TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 239, May 1978

ISOTOPY GROUPS

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ABSTRACT. For any mapping $f: V \to M$ (not necessarily an embedding), where V and M are differentiable manifolds without boundary of dimensions k and n, respectively, V compact, let $[V \subset M]_f = \pi_1(M^V, E, f)$, i.e., the set of isotopy classes of embeddings with a specific homotopy to f (E = space of embeddings). The purpose of this paper is to enumerate $[V \subset M]_f$. For example, if $k \geqslant 3$, $n = 2k$, and M is simply connected, $[S^k \subset M]_f$ corresponds to $\pi_2 M$ or $\pi_2 M \otimes Z_2$, depending on whether k is odd or even. In the metastable range, i.e., $3(k+1) > 2n$, a natural Abelian affine structure on $[V \subset M]_f$ is defined: if, further, f is an embedding $[V \subset M]_f$ is then an Abelian group. The set of isotopy classes of embeddings homotopic to f is the set of orbits of the obvious left action of $\pi_1(M^V, f)$ on $[V \subset M]_f$.

A spectral sequence is constructed converging to a theory $H^*(f)$. If $3(k+1) < 2n$, $H^0(f) \cong [V \subset M]_f$ provided the latter is nonempty. A single obstruction $\Gamma(f) \in H^1(f)$ is also defined, which must be zero if f is homotopic to an embedding; this condition is also sufficient if $3(k+1) < 2n$. The E_2 terms are cohomology groups of the reduced deleted product of V with coefficients in sheaves which are not even locally trivial. $[S^k \subset M]_f$ is specifically computed in terms of generators and relations if $n = 2k$, $k \geqslant 3$ (Theorem 6.0.2).

1. **Introduction.** In this paper, in some respects a sequel to [7], we attack the general problem of classifying, up to isotopy, embeddings of a compact k-manifold V in an n-manifold M in the metastable range, i.e., where $3(k+1) < 2n$.

Differentiable shall mean infinitely differentiable, manifold shall mean differentiable manifold without boundary (either compact or open), with a countable base; embedding shall mean differentiable embedding, and isotopy shall mean homotopy of embeddings.

Let $[V \subset M]$ denote the set of isotopy classes of embeddings of V in M. Computation of $[V \subset M]$ is the ultimate goal. Unfortunately, $[V \subset M]$ has no convenient algebraic structure. Thus we introduce a new object, $[V \subset M]_f$, where $f: V \to M$ is a specific map. A homotopy $f_t: V \to M$, $0 \leqslant t \leqslant 1$, is

Received by the editors September 28, 1973 and, in revised form, September 7, 1976.

AMS (MOS) subject classifications (1970). Primary 57D40, 55G35; Secondary 55H25.

Key words and phrases. Isotopy, obstructions to embedding, twisted cohomology.

[1]The author wishes to thank Robert D. Rigdon, who gave him invaluable help in reorganizing the pair fibration concepts.

called an embedding-homotopy (or e-homotopy) of f if $f_0 = f$ and f_1 is an embedding. If $f_{u,t}$: $V \to M$, for $0 \leq t$, $u \leq 1$ is a two-parameter homotopy such that, for all u, $f_{u,0} = f$ and $f_{u,1}$ is an embedding, we say that the e-homotopies $\{f_{0,t}\}$ and $\{f_{1,t}\}$ are isotopic. $[V \subset M]_f$ is then defined to be the set of isotopy classes of e-homotopies of f. If $\{f_t\}$ is an e-homotopy of f, let $[f_t] \in [V \subset M]_f$ be its isotopy class. It shall be shown that $[V \subset M]_f$ is an Abelian affine group in the metastable range, and an actual Abelian group if f is an embedding.

If $2k - n \leq 0$, the group $[S^k \subset M^n]_f$ is, in this paper, expressed in terms of generators and relations, involving only $\pi_1 M$ if $n = 2k + 1$, and involving both $\pi_1 M$ and $\pi_2 M$ if $n = 2k$. For $n > 2k + 1$, $[V^k \subset M^n]_f = 0$.

Generally (although not shown in this paper) the affine structure of $[V^k \subset M^n]_f$ in the metastable range depends only on the homotopy of M through dimension $2k - n + 2$, as well as on V.

The case $n = 2k + 1$ was done in [7], but the result was incorrectly stated. See Theorem 6.0.1 for the correct version. (Another error in [7], an invalid proof of Theorems 3.3.1 and 3.3.2, is corrected here in §8. The error was pointed out by the referee of this paper.) The case $n = 2k$ is computed for the first time in this paper, and involves evaluation of one nonzero differential and one nontrivial extension in a spectral sequence whose E_2 terms are cohomology groups of the reduced deleted product of S^k with coefficients in sheaves which do not, in general, have a local product structure. See Theorem 6.0.2 for the general result. Some specific cases are as follows (where f is any embedding):

THEOREM 1.0.1. *If $M = M^{2k}$ is simply connected, $k \geq 3$, then*:

$$[S^k \subset M]_f \cong \begin{cases} \pi_2 M \otimes Z_2 & \text{if } k \text{ is even,} \\ \pi_2 M & \text{if } k \text{ is odd.} \end{cases}$$

THEOREM 1.0.2. *If $k \geq 3$*:

$$[S^k \subset P^2 \times R^{2k-2}]_f \cong \begin{cases} Z_2 + Z_2 & \text{if } k \text{ is even,} \\ Z + Z + Z_2 & \text{if } k \text{ is odd.} \end{cases}$$

THEOREM 1.0.3. *If $k \geq 3$*:

$$[S^k \subset P^{2k}]_f \cong \begin{cases} 0 & \text{if } k \equiv 0 \bmod 4, \\ Z_4 & \text{if } k \equiv 1 \bmod 4, \\ Z_2 & \text{if } k \equiv 2 \bmod 4, \\ Z_2 + Z_2 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Let $\pi = \pi_1(M^V, f)$. We may define a left action:

$$\mu \colon \pi \times [V \subset M]_f \to [V \subset M]_f$$

as follows: if $\{f_t\}$ is an e-homotopy of f and if $\{f_t'\}$ is a self-homotopy of f representing $a \in \pi$, then $\mu(a, [f_t]) = [f_t'']$ where $f_t'' = f_{2t}$ or f_{2t-1}', depending on the value of t. The action μ respects the affine structure of $[V \subset M]_f$, i.e., for each $a \in \pi$, $\mu(a,)$ is an affine automorphism (cf. Theorem 2.7.1).

Now let ϕ: $[V \subset M]_f \to [V \subset M]$ be the forgetful function, i.e., if $\{f_t\}$ is an e-homotopy of f, $\phi[f_t] = [f_1]$, the isotopy class containing f_1. Trivially, the reader can convince himself of the following:

(i) Every element of $[V \subset M]$ lies in the image of ϕ for some choice of f.

(ii) If $x, y \in [V \subset M]_f$, then $\phi x = \phi y$ if and only if $\mu(a, x) = y$ for some $a \in \pi$.

Classification, up to isotopy, of embeddings of V in M in the stable range may thus be reduced to the following two problems:

I. Compute the affine structure of $[V \subset M]_f$, for each f: $V \to M$.

II. Determine the action μ of $\pi_1(M^V, f)$ on $[V \subset M]_f$.

It is to the first of these problems that this paper is addressed.

In §2, we use Haefliger's results [4] to define the affine structure on $[V \subset M]_f$. An alternative, equivalent definition of the affine structure is given in §4; this definition is much more geometric and easier to comprehend, but it is the Haefliger definition that yields to computation, by homotopy methods. In §3, a spectral sequence is defined, in §5, the sheaves we need are computed, and in §6 the spectral sequence is fully worked in the special cases.

Using somewhat different methods, Dax [2] and Salomonsen [12] have obtained results on the same and similar problems.

2. The affine structure of $[V \subset M]_f$.

2.1. *Pair fibrations and weak pair fibrations.* Following [11], we say that a map p: $(E, E_0) \to (X, X_0)$ is a *pair fibration* if for any given map h: $(Y, Y_0) \to (E, E_0)$ and every homotopy f_t: $(Y, Y_0) \to (X, X_0)$, $0 \leq t \leq 1$, such that $f_0 = ph$, there exists a homotopy h_t: $(Y, Y_0) \to (E, E_0)$ such that $h_0 = h$ and $ph_t = f_t$ for all t.

A *section* of a pair fibration is a map g: $(X, X_0) \to (E, E_0)$ such that $pg = 1$ (the identity). A *lifting* of f: $(Y, Y_0) \to (X, X_0)$ to p is a map h: $(Y, Y_0) \to (E, E_0)$ such that $ph = f$. Sections and liftings are said to be *homotopic* if they are homotopic *as* sections or liftings (i.e., fiber homotopic).

All the theorems in §§2 and 3 which deal with sections of pair fibrations are equivalent to corresponding theorems which deal with liftings, since a lifting of f to p corresponds to a section of $f^{-1}p$ (where f^{-1} is the usual pullback construction).

If (X, X_0) is a topological pair, we say that (A, A_0) is a *sub-pair* of (X, X_0) if $A \subset X$ and $A_0 = A \cap X_0$. Then $((X, X_0), (A, A_0))$ is a *pair of pairs.* If X is a C.W. complex, and X_0 and A are subcomplexes, it is a C.W. pair of pairs.

For the following definitions, let (as needed) f: $((Y, Y_0), (B, B_0)) \to$

$((X, X_0), (A, A_0))$ be a map of pairs of pairs, let $z\colon (A, A_0) \to (E, E_0)$ be a partial section of p, let $w\colon (B, B_0) \to (E, E_0)$ be a partial lifting of f to p.

DEFINITION 2.1.1. Let $[(X, X_0); p]$ be the set of homotopy classes of sections of p.

DEFINITION 2.1.2. Let $[(Y, Y_0); p]_f$ be the set of homotopy classes of liftings of f to p.

DEFINITION 2.1.3. Let $[(X, X_0), (A, A_0); p]^z$ be the set of rel(A, A_0) homotopy classes of sections of p which extend z. If p has a standard section, say $s\colon (X, X_0) \to (E, E_0)$, then we shall always presume (unless otherwise stated) that $z = s|(A, A_0)$ and suppress z in the notation.

DEFINITION 2.1.4. Let $[(Y, Y_0), (B, B_0); p]_f^w$ be the set of rel(B, B_0) homotopy classes of liftings of f to p which extend w. If p has a standard section, s, presume that $w = sf|(B, B_0)$ and suppress w in the notation.

The following lemma will be important later.

LEMMA 2.1.5. *If $((X, X_0), (A, A_0))$ is a* C.W. *pair of pairs, p is a pair fibration over (X, X_0), $\{z_t\}$ is a homotopy of partial sections of p over (A, A_0), there is a one-to-one correspondence $[(X, X_0), (A, A_0); p]^{z_0} \cong [(X, X_0), (A, A_0); p]^{z_1}$ such that, whenever $\{g_t\}$ is a homotopy of sections of p which extends $\{z_t\}$, $[g_0]$ (the homotopy class containing g_0) corresponds to $[g_1]$.*

PROOF. By induction on the skeleta of X. We omit the details.

2.2. *Fiberwise suspensions.* Choose standard spheres and balls $S^{N-1} \subset B^N$ for all $N \geqslant 0$ (where S^{-1} is empty) such that $S^0 * S^{N-1} = S^N$ and $S^0 * B^N = B^{N+1}$ (where $* =$ join). Let each sphere have a South pole and a North pole, preserved under inclusion $S^N \subset S^{N+1}$. The South pole will be considered to be the base point of each sphere.

If $p\colon E \to X$ is a fibration, let $S^N p\colon S_X^N E \to X$, where $S_X^N E = S^N *_X E$, the fiberwise join of $p\colon E \to X$ with the trivial fibration $X \times S^N \to X$. We give $S^N *_X E$ the strong topology, thus insuring that $S^N p$ is a fibration [5]. Note that $S^0 p = p$, and that for $N \geqslant 1$, $S^N p$ has two standard sections, the South polar and the North polar, denoted s_0 and s_1. If only one section is needed, s_0 shall be used. Note also that $Sp = S^1 p$ is the fiberwise two point suspension of p.

Now, for any $N \geqslant 0$, let $P^N S^N p\colon P_X^N S_X^N \to X$ be defined as follows. For each $x \in X$, let $(P^N S^N p)^{-1}\{x\}$ be the set of all maps $\sigma\colon B^N \to (S^N p)^{-1}\{x\}$ such that $\sigma|S^{N-1}$ is simply the identity (recall that each fiber of $S^N p$ contains a copy of S^{N-1}). Let $P_X^N S_X^N E$ then have the topology of a subspace of $(S_X^N E)^{B^N}$ with the compact-open topology. We can define a fiber-preserving inclusion $P_X^N S_X^N E \subset P_X^{N+1} S_X^{N+1} E$ by identifying each $\sigma \in P_X^N S_X^N E$ with $S^0 * \sigma$.

Now if $M \geqslant 0$, $N \geqslant 1$, let $\Omega^M S^N p\colon \Omega_X^M S_X^N E \to X$ be given by

$(\Omega^M S^N p)^{-1}\{x\} = \Omega^M (S^N p)^{-1}\{x\}$, the M-fold loop space, which we take to be the set of maps $(B^M, S^{M-1}) \to ((S^N p)^{-1}\{x\}, s_0 x)$, for any $x \in X$. We use the compact open topology for $\Omega^M_X S^N_X E$, and again there is an inclusion $\Omega^M_X S^M_X E \subset \Omega^{M+1}_X S^{N+1}_X E$.

The above constructions generalize naturally to pair fibrations. If $p: (E, E_0) \to (X, X_0)$ is a pair fibration, we define:

$$S^N p: \left(S^N_X E, S^N_{X_0} E_0\right) \to (X, X_0),$$

$$P^N S^N p: \left(P^N_X S^N_X E, P^N_{X_0} S^N_{X_0} E_0\right) \to (X, X_0),$$

$$\Omega^M S^N p: \left(\Omega^M_X S^N_X E, \Omega^M_{X_0} S^N_{X_0} E_0\right) \to (X, X_0).$$

2.3. *Groups and affine groups of sections and liftings*. Henceforth in this section, let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, and $p: (E, E_0) \to (X, X_0)$ a pair fibration.

THEOREM 2.3.1. *If* $N \geqslant 1$, $M \geqslant 1$, *then* $[(X, X_0), (A, A_0); \Omega^M S^N p]$ *is a group. Furthermore, it is Abelian if* $M \geqslant 2$.

We omit the proof of 2.3.1: simply follow, fiberwise, the usual proof that $[X, A; \Omega^M Y]$ is a group, Abelian if $M \geqslant 2$, if Y is any pointed space.

The affine structure. Let $N \geqslant 1$, and fix a partial section of $P^N S^N p$ over (A, A_0). We proceed to define a ternary operation, τ, on $[(X, X_0), (A, A_0); P^N S^N p]^z$.

Let U and L be the upper and lower hemispheres, respectively, of ∂B^N. Let B^N_1, B^N_2, and B^N_3 be copies of B^N, with upper and lower hemispheres U_1, L_1, U_2, etc. Now choose a homeomorphism $\phi: B^N \to W = (B^N_1 \cup B^N_2 \cup B^N_3)/\sim$, where "$\sim$" identifies U_1 with U_2 and L_2 with L_3, and $\phi: L \cong L_1$, $\phi: U \cong U_3$. (Note that ϕ has degree 1 in homology onto B^N_1 and B^N_3, and -1 onto B^N_2.) Now every section of $P^N S^N p$ may be thought of as a map $X \times B^N \to S^N_X E$ satisfying the appropriate conditions. If g_1, g_2, and g_3 are sections of $P^N S^N p$, we define a new section of $P^N S^N p$ called $g_1 \wedge g_2^{-1} \wedge g_3$, by commutativity of the following diagram for each $i = 1, 2, 3$:

$$\begin{array}{ccc} X \times B^N & \xrightarrow{g_1 \wedge g_2^{-1} \wedge g_3} & S^N_X E \\ \downarrow{\scriptstyle 1_X \times \phi} & & \uparrow{\scriptstyle g_i} \\ X \times W & \supset X \times B^N_i \cong & X \times B^N \end{array}$$

Now if g is any section, $g \wedge g^{-1} \wedge g$ is canonically homotopic to g itself. (To see this, let $\chi: W \to B^N$ be the "triple folding" map obtained by identifying each B^N_i with B^N. Now choose a rel S^{N-1} homotopy of $\chi \circ \phi: B^N \to B^N$ with the identity; this gives a homotopy of g with $g \wedge g^{-1} \wedge g = 1_X \times (\chi \circ \phi)$. This homotopy is functorial with respect to maps of pair fibrations.) Thus, since $z \wedge z^{-1} \wedge z$ is homotopic to z, we obtain, by Lemma

2.1.5, a one-to-one correspondence

$$\xi\colon \left[(X, X_0), (A, A_0); P^N S^N p\right]^{z \wedge z^{-1} \wedge z} \cong \left[(X, X_0), (A, A_0); P^N S^N p\right]^{z}.$$

Now define $\tau([g_1], [g_2], [g_3]) = [g_1][g_2]^{-1}[g_3] = \xi[g_1 \wedge g_2^{-1} \wedge g_3]$.

THEOREM 2.3.2. (I) *Under* τ, $[(X, X_0), (A, A_0); P^N S^N p]^z$ *is an affine group. If* $N > 2$, *it is an Abelian affine group.* (II) *If* $[(X, X_0), (A, A_0); P^N S^N p]^z$ *is nonempty, its left action group is* $[(X, X_0), (A, A_0); \Omega^N S^N p]$.

PROOF. It is a routine exercise to show that τ is associative and satisfies cancellation, and thus is an affine structure. To obtain (II), consider the map $\eta\colon B^N \to Y = (B_1^N \cup B_2^N)/\sim \cong S^N$ where "$\sim$" identifies S_1^{N-1} with S_2^{N-1}, and η maps all of S^{N-1} onto the common South pole, and is of degree 1 onto B_1^N, degree -1 onto B_2^N. Then let $[g_1][g_2]^{-1} = [g_1 \wedge g_2^{-1}]$, where, for $i = 1$ or 2, the following diagram commutes:

$$\begin{array}{cccc}
X \times B^N & \xrightarrow{g_1 \wedge g_2^{-1}} & & S_X^N E \\
\downarrow 1_X \times \eta & & & \uparrow \\
X \times Y & \supset X \times B_i^N & \cong & X \times B^N
\end{array}$$

Clearly $g_1 \wedge g_2^{-1}$ is a section of $\Omega^N S^N p$, and verification of the remaining details is routine. (Hint: $z \wedge z^{-1}$ is canonically homotopic to a trivial section.)

The commutativity of τ in the case $N > 2$ now follows from 2.3.1, and we are done with the proof of 2.3.2.

In certain cases which turn out to be important later there is a geometric definition of the affine structure. Suppose that U_1 and U_2 are open subsets of X such that $U_1 \cup U_2 = X$, and suppose that z is a partial section of $P^N S^N p$ over (A, A_0).

PROPOSITION 2.3.3. *If* g_1 *and* g_2 *are sections of* $\Omega^M S^N p$, *for* $M > 1$, $N > 1$, *such that* g_1 *and are both trivial on* (A, A_0), *and such that* $g_1 | U_2$ *is trivial and* $g_2 | U_1$ *is trivial, let* g_3 *be the section which agrees with* g_1 *on* U_1, *with* g_2 *on* U_2, *and is trivial elsewhere. Then* $[g_1][g_2] = [g_3]$.

PROOF. It is clear that g_3 is homotopic to $g_1 \wedge g_2$ on $X - U_1$ and also on $X - U_2$; they are both trivial on $U_1 \cap U_2$. Homotopies on the two closed sets may easily be chosen, and do not interfere.

PROPOSITION 2.3.4. *If* g_1, g_2, *and* g_3 *are sections of* $P^N S^N p$, *for* $N > 1$, *such that all agree with* z *on* (A, A_0), *and such that* g_1 *agrees with* g_2 *outside of* U_2, *and* g_3 *agrees with* g_2 *outside of* U_3, *let* g_4 *be the section where, for all* $x \in X$,

$$h_4(x) = \begin{cases} h_1(x) & \text{if } x \notin U_2, \\ h_2(x) & \text{if } x \in U_1 \cap U_2, \\ h_3(x) & \text{if } x \notin U_1. \end{cases}$$

Then $[h_1][h_2]^{-1}[h_3] = [h_4]$.

Proof. Note that $h_1 \wedge h_2^{-1} \wedge h_3$ and h_4 are homotopic, albeit via different standard homotopies, on each of the two sets U_1 and U_2. By choosing a Urysohn function $X \to I$ which is 0 on the complement of U_2 and 1 on the complement of U_1, these homotopies can be made to blend smoothly into one another over $U_1 \cap U_2$, since on that region h_1, h_2, h_3, and h_4 all agree.

2.4. *Equivalence of classes of sections.* Let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, and let $p: (E, E_0) \to (X, X_0)$ and $p': (E', E_0') \to (X, X_0)$ be pair fibrations. Let $\gamma: (E, E_0) \to (E', E_0')$ be a fiber map, and $z: (A, A_0) \to (E, E_0)$ a partial section of p. Finally, define $z' = \gamma \cdot z$, a partial section of p' over (A, A_0). We have an obvious function, induced by composition with γ:

$$\gamma_{\#}: \left[(X, X_0), (A, A_0); p\right]^z \to \left[(X, X_0), (A, A_0); p'\right]^{z'}.$$

Under certain conditions on the homotopy of the fibers, $\gamma_{\#}$ is one-to-one or onto.

Let $E_x = p^{-1}\{x\}$, for any $x \in X$, and let $(E_0)_x = E_x \cap E_0$. Similarly, let $E_x' = (p')^{-1}\{x\}$ and $(E_0')_x = E_x' \cap E_0'$.

Theorem 2.4.1. *Suppose that X/A is finite dimensional. Let $n = \dim(X/(X_0 \cup A))$, and $n_0 = \dim(X_0/A_0)$. (I) Suppose that, for all $x \in (X - (X_0 \cup A))$, $\gamma_*: \pi_k(E_x) \to \pi_k(E_x')$ is one-to-one and onto for all $0 \leq k < n$ and onto for $k = n$; and that, for all $x \in (X_0 - A_0)$, $\gamma_*: \pi_k(E_0)_x \to \pi_k(E_0')_x$ is one-to-one and onto for all $0 \leq k < n_0$, and onto for $k = n_0$. Then $\gamma_{\#}$ is onto. (II) Suppose that, for all $x \in (X - (X_0 \cup A))$, $\gamma_*: \pi_k(E_x) \to \pi_k(E_x')$ is one-to-one and onto for all $0 \leq k \leq n$ and onto for $k = n + 1$; and that $\gamma_*: \pi_k(E_0)_x \to \pi_k(E_0')_x$ is one-to-one and onto for all $0 \leq k \leq n_0$ and onto for $k = n_0 + 1$ for all $x \in (X_0 - A_0)$. Then $\gamma_{\#}$ is one-to-one.*

Proof. This theorem is simply a pair-fibration version of a well-known result in fibrations. (See, for example, Lemma 2.2 of [8].) The proof is unaltered by the pair nature, since it is done inductively one cell at a time.

Definition 2.4.2. We say that p is *n-connected* if, for all $k \leq n$, $\pi_k(E)_x = 0$ for all $x \in X$ and $\pi_k(E_0)_x = 0$ for all $x \in X_0$.

Corollary 2.4.3. *If p is n-connected and* $\dim(X/A) \leq 2n$, *then* $[(X, X_0), (A, A_0); p]^z$ *is an Abelian affine group.*

Proof. The inclusion $i: p \subset P^N S^N p$ induces isomorphism on the homotopy of the fibers up through dimension $2n$, and epimorphism in dimension $2n + 1$

(since each fiber of $P^N S^N p$ is of the same homotopy type as the N-fold loop space of the N-fold suspension of the corresponding fiber of p). Thus, by 2.4.1, $[(X, X_0), (A, A_0); p]^z \cong [(X, X_0), (A, A_0); P^N S^N p]^{iz}$ for all N. Apply 2.3.2, and we are done.

We remark that the Abelian affine structure on $[(X, X_0), (A, A_0); p]^z$, under the hypotheses of 2.4.3, can be alternatively defined in manner described in 2.3.4. Verification is trivial.

2.5. *The obstruction theory*. Let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, $p: (E, E_0) \to (X, X_0)$ a pair fibration, and z a partial section of p over (A, A_0).

DEFINITION 2.5.1. For any integer i, let

$$H^i((X, X_0), (A, A_0); p) = \lim_{N\to\infty} \left[(X, X_0), (A, A_0); \Omega^{N-i} S^N p\right],$$

the direct limit. Write $H^i((X, X_0); p)$ if A is empty.

We remark in passing that $H^*(\; ; p)$ is a cohomology theory in a certain sense; and satisfies a version of the Eilenberg-Steenrod axioms. In particular, if $f: ((Y, Y_0), (B, B_0)) \to ((X, X_0), (A, A_0))$ is a map of C.W. pairs of pairs, there is an induced homomorphism

$$f^*: H^i((X, X_0), (A, A_0); p) \to H^i\left((Y, Y_0), (B, B_0); f^{-1}p\right),$$

and if $q: (F, F_0) \to (X, X_0)$ is another pair fibration and $\gamma: (E, E_0) \to (F, F_0)$ a fiber preserving map, there is an induced homomorphism

$$\gamma_\#: H^i((X, X_0), (A, A_0); p) \to H^i((X, X_0), (A, A_0); q).$$

REMARK 2.5.2. If p is n-connected and $\dim(X/A) < 2n$, then $[(X, X_0), (A, A_0); p]^z$ (an Abelian affine group, by 2.4.3), if nonempty, has $H^0((X, X_0), (A, A_0); p)$ as its action group.

Thus, classification of sections of a pair fibration, under suitable dimensional restrictions, reduces to algebraic computation of a cohomology group. In the next section, we shall show how $H^*((X, X_0), (A, A_0); p)$ can be attacked by a familiar spectral sequence technique.

The single obstruction. We now consider the question of whether z can be extended to a section of p. Recall that Sp has two sections, s_0 and s_1, the South and North polar sections. Now $S_X E$ may be obtained from $E \times I$ by collapsing the ends in the appropriate manner (although with the strong, not the quotient topology). Let $u_t x = [zx, t] \in S_A E$ for all $x \in A$. By 2.1.5, $\{u_t\}$ gives a one-to-one correspondence:

$$\sigma: \left[(X, X_0), (A, A_0); Sp\right]^{s_1|A} \cong \left[(X, X_0), (A, A_0); Sp\right].$$

DEFINITION 2.5.3. Let $\gamma(p) = \sigma[s_1] \in [(X, X_0), (A, A_0); Sp]$, the *primitive single obstruction* to section of p extending z.

DEFINITION 2.5.4. Let $\Gamma(p) \in H^1((X, X_0), (A, A_0); p)$ be the image of $\gamma(p)$ in the direct limit, the *single obstruction* to section of p extending z.

THEOREM 2.5.5. (I) *There exists a section of p extending* $z \Rightarrow \gamma(p) = 0 \Rightarrow \Gamma(p) = 0$. (II) *Suppose that p is n-connected and* $\dim(X/A) \leq 2n + 1$. *Then* $\Gamma(p) = 0 \Rightarrow \gamma(p) = 0 \Rightarrow$ *there exists a section of p extending z.*

PROOF. (I) If g is an extension of z, the homotopy $\{u_t\}$ can obviously be extended to a homotopy of s_0 with s_1, hence $\sigma[s_1] = [s_0] = 0$. The second implication is obvious. (II) If $\Gamma(p) = 0$, then $\gamma(p) = 0$ by 2.4.1. Thus, there is a homotopy between s_0 and s_1, extending $\{u_t\}$, hence a section of PSp extending iz. By 2.4.1, $i_\#$: $[(X, X_0), (A, A_0); p]^z \to [(X, X_0), (A, A_0); PSp]^{iz}$ is onto, and we are done.

The single difference. Suppose now that $\{z_t\}$ is a homotopy of partial sections of p over (A, A_0). Suppose that g_0 and g_1 are sections of p extending z_0 and z_1, respectively. Recalling the construction in the proof of 2.3.2, we have $(ig_1) \wedge (ig_0)^{-1}$, a section of ΩPp. Now that section extends $(iz_1) \wedge (iz_0)^{-1}$ which is homotopic (by continuously varying the first index from 1 to 0) to $(iz_0) \wedge (iz_0)^{-1}$, which in turn is canonically null-homotopic. Thus, using 2.1.5, we can define:

DEFINITION 2.5.6. Let $\delta(g_0, g_1;\ \{z_t\}) \in [(X, X_0), (A, A_0); \Omega Pp]$, the *primitive single difference* be the element represented by $(ig_1) \wedge (ig_0)^{-1}$. If $z_t = z$ for all t, write $\delta(g_0, g_1; z)$.

DEFINITION 2.5.7. Let $\Delta(g_0, g_1;\ \{z_t\}) \in H^0((X, X_0), (A, A_0); p)$, the *single difference class* be the image of $\delta(g_0, g_1;\ \{z_t\})$ in the direct limit. Write $\Delta(g_0, g_1; z)$ if $z_t = z$ for all t.

THEOREM 2.5.8 (I) *The homotopy* $\{z_t\}$ *can be extended to a homotopy of* g_0 *with* $g_1 \Rightarrow \delta(g_0, g_1;\ \{z_t\}) = 0 \Rightarrow \Delta(g_0, g_1;\ \{z_t\}) = 0$. (II) *Suppose that p is n-connected for some n, and* $\dim(X/A) \leq 2n + 1$. *Then* $\Delta(g_0, g_1;\ \{z_t\}) = 0 \Rightarrow \delta(g_0, g_1;\ \{z_t\}) = 0 \Rightarrow$ *there exists a homotopy of* g_0 *with* g_1 *which extends* $\{z_t\}$.

PROOF. Similar to that of 2.5.5.

2.6. *Obstructions to embedding and isotopy.* Much of the following material is from [7]. If M is any manifold, of dimension n, let PM be the projective bundle associated with the tangent bundle of M, let $R^*M = (M^2 - \Delta M)/T$, where $T(x,y) = (y, x)$, and let $\mathbf{R}^*M = R^*M \cup PM$, a manifold with boundary PM. Let $({}^N\mathbf{R}^*M, {}^NPM) = (\mathbf{R}^*(M \times R^N), P(M \times R^N))$, and let $({}^\infty\mathbf{R}^*M, {}^\infty PM)$ be the obvious union with the weak topology. The inclusion $(\mathbf{R}^*M, PM) \subset ({}^\infty\mathbf{R}^*M, {}^\infty PM)$ we replace by a fibration of pairs in a standard manner: Let Y_M be the space of all paths σ: $I \to {}^\infty\mathbf{R}^*M$ such that $\sigma(1) \in \mathbf{R}^*M$ and let Z_M be the space of all paths σ: $I \to {}^\infty PM$ such that $\sigma(1) \in PM$, and let ζ_M: $(Y_M, Z_M) \to ({}^\infty\mathbf{R}^*M, {}^\infty PM)$ be evaluation at 0, an $(n-2)$-connected pair fibration.

Now let V be a compact manifold of dimension k, and let f: $V \to M$ be a differentiable map. Choose, once and for all, an embedding $\mathbf{i}$: $V \subset R^\infty$. Let

$F = \mathbf{R}^*(f, \mathrm{i})$: $(\mathbf{R}^*V, PV) \to ({}^\infty\mathbf{R}^*M, {}^\infty PM)$. We now have a diagram (basically diagram (3.2-1) of [7], with slightly changed notation):

$$
\text{(2.6-1)} \qquad \begin{array}{ccc} & & (Y_M, Z_M) \\ & & \downarrow \zeta_M \\ (\mathbf{R}^*V, PV) & \xrightarrow{F} & ({}^\infty\mathbf{R}^*M, {}^\infty PM) \end{array}
$$

Now define a function ϕ: $[V \subset M]_f \to [(\mathbf{R}^*V, PV); \zeta_M]_F$ as follows. If $\{f_t\}$ is an embedding homotopy of f, i.e., $f_0 = f$ and f_1 is an embedding, let $\Phi\{f_t\}$: $(\mathbf{R}^*V, PV) \to (Y_M, Z_M)$ be defined as follows. For any $0 \leqslant u \leqslant 1$ and any $\mathbf{r} \in \mathbf{R}^*V$, $\Phi\{f_t\}(\mathbf{r})(u) = \mathbf{R}^*(f_u, (1-u)\mathrm{i})(\mathbf{r})$. Simply then let $\phi[f_t]$ be the homotopy class containing $\Phi\{f_t\}$. It follows directly[2] from Theorems 3.3.1 and 3.3.2 of [7] that:

THEOREM 2.6.1. (I) *If* $2n \geqslant 3(k+1)$, ϕ *is onto*. (II) *If* $2n > 3(k+1)$, ϕ *is one-to-one.*

DEFINITION 2.6.2. For any integer i, let $H^i(f) = H^i((\mathbf{R}^*V, PV); F^{-1}\zeta_M)$.

DEFINITION 2.6.3. Let $\Gamma(f) = \Gamma(F) \in H^1(f)$, the *single obstruction* to homotopy of f with an embedding.

DEFINITION 2.6.4. Suppose that $\{f_t^1\}$ and $\{f_t^2\}$ are both e-homotopies of f. Then let $\Delta(\{f_t^1\}, \{f_t^2\}) = \Delta(\Phi\{f_t^1\}, \Phi\{f_t^2\}) \in H^0(f)$.

Now, from 2.5.2, 2.5.5, and 2.5.8, we immediately have:

THEOREM 2.6.5. *If* $2n > 3(k+1)$, $[V \subset M]_f$ *is an Abelian affine group, and, if nonempty, it has action group* $H^0(f)$.

THEOREM 2.6.6. *If* f *is homotopic to an embedding*, $\Gamma(f) = 0$. *The converse holds if* $2n \geqslant 3(k+1)$.

THEOREM 2.6.7. *Suppose* $\{f_t^1\}$ *and* $\{f_t^2\}$ *are embedding homotopies of* f. *If they are isotopic*, $\Delta(\Phi\{f_t^1\}, \Phi\{f_t^2\}) = 0$. *The converse holds if* $2n > 3(k+1)$.

Finally, the following remarks will show how $[V \subset M]_f$, up to isomorphism, depends only on the homotopy class of f. Let f_t: $V \to M$, $0 \leqslant t \leqslant 1$, be any homotopy, where f_0 and f_1 are differentiable. Let $\{f_t\}^*$: $[V \subset M]_{f_1} \to [V \subset M]_{f_0}$ be the function, where, if $\{g_t\}$ is an e-homotopy of f_1, $\{f_t\}^*[g_t] = [h_t]$, where $h_t = f_{2t}$ or g_{2t-1}, depending on the value of t. Clearly $\{f_t\}^*$ is one-to-one and onto, since $\{f_{1-t}\}^*$ is its two-sided inverse.

THEOREM 2.7.1. *If* $2n > 3(k+1)$, $\{f_t\}^*$ *is an isomorphism of affine groups.*

PROOF. Applying the polyhedral covering homotopy property (pair version)

[2]Theorems 3.3.1 and 3.3.2 of [7] are correct as stated in that paper, but the proofs are invalid, as has been kindly pointed out by the referree of this paper. A correction is contained here, as an appendix, §8.

of ζ_M, one may easily show that, for any $0 \leqslant u \leqslant 1$, $(i_u)^*$: $[(\mathbf{R}^* V \times I, PV \times I); \zeta_M]_{Fp} \to [(\mathbf{R}^* V, PV); \zeta_M]_F$ is an isomorphism, where $i_u(\mathbf{r}) = (\mathbf{r}, u)$ for all $\mathbf{r} \in \mathbf{R}^* V$ and $p(\mathbf{r}, u) = \mathbf{r}$. Very simply, checking definitions, one can see that $\{f_t\}^* = (i_0)^*((i_1)^*)^{-1}$, and we are done.

In a similar manner, we shall write $\{f_t\}^*$: $H^i(f_1) \cong H^i(f_0)$ for any integer i. We leave the details to the reader.

3. **A spectral sequence.** In this section, let $((X, X_0), (A, A_0))$ be a fixed C.W. pair of pairs, and let ζ: $(E, E_0) \to (X, X_0)$ be a pair fibration. We consider the problem of enumeration of sections of ζ from a spectral sequence viewpoint.

All pair fibrations shall be over (X, X_0).

3.1. *Homotopy sheaves*. Let ζ have a section s. We define $\pi_k(\zeta, s)$, the kth *homotopy sheaf* of ζ, to be the sheaf over X determined by a presheaf Π, where, if $U \subset X$ is open and $U_0 = U \cap X_0$, $\Pi(U) = [(U, U_0); \Omega^k\zeta]$. Hence, for any $x \in X$,

$$\pi_k(\zeta, s)_x = \begin{cases} \pi_k(E_x, sx) & \text{if } x \in X - X_0, \\ \pi_k((E_0)_x, sx) & \text{if } x \in X_0. \end{cases}$$

The total space of $\pi_k(\zeta, s)$ need not be Hausdorff.

If each E_x and each $(E_0)_x$ is k-simple, $\pi_k(\zeta)$ can be defined regardless of choice (or even existence) of a section. We leave the details to the reader. (Hint: it is sufficient to define $\Pi(U)$ for U contractible.)

Note that the concept of homotopy sheaf is simply a generalization of the usual local system of coefficients for a fibration.

Without any restrictions on ζ, we can always define $\pi_k^S(\zeta) = \operatorname{Lim}_{N\to\infty}\pi_k(\Omega^N S^N \zeta)$, the kth *stable homotopy sheaf*.

Let G be a sheaf over X. We say that ζ is an Eilenberg Mac Lane pair fibration of type (G, n) (we will write $\zeta = k(G, n)$) if:

(i) ζ has a section,

(ii) $\pi_n(\zeta) = G$,

(iii) $\pi_k(\zeta) = 0$ for all $k \neq n$.

It is important to note that G cannot simply be any sheaf of groups, Abelian if $n \geqslant 2$. Only for G satisfying special conditions will $k(G, n)$ exist.

In this context, Theorem 2.5.1 of [7] becomes:

THEOREM 3.1.1. *If $\zeta = k(G, n)$ for some sheaf of Abelian groups G, then* $[(X, X_0), (A, A_0); \zeta] \cong H^n(X, A; G)$.

3.2. *The fiber of a pair-fibration map.* Let ζ': $(E', E_0') \to (X, X_0)$ be another pair fibration. We say that γ is a pair fibration map, and simply write γ: $\zeta \to \zeta'$, if γ: $(E, E_0) \to (E', E_0')$ si simply a fiber-preserving map. Suppose that ζ has a section s. Then $s' = \gamma s$ is a section of ζ'. We define a pair fibration $\phi[\gamma]: (F, F_0) \to (X, X_0)$, called the *fiber* of γ, as follows: for each $x \in X$, F_x is the usual homotopy theoretic fiber of $E_x \to E'_x$, that is, $F_x = \{(e, \sigma) | e \in E_x, \sigma: I \to E'_x, \sigma(0) = s'x, \sigma(1) = \gamma e\}$, and $(F_0)_x$ is the homotopy theoretic fiber of $(E_0)_x \to (E'_0)_x$. Let $\lambda: \phi[\gamma] \to \zeta$ be given by $\lambda(e, \sigma) = e$, and let $\iota: \Omega\zeta' \to \phi[\gamma]$ be the obvious inclusion.

Identifying $\Omega\phi[\gamma]$ with $\phi[\Omega\gamma]$ in the obvious way, we have:

LEMMA 3.2.1. *The following long sequence is exact*:

$$\cdots \to [(X, X_0), (A, A_0); \Omega\zeta] \xrightarrow{(\Omega\gamma)_\#} [(X, X_0), (A, A_0); \Omega\zeta']$$
$$\xrightarrow{\iota_\#} [(X, X_0), (A, A_0); \phi[\gamma]]$$
$$\xrightarrow{\lambda_\#} [(X, X_0), (A, A_0); \zeta] \xrightarrow{\gamma_\#} [(X, X_0), (A, A_0); \zeta']$$

PROOF. Since the definition of fiber the natural definition for the category of pair-fibrations over (X, X_0), the details of the proof are routine and obvious.

Similarly, we have:

LEMMA 3.2.2. *The following sequence of homotopy sheaves is exact*:

$$\cdots \xrightarrow{\gamma_\#} \pi_k(\Omega\zeta') = \pi_{k-1}(\zeta') \xrightarrow{\iota_\#} \pi_k(\phi[\gamma]) \xrightarrow{\lambda_\#} \pi_k(\zeta) \xrightarrow{\gamma_\#} \pi_k(\zeta') \to \cdots.$$

3.3. *The homotopy killing constructions.*

The strong topology double mapping cylinder. Let A, B, and C be topological spaces, and $\alpha: B \to A$, $\beta: B \to C$ maps. Let $A \cup_\alpha (B \times I) \cup_\beta C$ be the space obtained from $A \cup (B \times I) \cup C$ by identifying $(x, 0)$ with αx and $(x, 1)$ with βx for all $x \in B$; but with the strong topology, not the quotient topology. Neighborhoods of points in $B \times (0, 1)$ are as usual in the product topology, while if $a \in A$, a basic neighborhood of $a \in W$ is of the form $U \cup [\alpha^{-1}U \times [0, \varepsilon)]$ for $\varepsilon > 0$ and U a neighborhood of a in A. Neighborhoods along C are similar. Equivalently, the strong topology is the strongest topology such that all of the following obvious projections are continuous:

$$A \cup_\alpha (B \times I) \cup_\beta C \to I,$$
$$A \cup_\alpha (B \times [0, 1)) \to A,$$
$$(B \times (0, 1]) \cup_\beta C \to C.$$
$$B \times (0, 1) \to B.$$

Now suppose that $a: (A, A_0) \to (X, X_0)$, $b: (B, B_0) \to (X, X_0)$, $c: (C, C_0) \to (X, X_0)$, $\alpha: (B, B_0) \to (A, A_0)$, and $\beta: (B, B_0) \to (C, C_0)$ are all pair fibrations.

LEMMA 3.3.1. $a \cup_\alpha (b \times I) \cup_\beta c$: $(A \cup_\alpha (B \times I) \cup_\beta C, A_0 \cup_\alpha (B_0 \times I) \cup_\beta C_0) \to (X, X_0)$ *is a pair fibration.*

PROOF. Omit. See Hall [5] for the proof of a special case, the fiberwise strong topology join.

Now suppose that, for all $n \geqslant 1$, we have pair fibrations a_n, b_n, c_n, α_n, and β_n (as in the hypotheses of 3.3.1), such that $a_{n+1} = a_n \cup_{\alpha_n} (b_n \times I) \cup_{\beta_n} c_n$ for all n. Let $a_\infty = \bigcup a_n$, with the weak topology.

LEMMA 3.3.2. *a_∞ is a pair fibration.*

PROOF. Essentially by induction on n, mimicking the proof of 3.3.1 at each step. We omit the details.

Let $E_X^{S^n} \to X$ be the fibration where, for each $x \in X$, $(E_X^{S^n})_x = (E_x)^{S^n}$, the space of maps $S^n \to E_x$, with the compact open topology. Let e: $E_X^{S^n} \times S^n \to E$ be the evaluation map, and let $K_X^n E = E \cup_e (E_X^{S^n} \times B^{n+1})$, with the strong topology. (Define as follows: Since B^{n+1} is the cone over S^n, $K_X^n E = E \cup_e (E_X^{S^n} \times S^n \times I) \cup_\pi E_X^{S^n}$, the strong topology double mapping cylinder, where π is projection.) Now let $K^n\zeta$: $(K_X^n E, K_{X_0}^n E_0) \to (X, X_0)$; by 3.2.1, a pair fibration.

THEOREM 3.3.3. (I) *If $k < n$, $i_\#$: $\pi_k(\zeta) \cong \pi_k(K^n\zeta)$.* (II) $\pi_n(K^n\zeta) = 0$.

PROOF. We first need a lemma.

LEMMA 3.3.4. *If Z is any simplicial complex of dimension less than or equal to n, and if Y is any space, then $[Z;\, Y] \to [Z;\, Y \cup_e (Y^{S^n} \times B^{n+1})]$ is onto.*

PROOF OF LEMMA. Let $* \in B^{n+1}$ be its center. If f: $Z \to Y \cup_e (Y^{S^n} \times B^{n+1})$, f can be deformed slightly so that its image does not intersect $Y^{S^n} \times \{*\}$. The complement of that subset collapses to Y, and we are done.

Returning to the proof of 3.3.3, we have immediately from the lemma that $\pi_k(\zeta) \to \pi_k(K^n\zeta)$ is onto for $k \leqslant n$ and one-to-one for $k < n$. Finally, $\pi_n(\zeta) \to \pi_n(K^n\zeta)$ is the zero map, since the construction attaches an $(n+1)$-cell to every possible map of S^n to every fiber.

DEFINITION 3.3.5. For $0 \leqslant n \leqslant m$, let $K^{n,m}\zeta$ be defined inductively by $K^{n,m}\zeta = K^n\zeta$, and $K^{n,m+1}\zeta = K^{m+1}(K^{n,m}\zeta)$. Finally, let $\mathbf{K}^n\zeta = \bigcup_m K^{n,m}\zeta$.

THEOREM 3.3.6. (I) *If $k < n$, $\pi_k(\zeta) \to \pi_k(\mathbf{K}^n\zeta)$.* (II) *If $k \geqslant n$, $\pi_k(\mathbf{K}^n\zeta) = 0$.*

PROOF. Direct from 3.3.3.

3.4. *The spectral sequence determined by a resolution.* Suppose that ζ has a section. We say that the following commutative diagram of sectioned pair fibrations and maps

(3.4-1)
$$\begin{tikzcd}
 & \cdots & \zeta \arrow[dl, "\beta_2"'] \arrow[d, "\beta_1"] \arrow[dr, "\beta_0"] & \\
\cdots \arrow[r, "\alpha_3"] & \zeta_2 \arrow[r, "\alpha_2"] & \zeta_1 \arrow[r, "\alpha_1"] & \zeta_0
\end{tikzcd}$$

is a *resolution* if, for each integer n, there exists an integer $N(n)$ such that $(\alpha_m)_\#: \pi_k(\zeta_m) \cong \pi_k(\zeta_{m-1})$ and $(\beta_m)_\#: \pi_k(\zeta) \cong \pi_k(\zeta_m)$ for all $k < n$ and all $m > N(n)$. We say that it is a *Postnikov resolution* if, for every $n > 0$, $(\beta_n)_\#: \pi_k(\zeta) \cong \pi_k(\zeta_n)$ for all $k < n$, and $\pi_k(\zeta_n) = 0$ for all $k > n$.

THEOREM 3.4.1. *If ζ has a section, ζ has a Postnikov resolution. Furthermore, if $\gamma: \zeta \to \zeta'$, γ induces a natural map of Postnikov resolutions.*

PROOF. For every integer $n \geq 0$, let $\zeta_n = \mathrm{K}^{n+1}\zeta$. Let the α_i and β_i be the inclusions. By 3.3.6, we are done.

LEMMA 3.4.2. *If* (3.4-1) *is a Postnikov resolution,* $\phi[\alpha_n] = k(\pi_n(\zeta), n)$.

PROOF. Directly from 3.2.2.

Suppose diagram (3.4-1) is given. We define (taking $\zeta_q: (X, X_0) \to (X, X_0)$ for $q < 0$):

$$\begin{aligned}
E_2^{p,q} &= \left[(X, X_0), (A, A_0); \Omega^{-p}\phi[\alpha_q]\right] \quad \text{for } p \leq 0,\\
D_2^{p,q} &= \left[(X, X_0), (A, A_0); \Omega^{-p}\zeta_q\right] \quad \text{for } p \leq 0,\\
i_2 &= (\Omega^{-p}\alpha_q)_\#: D_2^{p,q} \to D_2^{p,q-1} \quad \text{for } p \leq 0,\\
j_2 &= (\Omega^{-p-1}\iota)_\#: D_2^{p,q} \to E_2^{p+1,q+1} \quad \text{for } p \leq -1,\\
k_2 &= (\Omega^{-p}\lambda)_\#: E_2^{p,q} \to D_2^{p,q} \quad \text{for } p \leq 0.
\end{aligned}$$

By 3.2.1, we now have a bigraded exact couple (although there is an edge problem, since p must not be positive). Furthermore, from 3.1.1 and 3.4.2, we immediately have:

REMARK 3.4.3. If (3.4-1) is a Postnikov resolution, $E_2^{p,q} = H^{p+q}(X, A; \pi_q(\zeta))$.

Let $E_r^{p,q}$, $D_r^{p,q}$, $E_\infty^{p,q}$, and $D_\infty^{p,q}$ be obtained in the usual manner.

THEOREM 3.4.4. *If $p \leq -1$, the $E_\infty^{p,q}$ (for various q) give a composition series for* $[(X, X_0), (A, A_0); \Omega^{-p}\zeta]$, *provided* $\dim(X/A) < \infty$.

PROOF. Let $G^{p,q}$ be the kernel of $(\Omega^{-p}\beta_q)_\#: [(X, X_0), (A, A_0); \Omega^{-p}\zeta] \to D_2^{p,q}$. Thus, $G^{p,-1} = [(X, X_0), (A, A_0); \Omega^{-p}\zeta]$ and, for dimensional reasons, $G^{p,q} = 0$ for q sufficiently large. By standard spectral sequence arguments, $E_\infty^{p,q} = G^{p,q-1}/G^{p,q}$, and we are done.

If ζ is deloopable, i.e., $\zeta = \Omega\eta$, we can, by constructing a Postnikov resolution for η, obtain a spectral sequence converging to a composition series for $[(X, X_0), (A, A_0); \zeta]$. If ζ is infinitely deloopable, the exact couple can be constructed with no restrictions on the indices. Since computation of $H^i((X, X_0), (A, A_0); \zeta)$ for $\dim(X/A) < \infty$ involves classifying sections of $\Omega^N S^{i+N}\zeta$ for large N, that cohomology theory can also be obtained by a spectral sequence, where $E_2^{p,q} = H^{p+q}(X, A; \pi_q^S(\zeta))$. We leave the details to the reader.

4. A geometric interpretation of the affine structure. Let $f\colon V \to M$ be a map, where V is a compact manifold of dimension k, and M is a manifold of dimension n, where $2n > 3(k+1)$. We give a geometric interpretation of the affine group structure of $[V \subset M]_f$.

By Theorem 2.7.1, it is sufficient to consider the case where f is actually an embedding. We then consider $[V \subset M]_f$ to be an actual Abelian group with identity $[f]$, represented by the constant homotopy.

Let $\{f_t^1\}$, $\{f_t^2\}$ be e-homotopies of f such that

(i) f_t^1 and f_t^2 are differentiable for all $t \in I$.

(ii) For some disjoint closed sets $K_1, K_2 \subset V$, and for all $t \in I$, f_t^i agrees with f on $V - K_i$.

(iii) For any $x_1 \in K_1$, $x_2 \in K_2$, $x_3 \in V - (K_1 \cup K_2)$, and for any $t, u \in I$; $f_t^1(x_1)$, $f_t^2(x_2)$, and $f(x_3)$ are distinct.

Now let $f_t^3\colon V \to M$ be the e-homotopy of f such that, for all $x \in V$, $t \in I$:

$$f_t^3(x) = \begin{cases} f_t^1(x) & \text{if } x \in K_1, \\ f_t^2(x) & \text{if } x \in K_2, \\ f(x) & \text{otherwise.} \end{cases}$$

THEOREM 4.1. $[f_t^1] + [f_t^2] = [f_t^3]$.

PROOF. We need only show that

$$\left[\iota\Phi\{f_t^1\}\right]\left[\iota\Phi\{f\}\right]^{-1}\left[\iota\Phi\{f_t^2\}\right] = \left[\iota\Phi\{f_t^3\}\right] \in \left[(\mathbf{R}^* V, PV); PS\zeta_M\right]_F,$$

where $\iota\colon \zeta_M \subset PS\zeta_M$. Pick disjoint closed sets L_1, L_2 in V such that $K_i \subset \operatorname{Int} L_i$. Choose continuous functions $\rho_1, \rho_2\colon V \to I$ such that $\rho_i(x) = 1$ for all $x \in K_i$, and 0 for all $x \not\in L_i$. For each $i = 1, 2$, or 3, let $\Phi_{i,u}$, $0 \leqslant u \leqslant 1$, be the homotopy of liftings of F to ζ_M such that, for all u, $\Phi_{i,u}$ agrees with $\Phi\{f_t^i\}$ on PV, and such that for all $0 \leqslant t, u \leqslant 1$ and all $[x,y] \in R^* V$,

$$\Phi_{i,u}[x,y](t) = \left[\left(f_v^i(x), (1-t)\mathbf{i}(x)\right), \left(f_w^i(y), (1-t)\mathbf{i}(y)\right)\right]$$
$$\in R^*(M \times R^\infty) = {}^\infty R^* M,$$

where:

$$v = t\left(u\rho_i(x) + 1 - u\right) \quad \text{and} \quad w = t\left(u\rho_i(y) + 1 - u\right) \quad \text{if } i = 1 \text{ or } 2,$$

$$v = t\left(u\rho_j(x) + 1 - u\right) \quad \text{and} \quad w = t\left(u\rho_j(y) + 1 - u\right)$$
$$\text{if } i = 3, j = 1 \text{ or } 2, \text{ and } \rho_j(x)\rho_j(y) > 0,$$

$$v = w = t(1-u) \quad \text{if } i = 3, \text{ and } \rho_1(x)\rho_1(y) + \rho_2(x)\rho_2(y) = 0.$$

Note then that $\Phi_0^i = \Phi\{f_t^i\}$ for all $i = 1, 2,$ or 3, and by 2.3.4, letting $D_i = Q(L_i^2) \subset \mathbf{R}^*V$ for $i = 1$ or 2, we obtain: $[\iota\Phi_1^1][\iota\Phi\{f\}]^{-1}[\iota\Phi_1^2] = [\iota\Phi_1^3]$, and we are done.

Using a general position argument, based on the dimensional restriction, it is possible to show that any two e-homotopies of f are respectively isotopic to a pair of e-homotopies satisfying (i) through (iii) above. Thus Theorem 4.1 suffices to describe the affine structure on $[V \subset M]_f$.

In later calculations, the following result is useful:

THEOREM 4.2. *If f': $V \to M$ is another map such that $f|D$ is homotopic to $f'|D$ for some $D \subset V$ such that $V - D \cong R^n$, then $[V \subset M]_f \cong [V \subset M]_{f'}$, if both are nonempty.*

PROOF. Without loss of generality, f and f' are both embeddings, and $f'|D = f|D$. If $\alpha \in [V \subset M]_f$, we may, by a general position argument, choose an e-homotopy $\{f_t\}$ of f such that

(i) $[f_t] = \alpha$,

(ii) $f_t|(V - D) = f|(V - D)$ for all t,

(iii) $f_t(D) \cap f'(V - D)$ is empty for all t.

Now let $\phi(\alpha) = [f_t']$, where, for all $0 \leqslant t \leqslant 1, f_t'(x) = f_t(x)$ if $x \in D, f_t'(x) = f'(x)$ if $x \in (V - D)$. Using 4.1 and standard general position arguments, one may easily show that ϕ: $[V \subset M]_f \to [V \subset M]_{f'}$ is well defined, and is an isomorphism.

5. **The structure of** $\pi_n\zeta_M$. Let M be any connected n-manifold. In this section, we explicitly compute the sheaf $\pi_n\zeta_M$ over ${}^\infty\mathbf{R}^*M$, provided $n \geqslant 6$. (The sheaf $\pi_{n-1}\zeta_M$ was computed in [7], provided $n \geqslant 5$; we restate the results here. For $k < n - 1$, $\pi_k\zeta_M = 0$.)

5.1. *Building a twisted sheaf.* Suppose that X is any path-connected space with basepoint x_0, and S is a sheaf over X with a local product structure. Let S_x be the stalk of S over any $x \in X$. If $a \in \pi_1 X$, let α: $(I, \partial I) \to (X, x_0)$ be a loop representing a, and let $\tilde{\alpha}$: $I \times S_{x_0} \to S$ be a map such that $\tilde{\alpha}(t, s) \in S_{\alpha(t)}$ and $\tilde{\alpha}(1, s) = s$ for all $t \in I$, $s \in S_{x_0}$. Let $\langle a \rangle$: $S_{x_0} \to S_{x_0}$ be the automorphism where $\langle a \rangle(s) = \tilde{\alpha}(0, s)$. Let μ_S: $\pi_1 X \times S_{x_0} \to S_{x_0}$ be the left action where $\mu_S(a, s) = \langle a \rangle(s)$ for all $a \in \pi_1 X$, $s \in S_{x_0}$.

LEMMA 5.1.1. *Let X be path connected and locally simply connected, with basepoint. Let G be any group and let μ: $\pi_1 X \times G \to G$ be any left action. Then there exists a sheaf $S = S(G, \mu)$, with local product structure over X, such that $S_{x_0} = G$ and $\mu_S = \mu$. Furthermore, S is unique with these properties, up to isomorphism.*

PROOF. Let π: $Y \to X$ be the universal covering of X, and let ν: $\pi_1 X \times Y \to Y$ be the associated (left) action, where $\pi y_1 = y_2$ if and only if $\nu(a, y_2)$ for some $a \in \pi_1 X$. Then let $S(G, \mu)$ be the quotient space of $Y \times G$ obtained by identifying each (y, g) with $(\nu(a, y), \mu(a, g))$ for all $a \in \pi_1 X$. Trivially, the required conditions are satisfied. The proof of uniqueness is trivial and straightforward, hence we omit it.

5.2. *Preliminary definitions.* For any topological space X, let ΓX be the quotient space of $X \times X \times S^\infty$ obtained by identifying each (x, y, a) with $T(x, y, a) = (y, x, -a)$. If $f\colon X_1 \to X_2$ is any map, let $\Gamma f\colon \Gamma X_1 \to \Gamma X_2$ be the obvious map. Note that if $*$ is a single point space, $\Gamma * = p^\infty = S^\infty/T$. Let $\Gamma 0\colon \Gamma X \to P^\infty$, where $0\colon X \to *$ is the collapsing map. If $* \in X$ is a basepoint, let $j = \Gamma i\colon P^\infty \to \Gamma X$, where $i\colon * \to X$ is the inclusion. Let $\pi\colon X \times X \times S^\infty \to \Gamma X$ denote the 2-1 covering, and, by a slight abuse of notation, let $\pi\colon X \times X \to \Gamma X$ denote the (equivalent to the covering) inclusion, where $\pi(x, y) = [x, y, *]$, $* \in S^\infty$ a basepoint. We may consider $X \times P^\infty \subset \Gamma X$; if $x \in X$ and $a \in S^\infty$, identify $(x, [a])$ with $[x, x, a]$.

Let X be any topological space with basepoint, and let $\pi_k = \pi_k X$, written multiplicatively if $k = 1$, additively if $k > 1$. Let $\gamma_k\colon \pi_1 \times \pi_k \to \pi_k$ be the usual left action, determined by the map $\gamma_k\colon S^k \to S^1 \vee S^k$, where

$$\gamma_k[x_1, x_2, \ldots, x_k] = \begin{cases} [2x_1] \in S^1 & \text{if } 0 \leqslant x_1 \leqslant \frac{1}{2}, \\ [2x_1 - 1, x_2, \ldots, x_k] \in S^k & \text{if } \frac{1}{2} \leqslant x_1 \leqslant 1, \end{cases}$$

where $S^k = I^k/\partial I^k$. We let $x^a = \gamma_k(a, x)$ for all $a \in \pi_1$, $x \in \pi_k$. Let $\alpha_k\colon \pi_2 \otimes \pi_k \to \pi_{k+1}$, for $k \geqslant 2$, be the Whitehead product, determined by the map $\alpha_k\colon S^{k+1} \to S^2 \vee S^k$, where

$$\alpha_k(x_1, x_2, \ldots, x_{k+2}) = \begin{cases} [x_1, x_2] \in S^2 & \text{if } x_i \in \partial I \text{ for some } 3 \leqslant i \leqslant k+2, \\ [x_3, \ldots, x_{k+2}] \in S^k & \text{if } x_i \in \partial I \text{ for } i = 1 \text{ or } 2, \end{cases}$$

where $S^2 = I^2/\partial I^2$, $S^k = I^k/\partial I^k$, and $S^{k+1} = \partial I^{k+2}$.

If X is a manifold and $x \in \pi_k$, let $w_k x \in Z_2$ be the value of $f^* w_k X \in H^k(S^k; Z_2)$ (the Stiefel-Whitney class), where $f\colon S^k \to X$ represents x. If $a \in \pi_1$, let $(-1)^a = 1$ if $w_1 a = 0$, -1 otherwise.

Again, if X is a manifold, and if $a \in \pi_1$, $a^2 = 1$ (the identity), let $f_a\colon P^2 \to X$ be a map which sends the generator of $\pi_1 P^2$ to a, and elt $\tilde{a} = [f_a s] \in \pi_2$, where $s\colon S^2 \to P^2$ is the covering map onto the real projective plane. Let k_a be an integer such that the vector bundles $f_a^{-1}\tau$ and $k_a h$ are stably equivalent, where τ is the tangent bundle of X and h is the canonical line bundle over P^2. Note that the *pair* $(k_a, \tilde{a})$ has indeterminacy (because of the choice of f_a) $(4Z \oplus 0) + \operatorname{Im} \chi_a$, where $\chi_a x = (m w_2 x, x + x^a)$ for all $x \in \pi_2$, where $m\colon Z_2 \to Z_4$ is the monomorphism. (Hint: recall that $\tilde{K}^0(P^2) = Z_4$.)

Let G and N be groups, and let $\varphi\colon G \to \operatorname{Aut} N$ be any homomorphism. Let

$N \times_\varphi G$ be the semidirect product. Specifically, as a set, $N \times_\varphi G$ is the Cartesian product $N \times G$, with the operation $(n_1, g_1)(n_2, g_2) = (n_1\varphi(g_1)n_2, g_1 g_2)$. Note that if φ is trivial, $N \times_\varphi G$ is simply the direct sum. We always have $G \subset N \times_\varphi G$ and $N \lhd N \times_\varphi G$, where we identify N and G with $N \times \{1\}$ and $\{1\} \times G$, respectively.

For any (multiplicatively written) group G, let ZG be the group ring of G, which we represent as finite formal sums of elements of G. Thus, $G \subset ZG$.

5.3. *Structure of* $\pi_{n-1}\zeta_M$ *and* $\pi_n\zeta_M$. Now let $X = {}^\infty\mathbf{R}^*M$, and $X_0 = {}^\infty PM$. Once and for all, fix basepoints $* \in M$ and $* \in SM$ over $*$, and let $* \in PM$ also denote the image of $*$. Let $\pi_k = \pi_k M$, written multiplicatively if $k = 1$, additively otherwise. For consistency of notation, we introduce a group $T_2 = \{1, m\}$, a multiplicative group of two elements. Let $H_2 = \{0, \eta\} \cong Z_2$, the stable 1-stem in the homotopy of spheres, which we treat as a Z_2-module. For any group G, let $\Delta G \subset G \oplus G$ be the diagonal.

Lemma 5.3.1. (I) $\pi_1 X = (\pi_1 \oplus \pi_1) \times_\varphi T_2$, *where* $\varphi(m)(a, b) = (b, a)$ *for all* $a, b \in \pi_1$. (II) $\pi_1 X_0 = \Delta\pi_1 \oplus T_2$, *and* i: $X_0 \subset X$ *induces the inclusion of groups described in* 5.2.

Proof. Let ι: $M \times R^\infty \to R^\infty$ be any embedding, and let ρ: $\mathbf{R}R^\infty \to S^\infty$ be the equivalent retraction, determined by $\rho(u, v) = \|u - v\|^{-1}(u - v)$ for all $(u, v) \in RR^\infty$. Let $\psi = (p_1^2 Q, \rho\mathbf{R}\iota)$: $(\mathbf{R}(M \times R^\infty), S(M \times R^\infty)) \to (M \times M \times S^\infty, \Delta M \times S^\infty)$ which is equivariantly a homotopy equivalence of pairs, by an elementary obstruction theory argument. Passing to quotient spaces under the involution T, we have that the pair (X, X_0) is of the homotopy type of $(\Gamma M, M \times P^\infty)$.

We have a partially split exact sequence:

$$\begin{array}{ccccccc} 1 \to & \pi_1 M^2 & \to & \pi_1\Gamma M & \underset{j_*}{\overset{c_*}{\rightleftarrows}} & \pi_1 P^\infty & \to 1 \\ & \| & & \| & & \| & \\ & \pi_1 \oplus \pi_1 & & \pi_1 X & & T_2 & \end{array}$$

A simple deck-transformation argument then verifies that $\pi_1 M$ is the desired semi-direct product. (II) follows trivially.

Now let θ_M: $Y \to X$ and ρ_M: $Z \to X_0$ be the fibrations which (as functions) agree with the pair fibration ζ_M.

Lemma 5.3.2. (I) *If* $n > 5$, $\pi_{n-1}\theta_M = S(Z\pi_1, \mu')$, *where*

(i) $\mu'((b, c, 1), a) = (-1)^c bac^{-1}$,

(ii) $\mu'((b, c, m), a) = (-1)^n(-1)^{ac} ba^{-1}c^{-1}$ *for all* $a, b, c \in \pi_1$.

(II) *If* $n > 6$, $\pi_n\theta_M = S(Z\pi_1 \otimes (\pi_2 \oplus H_2), \mu)$, *where*

(i) $\mu((b, c, 1), a \otimes (x, \lambda\eta)) = (-1)^c bac^{-1} \otimes (x^c, \lambda\eta)$,

(ii) $\mu((b, c, m), a \otimes (x, \lambda\eta)) = (-1)^{n+1}(-1)^{ac} ba^{-1}c^{-1} \otimes (x^{c^{-1}a}, (\lambda + w_2 x)\eta)$ *for all* $a, b, c \in \pi_1$, $x \in \pi_2$, *and* $\lambda \in Z_2$.

We postpone the proof.

Identify Z with $Z\{1\} \subset Z\pi_1$. Let ν': $((Z\Delta\pi_1) \oplus T_2) \times Z \to Z$ and ν: $((Z\Delta\pi_1) \oplus T_2) \times (Z \otimes H_2) \to Z \otimes H_2$ be the restrictions of μ' and μ, respectively.

LEMMA 5.3.3. (I) *If* $n \geqslant 5$, $\pi_{n-1}\rho_M = S(Z, \nu')$. (II) *If* $n \geqslant 6$, $\pi_n\rho_M = S(Z \otimes H_2, \nu)$.

We leave the proof to the reader. (Hint: the fiber of ρ_M is S^{n-1}.)

One final lemma completes the description of $\pi_{n-1}\zeta_M$ and $\pi_n\zeta_M$.

LEMMA 5.3.4. *If* $\pi_k\rho_M$ *is a subsheaf of* $(\pi_k\theta_M)|X_0$, *then* $\pi_k\zeta_M$ *is the subsheaf of* $\pi_k\theta_M$ *whose stalks over* X_0 *agree with those of* $\pi_k\theta_M$, *and whose stalks over* $X - X_0$ *agree with those of* $\pi_k\rho_M$.

PROOF. Follows directly from the definition of $\pi_k\zeta_M$.

5.4. *Proof of Lemma* 5.3.2. Part (I) of 5.3.2 is simply Lemmas 3.4.2, 3.4.4, and 3.4.5 of [7]. (However, we have replaced the right actions of that paper by left actions.)

Now we saw in the proof of 3.4.2 of [7] that the fiber of θ_M is the fiber Φ of the inclusion $\tilde{M}^{00} \subset \tilde{M}$, where $\tilde{M}$ is the universal covering space of M, and $\tilde{M}^{00}$ is the universal covering space of $M^0 = M - \{*\}$. A straightforward Serre spectral sequence argument reveals that

(i) $\pi_{n-1}(\Phi) = Z\pi_1$,

(ii) $\pi_n(\Phi) = Z\pi_1 \otimes (\pi_2 \oplus H_2)$,

where composition with η: $S^n \to S^{n-1}$ is represented by $\otimes\ \eta$, and where, for each $a \in \pi_1$ and $x \in \pi_2$, $a \otimes x$ is represented by the map ϕ in the following homotopy commutative diagram, where $S^2 \xrightarrow{x} \tilde{M}^{00} \subset \tilde{M} \to M$ represents x:

$$\begin{array}{ccc} S^n & \xrightarrow{\phi} & \Phi \\ {\scriptstyle \alpha_{n-1}}\downarrow & & \downarrow \\ S^{n-1} \vee S^2 & \xrightarrow{a \vee x} & \tilde{M}^{00} \subset \tilde{M} \end{array}$$

The reader can easily verify that (II) need only be checked for the following five special cases (where $a, b, c \in \pi_1$; $x \in \pi_2$):

(i) $((b, 1, 1), a \otimes \eta) = ba \otimes \eta$,

(ii) $((1, 1, m), a \otimes \eta) = a^{-1} \otimes \eta$,

(iii) $((b, 1, 1), a \otimes x) = ba \otimes x$,

(iv) $((1, c, 1), 1 \otimes x) = (-1)^c c^{-1} \otimes x^c$,

(v) $((1, 1, m), 1 \otimes x) = (-1)^{n+1} \otimes ((w_2 x)\eta + x)$.

Now (i), (ii) follow immediately from (I), by composition with η, while the proof of (iii) involves (I), together with mere straightforward checking of maps.

To prove (iv), let $\gamma\colon I \to M$ be a loop representing c. Let $\psi_t\colon S^2 \to M$, $0 \leqslant t \leqslant 1$, be a homotopy such that $\psi_t(*) = \gamma(t)$ for all t, and ψ_1 represents x. Then ψ_0 represents x^c.

We briefly introduce a general construction. If Y is any pointed space, $B \subset Y$, and if $f\colon (I^k, \partial I^k) \to (Y, B)$ is a map such that $f(*) = *$ for some $* \in I^k$, let $\langle f \rangle\colon S^{k-1} = \partial I^k \to F$ ($F =$ homotopy theoretic fiber of the inclusion $B \subset Y$) be defined as follows: let $\rho\colon CS^{k-1} \cong I^k$ ($CX =$ cone over X) and let $\langle f \rangle(v) = \sigma_v \in F \subset Y^I$, where $\sigma_v(t) = f(\rho[v, t])$. By a slight modification, it is not necessary to assume $B \subset Y$, only $B \to Y$.

Returning to the proof, choose a homotopy $\theta_t\colon (I^n, S^{n-1}) \to (\tilde{M}^2, \tilde{R}M)$, where $\tilde{R}M$ is the universal covering space of $RM = M^2 - \Delta M$, such that:

(i) $p_2\theta_t(v) = \gamma(t)$ for all $t \in I$,

(ii) $\langle \theta_1 \rangle$ represents $1 \in Z\pi_1 = \pi_{n-1}(\Phi)$,

where $p_2\colon \tilde{M}^2 \to \tilde{M}$ is projection to the second factor. Clearly $\langle \theta_0 \rangle$ represents $(-1)^c c^{-1} \in Z\pi_1 = \pi_{n-1}(\Phi)$. Now, for all $t \in I$, we have a homotopy commutative diagram:

$$
\begin{tikzcd}
S^n \arrow[r] \arrow[dr, "\alpha_{n-1}"'] & H \arrow[r, "h_t"] \arrow[d] & \Phi \arrow[d] & \\
& S^2 \vee S^{n-1} \arrow[r, "\xi_t \vee \theta_t"] & RM \arrow[r, "p^2"] & RM \\
& \cap & \cap & \cap \\
& S^2 \vee I^n \arrow[r, "\xi_t \vee \theta_t"] & M^2 \arrow[r, "p^2"] & M^2
\end{tikzcd}
$$

where $p\colon \tilde{M} \to M$ is the covering map, $p^2\xi_t = i_1\psi_t$ for all t, $\xi_1(*) = *$, and β is chosen independent of t. By definition, $[\beta h_1] = 1 \otimes x$ and $\mu((1, c, 1), 1 \otimes x) = [\beta h_0] = (-1)^c(c^{-1} \otimes x^c)$, and (iv) is verified.

To prove (v), let $g\colon R^n \to M$ be a coordinate patch such that $g(0) = *$. Let $\kappa\colon (I^n, S^{n-1}) \to (R^n, R^n - \{0\})$ be the obvious orientation preserving equivalence. Then $\langle g\kappa \rangle = \pm 1 \in Z \subset Z\pi_1 = \pi_{n-1}(\Phi)$. We alter g if necessary to insure that $\langle g\kappa \rangle = 1$.

Let $(SM)_*$ be the fiber of $SM \to M$ over $*$. Choose a homeomorphism $\iota\colon S^{n-1} \to (SM)_*$ such that $\langle \lambda \rangle = 1 \in \pi_{n-1}(\Phi)$, where $\lambda\colon (I^n, S^{n-1}) \to (M^2, \mathbf{R}M)$ is chosen such that $\lambda(I^n) = * \in M^2$, and $\lambda | S^{n-1} = \iota$. When convenient, we shall identify S^{n-1} with $(SM)_*$.

Let $r\colon E \to S^2$ be the S^{n-1} bundle which is the pullback of $SM \to M$, where $\gamma\colon S^2 \to M$ classifies $x \in \pi_2$, and let $j\colon S^2 \to E$ be a section of r. Let $e\colon I^2 \times S^{n-1} \to E$ be a map such that the following diagram commutes:

$$
\begin{tikzcd}
L \times S^{n-1} \arrow[r, "p_2"] \arrow[d, phantom, "\cap" description] & S^{n-1} \arrow[r, "\iota"] \arrow[d, phantom, "\cap" description] & (SM)_\gamma \arrow[d, phantom, "\cap" description] \\
I^2 \times S^{n-1} \arrow[r, "e"] \arrow[d, shift right, "p_1"'] & E \arrow[r, "\tilde{\gamma}"] \arrow[d, shift right, "j"'] & SM \arrow[d] \\
I^2 \arrow[r, "c"] \arrow[u, shift right, "i_1"'] & S^2 \arrow[r, "\gamma"] \arrow[u, shift right, "r"'] & M
\end{tikzcd}
$$

where p_i is the projection on the ith factor, c collapses ∂I^2 to $* \in S^2$, $L = \{1\} \times I \cup I \times \partial I \subset I^2$, and i_1 is inclusion along the first factor. Let y_t: $I^2 \to M^2$, $0 \leqslant t \leqslant 1$, be the homotopy such that

$$
y_t(u, v) = \begin{cases} (\gamma[1-u, v], *) & \text{if } 0 \leqslant u \leqslant t, \\ (\gamma[1-t, v], \gamma[u-t, v]) & \text{if } t \leqslant u \leqslant 1. \end{cases}
$$

Let $W = (S^1 \times S^{n-1}) \cup (I^2 \times \{*\}) \subset I^2 \times S^{n-1}$, where $S^1 = \partial I^2$, and choose a homotopy $\tilde{y}_t$: $W \to \mathbf{R}M$, $0 \leqslant t \leqslant 1$, such that $\tilde{y}_t | S^1 \times S^{n-1} = \tilde{\gamma} e(J_t \times 1)$ for all t, where J_t: $S^1 \to I^2$ is given by $J_t(u, v) = (1 - tu, v)$ for all $(u, v) \in S^1 \subset I^2$; and such that $Q\tilde{y}_t | I^2 \times \{*\}$ and $y_t p_1 | I^2 \times \{*\}$ are homotopic, rel $S^1 \times \{*\}$, for all t; where Q: $\mathbf{R}M \to M^2$ is the usual quotient map.

The following diagram illustrates the next portion of the argument.

$$
\begin{tikzcd}
S^1 \times S^{n-1} \arrow[rr, "J_t"] \arrow[d, phantom, "\cap" description] & & I^2 \times S^{n-1} \arrow[rr, "\tilde{\gamma} e"] & & SM \arrow[d, phantom, "\cap" description] \\
W \arrow[rrrr, "\tilde{y}_t"] \arrow[dd, "p_1"] & & & & \mathbf{R}M \arrow[dd, "Q"] \\
& F \arrow[ul] \arrow[rr, "z_t"] & & \Phi \arrow[ur] & \\
I^2 \arrow[rrrr, "y_t"] & & & & M^2
\end{tikzcd}
$$

Let z_t: $F \to \Phi$, $0 \leqslant t \leqslant 1$, be the homotopy induced by $\{y_t\}$, $\{\tilde{y}_t\}$ (where F = fiber of p_1: $W \to I^2$). Let i_2: $S^{n-1} \subset W$ be inclusion along the second factor, and let $\omega = (d_\#)^{-1}[\alpha_{n-1}] \in \pi_n W = \pi_n F$, where d: $W \to S^2 \vee S^{n-1}$ is the map which collapses $S^1 \times \{*\} \subset W$. Clearly $(z_0)_\# = (-1)^n \mu((1, 1, m), 1 \otimes x)$; it remains to show only that $(z_1)_\# \omega = -(1 \otimes x) + (w_2 x) 1 \otimes \eta$.

Note that $\tilde{y}_1$: $S^1 \times \{*\} \to j(*) \in (SM)_* \subset \mathbf{R}M$, thus y^*: $S^2 \to \mathbf{R}M$, where $y^*[u, v] = y_1(u, v, *)$ is well defined. We then have a commutative diagram (where S^2 is identified with $jS^2 \subset E$):

$$\begin{tikzcd}
 & & I^2 \times S^{n-1} \arrow[ddr] & & & \\
F \arrow[rr] \arrow[ddr, "\epsilon"] \arrow[ddd, "z_1"'] & & W \arrow[u, hook] \arrow[rr, "p_1"] \arrow[ddr, "e(J_1 \times 1)"] \arrow[ddd, "\tilde{y}_1"'] & & I^2 \arrow[ddd, "y_1"'] \arrow[ddr, "J_1"] & \\
 & & & E & & \\
 & H \arrow[rr] \arrow[dl, "\nu"] & & S^2 \vee S^{n-1} \arrow[u, hook] \arrow[rr] \arrow[dl, "y^* \vee \iota"] & & S^2 \arrow[dl, "\gamma"] \\
\Phi & & \mathbf{R}M \arrow[rr, "Q"] & & M^2 &
\end{tikzcd}$$

where ε, ν are the appropriate maps on the fibers. Now $I^2 \times S^{n-1} = W \cup_w e^{n+1}$, where $w: S^n \to W$ classifies ω, and $E = S^2 \vee S^{n-1} \cup_\lambda e^{n+1}$, where $\lambda = e(J_1 \times 1)w$. Now $\mathrm{Sq}^2\phi = \gamma^* w_2 M \cup \phi$, and $\phi \cup \sigma = -e^{n+1}$ (since J_1 has degree -1), where $\phi \in H^{n-1}(E; Z)$ is the generator and $\sigma \in H^2(S^2; Z)$ is the fundamental class. Thus $[\lambda] = -[\alpha_{n-1}] + (w_2 x)[\eta i_2] \in \pi_n(S^2 \vee S^{n-1})$, and we are done.

6. **Explicit computation of** $[S^k \subset M^{2k}]_f$. In this section, $[S^k \subset M^n]_f$ is explicitly stated, in terms of generators and relations, where M is any manifold without boundary, $f: S^k \to M$ is any embedding, and $n = 2k + 1$, $n \geqslant 2$, or $n = 2k$, $k \geqslant 3$.

THEOREM 6.0.1. *If $k \geqslant 2$, $n = 2k + 1$, then $[S^k \subset M^n]_f$ is generated by elements y_a for all $a \in \pi_1 M$, subject only to the following relations*:

(i) $y_1 = 0$, *where* $1 \in \pi_1 M$ *is the identity*;

(ii) *For any* $a \in \pi_1 M$, $y_{a^{-1}} = (-1)^{k+1}(-1)^a y_a$.

THEOREM 6.0.2. *If $k \geqslant 3$, $n = 2k$, then $[S^k \subset M^n]_f$ is generated only by elements*:

(i) z_a *for all* $a \in \pi_1 M$ *such that* $a^2 = 1$ *and* $(-1)^k(-1)^a = 1$.

(ii) $y_a \otimes u$ *for all* $a \in \pi_1 M$ *and all* $u \in \pi_2 M \oplus H_2$, *where* $H_2 = \{0, \eta\}$ *is the stable* 1-*stem in the homotopy of spheres.*

And is subject only to the relations:

(iii) $y_a \otimes u + y_a \otimes v = y_a \otimes (u + v)$ *for any* $a \in \pi_1 M$, $u, v \in \pi_2 M \oplus H_2$.

(iv) $y_1 \otimes \eta = 0$.

(v) $z_1 = 0$.

(vi) $y_a \otimes \eta = y_{a^{-1}} \otimes \eta$ *for any* $a \in \pi_1 M$.

(vii) $y_a \otimes x + (-1)^a(-1)^k y_{a^{-1}} \otimes x^a + (w_2 x) y_a \otimes \eta = 0$ *for all* $a \in \pi_1 M$, $x \in \pi_2 M$.

(viii)

$$y_a \otimes \tilde{a} + \left(\binom{k-3}{2} + \binom{2k-k_a}{2}\right) y_a \otimes \eta = 0$$

for all $a \in \pi_1 M$ *such that* $a^2 = 1$ *and* $(-1)^a(-1)^k = -1$.

(ix)

$$2z_a = y_a \otimes \tilde{a} + \left(\binom{2k-k_a}{2} + \binom{k-3}{2} + \binom{k}{1}\right) y_a \otimes \eta$$

for all $a \in \pi_1 M$ *such that* $a^2 = 1$ *and* $(-1)^a(-1)^k = 1$.

Finally, these representations are natural, i.e.;

THEOREM 6.0.3. *Let* $g\colon M \subset M'$ *be an embedding, where* M' *is another manifold. Then* (I) *If* $k \geqslant 2$, $n = 2k+1$, $g_{\#} y_a = y_{g_* a}$ *for all* $a \in \pi_1 M$. (II) *If* $k \geqslant 3$, $n = 2k$, $g_{\#}(y_a \otimes x) = y_{g_* a} \otimes g_{\#} x$ *and* $g_{\#}(y_a \otimes \eta) = y_{g_* a} \otimes \eta$ *for all* $a \in \pi_1 M$, $x \in \pi_2 M$, *and* $g_{\#} z_a = z_{g_* a}$ *if* $a \in \pi_1 M$, $a^2 = 1$, *and* $(-1)^a(-1)^k = 1$.

6.0.1 and 6.0.2 are proved below (6.0.1 was incorrectly stated as Theorem 1.2.1 of [7], the result there is off by a sign), while we leave the proof of 6.0.3 to the reader. Basically, 6.0.3 follows from the fact that all of the constructions are natural. Examination of those constructions reveals that y_a, $y_a \otimes x$ and $y_a \otimes \eta$ are canonically chosen, while z_a has an indeterminacy, as defined below, the group generated by $y_a \otimes \tilde{a}$, $y_a \otimes \eta$, and $y_a \otimes x + y_a \otimes x^a$ for all $x \in \pi_2 M$. By making a once-and-for-all choice in each of the four universal examples (see the proof of 6.1.3), this indeterminacy could possibly be further reduced.

6.1. *Proof of* 6.0.1 *and* 6.0.2. Without loss of generality, f may be assumed inessential (cf. 4.2). Now, if $k \geqslant 2$, $n = 2k+1$, there is only one nonzero E_∞ term in the composition series for $[S^k \subset M]_f = H^0(f)$, namely $E_\infty^{0,n-1}$. Now $E_2^{0,n-1} = H^{n-1}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1}\zeta_M)$, and our result follows immediately from Lemma 6.1.1, below, where y_a corresponds to $[a]$ for all $a \in \pi_1 = \pi_1 M$.

Suppose now that $k \geqslant 3$, $n = 2k$. Let $\pi_1 = \pi_1 M$, $\pi_2 = \pi_2 M$. The only E_2 terms which play a role in the computation of $[S^k \subset M]_f$ are $E_2^{0,n-1} = H^{n-1}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1}\zeta_M)$, $E_2^{-1,n-1} = H^{n-2}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1}\zeta_M)$, and $E_2^{0,n} = H^n(\mathbf{R}^* S^k; F^{-1}\pi_n\zeta_M)$; and the only relevant differential is d_2 is the exact sequence:

$$E_2^{-1,n-1} \xrightarrow{d_2} E_2^{0,n} \xrightarrow{\lambda} H^0(f) \xrightarrow{\rho} E_2^{0,n-1} \to 0. \tag{6.1-1}$$

Now if $e\colon 0 \to A \to B \to C \to 0$ is any short exact sequence of Abelian groups, e determines a homomorphism $\Phi_2\colon K_2 \to A/2A$, where $K_2 = \{x \in C \mid 2x = 0\}$, as follows: for $x \in K_2$, $\Phi_2 x = \lambda^{-1} 2 \rho^{-1} x$. By Theorem 5.1 of [9], knowledge of Φ_2 suffices to determine B as an extension of C by A if $2C = 0$. By 6.1.1, $2E^{0,n-1} = 0$; then (for the sequence (6.1-1)) the groups are given by 6.1.1, and d_2 and Φ_2 are given by 6.1.3, and Theorem 6.0.2 follows immediately; where, for $a \in \pi_1$, $x \in \pi_2$, $\lambda[a \otimes x] = y_a \otimes x$, $\lambda[a \otimes \eta] = y_a \otimes \eta$; and, if $a^2 = 1$ and $(-1)^a(-1)^k = 1$, $\rho z_a = [a]$.

Recall that $H_2 = \{0, \eta\} \cong Z_2$.

For any integer r, $k \leqslant r \leqslant 2k$, let $K^r_{n-1} = Z\pi_1$ and $K^r_n = Z\pi_1 \otimes (\pi_2 \oplus H_2)$. For $r < k$ or $r > 2k$, let $K^r_{n-1} = K^r_n = 0$. Define $\delta^r_{n-1}\colon K^r_{n-1} \to K^{r+1}_{n-1}$ and $\delta^r_n\colon K^r_n \to K^{r+1}_n$ for all r as follows: for $k \leqslant r < 2k$, and for $a \in \pi_1$, $x \in \pi_2$:

(i) $\delta^r_{n-1}(a) = a + (-1)^{r+k+n+1}(-1)^a a^{-1}$,

(ii) $\delta^r_n(a \otimes x) = a \otimes x + (-1)^a(-1)^{r+k+n} a^{-1} \otimes x^a + (w_2 x) a^{-1} \otimes \eta$,

(iii) $\delta^r_n(a \otimes \eta) = a \otimes \eta + a^{-1} \otimes \eta$.

We now define $H^r_s = \operatorname{Ker} \delta^r_s / \operatorname{Im} \delta^r_s$ for $s = n - 1$ or n.

LEMMA 6.1.1. (I) *For all r, and for $s = n - 1$ or n,*

$$H^r\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\zeta_M\right) = H^r\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\theta_M\right) = H^r_s.$$

(II) *For all $r > k$, and for $s = n - 1$ or n, $j^*\colon H^r_s \to H^r(\mathbf{R}S^k; F^{-1}\pi_s\zeta_M)$ is onto, and*

$$\operatorname{Ker} j^* = \begin{cases} (Z \cap \operatorname{Ker} \delta^r_{n-1})/\operatorname{Im} \delta^{r-1}_{n-1} & \text{if } s = n-1, \\ (Z \otimes H_2 \cap \operatorname{Ker} \delta^r_n)/\operatorname{Im} \delta^{r-1}_n & \text{if } s = n. \end{cases}$$

PROOF (I). Trivially,

$$H^*\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\zeta_M\right) = H^*\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\theta_M\right),$$

since these two coefficient sheaves differ only on PS^k.

We now consider $S^k \subset R^{k+1}$ to be the unit sphere, and $\pi\colon S^k \to P^k$ the covering map onto real projective k-space. For any $x \in P^k$, let $[x] \subset R^{k+1}$ be the line through 0 determined by x. Let ν be the k-plane bundle over P^k such that, for each $x \in P^k$, $\nu_x = [x]^\perp \subset R^{k+1}$. Clearly, $\pi^{-1}\nu = \tau$, the tangent bundle of S^k, and $\nu \oplus h = (k + 1)$, the trivial $(k + 1)$-plane bundle, where h is the canonical line bundle over P^k and "$\oplus$" denotes Whitney sum. Thus $w_i\nu = u^i$ for all i, where $u \in H^1(P^k; Z_2)$ is the generator.

For any vector bundle ξ, let $E\xi$ and $S\xi$ be the total spaces of the associated disc and sphere bundles, respectively. We construct a commutative diagram of pairs

$$\begin{array}{ccc} (E\tau, S\tau) & \xrightarrow{\phi} & (\mathbf{R}S^k, SS^k) \\ \downarrow\pi & & \downarrow\pi \\ (E\nu, S\nu) & \xrightarrow{\Upsilon} & (\mathbf{R}^*S^k, PS^k) \end{array}$$

as follows. $E\tau = \{(u, v) \in S^k \times B^{k+1} | u \perp v\}$, where $B^{k+1} \subset R^{k+1}$ is the unit ball, while $SS^k = S\tau = \{(u, v) \in S^k \times S^k | u \perp v\}$. For any $(u, v) \in E\tau$, let

$$\phi(u, v) = \begin{cases} (\alpha_v v + u, \alpha_v v - u) \in RS^k & \text{if } (u, v) \notin S\tau, \\ (v, u) \in SS^k & \text{if } (u, v) \in S\tau, \end{cases}$$

where $\alpha_v = (1 - \|v\|^2)^{1/2}$, and let Υ be the unique map which makes the diagram commutative. Trivially, ϕ and Υ are both homeomorphisms of pairs. Thus $\mathbf{R}^*S^k/PS^k \cong M(\nu)$, the Thom complex of ν, whence $H^i(\mathbf{R}^*S^k, PS^k; Z_2) \cong Z_2$ if $k \leqslant i \leqslant 2k$, 0 otherwise; and $\mathrm{Sq}^j\colon H^i(\mathbf{R}^*S^k, PS^k; Z_2) \to H^{i+j}(\mathbf{R}^*S^k, PS^k; Z_2)$ is nonzero if and only if $k \leqslant i \leqslant i+j \leqslant 2k$ and $\binom{i-k-1}{j} = 1 \bmod 2$ (where we take $\binom{-1}{j} = 1 \bmod 2$ for all j). $\mathbf{R}S^k/SS^k$ may then be considered to have two cells, e_1^i and e_2^i, in each dimension $k \leqslant i \leqslant 2k$, and none in other dimensions, such that:

(i) $Te_1^i = e_2^i$ for all i, where $T\colon \mathbf{R}S^k \to \mathbf{R}S^k$ is the map which exchanges coordinates;

(ii) For all $k \leqslant i < 2k$ and $j = 1$ or 2,

$$\delta e_j^i = \begin{cases} e_1^{i+1} + e_2^{i+1} & \text{if } i-k \text{ is even,} \\ (-1)^{j+1}\left(e_1^{i+1} - e_2^{i+1}\right) & \text{if } i-k \text{ is odd.} \end{cases}$$

Then $\mathbf{R}^*S^k/PS^k$ has one cell in each dimension $k \leqslant i \leqslant 2k$, namely $e^i = \pi e_1^i = \pi e_2^i$, and no cells in other dimensions; and $\delta e^i = (1 + (-1)^{i-k})e^{i+1}$.

Now let G be any Abelian group and $e\colon G \to G$ an automorphism of order 2. Recall that $\pi_1(\mathbf{R}^*S^k) \cong T_2 = \{1, m\}$, a multiplicative group of order 2. Let $\mu_e\colon T_2 \times G \to G$ be the action where $\mu_e(m, x) = ex$ for all $x \in G$, and let $G^e = S(G, \mu_e)$, a sheaf over $\mathbf{R}^*S^k$. $H^*(\mathbf{R}^*S^k, PS^k; G^e)$ may be considered to be the equivariant cohomology of the pair $(\mathbf{R}S^k, SS^k)$ with coefficients in G under the action μ_e, specifically, the homology of the chain complex

$$G \xrightarrow{d^k} G \xrightarrow{d^{k+1}} G \to \cdots \to G \xrightarrow{d^{2k-1}} G$$

where, for $k \leqslant i < 2k$, $d^i x = x + (-1)^{i-k}ex$ for all $x \in G$. Letting $G_{n-1} = Z\pi_1$ and $ea = (-1)^a(-1)^n a^{-1}$ for all $a \in \pi_1$, and letting $G_n = Z\pi_1 \otimes (\pi_2 \oplus H_2)$ and $e(a \otimes (x + \lambda\eta)) = ((-1)^a(-1)^{n+1})a^{-1} \otimes x^a + a^{-1} \otimes (\lambda + w_2 x)\eta$ for all $a \in \pi_1$, $x \in \pi_2$, $\lambda \in Z_2$, we have by 5.3.3 that $G_s^e = F^{-1}\pi_s\theta_M$ for $s = n-1$ or n, and we are done with the proof of (I).

Proof (II). For convenience of notation, we agree to let $S_s = F^{-1}\pi_s\zeta_M$ and $Q_s = F^{-1}\pi_s\theta_M$ for $s = n-1$ or n. Let $L_s \subset S_s$ be the unique maximal subsheaf with a local product structure, specifically, $L_s = J_s^e$, where $J_{n-1} = Z \subset G_{n-1}$ and $J_n = Z \otimes H_2 \subset G_n$. Note that $G_s = J_s \oplus R_s$, for suitably chosen $R_s \subset G_s$, and T_2 acts independently on each direct summand: it follows that $S_s = L_s \oplus R_s^e$, a direct summation of sheaves. We also have that $L_s|PS^k = S_s|PS^k$ and $Q_s|R^*S^k = S_s|R^*S^k$, and:

$$L_n \cong Z_2;$$

$$L_{n-1} \cong \begin{cases} Z & \text{if } n \text{ is even,} \\ Z^T & \text{if } n \text{ is odd.} \end{cases}$$

Thus $i^*\colon H^r(\mathbf{R}^*S^k, PS^k; L_s) \to H^r(\mathbf{R}^*S^k, PS^k; S_s)$ is mono, its image exactly what we wish $\operatorname{Ker} j^*$ to be. We have a commutative diagram with exact rows (since $H^i(\mathbf{R}^*S^k; L_s) = 0$ for all $i > k$, because L_s has a local product structure and $\mathbf{R}^*S^k$ is of the homotopy type of P^k):

$$\begin{tikzcd}
H^{r-1}(PS^k; L_s) \arrow[r] \arrow[d, "\cong"'] & H^r(\mathbf{R}^*S^k, PS^k; L_s) \arrow[r] \arrow[d, "1-1"', "i^*"] & 0 \arrow[r] \arrow[d] & H^r(PS^k; L_s) \arrow[r] \arrow[d, "\cong"'] & H^{r+1}(\mathbf{R}^*S^k, PS^k; L_s) \arrow[r] \arrow[d, "1\text{-}1"', "i^*"] & 0 \\
H^{r-1}(PS^k; S_s) \arrow[r, "\delta"] & H^r(\mathbf{R}^*S^k, PS^k; S_s) \arrow[r, "j^*"] & H^r(\mathbf{R}^*S^k; S_s) \arrow[r, "i^*"] & H^r(PS^k; S_s) \arrow[r, "\delta"] & H^{r+1}(\mathbf{R}^*S^k, PS^k; S_s) &
\end{tikzcd}$$

By a simple diagram-chasing argument, we are done.

Notation. If $y \in \operatorname{Ker} \delta^r$, we shall let $[y] \in \operatorname{Ker} \delta^r/\operatorname{Im} \delta^{r-1}$ be the class represented by y.

In the following lemmas, we shall assume that $n = 2k$, $k \geqslant 3$.

We examine a particular case. For any integer s, let N_s^n be the total space of a real $(n-2)$-plane bundle ξ_x^{n-2} over P^2 such that $\xi_s^{n-2} \oplus 3h$ is stably equivalent to sh, the s-fold Whitney sum of the canonical line bundle, h. Clearly, $N_s^n = N_{s+4}^n$, since h has order 4 in K-theory. If τ_s is the tangent bundle of $N_s^n = N$, τ_s is stably equivalent to sh, hence $w_1 N = \binom{s}{1} m$ and $w_2 N = \binom{s}{2} m^2$, where $m \in H^1(N; Z_2)$ is the generator of that group.

Let $a \in \pi_1 N \cong T_2$ be the generator. Then $\tilde{a} \in \pi_2 N \cong Z$ is a generator, and $\tilde{a}^a = -\tilde{a}$. Let $f\colon S^k \to N$ be an embedding. Recall that Φ is the fiber of θ_N. We have by 5.3.2 that

$$F^{-1}\pi_{n-1}\theta_N \cong \begin{cases} Z + Z & \text{if } s \text{ is even,} \\ Z + Z^T & \text{if } s \text{ is odd,} \end{cases}$$

where the first generator is represented by $1 \in Z\pi_1 = \pi_{n-1}\Phi$, the second by $a \in \pi_{n-1}\Phi$; and also that

$$F^{-1}\pi_n\theta_N \cong \begin{cases} Z^T + Z_2 + Z + Z_2 & \text{if } s \text{ is even,} \\ Z^T + Z_2 + Z^T + Z_2 & \text{if } s \text{ is odd,} \end{cases}$$

where the generators are represented, respectively, by $1 \otimes a$, $1 \otimes \eta$, $a \otimes \tilde{a}$, $a \otimes \eta \in Z\pi_1 \otimes (\pi_2 \oplus H_2) = \pi_n\Phi$.

Let $B = \mathbf{R}^*S^k$. Since f is an embedding, F has a lifting to Y_N, thus $F^{-1}\theta_N$ has a section, i.e. $F^{-1}\theta_N$ is in the category $\mathcal{X}_B^+$ of B-sectioned fibrations. The following lemma is phrased in the notation of **[10]**.

LEMMA 6.1.2. *If $N = N_s^n$, then the first two stages of the Postnikov tower for $F^{-1}\theta_N$ are given by* ($B = \mathbf{R}^*S^k$):

$$\begin{tikzcd}
k_B(Z, n, m) \times k_B(Z_2, n) \times k_B\left(Z, n, \binom{s}{1}m\right) \times k_B(Z_2, n) \arrow[r, "\lambda"] & e_1 \arrow[d] & F^{-1}\theta_N \arrow[l, "P_1"'] \\
k_B(Z, n-1) \times k_B\left(Z, n-1, \binom{s}{1}m\right) = e_0 \arrow[r, "\alpha"] & {} &
\end{tikzcd}$$

$$k_B(Z, n+1, m) \times k_B(Z_2, n+1) \times k_B\left(Z, n+1, \binom{s}{1}m\right) \times k_B(Z_2, n+1)$$

where P_1 induces isomorphism in homotopy through dimension n, and where

$$\begin{aligned}\alpha^*\colon \iota_{n+1} \otimes 1 \otimes 1 \otimes &\mapsto 0,\\ 1 \otimes \iota_{n+1} \otimes 1 \otimes 1 &\mapsto (\mathrm{Sq}^2 + m^2)\iota_{n-1} \otimes 1,\\ 1 \otimes 1 \otimes \iota_{n+1} \otimes 1 &\mapsto 1 \otimes (\delta m)\iota_{n-1},\\ 1 \otimes 1 \otimes 1 \otimes \iota_{n+1} &\mapsto 1 \otimes \left(\mathrm{Sq}^2 + \tbinom{n-s}{2} m^2\right)\iota_{n-1},\end{aligned}$$

where δ is the Bokstein homomorphism associated with the coefficient sequence $Z \xrightarrow{2} Z \to Z_2$.

We postpone the proof.

LEMMA 6.1.3. *Let $a \in \pi_1$, $a^2 = 1$, and choose (jointly) $a \in \pi_2$, $0 \leq s = k_a < 4$ (cf. 5.2). Then in sequence* (6.1-1):

(I) *If* $(-1)^a(-1)^k = -1$, $d_2[a] = a \otimes \tilde{a} + (\binom{k-3}{2} + \binom{n-s}{2})a \otimes \eta$;

(II) *If* $(-1)^a(-1)^k = 1$, $\Phi_2[a] \in \operatorname{Coker} d_2$ *is represented by* $[a \otimes \tilde{a} + (\binom{k-3}{2} + \binom{n-s}{2} + \binom{s}{1})a \otimes \eta]$.

PROOF. We remind the reader that $(-1)^a = (-1)^s$, thus $\binom{s}{1} = \binom{k}{1} \bmod 2$ if $(-1)^a(-1)^k = 1$.

Without loss of generality, f is inessential, hence we may assume $f\colon S^k \to N_s^n \subset M$. Since all constructions involved are natural with respect to inclusions of manifolds of the same dimension, we may assume that $M = N_n^s$. We have a commutative diagram with exact rows, where the first, second and fourth vertical arrows represent onto maps, by 6.1.1:

$$\begin{array}{ccccccccc} H_{n-1}^{n-2} & \xrightarrow{d_2} & H_n^n & \xrightarrow{\lambda} & [\mathbf{R}^*S^k, PS^k; \theta_M]_F & \xrightarrow{\rho} & H_{n-1}^{n-1} & \to 0 \\ \text{onto}\downarrow j^* & & \text{onto}\downarrow j^* & & \downarrow j^* & & \text{onto}\downarrow j^* & \\ E_2^{-1,n-1} & \xrightarrow{d_2} & E_2^{0,n} & \xrightarrow{\lambda} & [S^k \subset M]_f & \xrightarrow{\rho} & E^{0,n-1} & \to 0 \end{array}$$

It is sufficient to compute d_2 and Φ_2 for the top row. Recall that $\cup\, m^2\colon H^{n-2}(\mathbf{R}^*S^k, PS^k; Z_2) \to H^n(\mathbf{R}^*S^k, PS^k; Z_2)$, while Sq^2, on the same group, is nonzero if and only if $\binom{k-3}{2} = 1 \bmod 2$. Part (I) now follows immediately from 6.1.1 and 6.1.2, while (II) follows from 6.1.1, 6.1.2, and the results of **[10]**.

6.2. *Proof of* 6.1.2. Let $p\colon \tilde{N} \to N$ be the universal covering of N; $\tilde{N} = S^2 \times R^{n-2}$. Since f may be assumed to be inessential, we may choose an embedding $\tilde{f}\colon S^k \to N$ such that $p\tilde{f} = f$. Note that $R^\infty \cong R^\infty \times R^\infty$. Thus we may choose an embedding P such that the following diagram commutes, where p_1 is projection:

$$\begin{array}{ccc} \tilde{N} \times R^\infty & \xrightarrow{P} & N \times R^\infty \\ \downarrow p_1 & & \downarrow p_1 \\ \tilde{N} & \xrightarrow{p} & N \end{array}$$

Let $\nabla N = (p^2)^{-1} \nabla N - \Delta \tilde{N} \subset R\tilde{N}$, and let $\nabla {}^*N \subset R^*\tilde{N}$ be the image of ∇N. Let $\tilde{R}N = R\tilde{N} - \nabla N$ and $\tilde{R}^*N = R^*N - \nabla {}^*N$. Consider the diagram (where $p^*[x, y] = [px, py]$ for all $[x, y] \in \tilde{R}^*N$):

$$\begin{array}{ccc} \tilde{R}^*N & \xrightarrow{p} & R^*N \\ \cap & & \\ R^*\tilde{N} & & \cap \\ \cap & & \\ R^*(\tilde{N} \times R^\infty) & \xrightarrow{R^*p} & R^*(N \times R^\infty) \end{array}$$

We have that Φ = fiber of $\tilde{R}^*N \subset R^*(\tilde{N} \times R^\infty)$ = fiber of $R^*N \subset R^*(N \times R^\infty)$.

LEMMA 6.2.1 (HAEFLIGER [4]). (I) *As an algebra over* Z_2, $H^* = H^*(R^*(\tilde{N} \times R^\infty); Z_2) = H^*(\Gamma S^2; Z_2)$ *is generated only by the elements*: $m \in H^1$, $\Gamma\sigma \in H^2$, *and* $\Sigma\sigma \in H^4$, *subject only to the relations*: $m\Gamma\sigma = (\Gamma\sigma)^2 = \Gamma\sigma\Sigma\sigma = (\Sigma\sigma)^2 = 0$.

(II) (R^*i): $H^* \to H^*(R^*N; Z_2)$ *is surjective, and its kernel is generated by* m^n *and* $m^{n-2}\Sigma\sigma$.

LEMMA 6.2.2. *The map* i^*: $H^*(R^*\tilde{N}; Z_2) \to H^*(\tilde{R}^*N; Z_2)$ *is injective, and its cokernel is generated over* Z_2 *by* $\omega \in H^{n-1}(\tilde{R}^*N; Z_2)$, $m\omega$, *and* $m^2\mu$. *Furthermore*, $(\Gamma\sigma)\omega = m^2\omega$, *and* $\mathrm{Sq}^i\omega = \binom{n-s}{i} m^i\omega$ *for all* i.

PROOF. We construct a map β: $R^*\tilde{N} \to \tilde{R}^*N$ such that $i\beta$ is homotopic to the identity on $R^*\tilde{N}$. Recall that $N = N_s^n$, the total space of $\xi_s^{n-2} = \xi$ over P^2. Choose a nonzero section χ^* of $\xi \otimes h$, and let χ: $S^2 \to S^2 \times (R^{n-2} - \{0\})\tilde{N}$ be the corresponding section of $\pi^{-1}(\xi \otimes h)$, the trivial n-bundle over S^2. Let p_1: $\tilde{N} \to S^2$, p_2: $\tilde{N} \to R^{n-2}$ be the projections. For all $0 \leqslant t \leqslant 1$, let β_t: $R^*\tilde{N} \to \tilde{R}^*N$ be defined as follows. If $[x, y] \in R^*\tilde{N}$, let $\rho = \rho(x, y) = \|p_1x - p_1y\|$, and let:

$$\beta_t[x, y] = \begin{cases} [x, y] \quad \text{if } 1 \leqslant \rho \leqslant 2, \\ \big[\big(p_1x, (1 + (\rho - 1)t)p_2x + (1 - \rho)tp_2(\chi(p_1x))\big), \\ \quad p_1y, \big(1 + (\rho - 1)tp_2y + (1 - \rho)tp_2(\chi(p_1y))\big)\big] \\ \hfill \text{if } 0 \leqslant \rho \leqslant 1. \end{cases}$$

Let $\beta = \beta_1$, which clearly has the desired property. Now the cofiber of the inclusion $\tilde{R}^*N \subset R^*\tilde{N}$ is the Thom complex of $\tau \otimes h$ over N, which we denote by TC, where τ is the tangent bundle of N. We have a commutative diagram with split exact columns, where each row is a long exact Thom-Gysin sequence, and all coefficients are Z_2:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & H(R^*\tilde{N}) \arrow[r, "\pi^*"] \arrow[d, shift left, "i^*"] & H(R\tilde{N}) \arrow[r, "\theta"] \arrow[d, shift left, "i^*"] & H^*(R^*\tilde{N}) \arrow[r, "\cup m"] \arrow[d, shift left, "i^*"] & H(R^*\tilde{N}) \arrow[r] \arrow[d, shift left, "i^*"] & \cdots \\
\cdots \arrow[r] & H^*(\tilde{R}^*N) \arrow[r, "\pi^*"] \arrow[u, shift left, "\beta^*"] \arrow[d, "\delta"'] & H^*(\tilde{R}N) \arrow[r, "\theta"] \arrow[u, shift left, "\beta^*"] \arrow[d, "\delta"'] & H^*(\tilde{R}^*N) \arrow[r, "\cup m"] \arrow[u, shift left, "\beta^*"] \arrow[d, "\delta"'] & H^*(\tilde{R}^*N) \arrow[r] \arrow[u, shift left, "\beta^*"] \arrow[d, "\delta"'] & \cdots \\
\cdots \arrow[r] & H^*(TC) \arrow[r, "\pi^*"] \arrow[d] & H^*(\tilde{N}/S^2) \arrow[r, "\theta"] \arrow[d] & H^*(TC) \arrow[r, "\cup m"] \arrow[d] & H^*(TC) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

We now define $\omega \in H^{n-1}(\tilde{R}^*N; Z_2)$ to be the unique element such that $\beta^*\omega = 0$ and $\delta\omega = U$, the Thom class of $\tau \otimes h$. Now $H^*(TC)$ has only generators U, mU, and m^2U over Z_2, and $\mathrm{Sq}^i U = \binom{n-s}{i} m^i U$ for all i, although $m^3U = 0$. By exactness of the third row, $\theta(\sigma \cup \pi^* U) = \theta((\sigma \otimes 1) \cup \pi^* U)$ must be m^2U. Also, $\theta(\sigma \otimes 1) = \Gamma\sigma$, thus $(\Gamma\sigma)\omega = \theta((\sigma \otimes 1) \cup \pi^*\omega) = m^2\omega$. 6.2.2 then follows trivially.

Finally, Lemma 6.1.2 follows from routine Serre spectral sequence computation of the relative Postnikov tower of the inclusion $\tilde{R}^*N \subset R^*(\tilde{N} \times R^\infty)$.

7. Affine groups.

Definition 7.1. An *affine group* is a set A together with an operation $\cdot_a$ for each $a \in A$, such that $(A, \cdot_a)$ is a group with identity a, and such that, for any $x, y, a, b \in A$, $x \cdot_a y = x \cdot_b (a)_b^{-1} \cdot_b y$, where $(a)_b^{-1}$ is the inverse of a under $\cdot_b$. In addition, we say that A is *Abelian* if $\cdot_a$ is commutative for all $a \in A$. (The operation shall then usually be denoted $+_a$.)

We remark that Becker's definition of affine group [1] is that of Abelian affine group.

An alternative definition, which is more natural to the constructions of this paper, is:

Definition 7.2. An *affine group* is a set together with a ternary operation τ on A such that (writing $\tau(x, y, z) = xy^{-1}z$)

(i) $vw^{-1}(xy^{-1}z) = (vw^{-1}x)y^{-1}z$ (associative law),

(ii) $xy^{-1}y = yy^{-1}x = x$ (cancellation law).

In addition, we say that A is *Abelian* if

(iii) $xy^{-1}z = zy^{-1}x$ (commutative law).

We leave the proof of the equivalence of 7.1 and 7.2 to the reader as an exercise (Hint: $x \cdot_a y = xa^{-1}y$). Note that the empty set is an affine group.

Letting A be any affine group, we define two equivalence relations on $A \times A$, R and L. For any $x, y, z, w \in A$, we say that $(x, y)R(z, w)$ if $y = xz^{-1}w$, and $(x, y)L(z, w)$ if $x = zw^{-1}y$. (Note that $R = L$ if A is commutative.) Let $A^R = A \times A/R$, and $A^L = A \times A/L$. If $x, y \in A$, let $x^{-1}y \in A^R$ and $xy^{-1} \in A^L$ denote the elements represented by the ordered pair (x, y). Both A^R and A^L are groups, under the operations $(x^{-1}y)(z^{-1}w) = x^{-1}(yz^{-1}w)$, and $(xy^{-1})(zw^{-1}) = (xy^{-1}z)w^{-1}$, respectively. We call A^R and A^L the right and left action groups of A, since A^R acts on A on the right, in the obvious manner suggested by the notation, and A^L acts on A on the left. We remark that A^R and A^L are isomorphic, canonically if A is Abelian, in which case we write $A^* = A^R = A^L$, and $x - y$ instead of xy^{-1} or $y^{-1}x$.

We remark that two affine groups are isomorphic if and only if their action groups are isomorphic. If G is any group, G may be taken to be an affine group by letting $xy^{-1}z$ have the usual meaning ($x - y + z$ if G Abelian), in which case $G \cong G^L$ by letting x correspond to $x1^{-1}$.

If we say that an affine group A is isomorphic to a group G, we shall mean that A is isomorphic to G considered as an affine group; equivalently, $A^R \cong G$ or $A^L \cong G$.

8. **Correction.** Recall diagram (2.6-1) in this paper, which is basically diagram (3.2-1) of [7]. We define a pair fibration $'\zeta_M: ('Y_M, 'Z_M) \to ({}^\infty\mathbf{R}M, {}^\infty PM)$ as follows: Let $'Y_M$ be the space consisting of all paths $\sigma: I \to {}^\infty\mathbf{R}^*M$ such that $\sigma(1) \in \mathbf{R}M$, and *either* $\sigma(0) \in {}^\infty PM$ *or* $\sigma(1) \not\in PM$. Let $'Z_M$ be the space consisting of all paths $\sigma: I \to {}^\infty PM$ such that the composition $I \to^\sigma {}^\infty PM \to M$ is constant. Let $'\zeta_M: ('Y_M, 'Z_M) \to ({}^\infty\mathbf{R}^*M, {}^\infty PM)$ be evaluation at 0, a pair fibration, and let $\iota: '\zeta_M \subset \zeta_M$ be the obvious inclusion.

LEMMA 8.1. $\iota_\#: [(\mathbf{R}^*V, PV); '\zeta_M]_F \cong [(\mathbf{R}^*V, PV); \zeta_M]_F$.

PROOF. Let $''\zeta_M: ('Y_M, Z_M) \to ({}^\infty\mathbf{R}^*M, {}^\infty PM)$. Now the inclusion $i: ('Y_M, Z_M) \subset (Y_M, Z_M)$ is a fiber homotopy equivalence of pairs. Let $U \subset {}^\infty\mathbf{R}^*M$ be a regular collar of ${}^\infty PM$, such that $U \cap \mathbf{R}M$ is a regular collar of PM. Let $r_t: {}^\infty\mathbf{R}^*M \to \mathbf{R}^*M$, $0 \leq t \leq 1$, with $r_0 =$ identity, be a strong deformation retraction of ${}^\infty\mathbf{R}^*M$ onto the complement of U such that $r_t(\mathbf{R}^*M) \subset \mathbf{R}^*M$ for all t. Let ρ: ${}^\infty\mathbf{R}^*M \to I$ be continuous such that $\rho^{-1}\{0\} = {}^\infty PM$, and let $(j\sigma)(t) = r_s(t)$ for all $\sigma \in Y_M$, $t \in I$, where $s = t\rho(\sigma(0))$. Clearly j is the identity on ${}^\infty PM$. The fact that j is a pair fiberwise homotopy inverse of i is trivial to verify. Thus $i_\#: [(\mathbf{R}^*V, PV); ''\zeta_M]_F \cong [(\mathbf{R}^*V, PV); \zeta_M]_F$. Now $\pi_k('\zeta_M) = \pi_k(''\zeta_M)$ for all k, since $'Z_M \to {}^\infty PM$ and $Z_M \to {}^\infty PM$ both have fibers of the homotopy type of S^{n-1}. By 2.4.1, we are done.

The error in the proof of Theorem 3.3.1 of [7] is the fact that if a lifting Φ

of F to ζ_M is given, the map $G[\Phi]$ in the diagram on p. 362 may not exist. If, on the other hand, Φ is a lifting of F to $'\zeta_M$ (which we may assume, by Lemma 8.1), existence of $G[\Phi]$ is assured. The remainder of the proof is valid.

Theorem 3.3.2 of [7] can be corrected in a similar manner.

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