}^*(f_u, (1-u)i)(r)$. Simply then let $\phi(f_i)$ be the homotopy class containing $\Phi(f_i)$. It follows directly\footnote{Theorems 3.3.1 and 3.3.2 of [7] are correct as stated in that paper, but the proofs are invalid, as has been kindly pointed out by the referee of this paper. A correction is contained here, as an appendix, §8.} from Theorems 3.3.1 and 3.3.2 of [7] that:

**Theorem 2.6.1.** (I) If $2n > 3(k + 1)$, $\phi$ is onto. (II) If $2n > 3(k + 1)$, $\phi$ is one-to-one.

**Definition 2.6.2.** For any integer $i$, let $H^i(f) = H^i((\mathbb{R}^* V, PV); \mathbb{I}_M)$.

**Definition 2.6.3.** Let $\Gamma(f) = \Gamma(F) \in H^1(f)$, the single obstruction to homotopy of $f$ with an embedding.

**Definition 2.6.4.** Suppose that $\{f_1^1\}$ and $\{f_2^2\}$ are both $e$-homotopies of $f$. Then let $\Delta(\{f_1^1\}, \{f_2^2\}) = \Delta(\Phi(f_1^1), \Phi(f_2^2)) \in H^0(f)$.

Now, from 2.5.2, 2.5.5, and 2.5.8, we immediately have:

**Theorem 2.6.5.** If $2n > 3(k + 1)$, $[V \subset M]_f$ is an Abelian affine group, and, if nonempty, it has action group $H^0(f)$.

**Theorem 2.6.6.** If $f$ is homotopic to an embedding, $\Gamma(f) = 0$. The converse holds if $2n \geq 3(k + 1)$.

**Theorem 2.6.7.** Suppose $\{f_1^1\}$ and $\{f_2^2\}$ are embedding homotopies of $f$. If they are isotopic, $\Delta(\Phi(f_1^1), \Phi(f_2^2)) = 0$. The converse holds if $2n > 3(k + 1)$.

Finally, the following remarks will show how $[V \subset M]_f$, up to isomorphism, depends only on the homotopy class of $f$. Let $f_t : V \to M$, $0 < t < 1$, be any homotopy, where $f_0$ and $f_1$ are differentiable. Let $\{f_t\}^* : [V \subset M]_{f_t} \to [V \subset M]_{f_0}$ be the function, where, if $\{g_t\}$ is an $e$-homotopy of $f_t$, $\{f_t\}^*[g_t] = [h_t]$, where $h_t = f_{2t}$ or $g_{2t-1}$, depending on the value of $t$. Clearly $\{f_t\}^*$ is one-to-one and onto, since $\{f_{1-t}\}^*$ is its two-sided inverse.

**Theorem 2.7.1.** If $2n > 3(k + 1)$, $\{f_t\}^*$ is an isomorphism of affine groups.

**Proof.** Applying the polyhedral covering homotopy property (pair version)
of \( \xi_M \), one may easily show that, for any \( 0 < u < 1 \), \((i_0)^*: \left[ (\mathbb{R}^* V \times I, PV \times I); \xi_M \right]_{FP} \to \left[ (\mathbb{R}^* V, PV); \xi_M \right]_F \) is an isomorphism, where \( i_u(r) = (r, u) \) for all \( r \in \mathbb{R}^* V \) and \( p(r, u) = r \). Very simply, checking definitions, one can see that \( \{f_i\}^* = (i_0)^* \left( (i_i)^* \right)^{-1} \), and we are done.

In a similar manner, we shall write \( \{f_i\}^*: H^i(f_1) \cong H^i(f_0) \) for any integer \( i \). We leave the details to the reader.

3. A spectral sequence. In this section, let \( ((X, X_0), (A, A_0)) \) be a fixed C.W. pair of pairs, and let \( \xi: (E, E_0) \to (X, X_0) \) be a pair fibration. We consider the problem of enumeration of sections of \( \xi \) from a spectral sequence viewpoint.

All pair fibrations shall be over \( (X, X_0) \).

3.1. Homotopy sheaves. Let \( \xi \) have a section \( s \). We define \( \pi_k(\xi, s) \), the \( k \)th homotopy sheaf of \( \xi \), to be the sheaf over \( X \) determined by a presheaf \( \Pi \), where, if \( U \subset X \) is open and \( U_0 = U \cap X_0 \), \( \Pi(U) = [(U, U_0); \Omega(S)] \). Hence, for any \( x \in X \),

\[
\pi_k(\xi, s) = \begin{cases} 
\pi_k(E_x, sx) & \text{if } x \in X - X_0, \\
\pi_k((E_0)_x, sx) & \text{if } x \in X_0.
\end{cases}
\]

The total space of \( \pi_k(\xi, s) \) need not be Hausdorff.

If each \( E_x \) and each \( (E_0)_x \) is \( k \)-simple, \( \pi_k(\xi) \) can be defined regardless of choice (or even existence) of a section. We leave the details to the reader. (Hint: it is sufficient to define \( \Pi(U) \) for \( U \) contractible.)

Note that the concept of homotopy sheaf is simply a generalization of the usual local system of coefficients for a fibration.

Without any restrictions on \( \xi \), we can always define \( \pi_k^S(\xi) = \lim_{N \to \infty} \pi_k(\Omega^N S N \xi) \), the \( k \)th stable homotopy sheaf.

Let \( G \) be a sheaf over \( X \). We say that \( \xi \) is an Eilenberg Mac Lane pair fibration of type \( (G, n) \) (we will write \( \xi = k(G, n) \)) if:

(i) \( \xi \) has a section,
(ii) \( \pi_n(\xi) = G, \)
(iii) \( \pi_k(\xi) = 0 \) for all \( k \neq n \).

It is important to note that \( G \) cannot simply be any sheaf of groups, Abelian if \( n > 2 \). Only for \( G \) satisfying special conditions will \( k(G, n) \) exist.

In this context, Theorem 2.5.1 of [7] becomes:

**Theorem 3.1.1.** If \( \xi = k(G, n) \) for some sheaf of Abelian groups \( G \), then \([[(X, X_0), (A, A_0); \xi]] \cong H^n(X, A; G)\).

3.2. The fiber of a pair-fibration map. Let \( \xi': (E', E'_0) \to (X, X_0) \) be another pair fibration. We say that \( \gamma \) is a pair fibration map, and simply write \( \gamma: \xi \to \xi' \), if \( \gamma: (E, E_0) \to (E', E'_0) \) is simply a fiber-preserving map. Suppose that \( \xi \) has a section \( s \). Then \( s' = \gamma s \) is a section of \( \xi' \). We define a pair
fibration \( \phi[\gamma]: (F, F_0) \to (X, X_0) \), called the fiber of \( \gamma \), as follows: for each \( x \in X \), \( F_x \) is the usual homotopy theoretic fiber of \( E_x \to E'_x \), that is, \( F_x = \{(e, o) \mid e \in E_x, \sigma: I \to E'_x, \sigma(0) = \gamma x, \sigma(1) = \gamma e\} \), and \( (F_0)_x \) is the homotopy theoretic fiber of \( (E_0)_x \to (E'_0)_x \). Let \( \lambda: \phi[\gamma] \to \xi \) be given by \( \lambda(e, o) = e \), and let \( \iota: \Omega \xi' \to \phi[\gamma] \) be the obvious inclusion.

Identifying \( \Omega \phi[\gamma] \) with \( \phi[\gamma] \) in the obvious way, we have:

**Lemma 3.2.1.** The following long sequence is exact:

\[
\cdots \to [(X, X_0), (A, A_0); \Omega \xi']^{(\Omega \gamma)} \xrightarrow{\iota_x} [(X, X_0), (A, A_0); \Omega \xi'] \xrightarrow{\lambda_x} [(X, X_0), (A, A_0); \phi[\gamma]] \xrightarrow{\gamma_x} [(X, X_0), (A, A_0); \xi'] \to \cdots
\]

**Proof.** Since the definition of fiber the natural definition for the category of pair-fibrations over \( (X, X_0) \), the details of the proof are routine and obvious.

Similarly, we have:

**Lemma 3.2.2.** The following sequence of homotopy sheaves is exact:

\[
\cdots \to \pi_k(\Omega \xi') = \pi_{k-1}(\xi') \xrightarrow{\iota_x} \pi_k(\phi[\gamma]) \xrightarrow{\lambda_x} \pi_k(\xi') \to \pi_k(\xi') \to \cdots
\]

3.3. The homotopy killing constructions.

The strong topology double mapping cylinder. Let \( A, B \), and \( C \) be topological spaces, and \( \alpha: B \to A \), \( \beta: B \to C \) maps. Let \( A \cup_a (B \times I) \cup_{\beta} C \) be the space obtained from \( A \cup (B \times I) \cup C \) by identifying \( (x, 0) \) with \( \alpha x \) and \( (x, 1) \) with \( \beta x \) for all \( x \in B \); but with the strong topology, not the quotient topology. Neighborhoods of points in \( B \times (0, 1) \) are as usual in the product topology, while if \( a \in A \), a basic neighborhood of \( a \in W \) is of the form \( U \cup [\alpha^{-1}U \times [0, \varepsilon)] \) for \( \varepsilon > 0 \) and \( U \) a neighborhood of \( a \) in \( A \). Neighborhoods along \( C \) are similar. Equivalently, the strong topology is the strongest topology such that all of the following obvious projections are continuous:

\[
A \cup_a (B \times I) \cup_{\beta} C \to I,
\]

\[
A \cup_a (B \times [0, 1)) \to A,
\]

\[
(B \times [0, 1]) \cup_{\beta} C \to C.
\]

\[
B \times (0, 1) \to B.
\]

Now suppose that \( a: (A, A_0) \to (X, X_0) \), \( b: (B, B_0) \to (X, X_0) \), \( c: (C, C_0) \to (X, X_0) \), \( \alpha: (B, B_0) \to (A, A_0) \), and \( \beta: (B, B_0) \to (C, C_0) \) are all pair fibrations.

**Lemma 3.3.1.** \( a \cup_a (b \times I) \cup_{\beta} c: (A \cup_a (B \times I) \cup_{\beta} C, A_0 \cup_a (B_0 \times I) \cup_{\beta} C_0) \to (X, X_0) \) is a pair fibration.

Now suppose that, for all \( n \geq 1 \), we have pair fibrations \( a_n, b_n, c_n, \alpha_n, \) and \( \beta_n \) (as in the hypotheses of 3.3.1), such that \( a_{n+1} = a_n \cup \alpha_n (b_n \times I) \cup \beta_n c_n \) for all \( n \). Let \( a_\infty = \bigcup a_n \), with the weak topology.

**Lemma 3.3.2.** \( a_\infty \) is a pair fibration.

**Proof.** Essentially by induction on \( n \), mimicking the proof of 3.3.1 at each step. We omit the details.

Let \( E_x S^n \to X \) be the fibration where, for each \( x \in X \), \( (E_x S^n)_x = (E_x)^S^n \), the space of maps \( S^n \to E_x \), with the compact open topology. Let \( e : E_x S^n \times S^n \to E \) be the evaluation map, and let \( K_x^n E = E \cup_e (E_x S^n \times B^{n+1}) \), with the strong topology. (Define as follows: Since \( B^{n+1} \) is the cone over \( S^n \), \( K_x^n E = E \cup_e (E_x S^n \times S^n \times I) \cup_e E_x S^n \), the strong topology double mapping cylinder, where \( \pi \) is projection.) Now let \( K^n \xi : (K_x^n E, K_x^n E_0) \to (X, X_0) \); by 3.2.1, a pair fibration.

**Theorem 3.3.3.** (I) If \( k < n \), \( i_k : \pi_k (\xi) \cong \pi_k (K^n \xi) \). (II) \( \pi_n (K^n \xi) = 0 \).

**Proof.** We first need a lemma.

**Lemma 3.3.4.** If \( Z \) is any simplicial complex of dimension less than or equal to \( n \), and if \( Y \) is any space, then \([Z; Y] \to [Z; Y \cup_e (Y \times S^n \times B^{n+1})]\) is onto.

**Proof of lemma.** Let \( * \in B^{n+1} \) be its center. If \( f : Z \to Y \cup_e (Y \times S^n \times B^{n+1}) \), \( f \) can be deformed slightly so that its image does not intersect \( Y \times S^n \times \{*\} \). The complement of that subset collapses to \( Y \), and we are done.

Returning to the proof of 3.3.3, we have immediately from the lemma that \( \pi_k (\xi) \to \pi_k (K^n \xi) \) is onto for \( k < n \) and one-to-one for \( k < n \). Finally, \( \pi_n (\xi) \to \pi_n (K^n \xi) \) is the zero map, since the construction attaches an \((n + 1)\)-cell to every possible map of \( S^n \) to every fiber.

**Definition 3.3.5.** For \( 0 < n < m \), let \( K^{n,m} \xi \) be defined inductively by \( K^{n,m} \xi = K^n \xi \), and \( K^{n,m+1} \xi = K^{m+1}(K^{n,m} \xi) \). Finally, let \( K^n \xi = \bigcup_m K^{n,m} \xi \).

**Theorem 3.3.6.** (I) If \( k < n \), \( \pi_k (\xi) \to \pi_k (K^n \xi) \). (II) If \( k \geq n \), \( \pi_k (K^n \xi) = 0 \).

**Proof.** Direct from 3.3.3.

3.4. The spectral sequence determined by a resolution. Suppose that \( \xi \) has a section. We say that the following commutative diagram of sectioned pair fibrations and maps

\[
\begin{array}{ccc}
\cdots & \to & \xi_1 \\
\beta_1 & \downarrow & \beta_2 \\
\alpha_2 & \downarrow & \alpha_3 \\
\cdots & \to & \xi_2 \end{array}
\]
is a resolution if, for each integer \( n \), there exists an integer \( N(n) \) such that
\[
\tau_k(\xi_m) \cong \tau_k(\xi_{m-1}) \text{ and } \beta_m(\xi) \cong \tau_k(\xi_m) \text{ for all } k < n \text{ and all } m > N(n).
\]
We say that it is a Postnikov resolution if, for every \( n > 0 \),
\[
\beta_m(\xi) \cong \tau_k(\xi_m) \text{ for all } k < n, \text{ and } \tau_k(\xi_n) = 0 \text{ for all } k > n.
\]

**Theorem 3.4.1.** If \( \xi \) has a section, \( \xi \) has a Postnikov resolution. Furthermore, if \( \gamma : \xi \to \xi' \), \( \gamma \) induces a natural map of Postnikov resolutions.

**Proof.** For every integer \( n > 0 \), let \( \xi_n = K^n+1\xi. \) Let the \( \alpha_i \) and \( \beta_i \) be the inclusions. By 3.3.6, we are done.

**Lemma 3.4.2.** If (3.4-1) is a Postnikov resolution, \( \phi[\alpha_n] = k(\pi_n(\xi), n) \).

**Proof.** Directly from 3.2.2.

Suppose diagram (3.4-1) is given. We define (taking \( \gamma : (X, X_0) \to (X, X_0) \) for \( q < 0 \):
\[
E_{p,q}^2 = \left[ (X, X_0), (A, A_0); \Omega^{-p}\phi[\alpha_q] \right] \text{ for } p < 0,
\]
\[
D_{p,q}^2 = \left[ (X, X_0), (A, A_0); \Omega^{-p}\xi_q \right] \text{ for } p < 0,
\]
\[
i_2 = (\Omega^{-p}\alpha_q)_\# : D_{p,q}^2 \to D_{p,q}^2 \text{ for } p < 0,
\]
\[
j_2 = (\Omega^{-p-1}\iota)_\# : D_{p,q}^2 \to E_{p+1,q+1}^2 \text{ for } p < -1,
\]
\[
k_2 = (\Omega^{-p}\lambda)_\# : E_{p,q}^2 \to D_{p,q}^2 \text{ for } p < 0.
\]

By 3.2.1, we now have a bigraded exact couple (although there is an edge problem, since \( p \) must not be positive). Furthermore, from 3.1.1 and 3.4.2, we immediately have:

**Remark 3.4.3.** If (3.4-1) is a Postnikov resolution, \( E_{p,q}^2 = H^{p+q}(X, A; \tau_q(\xi)) \).

Let \( E_r^p, D_r^p, E_\infty^p, \) and \( D_\infty^p \) be obtained in the usual manner.

**Theorem 3.4.4.** If \( p < -1 \), the \( E_{\infty}^p \) (for various \( q \)) give a composition series for \( [(X, X_0), (A, A_0); \Omega^{-p}\xi] \), provided \( \dim(X/A) < \infty \).

**Proof.** Let \( G_{-1}^p \) be the kernel of \( (\Omega^{-p}\iota)_\# : [(X, X_0), (A, A_0); \Omega^{-p}\xi] \to D_{p,q}^2 \). Thus, \( G_{p,-1} = [(X, X_0), (A, A_0); \Omega^{-p}\xi] \) and, for dimensional reasons, \( G_{p,q}^p = 0 \) for \( q \) sufficiently large. By standard spectral sequence arguments, \( E_{\infty}^p = G_{p+1,-1}/G_{p,q}^p \), and we are done.

If \( \xi \) is deloopable, i.e., \( \xi = \Omega \eta \), we can, by constructing a Postnikov resolution for \( \eta \), obtain a spectral sequence converging to a composition series for \( [(X, X_0), (A, A_0); \xi] \). If \( \xi \) is infinitely deloopable, the exact couple can be constructed with no restrictions on the indices. Since computation of \( H^i((X, X_0), (A, A_0); \xi) \) for \( \dim(X/A) < \infty \) involves classifying sections of \( \Omega^NS^{i+N}\xi \) for large \( N \), that cohomology theory can also be obtained by a
spectral sequence, where $E_2^{pq} = H^{p+q}(X, A; \pi_q^*(1))$. We leave the details to the reader.

4. A geometric interpretation of the affine structure. Let $f: V \to M$ be a map, where $V$ is a compact manifold of dimension $k$, and $M$ is a manifold of dimension $n$, where $2n > 3(k + 1)$. We give a geometric interpretation of the affine group structure of $[V \subset M]$.

By Theorem 2.7.1, it is sufficient to consider the case where $f$ is actually an embedding. We then consider $[V \subset M]$ to be an actual Abelian group with identity $[f_0]$, represented by the constant homotopy.

Let $\{f_1^t\}$, $\{f_2^t\}$ be $e$-homotopies of $f$ such that

(i) $f_1^t$ and $f_2^t$ are differentiable for all $t \in I$.

(ii) For some disjoint closed sets $K_1, K_2 \subset V$, and for all $t \in I$, $f_1^t$ agrees with $f$ on $V - K_i$.

(iii) For any $x_1 \in K_1$, $x_2 \in K_2$, $x_3 \in V - (K_1 \cup K_2)$, and for any $t, u \in I$; $f_1^t(x_1), f_2^u(x_2)$, and $f(x_3)$ are distinct.

Now let $f_3^t: V \to M$ be the $e$-homotopy of $f$ such that, for all $x \in V, t \in I$:

$$f_3^t(x) = \begin{cases} f_1^t(x) & \text{if } x \in K_1, \\ f_2^u(x) & \text{if } x \in K_2, \\ f(x) & \text{otherwise.} \end{cases}$$

**Theorem 4.1.** $[f_1^t] + [f_2^u] = [f_3^t]$.

**Proof.** We need only show that

$$\left[ i\Phi\{f_1^t\} \right] \left[ i\Phi\{f_2^u\} \right]^{-1} = \left[ i\Phi\{f_3^t\} \right] \in \left[ \langle R^*V, PV \rangle; PS_{SM} \right]_F,$$

where $\iota: \xi_M \subset PS_{SM}$. Pick disjoint closed sets $L_1, L_2$ in $V$ such that $K_i \subset \text{Int } L_i$. Choose continuous functions $p_1, p_2: V \to I$ such that $p_i(x) = 1$ for all $x \in K_i$, and $0$ for all $x \notin L_i$. For each $i = 1, 2$, or $3$, let $\Phi_{i, u}, 0 < u < 1$, be the homotopy of liftings of $F$ to $\xi_M$ such that, for all $u$, $\Phi_{i, u}$ agrees with $\Phi(f^i)$ on $PV$, and such that for all $0 < t, u < 1$ and all $[x, y] \in R^*V$,

$$\Phi_{i, u}[x, y](t) = [(f^i_v(x), (1 - t)i(x)), (f^i_w(y), (1 - t)i(y))] \in R^*(M \times R^\infty) = \infty R^*M,$$

where:

- $v = t(u_{p_1}(x) + 1 - u)$ and $w = t(u_{p_1}(y) + 1 - u)$ if $i = 1$ or $2$,
- $v = t(u_{p_2}(x) + 1 - u)$ and $w = t(u_{p_2}(y) + 1 - u)$ if $i = 3$, $j = 1$ or $2$, and $p_j(x)p_j(y) > 0$,
- $v = w = t(1 - u)$ if $i = 3$, and $p_1(x)p_1(y) + p_2(x)p_2(y) = 0$. 

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Note then that $\Phi_0 = \Phi(f_i)$ for all $i = 1, 2, \text{ or } 3$, and by 2.3.4, letting $D_i = Q(L^2) \subset \mathbb{R}^* V$ for $i = 1$ or 2, we obtain: $\lbrack i \Phi \{ f \} \rbrack^{-1} [i \Phi_2] = [i \Phi_1]$, and we are done.

Using a general position argument, based on the dimensional restriction, it is possible to show that any two $\epsilon$-homotopies of $f$ are respectively isotopic to a pair of $\epsilon$-homotopies satisfying (i) through (iii) above. Thus Theorem 4.1 suffices to describe the affine structure on $[V \subset M]_\epsilon$.

In later calculations, the following result is useful:

Theorem 4.2. If $f': V \rightarrow M$ is another map such that $f\lbrack D \rbrack$ is homotopic to $f'|D$ for some $D \subset V$ such that $V - D \cong \mathbb{R}^n$, then $[V \subset M]_{f'} \cong [V \subset M]_f$, if both are nonempty.

Proof. Without loss of generality, $f$ and $f'$ are both embeddings, and $f'|D = f\lbrack D \rbrack$. If $\alpha \in [V \subset M]_\epsilon$, we may, by a general position argument, choose an $\epsilon$-homotopy $\{f_t\}$ of $f$ such that

(i) $[f_0] = \alpha$,

(ii) $f_t(V - D) = f\lbrack (V - D) \rbrack$ for all $t$,

(iii) $f_t(D) \cap f'_t(V - D)$ is empty for all $t$.

Now let $\phi(\alpha) = [f'_0]$, where, for all $0 < t < 1$, $f'_t(x) = f_t(x)$ if $x \in D$, $f'_t(x) = f'(x)$ if $x \in (V - D)$. Using 4.1 and standard general position arguments, one may easily show that $\phi: [V \subset M]_f \rightarrow [V \subset M]_{f'}$ is well defined, and is an isomorphism.

5. The structure of $\pi_n \xi_M$. Let $M$ be any connected $n$-manifold. In this section, we explicitly compute the sheaf $\pi_n \xi_M$ over $\mathcal{O}^\infty \mathbb{R}^* M$, provided $n > 6$. (The sheaf $\pi_{n-1} \xi_M$ was computed in [7], provided $n > 5$; we restate the results here. For $k < n - 1$, $\pi_k \xi_M = 0$.)

5.1. Building a twisted sheaf. Suppose that $X$ is any path-connected space with basepoint $x_0$, and $S$ is a sheaf over $X$ with a local product structure. Let $S_x$ be the stalk of $S$ over any $x \in X$. If $a \in \pi_1 X$, let $\alpha$: $(I, \partial I) \rightarrow (X, x_0)$ be a loop representing $a$, and let $\alpha$: $I \times S_{x_0} \rightarrow S$ be a map such that $\alpha(t, s) \in S_{\alpha(t)}$ and $\alpha(1, s) = s$ for all $t \in I$, $s \in S_{x_0}$. Let $\langle a \rangle$: $S_{x_0} \rightarrow S_{x_0}$ be the automorphism where $\langle a \rangle(s) = \alpha(0, s)$. Let $\mu_S$: $\pi_1 X \times S_{x_0} \rightarrow S_{x_0}$ be the left action where $\mu_S(a, s) = \langle a \rangle(s)$ for all $a \in \pi_1 X$, $s \in S_{x_0}$.

Lemma 5.1.1. Let $X$ be path connected and locally simply connected, with basepoint. Let $G$ be any group and let $\mu$: $\pi_1 X \times G \rightarrow G$ be any left action. Then there exists a sheaf $S = S(G, \mu)$, with local product structure over $X$, such that $S_{x_0} = G$ and $\mu_S = \mu$. Furthermore, $S$ is unique with these properties, up to isomorphism.

Proof. Let $\pi$: $Y \rightarrow X$ be the universal covering of $X$, and let $\nu$: $\pi_1 X \times Y \rightarrow Y$ be the associated (left) action, where $\nu y_1 = y_2$ if and only if $\nu(a, y_2)$ for
some \( a \in \pi_1 X \). Then let \( S(G, \mu) \) be the quotient space of \( Y \times G \) obtained by identifying each \((y, g)\) with \((\nu(a, y), \mu(a, g))\) for all \( a \in \pi_1 X \). Trivially, the required conditions are satisfied. The proof of uniqueness is trivial and straightforward, hence we omit it.

5.2. Preliminary definitions. For any topological space \( X \), let \( \Gamma X \) be the quotient space of \( X \times X \times S^\infty \) obtained by identifying each \((x, y, a)\) with \( T(x, y, a) = (y, x, -a) \). If \( f: X_1 \to X_2 \) is any map, let \( \Gamma f: \Gamma X_1 \to \Gamma X_2 \) be the obvious map. Note that if \( * \) is a single point space, \( \Gamma * = p^\infty = S^\infty / T \). Let \( \Gamma 0: \Gamma X \to P^\infty \), where \( 0: X \to * \) is the collapsing map. If \( * \in X \) is a basepoint, let \( j = \Gamma i: P^\infty \to \Gamma X \), where \( i: * \to X \) is the inclusion. Let \( \pi: X \times X \times S^\infty \to \Gamma X \) denote the 2-1 covering, and, by a slight abuse of notation, let \( \pi: X \times X \to \Gamma X \) denote the (equivalent to the covering) inclusion, where \( \pi(x, y, \cdot) = [x, y, \cdot], \cdot \in S^\infty \) a basepoint. We may consider \( X \times P^\infty \subset \Gamma X \); if \( x \in X \) and \( a \in S^\infty \), identify \([x, a] \) with \([x, x, a] \).

Let \( X \) be any topological space with basepoint, and let \( \pi_k = \pi_k X \), written multiplicatively if \( k = 1 \), additively if \( k > 1 \). Let \( \gamma_k: \pi_1 \times \pi_k \to \pi_k \) be the usual left action, determined by the map \( \gamma_k: S^k \to S^1 \), where

\[
\gamma_k \left( x_1, x_2, \ldots, x_k \right) = \begin{cases} 
[2x_1] \in S^1 & \text{if } 0 < x_1 < \frac{1}{2}, \\
[2x_1 - 1, x_2, \ldots, x_k] \in S^k & \text{if } \frac{1}{2} < x_1 < 1,
\end{cases}
\]

where \( S^k = I^k / \partial I^k \). We let \( x^a = \gamma_k(a, x) \) for all \( a \in \pi_1, x \in \pi_k \). Let \( \alpha_k: \pi_2 \otimes \pi_k \to \pi_{k+1} \), for \( k \geq 2 \), be the Whitehead product, determined by the map \( \alpha_k: S^2 \to S^k \), where

\[
\alpha_k \left( x_1, x_2, \ldots, x_{k+2} \right) = \begin{cases} 
[x_1, x_2] \in S^2 & \text{if } x_i \in \partial I \text{ for some } 3 < i < k + 2, \\
[x_3, \ldots, x_{k+2}] \in S^k & \text{if } x_i \in \partial I \text{ for } i = 1 \text{ or } 2,
\end{cases}
\]

where \( S^2 = I^2 / \partial I^2 \), \( S^k = I^k / \partial I^k \), and \( S^{k+1} = \partial I^{k+2} \).

If \( X \) is a manifold and \( x \in \pi_k \), let \( w_k x \in Z_2 \) be the value of \( f^* w_k X \in H^k(S^k; Z_2) \) (the Stiefel-Whitney class), where \( f: S^k \to X \) represents \( x \). If \( a \in \pi_1 \), let \((-1)^a = 1 \) if \( w_1 a = 0 \), \(-1 \) otherwise.

Again, if \( X \) is a manifold, and if \( a \in \pi_1, a^2 = 1 \) (the identity), let \( f_a: P^2 \to X \) be a map which sends the generator of \( \pi_1 P^2 \) to \( a \), and let \( \tilde{a} = [f_a s] \in \pi_2 \), where \( s: S^2 \to P^2 \) is the covering map onto the real projective plane. Let \( k_a \) be an integer such that the vector bundles \( f_a^{-1} \tau \) and \( k_a h \) are stably equivalent, where \( \tau \) is the tangent bundle of \( X \) and \( h \) is the canonical line bundle over \( P^2 \).

Note that the pair \((k_a, \tilde{a})\) has indeterminacy (because of the choice of \( f_a \)) \((4Z \oplus 0) + \text{Im} \chi_a \), where \( \chi_a x = (m w_2 x, x + x^a) \) for all \( x \in \pi_2 \), where \( m: Z_2 \to Z_4 \) is the monomorphism. (Hint: recall that \( K^0(P^2) = Z_4 \).)

Let \( G \) and \( N \) be groups, and let \( \phi: G \to \text{Aut } N \) be any homomorphism. Let
\( N \times_\varphi G \) be the semidirect product. Specifically, as a set, \( N \times_\varphi G \) is the Cartesian product \( N \times G \) with the operation \((n_1, g_1)(n_2, g_2) = (n_1\varphi(g_1)n_2, g_1g_2)\). Note that if \( \varphi \) is trivial, \( N \times_\varphi G \) is simply the direct sum.

We always have \( G \subset N \times_\varphi G \) and \( N \triangleleft N \times_\varphi G \), where we identify \( N \) and \( G \) with \( N \times \{1\} \) and \( \{1\} \times G \), respectively.

For any (multiplicatively written) group \( G \), let \( ZG \) be the group ring of \( G \), which we represent as finite formal sums of elements of \( G \). Thus, \( G \subset ZG \).

5.3. Structure of \( \pi_{n-1}S^p \) and \( \pi_{n-2}S^p \). Now let \( X = \infty R^*M \), and \( X_0 = \infty P^*M \). Once and for all, fix basepoints \( * \in M \) and \( * \in SM \) over \( * \), and let \( * \in PM \) also denote the image of \( * \). Let \( \pi_k = \pi_kM \), written multiplicatively if \( k = 1 \), additively otherwise. For consistency of notation, we introduce a group \( T_2 = \{1, m\} \), a multiplicative group of two elements. Let \( H_2 = \{0, \eta\} = Z_2 \), the stable 1-stem in the homotopy of spheres, which we treat as a \( Z_2 \)-module.

For any group \( G \), let \( \Delta G \subset G \oplus G \) be the diagonal.

**Lemma 5.3.1.** (I) \( \pi_1X = (\pi_1 \oplus \pi_1) \times_\varphi T_2 \), where \( \varphi(m)(a, b) = (b, a) \) for all \( a, b \in \pi_1 \). (II) \( \pi_1X_0 = \Delta \pi_1 \oplus T_2 \), and \( i: X_0 \subset X \) induces the inclusion of groups described in 5.2.

**Proof.** Let \( \iota: M \times R^\infty \to R^\infty \) be any embedding, and let \( \rho: RR^\infty \to S^\infty \) be the equivalent retraction, determined by \( \rho(u, v) = \|u - v\|^{-1}(u - v) \) for all \( (u, v) \in RR^\infty \). Let \( \psi = (\rho^2, \rho R) : (R(M \times R^\infty), S(M \times R^\infty)) \to (M \times M \times S^\infty, \Delta M \times S^\infty) \) which is equivariantly a homotopy equivalence of pairs, by an elementary obstruction theory argument. Passing to quotient spaces under the involution \( T \), we have that the pair \( (X, X_0) \) is of the homotopy type of \((\Gamma M, M \times P^\infty)\).

We have a partially split exact sequence:

\[
1 \to \pi_1M^2 \to \pi_1\Gamma M \xrightarrow{\varphi} \pi_1P^\infty \to 1
\]

A simple deck-transformation argument then verifies that \( \pi_1M \) is the desired semi-direct product. (II) follows trivially.

Now let \( \theta_M: Y \to X \) and \( \rho_M: Z \to X_0 \) be the fibrations which (as functions) agree with the pair fibration \( S^p \).

**Lemma 5.3.2.** (I) If \( n > 5 \), \( \pi_{n-1}\theta_M = S(Z\pi_1, \mu') \), where

(i) \( \mu'((b, c, 1), a) = (-1)^{bc-1}a \),

(ii) \( \mu'((b, c, m), a) = (-1)^c(-1)^{bc-1}c^{-1} \) for all \( a, b, c \in \pi_1 \).

(II) If \( n > 6 \), \( \pi_n\theta_M = S(Z\pi_1 \otimes (\pi_2 \oplus H_2), \mu) \), where

(i) \( \mu((b, c, 1), a \otimes (x, \lambda)) = (-1)^{bc-1} \otimes (x^c, \lambda) \),

(ii) \( \mu((b, c, m), a \otimes (x, \lambda)) = (-1)^{c+1}(-1)^{bc-1}c^{-1} \otimes (x^{-1}c, (\lambda + \omega_2x)\eta) \) for all \( a, b, c \in \pi_1, x \in \pi_2, \) and \( \lambda \in Z_2 \).
We postpone the proof.

Identify $Z$ with $Z_{(1)} \subset Z\pi_1$. Let $\nu': ((Z\Delta \pi_1) \oplus T_2) \times Z \to Z$ and $\nu: ((Z\Delta \pi_1) \oplus T_2) \times (Z \otimes H_2) \to Z \otimes H_2$ be the restrictions of $\mu'$ and $\mu$, respectively.

**Lemma 5.3.3.** (I) If $n > 5$, $\pi_{n-1}\rho_M = S(Z, \nu')$. (II) If $n > 6$, $\pi_n\rho_M = S(Z \otimes H_2, \nu)$.

We leave the proof to the reader. (Hint: the fiber of $\rho_M$ is $S^{n-1}$.)

One final lemma completes the description of $\pi_{n-1}\xi_M$ and $\pi_n\xi_M$.

**Lemma 5.3.4.** If $\pi_k\rho_M$ is a subsheaf of $(\pi_k\theta_M)|_{X_0}$, then $\pi_k\xi_M$ is the subsheaf of $\pi_k\theta_M$ whose stalks over $X_0$ agree with those of $\pi_k\theta_M$, and whose stalks over $X - X_0$ agree with those of $\pi_k\rho_M$.

**Proof.** Follows directly from the definition of $\pi_k\xi_M$.

5.4. **Proof of Lemma 5.3.2.** Part (I) of 5.3.2 is simply Lemmas 3.4.2, 3.4.4, and 3.4.5 of [7]. (However, we have replaced the right actions of that paper by left actions.)

Now we saw in the proof of 3.4.2 of [7] that the fiber of $\theta_M$ is the fiber $\Phi$ of the inclusion $\tilde{M}^0 \subset \tilde{M}$, where $\tilde{M}$ is the universal covering space of $M$, and $\tilde{M}^0$ is the universal covering space of $M^0 = M - \{\ast\}$. A straightforward Serre spectral sequence argument reveals that

(i) $\pi_{n-1}(\Phi) = Z\pi_1$,
(ii) $\pi_n(\Phi) = Z\pi_1 \otimes (\pi_2 \otimes H_2)$,

where composition with $\eta: S^n \to S^{n-1}$ is represented by $\otimes \eta$, and where, for each $a \in \pi_1$ and $x \in \pi_2$, $a \otimes x$ is represented by the map $\phi$ in the following homotopy commutative diagram, where $S^2 \xrightarrow{x} \tilde{M}^0 \subset \tilde{M} \to M$ represents $x$:

\[
\begin{array}{ccc}
S^n & \xrightarrow{\phi} & \Phi \\
\downarrow & & \downarrow \\
S^{n-1} \vee S^2 & \xrightarrow{a \vee x} & \tilde{M}^0 \subset \tilde{M}
\end{array}
\]

The reader can easily verify that (II) need only be checked for the following five special cases (where $a, b, c \in \pi_1; x \in \pi_2$):

(i) $((b, 1, 1), a \otimes \eta) = ba \otimes \eta$,
(ii) $((1, 1, m), a \otimes \eta) = a^{-1} \otimes \eta$,
(iii) $((b, 1, 1), a \otimes x) = ba \otimes x$,
(iv) $((1, c, 1), 1 \otimes x) = (-1)^c e^{-1} \otimes xc$,
(v) $((1, 1, m), 1 \otimes x) = (-1)^{m+1} \otimes (w_2x\eta + x)$.

Now (i), (ii) follow immediately from (I), by composition with $\eta$, while the proof of (iii) involves (I), together with mere straightforward checking of maps.
To prove (iv), let \( \gamma: I \to M \) be a loop representing \( c \). Let \( \psi_t: S^2 \to M \), \( 0 < t < 1 \), be a homotopy such that \( \psi_t(*) = \gamma(t) \) for all \( t \), and \( \psi_1 \) represents \( x \). Then \( \psi_0 \) represents \( x^c \).

We briefly introduce a general construction. If \( Y \) is any pointed space, \( B \subset Y \), and if \( f: (I^k, \partial I^k) \to (Y, B) \) is a map such that \( f(*) = * \) for some \( * \in I^k \), let \( \langle f \rangle: S^{k-1} = \partial I^k \to F \) (\( F = \) homotopy theoretic fiber of the inclusion \( B \subset Y \)) be defined as follows: let \( \rho: CS^{k-1} \cong I^k \) (\( CX = \) cone over \( X \)) and let \( \langle f \rangle(v) = \sigma_v \in F \subset Y' \), where \( \sigma_v(t) = f(\rho(v, t)) \). By a slight modification, it is not necessary to assume \( B \subset Y \), only \( B \to Y \).

Returning to the proof, choose a homotopy \( \theta_t: (I^n, S^{n-1}) \to (\tilde{M}^2, \tilde{R}M) \), where \( \tilde{R}M \) is the universal covering space of \( RM = M^2 - \Delta M \), such that:

(i) \( p_2\theta_t(v) = \gamma(t) \) for all \( t \in I \),

(ii) \( \langle \theta_t \rangle \) represents \( 1 \in \mathbb{Z}_{2n} = \pi_{2n-1}(\Phi) \),

where \( p_2: \tilde{M}^2 \to \tilde{M} \) is projection to the second factor. Clearly \( \langle \theta_0 \rangle \) represents \( (-1)^{c-1} \in \mathbb{Z}_{2n} = \pi_{2n-1}(\Phi) \). Now, for all \( t \in I \), we have a homotopy commutative diagram:

\[
\begin{array}{cccc}
S^n & \xrightarrow{H} & H & \xrightarrow{\Phi} \\
\downarrow{\alpha_{n-1}} & & \downarrow & \\
S^2 \vee S^{n-1} & \xrightarrow{\xi_t \vee \theta_t} & \tilde{R}M & \xrightarrow{p^2} \tilde{R}M \\
\cap & & \cap & \\
S^2 \vee I^n & \xrightarrow{\xi_t \vee \theta_t} & M^2 & \xrightarrow{p^2} M^2
\end{array}
\]

where \( p: \tilde{M} \to M \) is the covering map, \( p^2\xi_t = i_1\psi_t \) for all \( t \), \( \xi_t(*) = * \), and \( \beta \) is chosen independent of \( t \). By definition, \( [\beta h_i] = 1 \otimes x \) and \( \mu((1, c, 1), 1 \otimes x) = [\beta h_0] = (-1)^c(1 \otimes x^2) \), and (iv) is verified.

To prove (v), let \( g: R^n \to M \) be a coordinate patch such that \( g(0) = * \). Let \( \kappa: (I^n, S^{n-1}) \to (R^n, R^n - \{0\}) \) be the obvious orientation preserving equivalence. Then \( \langle g\kappa \rangle = \pm 1 \in Z \subset Z_{2n} = \pi_{2n-1}(\Phi) \). We alter \( g \) if necessary to insure that \( \langle g\kappa \rangle = 1 \).

Let \( (SM)_* \) be the fiber of \( SM \to M \) over \( * \). Choose a homeomorphism \( \iota: S^{n-1} \to (SM)_* \) such that \( \langle \lambda \rangle = 1 \in \pi_{n-1}(\Phi) \), where \( \lambda: (I^n, S^{n-1}) \to (M^2, RM) \) is chosen such that \( \lambda(I^n) = * \in M^2 \), and \( \lambda|S^{n-1} = \iota \). When convenient, we shall identify \( S^{n-1} \) with \( (SM)_* \).

Let \( r: E \to S^2 \) be the \( S^{n-1} \) bundle which is the pullback of \( SM \to M \), where \( \gamma: S^2 \to M \) classifies \( x \in \pi_2 \), and let \( f: S^2 \to E \) be a section of \( r \). Let \( e: I^2 \times S^{n-1} \to E \) be a map such that the following diagram commutes:
where $p_i$ is the projection on the $i$th factor, $c$ collapses $\partial I^2$ to $\ast \in S^2$, $L = \{1\} \times I \cup I \times \partial I \subset I^2$, and $i_1$ is inclusion along the first factor. Let $y_i: I^2 \to M^2$, $0 < t < 1$, be the homotopy such that

$$y_i(u, v) = \begin{cases} 
(\gamma[1 - u, v], \ast) & \text{if } 0 < u < t, \\
(\gamma[1 - t, v], \gamma[u - t, v]) & \text{if } t < u < 1.
\end{cases}$$

Let $W = (S^1 \times S^{n-1}) \cup (I^2 \times \{\ast\}) \subset I^2 \times S^{n-1}$, where $S^1 = \partial I^2$, and choose a homotopy $\tilde{y}_i: W \to RM$, $0 < t < 1$, such that $\tilde{y}_i|S^1 \times S^{n-1} = \gamma e(J_t \times 1)$ for all $t$, where $J_t: S^1 \to I^2$ is given by $J_t(u, v) = (1 - t, v)$ for all $(u, v) \in S^1 \subset I^2$; and such that $Q\tilde{y}_i|I^2 \times \{\ast\}$ and $y_i p_1|I^2 \times \{\ast\}$ are homotopic, rel $S^1 \times \{\ast\}$, for all $t$; where $Q: RM \to M^2$ is the usual quotient map.

The following diagram illustrates the next portion of the argument.

Let $z_i: F \to \Phi$, $0 < t < 1$, be the homotopy induced by $\{y_i\}$, $\{\tilde{y}_i\}$ (where $F = \text{fiber of } p_1: W \to I^2$). Let $i_2: S^{n-1} \subset W$ be inclusion along the second factor, and let $\omega = (d_{\omega})^{-1}[\alpha_{n-1}] \in \pi_n W = \pi_n F$, where $d: W \to S^2 \vee S^{n-1}$ is the map which collapses $S^1 \times \{\ast\} \subset W$. Clearly $(z_0)\omega = -(1 \otimes \eta)$; it remains to show only that $(z_1)\omega = -((1 \otimes 1) + (w_2 x) \otimes \eta)$.

Note that $\tilde{y}_i: S^1 \times \{\ast\} \to j(*) \in (SM)_* \subset RM$, thus $y^*: S^2 \to RM$, where $y^*[u, v] = y_i(u, v, \ast)$ is well defined. We then have a commutative diagram (where $S^2$ is identified with $jS^2 \subset E$):
where $e, \nu$ are the appropriate maps on the fibers. Now $I^2 \times S^{n-1} = W \cup \nu e^{n+1}$, where $\nu: S^n \to W$ classifies $\omega$, and $E = S^2 \vee S^{n-1} \cup \lambda e^{n+1}$, where $\lambda = e(J_1 \times 1)w$. Now $Sq^2\phi = \gamma^* w_1 M \cup \phi$, and $\phi \cup \sigma = - e^{n+1}$ (since $J_1$ has degree $-1$), where $\phi \in H^{n-1}(E; Z)$ is the generator and $\sigma \in H^2(S^2; Z)$ is the fundamental class. Thus $[\lambda] = -[\alpha_{n-1}] + (w_2x)[\eta_2] \in \pi_n(S^2 \vee S^{n-1})$, and we are done.

6. Explicit computation of $[S^k \subset M^{2k}]_f$. In this section, $[S^k \subset M^n]_f$ is explicitly stated, in terms of generators and relations, where $M$ is any manifold without boundary, $f: S^k \to M$ is any embedding, and $n = 2k + 1$, $n > 2$, or $n = 2k$, $k > 3$.

**Theorem 6.0.1.** If $k > 2$, $n = 2k + 1$, then $[S^k \subset M^n]_f$ is generated by elements $y_a$ for all $a \in \pi_1 M$, subject only to the following relations:

(i) $y_1 = 0$, where $1 \in \pi_1 M$ is the identity;

(ii) For any $a \in \pi_1 M$, $y_{a^{-1}} = (-1)^{k+1}(-1)^y a$.

**Theorem 6.0.2.** If $k > 3$, $n = 2k$, then $[S^k \subset M^n]_f$ is generated only by elements:

(i) $z_a$ for all $a \in \pi_1 M$ such that $a^2 = 1$ and $(-1)^k(-1)^a = 1$.

(ii) $y_a \otimes u$ for all $a \in \pi_1 M$ and all $u \in \pi_2 M \oplus H_2$, where $H_2 = \{0, \eta\}$ is the stable 1-stem in the homotopy of spheres.

And is subject only to the relations:

(iii) $y_a \otimes u + y_a \otimes v = y_a \otimes (u + v)$ for any $a \in \pi_1 M$, $u, v \in \pi_2 M \oplus H_2$.

(iv) $y_1 \otimes \eta = 0$.

(v) $z_1 = 0$.

(vi) $y_a \otimes \eta = y_{a^{-1}} \otimes \eta$ for any $a \in \pi_1 M$.

(vii) $y_a \otimes x + (-1)^a(-1)^y_{a^{-1}} \otimes x^a + (w_2x)y_a \otimes \eta = 0$ for all $a \in \pi_1 M$, $x \in \pi_2 M$.

(viii) $y_a \otimes \bar{a} + ((k^2 - 3) + (k^2 - 4)) y_\bar{a} \otimes \eta = 0$.
for all $a \in \pi_1M$ such that $a^2 = 1$ and $(-1)^n(-1)^k = -1$.

(ix) $$2z_a = y_a \otimes \tilde{a} + \left(\binom{2k-k}{2} + \binom{k-3}{2} + \binom{n}{1}\right)y_a \otimes \eta$$

for all $a \in \pi_1M$ such that $a^2 = 1$ and $(-1)^n(-1)^k = 1$.

Finally, these representations are natural, i.e.;

**Theorem 6.0.3.** Let $g: M \subset M'$ be an embedding, where $M'$ is another manifold. Then (I) If $k > 2$, $n = 2k + 1$, $g^\#y_a = y_{g\#a}$ for all $a \in \pi_1M$. (II) If $k > 3$, $n = 2k$, $g^\#(y_a \otimes x) = y_{g\#a} \otimes g\#x$ and $g^\#(y_a \otimes \eta) = y_{g\#a} \otimes \eta$ for all $a \in \pi_1M, x \in \pi_2M$, and $g\#z_a = z_{g\#a}$ if $a \in \pi_1M, a^2 = 1$, and $(-1)^n(-1)^k = 1$.

6.0.1 and 6.0.2 are proved below (6.0.1 was incorrectly stated as Theorem 1.2.1 of [7], the result there is off by a sign), while we leave the proof of 6.0.3 to the reader. Basically, 6.0.3 follows from the fact that all of the constructions are natural. Examination of those constructions reveals that $y_a, y_a \otimes \tilde{a}, y_a \otimes \eta$, and $y_a \otimes x + y_a \otimes x^a$ are canonically chosen, while $z_a$ has an indeterminacy, as defined below, the group generated by $y_a \otimes \tilde{a}, y_a \otimes \eta$, and $y_a \otimes x + y_a \otimes x^a$ for all $x \in \pi_2M$. By making a once-and-for-all choice in each of the four universal examples (see the proof of 6.1.3), this indeterminacy could possibly be further reduced.

6.1. **Proof of 6.0.1 and 6.0.2.** Without loss of generality, $f$ may be assumed inessential (cf. 4.2). Now, if $k > 2$, $n = 2k + 1$, there is only one nonzero $E_\infty$ term in the composition series for $[S^k \subset M]_f = H^0(f)$, namely $E_\infty^{0,n-1}$. Now $E_2^{0,n-1} = H^{n-1}(R^*S^k; \mathbb{F})$, and our result follows immediately from Lemma 6.1.1, below, where $y_a$ corresponds to $[a]$ for all $a \in \pi_1 = \pi_1M$.

Suppose now that $k > 3$, $n = 2k$. Let $\pi_1 = \pi_1M, \pi_2 = \pi_2M$. The only $E_2$ terms which play a role in the computation of $[S^k \subset M]_f$ are $E_2^{0,n-1} = H^{n-1}(R^*S^k; \mathbb{F}), E_2^{1,n-1} = H^{n-2}(R^*S^k; \mathbb{F}), E_2^{0,n} = H^n(R^*S^k; \mathbb{F})$, and the only relevant differential is $d_2$ is the exact sequence:

\[(6.1-1) \quad E_2^{1,n-1} \xrightarrow{d_2} E_2^{0,n} \xrightarrow{\lambda} H^0(f) \xrightarrow{p} E_2^{0,n-1} \rightarrow 0.\]

Now if $e: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is any short exact sequence of Abelian groups, $e$ determines a homomorphism $\Phi_2: K_2 \rightarrow A/2A$, where $K_2 = \{ x \in C \mid 2x = 0 \}$, as follows: for $x \in K_2$, $\Phi_2x = \lambda^{-1}p^{-1}x$. By Theorem 5.1 of [9], knowledge of $\Phi_2$ suffices to determine $B$ as an extension of $C$ by $A$ if $2C = 0$. By 6.1.1, $2E_2^{0,n-1} = 0$; then (for the sequence (6.1-1)) the groups are given by 6.1.1, and $d_2$ and $\Phi_2$ are given by 6.1.3, and Theorem 6.0.2 follows immediately; where, for $a \in \pi_1, x \in \pi_2, \lambda[a \otimes x] = y_a \otimes x, \lambda[a \otimes \eta] = y_a \otimes \eta$; and, if $a^2 = 1$ and $(-1)^n(-1)^k = 1$, $\rho z_a = [a]$. 

**ISOTOPY GROUPS**
Recall that $H_2 = \{0, \eta\} \approx \mathbb{Z}_2$.

For any integer $r$, $k < r < 2k$, let $K_{r-1} = Z_{\pi_1}$ and $K_r = Z_{\pi_1} \otimes (\pi_2 \oplus H_2)$. For $r < k$ or $r > 2k$, let $K_{r-1} = K_r = 0$. Define $\delta_{r-1}': K_{r-1} \rightarrow K_{r-1}'$ and $\delta_r': K_r \rightarrow K_r'$ for all $r$ as follows: for $k < r < 2k$, and for $a \in \pi_1$, $x \in \pi_2$:

(i) $\delta_{r-1}^\prime(a) = a + ( -1)^{r+k+n+1}(-1)^r a^{-1}$,

(ii) $\delta_r^\prime(a \otimes x) = a \otimes x + ( -1)^{r+k+n}a^{-1} \otimes x^a + (w_2 x) a^{-1} \otimes \eta$,

(iii) $\delta_r^\prime(a \otimes \eta) = a \otimes \eta + a^{-1} \otimes \eta$.

We now define $H'_s = \text{Ker } \delta_s' / \text{Im } \delta_s'$ for $s = n - 1$ or $n$.

**LEMMA 6.1.1.** (I) For all $r$, and for $s = n - 1$ or $n$,

$$H^r(\mathbb{R}^*S^k, \mathbb{P}S^k; F^{-1}\pi_2'\theta_M) = H^r(\mathbb{R}^*S^k, \mathbb{P}S^k; F^{-1}\pi_2'\theta_M) = H'_s.$$

(II) For all $r > k$, and for $s = n - 1$ or $n$, $j^*: H'_s \rightarrow H^r(\mathbb{R}^*S^k; F^{-1}\pi_2'\theta_M)$ is onto, and

$$\text{Ker } j^* = \begin{cases} (Z \cap \text{Ker } \delta_{r-1}') / \text{Im } \delta_{r-1}' & \text{if } s = n - 1, \\ (Z \otimes H_2 \cap \text{Ker } \delta_r') / \text{Im } \delta_r' & \text{if } s = n. \end{cases}$$

**PROOF (I).** Trivially,

$$H^*(\mathbb{R}^*S^k, \mathbb{P}S^k; F^{-1}\pi_2'\theta_M) = H^*(\mathbb{R}^*S^k, \mathbb{P}S^k; F^{-1}\pi_2'\theta_M),$$

since these two coefficient sheaves differ only on $\mathbb{P}S^k$.

We now consider $S^k \subset R^{k+1}$ to be the unit sphere, and $\pi: S^k \rightarrow P^k$ the covering map onto real projective $k$-space. For any $x \in P^k$, let $[x] \subset R^{k+1}$ be the line through 0 determined by $x$. Let $\nu$ be the $k$-plane bundle over $P^k$ such that, for each $x \in P^k$, $\nu_x = [x]_\perp \subset R^{k+1}$. Clearly, $\pi^{-1}\nu = \tau$, the tangent bundle of $S^k$, and $\nu \oplus h = (k + 1)$, the trivial $(k + 1)$-plane bundle, where $h$ is the canonical line bundle over $P^k$ and "\oplus" denotes Whitney sum. Thus $\nu f = u f$ for all $f$, where $u \in H^1(P^k; Z_2)$ is the generator.

For any vector bundle $\xi$, let $E\xi$ and $S\xi$ be the total spaces of the associated disc and sphere bundles, respectively. We construct a commutative diagram of pairs

$$(E\tau, S\tau) \xrightarrow{\phi} (RS^k, SS^k) \xrightarrow{\pi} (Ev, S\nu) \xrightarrow{\tau} (R^*S^k, PS^k)$$

as follows. $E\tau = \{(u, v) \in S^k \times B^{k+1}[u \perp v]\}$, where $B^{k+1} \subset R^{k+1}$ is the unit ball, while $SS^k = S\tau = \{(u, v) \in S^k \times S^k|u \perp v\}$. For any $(u, v) \in E\tau$, let

$$\phi(u, v) = \begin{cases} (\alpha_v u + u, \alpha_v v - u) \in RS^k & \text{if } (u, v) \in S\tau, \\ (v, u) \in SS^k & \text{if } (u, v) \in S\tau, \end{cases}$$
where $\alpha_v = (1 - \|v\|^2)^{1/2}$, and let $T$ be the unique map which makes the diagram commutative. Trivially, $\phi$ and $T$ are both homeomorphisms of pairs. Thus $R^*S^k/PS^k \cong M(\nu)$, the Thom complex of $\nu$, whence $H^i(R^*S^k, PS^k; Z_2) \cong Z_2$ if $k < i < 2k$, 0 otherwise; and $Sq^j: H^i(R^*S^k, PS^k; Z_2) \to H^{i+j}(R^*S^k, PS^k; Z_2)$ is nonzero if and only if $k < i < i + j < 2k$ and $(-1)^{k-1} = 1 \mod 2$ (where we take $(-1)^1 = 1 \mod 2$ for all $j$). $RS^k/SS^k$ may then be considered to have two cells, $e_1$ and $e_2$, in each dimension $k < i < 2k$, and none in other dimensions, such that:

(i) $Te^i_j = e^j_j$ for all $i$, where $T: RS^k \to RS^k$ is the map which exchanges coordinates;

(ii) For all $k < i < 2k$ and $j = 1$ or $2$,

$$\delta e^i_j = \begin{cases} e^{i+1} + e^{j+1} & \text{if } i - k \text{ is even,} \\ (-1)^j (e^{i+1} - e^{j+1}) & \text{if } i - k \text{ is odd.} \end{cases}$$

Then $R^*S^k/PS^k$ has one cell in each dimension $k < i < 2k$, namely $e^i = \pi e^i = \pi e^i$, and no cells in other dimensions; and $\delta e^i = (1 + (-1)^{i-k})e^{i+1}$.

Now let $G$ be any Abelian group and $e: G \to G$ an automorphism of order 2. Recall that $\pi_1(R^*S^k) \cong T_2 = \{1, m\}$, a multiplicative group of order 2. Let $\mu_e: T_2 \times G \to G$ be the action where $\mu_e(m, x) = ex$ for all $x \in G$, and let $G^e = S(G, \mu_e)$, a sheaf over $R^*S^k$. $H^e(R^*S^k, PS^k; G^e)$ may be considered to be the equivariant cohomology of the pair $(RS^k, SS^k)$ with coefficients in $G$ under the action $\mu_e$, specifically, the homology of the chain complex

$$G \to G \to \cdots \to G \to G \to G$$

where, for $k < i < 2k$, $d^i x = x + (-1)^{i-k}ex$ for all $x \in G$. Letting $G_{n-1} = Z\pi_1$ and $ea = (-1)^a (-1)^{a-1}$ for all $a \in \pi_1$, and letting $G_n = Z\pi_1 \otimes (\pi_2 \otimes H_2)$ and $e(a \otimes (x + \lambda)) = ((-1)^a (-1)^n) a^{-1} \otimes x^a + a^{-1} \otimes (\lambda + w_2 x)\eta$ for all $a \in \pi_1$, $x \in \pi_2$, $\lambda \in Z_2$, we have by 5.3.3 that $G^e_s = F^{-1}\pi s \theta_M$ for $s = n - 1$ or $n$, and we are done with the proof of (I).

PROOF (II). For convenience of notation, we agree to let $S_s = F^{-1}\pi s \theta_M$ and $Q_s = F^{-1}\pi s \theta_M$ for $s = n - 1$ or $n$. Let $L_s \subset S_s$ be the unique maximal subsheaf with a local product structure, specifically, $L_s = J_s$, where $J_{n-1} = Z \subset G_{n-1}$ and $J_n = Z \otimes H_2 \subset G_n$. Note that $G_s = J_s \otimes R_s$, for suitably chosen $R_s \subset G_s$, and $T_2$ acts independently on each direct summand: it follows that $S_s = L_s \otimes R_s$, a direct summation of sheaves. We also have that $L_s|PS^k = S_s|PS^k$ and $Q_s|R^*S^k = S_s|R^*S^k$, and:

$$L_n \cong Z_2;$$
$$L_{n-1} = \begin{cases} Z & \text{if } n \text{ is even,} \\ Z^T & \text{if } n \text{ is odd.} \end{cases}$$

Thus $i^*: H'(R^*S^k, PS^k; L_s) \to H'(R^*S^k, PS^k; S_s)$ is mono, its image
exactly what we wish $\text{Ker} j^*$ to be. We have a commutative diagram with exact rows (since $H^i(R^*S^k; L_i) = 0$ for all $i > k$, because $L_i$ has a local product structure and $R^*S^k$ is of the homotopy type of $P^k$):

$$
\begin{array}{cccc}
H^{r-1}(PS^k; L_i) & \longrightarrow & H^r(R^*S^k, PS^k; L_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^{r-1}(PS^k; S_j) & \xrightarrow{\delta} & H^r(R^*S^k, PS^k; S_j) & \xrightarrow{\delta} & H^{r+1}(R^*S^k, PS^k; S_j)
\end{array}
$$

By a simple diagram-chasing argument, we are done.

**Notation.** If $y \in \text{Ker } \delta^r$, we shall let $[y] \in \text{Ker } \delta^r/\text{Im } \delta^{r-1}$ be the class represented by $y$.

In the following lemmas, we shall assume that $n = 2k$, $k > 3$.

We examine a particular case. For any integer $s$, let $N^s$ be the total space of a real $(n - 2)$-plane bundle $\xi^{n-2}$ over $P^2$ such that $\xi^{n-2} \oplus 3h$ is stably equivalent to $sh$, the $s$-fold Whitney sum of the canonical line bundle, $h$. Clearly, $N^s = N^{s+4}$, since $h$ has order 4 in $K$-theory. If $\tau_s$ is the tangent bundle of $N^s = N$, $\tau_s$ is stably equivalent to $sh$, hence $w_1N = (\tau)m$ and $w_2N = (\tau)m^2$, where $m \in H^1(N; Z_2)$ is the generator of that group.

Let $a \in \pi_1N \cong T_2$ be the generator. Then $\tilde{a} \in \pi_2N \cong Z$ is a generator, and $\tilde{a}^a = - \tilde{a}$. Let $f: S^k \to N$ be an embedding. Recall that $\Phi$ is the fiber of $\theta_N$. We have by 5.3.2 that

$$
F^{-1}\pi_{n-1}\theta_N \cong \begin{cases} 
Z + Z & \text{if } s \text{ is even,} \\
Z + Z & \text{if } s \text{ is odd,}
\end{cases}
$$

where the first generator is represented by $1 \in Z\pi_1 = \pi_{n-1}\Phi$, the second by $a \in \pi_{n-1}\Phi$; and also that

$$
F^{-1}\pi_n\theta_N \cong \begin{cases} 
Z^T + Z_2 + Z_2 + Z_2 & \text{if } s \text{ is even,} \\
Z^T + Z_2 + Z^T + Z_2 & \text{if } s \text{ is odd,}
\end{cases}
$$

where the generators are represented, respectively, by $1 \otimes a$, $1 \otimes \eta$, $a \otimes \tilde{a}$, $a \otimes \eta \in Z\pi_1 \otimes (\pi_2 \oplus H_2) = \pi_n\Phi$.

Let $B = R^*S^k$. Since $f$ is an embedding, $F$ has a lifting to $Y_N$, thus $F^{-1}\theta_N$ has a section, i.e. $F^{-1}\theta_N$ is in the category $\mathfrak{C}_{+}^B$ of $B$-sectioned fibrations. The following lemma is phrased in the notation of [10].

**Lemma 6.1.2.** If $N = N^s$, then the first two stages of the Postnikov tower for $F^{-1}\theta_N$ are given by ($B = R^*S^k$):

$$
k_b(Z, n, m) \times k_b(Z_2, n) \times k_b(Z, n, (\tau)m) \times k_b(Z_2, n) \xrightarrow{\lambda} \epsilon_1 \xrightarrow{P_1} F^{-1}\theta_N
$$

$$
k_b(Z, n - 1) \times k_b(Z, n - 1, (\tau)m) = \epsilon_0 \xrightarrow{\alpha}
$$

$$
k_b(Z, n + 1, m) \times k_b(Z_2, n + 1) \times k_b(Z, n + 1, (\tau)m) \times k_b(Z_2, n + 1)
$$
where \( P_i \) induces isomorphism in homotopy through dimension \( n \), and where
\[
\alpha^*: \tau_{n+1} \otimes 1 \otimes 1 \to 0,
\]
\[
1 \otimes \tau_{n+1} \otimes 1 \otimes 1 \to (\text{Sq}^2 + m^2)\tau_{n-1} \otimes 1,
\]
\[
1 \otimes 1 \otimes \tau_{n+1} \otimes 1 \to 1 \otimes (\delta m)_{n-1},
\]
\[
1 \otimes 1 \otimes 1 \otimes \tau_{n+1} \to 1 \otimes (\text{Sq}^2 + (\tau_{2-1})^2 m^2)_{n-1},
\]
where \( \delta \) is the Bokstein homomorphism associated with the coefficient sequence \( Z \to 2 \to Z \).

We postpone the proof.

**Lemma 6.1.3.** Let \( a \in \pi_1, a^2 = 1 \), and choose (jointly) \( a \in \pi_2, 0 < s = k_a < 4 \) (cf. 5.2). Then in sequence (6.1-1):

(I) If \((-1)^s(-1)^k = -1\), \( d_2[a] = a \otimes \tilde{a} + (\text{Sq}^2) + (\tau_{2-1}s)\alpha \otimes \eta; \)

(II) If \((-1)^s(-1)^k = 1\), \( \Phi_2[a] \in \text{Coker} d_2 \) is represented by \( [a \otimes \tilde{a} + (\text{Sq}^2) + (\tau_{2-1}s) + (\text{Sq}^2)(\tilde{a})^s] \alpha \otimes \eta \).

**Proof.** We remind the reader that \((-1)^s = (-1)^t\), thus \( (\tau) = (\tilde{a}) \mod 2 \) if \((-1)^s(-1)^k = 1\).

Without loss of generality, \( f \) is inessential, hence we may assume \( f: S^k \to N_s^n \subset M \). Since all constructions involved are natural with respect to inclusions of manifolds of the same dimension, we may assume that \( M = N_s^n \).

We have a commutative diagram with exact rows, where the first, second and fourth vertical arrows represent onto maps, by 6.1.1:

\[
\begin{array}{cccc}
H^2_{n-1} & \xrightarrow{d_2} & H^0_n & \xrightarrow{\lambda} \left[ R^* S^k, PS^k; \theta_M \right] & \xrightarrow{p} & H^0_{n-1} & \to 0 \\
onto & \onto & j^* & \downarrow & \downarrow & j^* & \onto & \onto
\end{array}
\]

It is sufficient to compute \( d_2 \) and \( \Phi_2 \) for the top row. Recall that
\[
\cup m^2: H^{n-2}(R^* S^k, PS^k; \mathbb{Z}_2) \to H^n(R^* S^k, PS^k; \mathbb{Z}_2),
\]
while \( \text{Sq}^2 \), on the same group, is nonzero if and only if \((\tau_{2-1}) = 1 \mod 2\). Part (I) now follows immediately from 6.1.1 and 6.1.2, while (II) follows from 6.1.1, 6.1.2, and the results of [10].

**6.2. Proof of 6.1.2.** Let \( p: \tilde{N} \to N \) be the universal covering of \( N \); \( \tilde{N} = S^2 \times R^{n-2} \). Since \( f \) may be assumed to be inessential, we may choose an embedding \( \tilde{f}: S^k \to \tilde{N} \) such that \( pf = f \). Note that \( R^\infty \simeq R^\infty \times R^\infty \). Thus we may choose an embedding \( P \) such that the following diagram commutes,

where \( p_1 \) is projection:

\[
\begin{array}{c}
\tilde{N} \times R^\infty \xrightarrow{p} N \times R^\infty \\
\downarrow p_1 & \downarrow p_1 \\
\tilde{N} & \rightarrow N
\end{array}
\]
Let $\nabla N = (p^2)^{-1} \nabla N - \Delta \tilde{N} \subset R\tilde{N}$, and let $\nabla^* N \subset R^*\tilde{N}$ be the image of $\nabla N$. Let $\tilde{R}N = R\tilde{N} - \nabla N$ and $\tilde{R}^*N = R^*\tilde{N} - \nabla^* N$. Consider the diagram

$\xymatrix{ R^*(\tilde{N} \times R^\infty) \ar[r]^{R^*p} \ar@{^{(}->}[d] & R^*(N \times R^\infty) \ar@{^{(}->}[d] \\
 R^*\tilde{N} \ar[r] & R^*N}$

We have that $\Phi = \text{fiber of } R^*N \subset R^*(\tilde{N} \times R^\infty) = \text{fiber of } R^*N \subset R^*(N \times R^\infty)$.

**Lemma 6.2.1 (Haefliger [4]).** (I) As an algebra over $\mathbb{Z}_2$, $H^\ast = H^\ast(R^\ast(\tilde{N} \times R^\infty); \mathbb{Z}_2) = H^\ast(TS^2; \mathbb{Z}_2)$ is generated only by the elements: $m \in H^1$, $\Gamma \sigma \in H^2$, and $\Sigma \sigma \in H^4$, subject only to the relations: $m \Gamma \sigma = (\Gamma \sigma)^2 = \Gamma \sigma \Sigma \sigma = (\Sigma \sigma)^2 = 0$.

(II) $(R^\ast i) : H^\ast \to H^\ast(R^\ast N; \mathbb{Z}_2)$ is surjective, and its kernel is generated by $m^2$ and $m^{-2} \Sigma \sigma$.

**Lemma 6.2.2.** The map $i^\ast : H^\ast(R^\ast \tilde{N}; \mathbb{Z}_2) \to H^\ast(R^\ast N; \mathbb{Z}_2)$ is injective, and its cokernel is generated over $\mathbb{Z}_2$ by $\omega \in H^{n-1}(R^\ast N; \mathbb{Z}_2)$, $m \omega$, and $m^2 \mu$. Furthermore, $(\Gamma \sigma) \omega = m^2 \omega$, and $Sq^i \omega = (\xi_i \omega)m\omega$ for all $i$.

**Proof.** We construct a map $\beta : R^\ast \tilde{N} \to R^\ast N$ such that $i\beta$ is homotopic to the identity on $R^*\tilde{N}$. Recall that $N = N^n$, the total space of $\xi_i^{n-2} = \xi$ over $P^2$. Choose a nonzero section $\chi^\ast$ of $\xi \otimes h$, and let $\chi : S^2 \to S^2 \times (\pi^{n-2} - \{0\})\tilde{N}$ be the corresponding section of $\pi^{-1}(\xi \otimes h)$, the trivial $n$-bundle over $S^2$. Let $p_1 : \tilde{N} \to S^2$, $p_2 : \tilde{N} \to \pi^{n-2}$ be the projections. For all $0 < \epsilon < 1$, let $\beta_\epsilon : R^\ast \tilde{N} \to R^\ast N$ be defined as follows. If $[x,y] \in R^\ast \tilde{N}$, let $\rho = \rho(x,y) = \|p_1x - p_1y\|$, and let:

$$
\beta_\epsilon[x,y] = \begin{cases} [x,y] & \text{if } 1 < \rho < 2, \\
(p_1x, (1 + (\rho - 1)\epsilon)p_2x + (1 - \rho)p_2(\chi(p_1x)))(1, \rho_2x, (1 + (\rho - 1)\epsilon)p_2y + (1 - \rho)p_2(\chi(p_1y))) & \text{if } 0 < \rho < 1.
\end{cases}
$$

Let $\beta = \beta_1$, which clearly has the desired property. Now the cofiber of the inclusion $R^\ast N \subset R^\ast \tilde{N}$ is the Thom complex of $\tau \otimes h$ over $N$, which we denote by $TC$, where $\tau$ is the tangent bundle of $N$. We have a commutative diagram with split exact columns, where each row is a long exact Thom-Gysin sequence, and all coefficients are $\mathbb{Z}_2$: 
We now define $\omega \in H^{n-1}(R^*N; \mathbb{Z}_2)$ to be the unique element such that $\beta^*\omega = 0$ and $\delta \omega = U$, the Thom class of $\tau \otimes h$. Now $H^*(TC)$ has only generators $U$, $mU$, and $m^2U$ over $\mathbb{Z}_2$, and $Sq^iU = \binom{i-1}{2}m^iU$ for all $i$, although $m^3U = 0$. By exactness of the third row, $\theta((\sigma \cup \pi^*U) = \theta((\sigma \otimes 1) \cup \pi^*U)$ must be $m^2U$. Also, $\theta(\sigma \otimes 1) = 1_\sigma$, thus $(1_\sigma)\omega = \theta((\sigma \otimes 1) \cup \pi^*\omega) = m^2\omega$. 6.2.2 then follows trivially.

Finally, Lemma 6.1.2 follows from routine Serre spectral sequence computation of the relative Postnikov tower of the inclusion $\tilde{R}^*N \subset R^*(N \times R^\infty)$.

7. Affine groups.

**Definition 7.1.** An affine group is a set $A$ together with an operation $\cdot_a$ for each $a \in A$, such that $(A, \cdot_a)$ is a group with identity $a$, and such that, for any $x, y, a, b \in A$, $x \cdot_a y = x \cdot_b (a)_{b^{-1}} \cdot_y$, where $(a)_{b^{-1}}$ is the inverse of $a$ under $\cdot_b$. In addition, we say that $A$ is Abelian if $\cdot_a$ is commutative for all $a \in A$. (The operation shall then usually be denoted $+_a$.)

We remark that Becker’s definition of affine group [1] is that of Abelian affine group.

An alternative definition, which is more natural to the constructions of this paper, is:

**Definition 7.2.** An affine group is a set together with a ternary operation $\tau$ on $A$ such that (writing $\tau(x, y, z) = xy^{-1}z$)

(i) $vw^{-1}(xy^{-1}z) = (vw^{-1}x)y^{-1}z$ (associative law),
(ii) $xy^{-1}y = yy^{-1}x = x$ (cancellation law).

In addition, we say that $A$ is Abelian if

(iii) $xy^{-1}z = zy^{-1}x$ (commutative law).

We leave the proof of the equivalence of 7.1 and 7.2 to the reader as an exercise (Hint: $x \cdot_a y = xa^{-1}y$). Note that the empty set is an affine group.
Letting $A$ be any affine group, we define two equivalence relations on $A \times A$, $R$ and $L$. For any $x, y, z, w \in A$, we say that $(x, y)R(z, w)$ if $y = xz^{-1}w$, and $(x, y)L(z, w)$ if $x = zw^{-1}y$. (Note that $R = L$ if $A$ is commutative.) Let $AR = A \times A/R$, and $AL = A \times A/L$. If $x, y \in A$, let $x^{-1}y \in AR$ and $xy^{-1} \in AL$ denote the elements represented by the ordered pair $(x, y)$. Both $AR$ and $AL$ are groups, under the operations $(x^{-1}y)(z^{-1}w) = x^{-1}(yz^{-1}w)$, and $(xy^{-1})(zw^{-1}) = (xy^{-1}z)w^{-1}$, respectively. We call $AR$ and $AL$ the right and left action groups of $A$, since $AR$ acts on $A$ on the right, in the obvious manner suggested by the notation, and $AL$ acts on $A$ on the left.

We remark that $AR$ and $AL$ are isomorphic, canonically if $A$ is Abelian, in which case we write $A^* = AR = AL$, and $x - y$ instead of $xy^{-1}$ or $y^{-1}x$.

We remark that two affine groups are isomorphic if and only if their action groups are isomorphic. If $G$ is any group, $G$ may be taken to be an affine group by letting $xy^{-1}z$ have the usual meaning ($x - y + z$ if $G$ Abelian), in which case $G \cong G^L$ by letting $x$ correspond to $x1^L$.

If we say that an affine group $A$ is isomorphic to a group $G$, we shall mean that $A$ is isomorphic to $G$ considered as an affine group; equivalently, $AR \cong G$ or $AL \cong G$.

8. Correction. Recall diagram (2.6-1) in this paper, which is basically diagram (3.2-1) of [7]. We define a pair fibration $\xi_M: (Y_M, Z_M) \to (\infty R^* M, \infty P M)$ as follows: Let $Y_M$ be the space consisting of all paths $\sigma: I \to \infty R^* M$ such that $\sigma(1) \in R M$, and either $\sigma(0) \in \infty P M$ or $\sigma(1) \notin P M$. Let $Z_M$ be the space consisting of all paths $\sigma: I \to \infty P M$ such that the composition $I \to \infty P M \to M$ is constant. Let $\xi_M: (Y_M, Z_M) \to (\infty R^* M, \infty P M)$ be evaluation at 0, a pair fibration, and let $i: \xi_M \subset \xi_M$ be the obvious inclusion.

**Lemma 8.1.** $i#: [(R^* V, PV); \xi_M]_F \cong [(R^* V, PV); \xi_M]_F$.

**Proof.** Let $\xi_M: (Y_M, Z_M) \to (\infty R^* M, \infty P M)$. Now the inclusion $i: (Y_M, Z_M) \subset (Y_M, Z_M)$ is a fiber homotopy equivalence of pairs. Let $U \subset \infty R^* M$ be a regular collar of $\infty P M$, such that $U \cap R M$ is a regular collar of $P M$. Let $r_t: \infty R^* M \to R^* M$, $0 < t < 1$, with $r_0 = \text{identity}$, be a strong deformation retraction of $\infty R^* M$ onto the complement of $U$ such that $r_t(R^* M) \subset R^* M$ for all $t$. Let $\rho: \infty R^* M \to I$ be continuous such that $\rho^{-1}(0) = \infty P M$, and let $(j*)(i) = r_i(t)$ for all $\sigma \in Y_M$, $t \in I$, where $s = tr_0(\sigma(0))$. Clearly $j$ is the identity on $\infty P M$. The fact that $j$ is a pair fiberwise homotopy inverse of $i$ is trivial to verify. Thus $i#: [(R^* V, PV); \xi_M]_F \cong [(R^* V, PV); \xi_M]_F$. Now $\pi_k(\xi_M) = \pi_k(\xi_M)$ for all $k$, since $Z_M \to \infty P M$ and $Z_M \to \infty P M$ both have fibers of the homotopy type of $S^{n-1}$. By 2.4.1, we are done.

The error in the proof of Theorem 3.3.1 of [7] is the fact that if a lifting $\Phi$
of $F$ to $\zeta_M$ is given, the map $G[\Phi]$ in the diagram on p. 362 may not exist. If, on the other hand, $\Phi$ is a lifting of $F$ to $\zeta_M$ (which we may assume, by Lemma 8.1), existence of $G[\Phi]$ is assured. The remainder of the proof is valid.

Theorem 3.3.2 of [7] can be corrected in a similar manner.

BIBLIOGRAPHY


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