TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 239, May 1978

A CHAIN FUNCTOR FOR BORDISM(1)

BY

STANLEY O. KOCHMAN

ABSTRACT. Chains of differential graded abelian monoids are defined for bordism and cobordism theories. These chains are used to define matric Massey products and can be filtered so as to define the Adams spectral sequence. From this point of view, we prove three basic theorems which show how Massey products behave in the Adams spectral sequence.

1. **Introduction.** The fundamental difference between singular homology and generalized homology is that the former is defined as the homology of a chain functor of differential graded abelian groups. This point of view has proved to be useful for understanding examples and proving theorems. Unfortunately no generalized homology theory can be defined as the homology of a chain functor of differential graded abelian groups. (See R. O. Burdick, P. E. Conner and E. E. Floyd [6].) However, a generalized homology theory can be defined as the homology of a chain functor with a weaker algebraic structure than that of an abelian group. Thus D. W. Anderson [4] defines chain functors of differential graded groups and A. Kock, L. Kristensen and I. Madsen [11] define chain functors of differential graded loops.

In §2 we define a differential graded abelian monoid. In §4 we define a chain functor of differential graded abelian monoids for any bordism theory. These chains are generated by manifolds with rigid structure which we define in §2. In §5 we use Spanier-Whitehead duality to define a cochain functor of differential graded abelian monoids with an associative product for any cobordism theory.

The applications of this theory appear in §§6, 7, 8 and in [10]. In §6 we show how the Atiyah-Hirzebruch and Adams spectral sequences for cobordism can be defined from filtrations on the cochain functor of §5. In §6 we develop a theory of cobordism Massey products which translates to G. J. Porter's higher products [17] under the Thom-Pontrjagin isomorphism. In §8 we use the point of view of §6 to show that cobordism Massey products behave exactly like their algebraic cousins [14] in the Adams spectral sequence. In [10] we will use the results of §8 to compute Ω^*_{Sp} by analyzing the classical mod two Adams spectral sequence [1]. The theorems of §8 also apply

Received by the editors August 20, 1975 and, in revised form, March 9, 1976.
AMS (MOS) subject classifications (1970). Primary 55B20, 55G30, 55H15, 55J15.
(1)The research described in this paper was partially supported by NSF Grant GP-38541.

to the classical and BP^* Adams spectral sequences for Π_*^S.

In §9 we will show that the results of §§4, 5, 7 and 8 apply to Brown-Peterson homology.

We summarize the notation which will be used throughout this paper. R is a subring of the rational numbers and N is the monoid of all nonnegative numbers in R. We work in a partially coordinate-free setting using the content and language of J. P. May [15, Chapters I, II]. Thus $\perp$ means "is orthogonal to", $+$ refers to internal direct sum, $\uplus$ means disjoint union and $*$ refers to one-point compactification. $\mathbf{R}^\infty$ is the inner product space with orthonormal basis $\{b_k | 1 \leq k < \infty\}$ and $\mathbf{R}^N$ is the vector space with basis $\{b_1, \ldots, b_N\}$. $\mathbf{R}_+^N = \mathbf{R}b_1 + \cdots + \mathbf{R}b_{N-1} + \mathbf{R}^+b_N$, $\mathbf{R}_-^N = \mathbf{R}_1 + \cdots + \mathbf{R}b_{N-1} + \mathbf{R}^-b_N$, $S^N = (\mathbf{R}^N)^*$, $S_+^N = (\mathbf{R}_+^N)^*$ and $S_-^N = (\mathbf{R}_-^N)^*$. $\mathfrak{B}$ always denotes a (B, f)-structure. (See [12] or Definition 3.1.) $\mathfrak{B}$ determines a bordism theory $\mathfrak{B}_*$ and a cobordism theory $\mathfrak{B}^*$ in the usual way [5].

The author expresses his appreciation to E. H. Brown, D. S. Kahn, J. P. May and L. Smith for stimulating conversations and helpful suggestions.

2. **Differential graded abelian monoids.** A differential graded abelian monoid, or DGM for short, is the algebraic model for bordism chains and cobordism cochains. A DGM is similar to a chain complex with abelian groups replaced by abelian monoids. In addition, the geometric technicalities make it convenient to allow a DGM to consist of a set of graded abelian monoids with isomorphic homology in each degree. We require that the homology of a DGM be an abelian group by means of a "chain map" τ and a "chain homotopy" σ from $\tau + 1$ to zero. Thus in homology $\tau_* = -1$.

We will also define a pairing of DGMs. This definition is the algebraic model for the bordism chain cross-product and the cobordism cochain cup-product which we define in §5.

We call an abelian monoid M an N-module if for all $m \in M$ and all positive integers n with $1/n \in N$ the equation $nx = m$ has a unique solution for $x \in M$. Let M'' be a submonoid of an abelian monoid M'. Recall that M'/M'' is defined as the abelian monoid M' modulo the following equivalence relation: $m_1 \sim m_2$ if there are $q_1, q_2 \in M''$ with $m_1 + q_1 = m_2 + q_2$. If M' and M'' are N-modules then M'/M'' is also an N-module. For example, $H_m[M_\lambda]$ below is given by Kernel ∂/ Image ∂. Since $H_m[M_\lambda]$ is an N-module and an abelian group, it is also an R-module.

Definition 2.1. (a) A differential graded abelian monoid (DGM) has the following structure:

(1) There is a Z-graded indexing set $I = \{I_n | n \in Z\}$.

(2) For each $\lambda \in I_n$ we have a "complex":

$$M_\lambda = \left[M_{n+1,\lambda} \xrightarrow{\partial} M_{n,\lambda} \xrightarrow{\partial} M_{n-1,\lambda} \xrightarrow{\partial} M_{n-2,\lambda} = 0 \right].$$

Each $M_{m,\lambda}$ is an abelian monoid with cancellation and an N-module such that only zero has an additive inverse. ∂ is a linear map such that $\partial^2 = 0$.

(3) For each $\lambda \in I_n$ there are linear maps τ: $M_{m,\lambda} \to M_{m,\lambda}$, $m \in \{n - 1, n, n + 1\}$, and σ: $M_{q,\lambda} \to M_{q+1,\lambda}$, $q \in \{n - 1, n\}$ such that $\partial\tau = \tau\partial$, $\sigma^2 = 0$, $\tau\sigma = \sigma$ and $\sigma\partial + \partial\sigma = \tau + 1$.

(4) For $\alpha, \beta, \gamma \in I_n$ there are group isomorphisms $\Psi_{\alpha,\beta}$: $H_n[M_\alpha] \cong H_n[M_\beta]$ such that $\Psi_{\alpha,\alpha} = 1$, $\Psi_{\beta,\alpha} = \Psi_{\alpha,\beta}^{-1}$ and $\Psi_{\beta,\gamma} \circ \Psi_{\alpha,\beta} = \Psi_{\alpha,\gamma}$. Thus $H_n[M]$ is well defined.

(b) We call M an absolute DGM if in addition to (1)–(4) above there are group isomorphisms Ψ_α: $H_{n-1}[M_\alpha] \cong H_{n-1}[M]$ for all $\alpha \in I_n$.

(c) A generating set G for a (absolute) DGM M consists of subsets $G_{m,\lambda}$ of each $M_{m,\lambda}$ which are closed under ∂, τ, σ and span $M_{m,\lambda}$ over N. Furthermore, for all $\lambda \in I_n$ the cycles of $G_{m,\lambda}$ span $H_n[M_\lambda]$ (and $H_{n-1}[M_\lambda]$) over R.

(d) A map of (absolute) DGMs f: $M' \to M''$ consists of DGMs M' and M'' indexed on I and linear maps $f_{m,\lambda}$: $M'_{m,\lambda} \to M''_{m,\lambda}$ for all $\lambda \in I_n$, $m \in \{n - 1, n, n + 1\}$. We require that the $f_{m,\lambda}$ commute with τ, ∂ and σ, and that the $f_{m,\lambda}$, commute with the $\Psi_{\alpha,\beta}$ (and Ψ_α). If all the $f_{m,\lambda}$ are injective maps then we call M' a sub DGM of M''.

NOTES. (1) A quotient of DGMs may not be a DGM because of condition (4).

(2) The set of singular manifolds will be a generating set for the absolute DGM of bordism chains.

(3) If M is a (absolute) DGM indexed on I then we also call the cocomplex M^* indexed on I^* a (absolute) DGM. For $n \in Z$, $I_n^* = I_{-n}$ and for $\lambda \in I_n^*$, $m \in \{n + 1, n, n - 1, n - 2\}$ we let $M_{m,\lambda}^* = M_{-m,\lambda}$. We let ∂, τ, σ induce linear maps ∂^*, τ^*, σ^* on M_λ^* of degrees $+1$, $0, -1$, respectively. If f: $M' \to M''$ is a map of (absolute) DGMs then we also call f^*: $M'^* \to M''^*$ a map of (absolute) DGMs.

Let A, B, C be DGMs with generating sets. A pairing μ which realizes a pairing μ_*: $H[A] \otimes_R H[B] \to H[C]$ will be defined to be a pairing μ of generating sets which induces μ_* in homology. The domain of definition S_μ of μ need not be all of $G^A \times G^B$, but S_μ is large enough to induce all of μ_*. This condition is imposed on us by the geometric fact that the product of two manifolds in $\mathbf{R}^\infty$ is defined only if the manifolds lie in orthogonal subspaces.

DEFINITION 2.2. (a) Let A, B, C be DGMs with generating sets G^A, G^B, G^C. Let an R-module homomorphism μ_*: $H[A] \otimes_R H[B] \to H[C]$ are given. A pairing μ of DGMs which realizes μ_* consists of a set $S_\mu \subset G^A \times G^B \times I^C$ and a function μ: $S_\mu \to G^C$ such that:

(1) If $(a, b, \xi) \in S_\mu$ then $\mu(a, b, \xi) \in G_{p,\xi}^C$ where $p = \deg a + \deg b$.

(2) If $m, n \in Z$ then there are $\lambda \in I_m^A$, $\gamma \in I_n^B$ and $\xi \in I_{m+n}^C$ such that $(G_{m,\lambda}^A \times G_{n,\gamma}^B \times \{\xi\}) \cap S_\mu \neq \emptyset$. Furthermore, if $(G_{m,\lambda}^A \times G_{n,\gamma}^B \times \{\xi\}) \cap S_\mu \neq$

$\varnothing$ then

$$G^A_{m,\lambda} \times G^B_{n,\gamma} = \{(a, b) | (a, b, \xi') \in S_\mu \text{ for some } \xi' \in I^C_{m+n}\}.$$

(3) If $(a, b, \xi) \in S_\mu$ then $(\tau a, b, \xi), (a, \tau b, \xi) \in S_\mu$ and $\mu(\tau a, b, \xi) = \mu(a, \tau b, \xi) = \tau\mu(a, b, \xi)$.

(4) If $(\sigma a, b, \xi)$ or $(a, \sigma b, \xi)$ is in S_μ then $(a, b, \xi) \in S_\mu$. In the first case $\mu(\sigma a, b, \xi) = \sigma\mu(a, b, \xi)$, while in the second case $\mu(a, \sigma b, \xi) = \sigma\mu(a, b, \xi)$.

(5) If $(a, b, \xi) \in S_\mu$ then $(\partial a, b, \xi), (a, \partial b, \xi), (\partial a, \partial b, \xi) \in S_\mu$ and

$$\partial\mu(a, b, \xi) + \sigma\mu(\partial a, \partial b, \xi) = \mu(\partial a, b, \xi) + \tau^{\deg a}\mu(a, \partial b, \xi).$$

(6) If $(a, b, \xi) \in S_\mu$ with $\partial a = 0$, $\partial b = 0$ then $\mu_*([a] \otimes [b]) = [\mu(a, b, \xi)]$.

(b) If $A = B = C$ in the preceding definition then A is called a differential graded monoidal algebra or DGMA for short.

(c) A map of paired DGMs f: $(A, B, C, \mu) \to (A', B', C', \mu')$ consists of maps of DGMs f^A: $A \to A'$, f^B: $B \to B'$ and f^C: $C \to C'$ such that:

(1) $\mu'_* \circ (f^A_* \otimes f^B_*) = f^C_* \circ \mu_*$,

(2) $(f^A \times f^B \times 1)(S_\mu) \subset S_{\mu'}$,

(3) if $(a, b, \xi) \in S_\mu$ then $f^C\mu(a, b, \xi) = \mu'(f^A(a), f^B(b), \xi)$.

3. **Manifolds with rigid structure.** The set of all manifolds $\mathfrak{M}$ with rigid $\mathfrak{B}$-structure, together with all $\sigma(\mathfrak{M})$, $\partial\sigma(\mathfrak{M})$, will be a generating set for the absolute DGM of bordism chains. A manifold with rigid $\mathfrak{B}$-structure differs from the standard notion [12] of a manifold with $\mathfrak{B}$-structure in two ways. First, a rigid manifold $\mathfrak{M}$ is a fixed subset of $\mathbf{R}^\infty$ rather than a manifold with an isotopy class of embeddings in $\mathbf{R}^\infty$. Secondly, a rigid $\mathfrak{B}$-structure on M is a fixed $\mathfrak{B}$-structure on a fixed tubular neighborhood of M rather than an equivalence class of $\mathfrak{B}$-structures on tubular neighborhoods of M.

We define the three basic operations of boundary, product, and union of manifolds with rigid $\mathfrak{B}$-structure. We then define the Pontrjagin construction $\mathcal{P}$ of a manifold M with rigid $\mathfrak{B}$-structure as a specific map. The standard definition would give $\mathcal{P}(M)$ as a homotopy class of maps. Under $\mathcal{P}$ the three basic operations above correspond to restriction of a map to the boundary of its domain, the smash product of two maps, and the union of two maps which agree on the intersection of their domains.

We recall the definition of a (B, f)-structure [12] in the language of J. P. May [15].

DEFINITION 3.1. A (B, f)-structure $(\mathfrak{B}, \mathfrak{T}, \omega)$, or $\mathfrak{B}$ for short, consists of an $\mathcal{G}$-functor $(\mathfrak{B}, \omega)$ and a morphism of $\mathcal{G}$-functors $\mathfrak{T}$: $\mathfrak{B} \to BO$ such that each $\mathfrak{T}(V)$ is a fibration. ($(BO, 1, \omega)$ is defined in Examples 3.2 below.) A map of (B, f)-structures G: $\mathfrak{B} \to \mathfrak{B}'$ is a morphism of $\mathcal{G}$-functors such that $\mathfrak{T} = \mathfrak{T}' \circ G$.

EXAMPLES 3.2. Define the (B, f)-structure BO by $BO(V) = BO(V^R_H)$ and

$\mathfrak{T} = 1$. (The notation is taken from [15, Chapter I, §1].) If G is a classical group then BG, with $\mathfrak{T}$ the natural projection, is a (B, f)-structure. Recall that $BG(V) = O(V_H^R \times V_H^R)/G(V_H^K) \times O(V_H^R)$ where K is R, C or H. In particular, framed structures Bfr are given by $G = e$. Natural projections define the following maps of (B, f)-structures:

$$Bfr \to BSp \to BSU \to BU \to BSO \to BO.$$

It is customary for purposes of bordism theory to consider manifolds embedded in $\mathbf{R}^N$, $\mathbf{R}_+^N$ if $\partial M = \varnothing$, $\partial M \neq \varnothing$, respectively. This convention, however, is too restrictive to define Massey products. We therefore define domains to replace $\mathbf{R}^N$ and $\mathbf{R}_+^N$. The subspace U of $\mathbf{R}^\infty$ which is included in the structure of a domain will be used to determine the classifying bundle for manifolds in the domain. A set of domains with nonempty boundary will be the indexing set J for our DGMs.

DEFINITION 3.3. (a) Let V be a finite dimensional subspace of $\mathbf{R}^\infty$. A k-dimensional domain in V, $\mathfrak{D} = (D, \varphi, U)$, consists of: (1) a noncompact submanifold with corners D of V [7],

(2) a homeomorphism φ: $S^k \to D^*$, φ: $S_+^k \to D^*$ for $\partial D = \varnothing$, $\partial D \neq \varnothing$, respectively,

(3) a subspace U of V.

(b) Let $\mathfrak{D}$ be a k-dimensional domain in V with $\partial D \neq \varnothing$.

Then there are $(k-1)$-dimensional domains $\partial\mathfrak{D}$, $\mathfrak{D}_+$, $\mathfrak{D}_-$ in V and a $(k-2)$-dimensional domain $\mathfrak{D}_0$ in V. $\partial\mathfrak{D} = (\partial D, \varphi|S^{k-1}, U)$, $\mathfrak{D}_+ = (\varphi(\mathbf{R}_+^{k-1}), \varphi|S_+^{k-1}, U)$, $\mathfrak{D}_- = (\varphi(\mathbf{R}_-^{k-1}), (\varphi|S_-^{k-1}) \circ \gamma^*, U)$ and $\mathfrak{D}_0 = (\varphi(\mathbf{R}^{k-2}), \varphi|S^{k-2}, U)$. γ: $\mathbf{R}^k \to \mathbf{R}^k$ by

$$\gamma(t_1 b_1 + \cdots + t_k b_k) = t_1 b_1 + \cdots + t_{k-1} b_{k-1} - t_k b_k.$$

The homeomorphism φ above will be used to interpret the Pontrjagin construction from a homotopy theoretic viewpoint. We are now ready to define a manifold with rigid $\mathfrak{B}$-structure.

DEFINITION 3.4. $\mathfrak{M} = (M^n, (-1)^e, \mathfrak{D}, T, F)$, a manifold with rigid $\mathfrak{B}$-structure in V, is given by the following data:

(1) $\mathfrak{D}$ is a k-dimensional domain in V.

(2) M is a compact smoothable n-dimensional PL-submanifold of D with $k - n \equiv 0 \bmod 4$.

(3) $\partial M = M \cap \partial D$.

(4) T is an open neighborhood of M in D. ν: $T \to M$ is a normal bundle of M in D with $T \cap \partial M = \nu^{-1}(\partial M)$.

(5)
$$\begin{tikzcd} T \arrow[r, "F"] \arrow[d, "\nu"] & E(U) \arrow[d, "\zeta_U"] \\ M \arrow[r, "F_0"] & B(U) \end{tikzcd}$$

F is a map of vector bundles. ζ_U is the pullback along $\mathcal{T}(U)$ of the associated U_H^R-bundle to the universal principal $O(U_H^R)$-bundle.

(6) $(-1)^\varepsilon$ is a formal sign to keep track of the orientation of $\mathfrak{M}$.

Observe that we make no identifications between 4-tuples $\mathfrak{M}$ and $\mathfrak{M}'$ which differ in any way whatsoever. We next give five operations on rigid manifolds.

DEFINITION 3.5. Let $\mathfrak{M}$, $\mathfrak{M}'$ be manifolds with rigid $\mathfrak{B}$-structures in V, V', respectively.

(a) Let H: $\mathfrak{B} \to \mathfrak{B}'$ be a map of (B, f)-structures. Then $H_*(\mathfrak{M}) = (M, (-1)^\varepsilon, \mathfrak{D}, T, H(U) \circ F)$ is a manifold with rigid $\mathfrak{B}'$-structure.

(b) The boundary of $\mathfrak{M}$ is the manifold with rigid $\mathfrak{B}$-structure in V given by

$$\partial \mathfrak{M} = \left(\partial M, (-1)^{\varepsilon+k}, \partial \mathfrak{D}, T \cap \partial D, F|T \cap \partial D\right)$$
$$\text{where } \partial \mathfrak{D} = \left(\partial D, \varphi|S^{k-1}, U\right).$$

(c) The negative of $\mathfrak{M}$ is the manifold with rigid $\mathfrak{B}$-structure in V given by $\tau\mathfrak{M} = (\mathfrak{M}, (-1)^{\varepsilon+1}, \mathfrak{D}, T, F)$.

(d) If $V \perp V'$ then

$$\mathfrak{M} \times \mathfrak{M}^1 = \left(M \times M', (-1)^\delta, \mathfrak{D} \times \mathfrak{D}', T \times T', \omega \circ (F \times F')\right)$$

is a manifold with rigid $\mathfrak{B}$-structure in $V + V'$. $\delta = \varepsilon + \varepsilon' + nk'$ if $\partial \mathfrak{M}' = \varnothing$, while $\delta = \varepsilon + \varepsilon' + nk' + k$ if $\partial \mathfrak{M}' \neq \varnothing$. $\mathfrak{D} \times \mathfrak{D}' = (D \times D', J^* \circ (\varphi \wedge \varphi'), U + U')$. J: $\mathbf{R}_\alpha^k \times \mathbf{R}_{\alpha'}^{k'} \to \mathbf{R}_\beta^{k+k'}$, with α, α', β nothing or plus, is the juxtaposition map followed if $\alpha = \alpha' = \beta = +$ by rotation on the last two coordinates.

(e) $\mathfrak{M} \cup \mathfrak{M}'$ is defined as a manifold with rigid $\mathfrak{B}$-structure in V if all of the following conditions hold:

(1) $V = V'$, $U = U'$, $n = n'$ and $(-1)^{\varepsilon+\varepsilon'} = -1$.

(2) $D \cap D' = \partial D = \partial D' \neq \varnothing$ and $\varphi|S^{k-1} = \varphi'|S^{k-1}$.

(3) $M \cap M' = \partial M = \partial M'$.

(4) $T \cap (D \cap D') = T' \cap (D \cap D') = T \cap T'$, $\nu|T \cap T' = \nu'|T \cap T'$ and $F|T \cap T' = F'|T \cap T'$. If these conditions hold then

$$\mathfrak{M} \cup \mathfrak{M}' = \left(M \cup M', (-1)^\varepsilon, \mathfrak{D} \cup \mathfrak{D}', T \cup T', F \cup F'\right).$$

$\mathfrak{D} \cup \mathfrak{D}' = (D \cup D', \varphi \cup (\varphi'\gamma^*), U)$ where γ is defined in 3.3(b).

Observe that $\mathfrak{M} \times \mathfrak{M}'$ is defined using internal direct sum rather than Cartesian product. Thus there are no differences as sets between $M \times M'$ and $M' \times M$. Also note that $\mathfrak{M} \cup \mathfrak{M}'$ is defined if and only if $\mathfrak{M}' \cup \mathfrak{M}$ is defined. However, $\mathfrak{M} \cup \mathfrak{M}'$ and $\mathfrak{M}' \cup \mathfrak{M}$ have opposite orientation.

DEFINITION 3.6. Let $\mathfrak{M}$ be a manifold with rigid $\mathfrak{B}$-structure in V. Define the Pontrjagin construction $\mathcal{P}(\mathfrak{M})$ as the following composite map:

$$D^* \xrightarrow{\kappa} D(\nu)/S(\nu) \xrightarrow{MF} D(\zeta_U)/S(\zeta_U) \equiv M\mathfrak{B}(U).$$

κ is the collapsing map. $D(\nu)$, $D(\zeta_U)[S(\nu), S(\zeta_U)]$ denote the total spaces of the disc [sphere] bundles associated to ν, ζ_U, respectively. MF is the map induced on quotient spaces by $F|D(\nu)$.

In the language of J. P. May [15, Chapter II] $M\mathfrak{B}$ is a prespectrum. Thus, $(-1)^\varepsilon \mathcal{P}(\mathfrak{M}) \circ \varphi$ determines an element in $\pi_{k-\dim U}(M\mathfrak{B})$ if $\partial\mathfrak{M} = \varnothing$, or determines that an element in $\pi_{k-\dim U-1}(M\mathfrak{B})$ is zero if $\partial\mathfrak{M} \neq \varnothing$. The following theorem is an immediate consequence of the preceding definitions.

THEOREM 3.7. *Let $\mathfrak{M}$, $\mathfrak{M}'$ be manifolds with rigid $\mathfrak{B}$-structures.*

(a) *If H: $\mathfrak{B} \to \mathfrak{B}'$ is a map of $\mathfrak{B}$-structures then $\mathcal{P}(H_*(\mathfrak{M})) = MH(U) \circ \mathcal{P}(\mathfrak{M})$.*

(b) $\mathcal{P}(\partial\mathfrak{M}) = \mathcal{P}(\mathfrak{M})|\partial D$.

(c) *If $\mathfrak{M} \times \mathfrak{M}'$ is defined then $\mathcal{P}(\mathfrak{M} \times \mathfrak{M}') = \mathcal{P}(\mathfrak{M}) \wedge \mathcal{P}(\mathfrak{M}')$.*

(d) *If $\mathfrak{M} \cup \mathfrak{M}'$ is defined then $\mathcal{P}(\mathfrak{M} \cup \mathfrak{M}') = \mathcal{P}(\mathfrak{M}) \cup \mathcal{P}(\mathfrak{M}')$.*

4. Bordism chains. We use manifolds with rigid $\mathfrak{B}$-structure to construct a functor $\mathcal{C}\mathfrak{B}_*(\ ;R)$ from the category of CW pairs to the category of DGMs indexed on J such that $H_*[\mathcal{C}\mathfrak{B}(\ ;R)]$ is naturally isomorphic to $\mathfrak{B}_*(\ ;R)$. This definition extends from CW complexes to spectra. We omit the details, however, because they involve redoing this section from a second viewpoint so that chain level suspension maps can be defined.

The functor $\mathcal{C}\mathfrak{B}_*(\ ;R)$ will merit the title of a chain functor because for (X, A) a CW pair, $\mathcal{C}\mathfrak{B}_*(A;R)$ will be a sub DGM of $\mathcal{C}\mathfrak{B}_*(X;R)$. Furthermore, this inclusion will induce the long exact sequence of the pair (X, A) in homology. The latter fact is a nontrivial matter. A short exact sequence of DGMs does not always induce a long exact sequence in homology. (See Example 9.2.)

DEFINITION 4.1. A monoid valued chain functor $\mathcal{C}_*$ for a homology theory h_* is a functor from the category of CW complexes to the category of absolute DGMs indexed on I. For all CW pairs (X, A), $\lambda \in I_n$, and $m \in \{n-1, n, n+1\}$, the following conditions hold:

(1) $\mathcal{C}_*(A)$ is a sub DGM of $\mathcal{C}_*(X)$, and $\mathcal{C}_*(X, A) \equiv \mathcal{C}_*(X)/\mathcal{C}_*(A)$ is a DGM.

(2) There is a natural isomorphism Φ: $H_n[\mathcal{C}(\ )_\lambda] \to h_n(\ )$ from the category of CW pairs to the category of abelian groups.

(3) The following diagram commutes:

$$\begin{tikzcd}
H_n[\mathcal{C}(A)_\lambda] \arrow[r, "i_*"] \arrow[d, "\Phi", "\cong"'] & H_n[\mathcal{C}(X)_\lambda] \arrow[r, "j_*"] \arrow[d, "\Phi", "\cong"'] & H_n[\mathcal{C}(X,A)_\lambda] \arrow[r, "\partial"] \arrow[d, "\Phi", "\cong"'] & H_{n-1}[\mathcal{C}(A)_\lambda] \arrow[r, "i_*"] \arrow[d, "\Phi", "\cong"'] & H_{n-1}[\mathcal{C}(X)_\lambda] \arrow[d, "\Phi", "\cong"'] \\
h_n(A) \arrow[r, "i_*"] & h_n(X) \arrow[r, "j_*"] & h_n(X,A) \arrow[r, "\partial"] & h_{n-1}(A) \arrow[r, "i_*"] & h_{n-1}(X)
\end{tikzcd}$$

The bottom row is part of the long exact sequence of the CW pair (X, A), while the maps in the top row are induced in the usual way from the data given in (1) above.

We are finally ready to define the bordism chains of a space. The definition is analogous to that of singular chains. The singular simplexes and the integers, however, are replaced by manifolds with rigid $\mathfrak{B}$-structures and N, respectively. In addition we will impose a cutting and pasting relation. (See relation (3) below.) This relation is essential in proving that $H_n[\mathcal{C}\mathfrak{B}(X, A)_\lambda] \cong \mathfrak{B}_n(X, A)$ and that the boundary formula of Definition 2.2 holds for the pairings of §5. The elements $\sigma(\mathfrak{M})$ below can be thought of as formal copies of $[0, 1] \times \mathfrak{M}$.

For the remainder of this paper J will denote the following specific indexing set. J_n is the set of all k-dimensional domains $\mathfrak{D}$ such that $\partial\mathfrak{D} \neq \varnothing$, $k > 2n + 2$ and $k - n \equiv 1 \bmod 4$.

DEFINITION 4.2. (a) Let X be a space, $\mathfrak{D} \in J_n$ and $m \in \{n - 1, n, n + 1\}$. Let $\mathfrak{M}^m$, $\mathfrak{M}^{m-1}$ denote manifolds with rigid $\mathfrak{B}$-structure and domains $\mathfrak{D}$, $\partial\mathfrak{D}$, $\mathfrak{D}_+$, $\mathfrak{D}_-$ or $\mathfrak{D}_0$. g: $M \to X$ and h: $N \to X$ are continuous maps. $\mathcal{C}\mathfrak{B}_m(X;\ R)_{\mathfrak{D}}$ is a free "N-module" $F_m(X;\ R)_{\mathfrak{D}}$ modulo five relations. $F_m(X;\ R)_{\mathfrak{D}}$ has as basis the set of all $(\mathfrak{M}^m, g)$, $\sigma(\mathfrak{M}^{m-1}, h)$ if $m = n + 1$; the set of all $(\mathfrak{M}^m, g)$, $\sigma(\mathfrak{M}^{m-1}, h)$, $\partial\sigma(\mathfrak{M}^m, g)$ if $m = n$; or the set of all $(\mathfrak{M}^m, g)$, $\partial\sigma(\mathfrak{M}^m, g)$ if $m = n - 1$. We call $(\mathfrak{M}^m, g)$ a manifold with rigid $\mathfrak{B} \times X$-structure. Furthermore, ∂, τ and σ are defined on the bases of $F_*(X;\ R)_{\mathfrak{D}}$ by $\partial(\mathfrak{M}, g) = (\partial\mathfrak{M}, g|\partial\mathfrak{M})$, $\partial\partial\sigma(\mathfrak{M}, g) = 0$, $\tau(\mathfrak{M}, g) = (\tau\mathfrak{M}, g)$, $\tau\sigma(\mathfrak{M}, g) = \sigma(\mathfrak{M}, g)$, $\tau\partial\sigma(\mathfrak{M}, g) = \partial\sigma(\mathfrak{M}, g)$, $\sigma\sigma(\mathfrak{M}, g) = 0$ and $\sigma\partial\sigma(\mathfrak{M}, g) = 2\sigma(\mathfrak{M}, g)$. We impose the following relations on $F_m(X;\ R)_{\mathfrak{D}}$ to define $\mathcal{C}\mathfrak{B}_m(X;\ R)_{\mathfrak{D}}$.

RELATION 1. $(\mathfrak{M} \uplus \mathfrak{M}', g \uplus g') \sim (\mathfrak{M}, g) + (\mathfrak{M}', g')$ if $T \cap T' = \varnothing$ and $\varepsilon = \varepsilon'$.

RELATION 2. $\partial\sigma(\mathfrak{M}, g) + \sigma\partial(\mathfrak{M}, g) \sim (\mathfrak{M}, g) + \tau(\mathfrak{M}, g)$.

RELATION 3. $(\mathfrak{M}' \cup \mathfrak{M}'', g' \cup g'') + \sigma(\mathfrak{M}' \cap \mathfrak{M}'', g'|M' \cap M'') \sim (\mathfrak{M}', g') + (\mathfrak{M}'', g'')$ if $m = n$, $\mathfrak{M}' \cup \mathfrak{M}''$ is defined, $\mathfrak{M}'$ has domain $\mathfrak{D}_+$, $\mathfrak{M}''$ has domain $\mathfrak{D}_-$ and $g'|M' \cap M'' = g''|M' \cap M''$.

RELATION 4. If $a \sim b$ then $\sigma(a) \sim \sigma(b)$.

RELATION 5. If $a + c \sim b + c$ then $a \sim b$, i.e., cancellation holds in $\mathcal{C}\mathfrak{B}_m(X;\ R)_{\mathfrak{D}}$.

(b) The images $G_m(X)_{\mathfrak{D}}$ in $\mathcal{C}\mathfrak{B}_m(X;\ R)_{\mathfrak{D}}$ of the above bases of the $F_m(X;\ R)_{\mathfrak{D}}$ define a generating set G for the DGM $\mathcal{C}\mathfrak{B}_*(X;\ R)$.

(c) Let f: $X \to Y$ be a continuous map. A unique linear map f_*: $\mathcal{C}\mathfrak{B}_m(X;\ R)_{\mathfrak{D}} \to \mathcal{C}\mathfrak{B}_m(Y;\ R)_{\mathfrak{D}}$ is induced by

$$f_*(\mathfrak{M}, g) = (\mathfrak{M}, f \circ g), \qquad f_*\partial = \partial f_* \quad \text{and} \quad f_*\sigma = \sigma f_*.$$

(d) Let H: $\mathfrak{B} \to \mathfrak{B}'$ be a map of (B, f)-structures. There is a unique linear

map

$$H_*: \mathcal{C}\mathfrak{B}_m(X; R)_{\mathfrak{D}} \to \mathcal{C}\mathfrak{B}'_m(X; R)_{\mathfrak{D}}$$

induced by

$$H_*(\mathfrak{M}, g) = \big(H_*(\mathfrak{M}), g\big), \qquad H_*\partial = \partial H_* \quad \text{and} \quad H_*\sigma = \sigma H_*.$$

The following theorem says that the preceding definition of $\mathcal{C}\mathfrak{B}_*(X; R)$ has all of the algebraic structure of an absolute DGM required by Definition 2.1.

THEOREM 4.3. *$\mathcal{C}\mathfrak{B}_*(\ ; R)$ is a functor from the category of topological spaces to the category of absolute DGMs indexed on J. G is a natural generating set for $\mathcal{C}\mathfrak{B}_*(\ ; R)$. If H: $\mathfrak{B} \to \mathfrak{B}'$ is a map of (B, f)-structures then H_*: $\mathcal{C}\mathfrak{B}_*(\ ; R) \to \mathcal{C}\mathfrak{B}'_*(\ ; R)$ is a natural transformation.*

PROOF. The required isomorphisms $\Psi_{\alpha,\beta}$ and Ψ_α will be constructed in Theorem 4.4. The only other nontrivial point is that in each $\mathcal{C}\mathfrak{B}_m(X; R)_{\mathfrak{D}}$ only zero has an additive inverse. If $x + y = 0$ in $\mathcal{C}\mathfrak{B}_m(X; R)_{\mathfrak{D}}$ then we can find x', y', e, α, β in $F_m(X; R)_{\mathfrak{D}}$ such that $\{x'\} = x$, $\{y'\} = y$, $\alpha + \sigma(\beta) \sim x' + y' + e$ by Relation (2), and $\alpha + \sigma(\beta) \sim e = \gamma + \sigma(\delta)$ by Relations (3) and (4). Thus, the summands of α are cut and pasted to form γ. Triangulating each summand of $\alpha, \beta, \gamma, \delta$ shows that the number of m-simplexes of α equals the number of m-simplexes of γ. Hence $\alpha = \gamma = 0$ and $\sigma(\beta) = x' + y' + e$. Thus $\beta + \sigma(\lambda) \sim \delta + \sigma(\xi)$ and the preceding argument implies that $\beta = 0$. Hence $x' = y' = 0$.

We will prove next that $\mathcal{C}\mathfrak{B}_*(\ ; R)$ is a chain functor for the bordism theory $\mathfrak{B}_*(\ ; R)$. Recall from [2] that $\mathfrak{B}_*(\ ; R) \cong \mathfrak{B}_*(\ ) \otimes_Z R$. We will use M. Atiyah's definition [5] of $\mathfrak{B}_n(X)$ as the abelian group of bordism classes of singular manifolds (M^n, g). M^n is a closed smooth manifold with $\mathfrak{B}$-structure as in [12], and g: $M \to X$. Addition in $\mathfrak{B}_n(X)$ is induced by disjoint union of manifolds. We define $\mathfrak{B}_*(X, A)$ as the relative bordism group of the pair of "(B, f)-structures" $(\mathfrak{B} \times X, \mathfrak{B} \times A)$ as in [19, p. 9].

THEOREM 4.4. *$\mathcal{C}\mathfrak{B}_*(\ ; R)$ is a chain functor for the bordism theory $\mathfrak{B}_*(\ ; R)$.*

PROOF. The general case follows from the case $R = Z$ by a direct limit argument. We begin by producing the isomorphism Φ in the absolute case. Every cycle in $\mathcal{C}\mathfrak{B}_*(X)$ is a linear combination of closed manifolds modulo Image $\partial\sigma$. Hence Φ is induced by $\Phi[(\mathfrak{M}, g)] = \{(\overline{M}, g \circ \varphi)\}$. $\overline{M}$ is $\varphi^{-1}(M)$ with $\mathfrak{B}$-structure determined by the rigid $\mathfrak{B}$-structure of $\mathfrak{M}$. Define Ψ: $\mathfrak{B}_m(X) \to H_m[\mathcal{C}\mathfrak{B}(X)_{\mathfrak{D}}]$ by $\Psi\{(M, g)\} = [(\mathfrak{M}, g \circ \varphi^{-1})]$. $\mathfrak{M}$ is $\varphi(M)$ with any rigid $\mathfrak{B}$-structure which determines the $\mathfrak{B}$-structure of M. Clearly $\Phi \circ \Psi = 1$ and $\Psi \circ \Phi = 1$. Thus the proof of the absolute case will be

complete if Φ and Ψ are well defined. We leave this verification for Ψ to the reader.

The main problem in showing that Φ is well defined is to show that the algebraic cutting and pasting relation can be realized geometrically by bordism relations. That is, if $A_1, \ldots, A_p, B_1, \ldots, B_q$ are connected closed manifolds with rigid $\mathfrak{B} \times X$-structure such that $\underline{A}_1 + \cdots + \underline{A}_p = \underline{B}_1 + \cdots + \underline{B}_q$ in $\mathcal{C}\mathfrak{B}_n(X)_{\mathfrak{D}}$ then we must show that $\bar{A}_1 \cup \cdots \cup \bar{A}_p \sim \bar{B}_1 \cup \cdots \cup \bar{B}_q$. This cutting and pasting by Relation 3 can be accomplished geometrically by an iteration of cutting and pasting along one connected manifold. There are two cases. The first case is the case $p = q = 2$ of the general problem. In the second case each of two manifolds A, E is cut to form a manifold with boundary. These two manifolds are then pasted together to form a connected manifold B. In Figure 1 we produce a cobordism W which shows that $\bar{A}_1 \cup \bar{A}_2 \sim \bar{B}_1 \cup \bar{B}_2$ in Case 1 or $A \cup E \sim B$ in Case 2.

The definition of Ψ extends to the relative case. This map will be an isomorphism by applying the Five Lemma to the diagram of Definition 4.1(3). We must check that the top row of that diagram is an exact sequence. We leave most of this verification to the reader and prove here that Kernel $j_* \subset$ Image i_*. Let $j_*[(\mathfrak{M}, g)] = 0$. Then in $\mathcal{C}\mathfrak{B}_n(X)_{\mathfrak{D}}$,

$$(\mathfrak{M}, g) + i(a) + \partial(x) = i(a') + \partial(y).$$

Hence $\partial(a) = \partial(a')$ in $\mathcal{C}\mathfrak{B}_{n-1}(A)_{\mathfrak{D}}$ where Relation 3 does not apply. Let ε denote plus or minus and let $(\mathfrak{N}^{n-1}, h) \in \mathcal{C}\mathfrak{B}_{n-1}(A)_{\mathfrak{D}}$ be a connected manifold with rigid $\mathfrak{B} \times A$-structure. Then the number of connected components of a, a' with domain $\mathfrak{D}_\varepsilon$ and a boundary component equal to $(\mathfrak{N}^{n-1}, h)$ are the same. Thus we can find $a_0 \in \mathcal{C}\mathfrak{B}_n(A)_{\mathfrak{D}}$ with $\partial(a_0) = \tau\partial(a)$ and $a + a_0 - \sigma\partial(a_0) = C$, $a' + a_0 - \sigma\partial(a_0) = C'$ defined in $\mathcal{C}\mathfrak{B}_n(A)_{\mathfrak{D}}$. Then $[(\mathfrak{M}, g)] = i_*[C] - i_*[C']$ is in Image i_*.

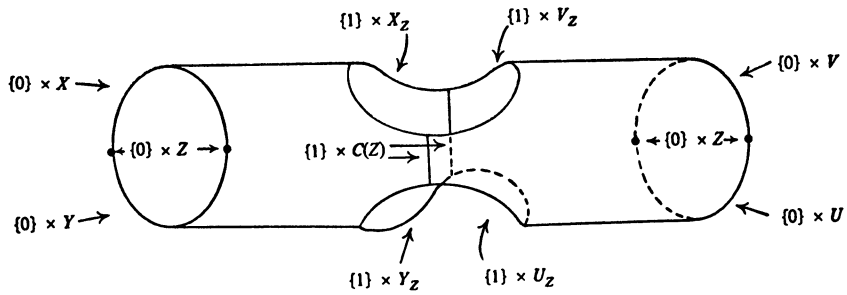
COROLLARY 4.5. *Let (X, A) be a CW pair, and let $(\mathfrak{M}, g)$ be a manifold with rigid $\mathfrak{B} \times X$-structure such that $g(\partial M) \subset A$. Then $(\mathfrak{M}, g)$ is homologous to zero in $\mathcal{C}\mathfrak{B}_*(X, A)$ if and only if $(\mathfrak{M}, g)$ is relatively cobordant to zero via rigid manifolds.*

COROLLARY 4.6. *Let X be a based CW complex. Define the reduced bordism chains of X by*

$$\widetilde{\mathcal{C}\mathfrak{B}}_m(X; R)_{\mathfrak{D}} = \mathcal{C}\mathfrak{B}_m(X,*; R)_{\mathfrak{D}}$$

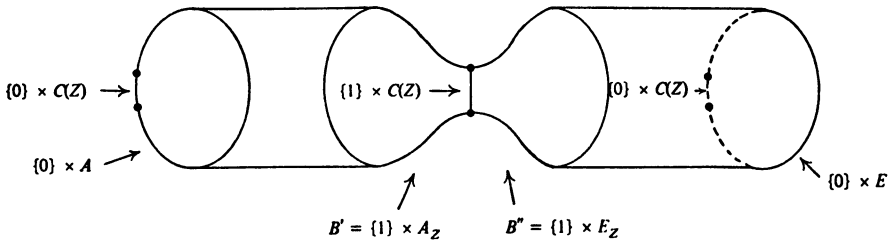
where $\mathfrak{D} \in J_n$, $m \in \{n-1, n, n+1\}$. Then $\widetilde{\mathcal{C}\mathfrak{B}}_*(X; R)$ is an absolute DGM indexed on J, and there is a natural isomorphism from $H_n[\widetilde{\mathcal{C}\mathfrak{B}}_*(\ ; R)]$ to $\tilde{\mathfrak{B}}_n(\ ; R)$.

Case 1



$$\bar{A}_1 = X \cup Y, \quad \bar{B}_1 = X_Z \cup V_Z,$$
$$\bar{A}_2 = U \cup V, \quad \bar{B}_2 = Y_Z \cup U_Z,$$
$$Z = X \cap Y = U \cap V = X_Z \cap V_Z = Y_Z \cap U_Z,$$
$$C(Z) = \text{two sided collar of } Z \text{ in } \bar{A}_i \quad \text{for } i = 1, 2,$$
$$S_Z = S - \operatorname{Int} C(Z) \quad \text{for } S = X, Y, U, V,$$
$$W = (I \times \bar{A}_1) \cup (I \times \bar{A}_2).$$

Case 2



$$B = B' \cup B'', \quad \{0, 1\} \times Z = B' \cap B'',$$
$$C(Z) = \text{two sided collar of } Z \text{ in } A, E,$$
$$S_Z = S - \operatorname{Int} C(Z) \quad \text{for } S = A, E,$$
$$W = (I \times A) \cup (I \times E).$$

FIGURE 1. The cobordism W

PROOF. The only nontrivial assertion is that $\widetilde{\mathcal{CB}}_*(X)$ is an absolute DGM. Let $\mathfrak{D} \in J_n$.

$$\begin{array}{ccccccc}
0 \to \mathfrak{B}_{n-1}(*; R) & \longrightarrow & \mathfrak{B}_{n-1}(X; R) & \longrightarrow & \mathfrak{B}_{n-1}(X, *; R) \to 0 \\
\cong \downarrow \Psi_* & & \cong \downarrow \Psi_* & & \\
H_{n-1}[\mathcal{CB}(*; R)_{\mathfrak{D}}] & \xrightarrow{i_*} & H_{n-1}[\mathcal{CB}(X; R)_{\mathfrak{D}}] & \xrightarrow{j_*} & H_{n-1}[\widetilde{\mathcal{CB}}(X; R)_{\mathfrak{D}}]
\end{array}$$

The above diagram is commutative, and the top row is exact. j_* is onto because ∂ is zero on $\widetilde{\mathcal{C}\mathcal{B}}_{n-1}(X;\ R)_{\mathfrak{D}}$, and clearly Kernel $j_* =$ Image i_*. Hence the above diagram induces the desired isomorphism.

5. **Cobordism cochains and products.** Let $\mathcal{S}_p$ be the category of triples $[(X, Y), D(X/Y), u]$ where (X, Y) is a finite CW pair and u: $(X/Y) \wedge D(X/Y) \to S^p$ is a p-duality map. (See [8].) A morphism $F = (f, Df)$ in $\mathcal{S}_p$ from $[(X, Y), D(X/Y), u]$ to $[(X', Y'), D(X'/Y'), u']$ consists of maps f: $X/Y \to X'/Y'$ and Df: $D(X'/Y') \to D(X/Y)$ such that the following diagram homotopy commutes.

$$\begin{tikzcd} X/Y \wedge D(X'/Y') \arrow[r, "1 \wedge Df"] \arrow[d, "f \wedge 1"'] & X/Y \wedge D(X/Y) \arrow[d, "u"] \\ X'/Y' \wedge D(X'/Y') \arrow[r, "u'"] & S^p \end{tikzcd}$$

We construct a cochain functor for $\mathcal{B}^*$ from $\mathcal{S}_p$ to the category of absolute DGMs.

DEFINITION 5.1. Let $J^{(p)}$ be the indexing set given by $J_n^{(p)} = J_{p-n}$ for $p \geqslant n$. For $[(X, Y), D(X/Y), u] \in \mathcal{S}_p$, $\mathfrak{D} \in J_n^{(p)}$ and $m \in \{n-1, n, n+1\}$ define

$$\mathcal{C}\mathcal{B}^m(X, Y; R)_{\mathfrak{D}} = \widetilde{\mathcal{C}\mathcal{B}}_{p-m}(D(X/Y); R)_{\mathfrak{D}}.$$

If $F = (f, Df)$ is a morphism in $\mathcal{S}_p$ then define $F^* = Df_*$. As special cases, we have

$$\widetilde{\mathcal{C}\mathcal{B}}^*(X; R) = \mathcal{C}\mathcal{B}^*(X, *; R) \quad \text{and} \quad \mathcal{C}\mathcal{B}^*(X; R) = \mathcal{C}\mathcal{B}^*(X^+, *; R)$$

where $X^+ = X \cup \{*\}$.

THEOREM 5.2. *$\mathcal{C}\mathcal{B}^*(\ ;\ R)$ is a functor from $\mathcal{S}_p$ to the category of absolute DGMs indexed on $J^{(p)}$. $H^n[\mathcal{C}\mathcal{B}^*(\ ;\ R)]$ is naturally isomorphic to $\mathcal{B}^n(\ ;\ R)$. The natural generating set of $\widetilde{\mathcal{C}\mathcal{B}}_*(\ ;\ R)$ defines a natural generating set for $\mathcal{C}\mathcal{B}^*(\ ;\ R)$.*

PROOF. This theorem follows from Corollary 4.6 and the basic properties of Spanier-Whitehead dualtiy.

We now have all the necessary definitions to realize the bordism cross-product, the bordism smash-product and the cobordism cup-product as pairings of DGMs.

THEOREM 5.3. (a) *Consider the cross-product*

$$\mu_*: \mathcal{B}_*(X; R) \underset{R}{\otimes} \mathcal{B}_*(Y; R) \to \mathcal{B}_*(X \times Y; R)$$

for CW complexes X, Y. Then μ_* can be realized by a natural pairing μ of

DGMs such that if $((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) \in S_\mu$ *then*

$$\mu((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) = (\mathfrak{M}' \times \mathfrak{M}'', g' \times g'').$$

(b) *Consider the smash-product*

$$\mu_*: \tilde{\mathfrak{B}}_*(X; R) \underset{R}{\otimes} \tilde{\mathfrak{B}}_*(Y; R) \to \tilde{\mathfrak{B}}_*(X \wedge Y; R)$$

for based CW complexes X, Y. *Then* μ_* *can be realized by a natural pairing* μ *of DGMs such that if* $((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) \in S_\mu$ *then*

$$\mu((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) = (\mathfrak{M}' \times \mathfrak{M}'', \pi \circ (g' \times g'')).$$

$\pi: X \times Y \to X \wedge Y$ *is the natural projection.*

(c) *Let* $[X, DX, u] \in \mathfrak{S}_p$ *and let* $(\Delta, D\Delta)$ *be a morphism in* $\mathfrak{S}_{2p}$ *from* $[X, S^pDX, S^pu]$ *to* $[X \wedge X, DX \wedge DX, u \wedge u]$ *with* Δ *the diagonal map. Consider the cup-product*

$$\mu^*: \tilde{\mathfrak{B}}^*(X; R) \underset{R}{\otimes} \tilde{\mathfrak{B}}^*(X; R) \to \tilde{\mathfrak{B}}^*(X; R).$$

μ^* *can be realized by a pairing* μ *of DGMs from* $\widetilde{\mathcal{C}\mathfrak{B}}^*(X; R)$, $\widetilde{\mathcal{C}\mathfrak{B}}^*(X; R)$ *indexed on* $J^{(p)}$ *to* $\widetilde{\mathcal{C}\mathfrak{B}}^*(X \wedge X; R)$ *indexed on* $J^{(2p)}$. S_μ *is defined from the case* $\bar{\mu}$ *of* (b) *where* $X = Y = DX$, *and* $\mu = \Delta^* \circ \bar{\mu}$. *By abuse of language we call* $\widetilde{\mathcal{C}\mathfrak{B}}^*(X; R)$ *a DGMA.*

PROOF. (a) S_μ contains the following elements.

(1) $((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) \in S_\mu$ if $\mathfrak{D} \times \mathfrak{D}'$ is defined, $\partial\mathfrak{M}' \neq \varnothing$, $\partial\mathfrak{M}'' \neq \varnothing$, $D' \times D'' = D$, $\partial D' \times D'' = D_+$ and $D' \times \partial D'' = D_-$.

(2) $((\mathfrak{M}', g'), (\mathfrak{M}'', g''), \mathfrak{D}) \in S_\mu$ if $\mathfrak{D}' \times \mathfrak{D}''$ is defined, $\partial\mathfrak{M}' = \varnothing$ or $\partial\mathfrak{M}'' = \varnothing$, and $D' \times D'' \in \{D, \partial D, D_+, D_-, D_0\}$.

(3) $(\varphi(\mathfrak{M}', g'), \psi(\mathfrak{M}'', g''), \mathfrak{D}) \in S_\mu$ if $\mathfrak{D}' \times \mathfrak{D}''$ is defined, $\partial\mathfrak{M}' = \varnothing$ or $\partial\mathfrak{M}'' = \varnothing$, $D' \times D'' \in \{\partial D, D_+, D_-, D_0\}$ and $\varphi, \psi \in \{1, \sigma, \partial\sigma\}$. μ is defined by the condition given in (a) above and by Definition 2.2(a) (3), (4), (5).

(b) The definitions of S_μ and μ are similar to the definitions above.

(c) These assertions follow from (b) and the fact that $\mu^* = (D\Delta)_* \circ \bar{\mu}_*$.

Observe that the cobordism cross-product and the cobordism smash-product can also be realized by cochain pairings.

6. **Spectral sequences.** Spectral sequences which occur in the study of generalized homology theories are defined from exact couples. If such a spectral sequence involved bordism theories, we would hope that the spectral sequence could be defined by a filtered DGM. However, a short exact sequence of DGMs does not automatically determine a long exact sequence in homology. Thus, in general a filtered DGM does not determine an exact couple as in [13, Chapter XI, Theorem 5.4] and hence does not determine a spectral sequence. However, in Definition 6.1 we list additional conditions

that a filtered DGM must satisfy. In Theorem 6.2 we show that such a filtered DGM does determine a spectral sequence. We then show that both the Atiyah-Hirzebruch and Adams spectral sequences for cobordism are defined by a filtered DGM.

All spectral sequences in this section are cohomology spectral sequences. Definition 6.1 and Theorem 6.2 have obvious analogues for homology spectral sequences. Theorems 6.3 and 6.4 have analogues with bordism replacing cobordism. We leave the formulation of these results to the reader.

DEFINITION 6.1. M is a filtered DGM indexed on I if there are absolute DGMs F^pM for $p \geqslant 0$, such that:

(a) For $p \geqslant 0$, $F^{p+1}M$ is a sub DGM of F^pM.

(b) $F^0M = M$ and $\bigcap_{p=0}^{\infty} F^pM = 0$.

(c) For $p \geqslant 0$ and $r \geqslant 1$, $F^pM/F^{p+r}M$ is a DGM and $F^pM \to F^pM/F^{p+r}M$ is a map of DGMs.

(d) For all $p \geqslant 0$, $r \geqslant 1$ and $\lambda \in I_n$, the following short exact sequence of DGMs induces five terms of a long exact sequence in homology.

$$0 \to F^{p+r}M \to F^pM \to F^pM/F^{p+r}M \to 0.$$

(e) Let $\lambda \in I_n$, $m \in \{n, n+1\}$ and $x \in F^pM_{m,\lambda}$ such that $d(x) = 0$. If $[x] = 0$ in $H^m[F^pM]$ then $x \in d(F^pM)$.

(f) Let $\lambda \in I_n$ and $x \in F^pM_{n,\lambda}$ such that $d\{x\} = 0$ in $F^pM_{n,\lambda}/F^{p+r}M_{n,\lambda}$. If $[\{x\}] = 0$ in $H^n[F^pM/F^{p+r}M]$ then $x \in d(F^pM)$ modulo $F^{p+r}M$.

Let M be a DGMA with generating set G. We call M a filtered DGMA if M is a filtered DGM such that:

(g) For $p \geqslant 0$ and $r \geqslant 1$, $G \cap F^pM$ is a generating set for F^pM and $G \cap F^pM$ projects to a generating set for $F^pM/F^{p+r}M$.

(h) If $a \in F^pM$, $b \in F^qM$ and $(a, b, \xi) \in S_\mu$ then $\mu(a, b, \xi) \in F^{p+q}M$.

THEOREM 6.2. *Let M be a filtered DGM. Then $\mathcal{E}$ is an exact couple of abelian groups.*

$$\mathcal{E} = \begin{tikzcd} \sum\limits_{p \geqslant -1} H[F^{p+1}M] \arrow[rr, "i"] & & \sum\limits_{p \geqslant 0} H[F^pM] \arrow[dl, "j"] \\ & \sum\limits_{p \geqslant 0} H[F^pM/F^{p+1}M] \arrow[ul, "\partial"] & \end{tikzcd}$$

Let $\{E_r\}_{r \geqslant 1}$ be the spectral sequence determined by $\mathcal{E}$.

(a) *For $\lambda \in I_n$ define $Z_r^{p,n-p} = \{x \in F^pM_{n,\lambda} | d(x) \in F^{p+r}M_{n+1,\lambda}\}$ and $B_r^{p,n-p} = \{d(y) \in F^pM_{n,\lambda} | Y \in F^{p-r+1}M_{n-1,\lambda}\} + F^{p+1}M_{n,\lambda} \cap Z_r^{p,n-p}$. Then $E_r^{p,n-p} = Z_r^{p,n-p}/B_r^{p,n-p}$. Furthermore, if $x \in Z_r^{p,n-p}$ with $\{x\} = 0$ in $E_r^{p,n-p}$ then $x \in d(F^{p-r+1}M)$ modulo $F^{p+1}M$.*

(b) *If* $x \in Z_r^{p,*}$ *then* $d_r\{x\} = \{d(x)\}$ *in* $E_r^{p+r,*}$. *Furthermore, if* $\{x\} \in E_r^{p,*}$ *is in* Image d_r *then there is* $y \in Z_r^{p-r,*}$ *with* $x = d(y)$ *modulo* $F^{p+1}M$.

(c) E_∞ *is the associated graded R-module of* $H[M]$ *with respect to the filtration*

$$F^pH[M] = \text{Image}(H[F^pM] \to H[M]).$$

(d) *If M is a filtered DGMA then*:

(i) $\{E_r\}_{r\geq 1}$ *is a spectral sequence of R-algebras.*

(ii) *For* $p \geq 0$ *and* $r \geq 1$, $\{\{x\} | x \in G \cap Z_r^{p,*}\}$ *spans* $E_r^{p,*}$ *over R.*

(iii) *If* $x, y \in G \cap Z_r$ *and* $(x, y, \xi) \in S_\mu$ *then* $\{x\} \cdot \{y\} = \{\mu(x, y, \xi)\}$ *in* E_r.

PROOF. Recall that $E_r = \overline{Z}_r/\overline{B}_r$ where $\overline{Z}_r = \partial^{-1}(\text{Image } i^{r-1})$ and $\overline{B}_r = j(\text{Kernel } i^{r-1})$. (See [13, Chapter XI, §5].) By Definition 6.1(d) and [9, Theorem I.10.2] we have the following long exact sequence.

$$\cdots \xrightarrow{\partial'} H^n[F^{p+1}M/F^{p+r}M] \xrightarrow{i'} H^n[F^pM/F^{p+r}M]$$
$$\xrightarrow{j'} H^n[F^pM/F^{p+1}M] \xrightarrow{\partial'} \cdots.$$

Define an isomorphism Γ: $E_r^{p,*} \to Z_r^{p,*}/B_r^{p,*}$ by $\Gamma(x) = \{y\}$ if $j'[\{y\}] = x$. This proves (a) and the remainder of the proof follows from Definition 6.1 by standard arguments.

The Atiyah-Hirzebruch spectral sequence for $\mathfrak{B}^*$ can now be defined from a filtered DGM exactly as the Serre spectral sequence is defined for singular cohomology.

THEOREM 6.3. *Let* $F \to^i E \to^p B$ *be a Serre fibration of finite CW complexes. Assume that B is connected and that* $\pi_1(B)$ *acts trivially on* $H_*(F)$. *Define*

$$F^n\mathcal{C}\mathfrak{B}^*(E; R) = \text{Image}\left[\mathcal{C}\mathfrak{B}^*\left(E, p^{-1}(B^{(n-1)}); R\right) \to \mathcal{C}\mathfrak{B}^*(E; R)\right].$$

Then $\mathcal{C}\mathfrak{B}^*(E; R)$ *is a filtered DGM which defines the Atiyah-Hirzebruch spectral sequence* [2, *Part* III, §7]:

$$E_2^{s,t} = H^s\left(B; \mathfrak{B}^t(F; R)\right) \Rightarrow \mathfrak{B}^{s+t}(E; R).$$

PROOF. We show that

$$H[F^n\widetilde{\mathcal{C}\mathfrak{B}}^*(E; R)] \cong \mathfrak{B}^*\left(E, p^{-1}(B^{(n-1)}); R\right)$$

and

$$H[F^k\mathcal{C}\mathfrak{B}^*(E; R)/F^{k+r}\mathcal{C}\mathfrak{B}^*(E; R)] \cong \mathfrak{B}^*\left(p^{-1}(B^{(k+r)}), p^{-1}(B^{(k)}); R\right).$$

These isomorphisms imply that $\mathcal{C}\mathfrak{B}^*(E; R)$ is a filtered DGM. It determines the spectral sequence of the exact couple which defines the Atiyah-Hirzebruch spectral sequence [2]. Let dim $B = N$ and use iterated mapping cylinders to insure that

$$
\begin{aligned}
* &= D\big(E/p^{-1}(B^{(N)})\big) \subset \cdots \subset D\big(E/p^{-1}(B^{(k)})\big) \\
(*) \qquad &\subset D\big(E/p^{-1}(B^{(k-1)})\big) \subset \cdots \subset D\big(E/p^{-1}(B^{(0)})\big) \\
&= DE \subset D(E^+).
\end{aligned}
$$

Hence $F^n\mathcal{C}\mathfrak{B}^*(E;\ R) \cong \widetilde{\mathcal{C}\mathfrak{B}}^*(E/p^{-1}(B^{(n-1)});\ R)$ and $H[F^n\mathcal{C}\mathfrak{B}^*(E;\ R)] \cong \mathfrak{B}^*(E, p^{-1}(B^{(n-1)});\ R)$ as asserted above. For $N \geqslant k + r > k \geqslant 0$ we see from $(*)$ that

$$
\begin{aligned}
&H\big[F^k\mathcal{C}\mathfrak{B}^*(E;\ R)/F^{k+r}\mathcal{C}\mathfrak{B}^*(E;\ R)\big] \\
&\quad \cong H\big[\widetilde{\mathcal{C}\mathfrak{B}}_*\big(D\big(E/p^{-1}(B^{(k)})\big);\ R\big)/\widetilde{\mathcal{C}\mathfrak{B}}_*\big(D\big(E/p^{-1}(B^{(k+r)})\big);\ R\big)\big] \\
&\quad \cong \mathfrak{B}_*\big(D\big(E/p^{-1}(B^{(k)})\big), D\big(E/p^{-1}(B^{(k+r)})\big);\ R\big) \\
&\quad \cong \mathfrak{B}^*\big(p^{-1}(B^{(k+r)}), p^{-1}(B^{(k)});\ R\big).
\end{aligned}
$$

Consider the generalized Adams spectral sequence [2, Part III, §15] which converges to $\mathfrak{B}^*(F)^E$. For p a prime, we also consider the classical mod p Adams spectral sequence which converges to $\mathfrak{B}^*(F)$ modulo all q-torsion for q prime, $q \neq p$. We consolidate our notation by denoting the preceding quotient by $\mathfrak{B}^*(F)^{K(Z_p)}$. Thus the classical mod p Adams spectral sequence becomes the "case" $E = K(Z_p)$ of the generalized Adams spectral sequence. The cochains of §5 are not suitable for defining this spectral sequence. We replace them by cochains which depend on a fixed Adams resolution Y_0. These cochains are a filtered DGMA if Y_0 realizes a differential algebra C and $Y_0 \wedge Y_0$ realizes $C \otimes C$. These hypotheses can be met in the following two cases. The first case is $E = K(Z_p)$, p is prime and C is the cobar construction. The second case is $\mathfrak{B} = \mathfrak{B}fr$, $F = *$ and $Y_p = \bar{E}^p$ as in [2, p. 319]. The product on this C is the shuffle product, which is associative. This spectral sequence converges to $(\pi_*^S)^E$.

THEOREM 6.4. *Let E be a ring spectrum, and let F be a finite CW complex. Consider the Adams spectral sequence of algebras*:

$$
(*) \qquad E_2 = \operatorname{Ext}_{E_*(E)}\big(E_*(F),\ E_*(M\mathfrak{B})\big) \Rightarrow \mathfrak{B}^*(F)^E.
$$

There are two cases:

(1) $(*)$ *may be the classical Adams spectral sequence* [1] *with $E = K(Z_p)$ and p prime. In this case we define $R = Z$.*

(2) $(*)$ *may be the generalized Adams spectral sequence if E, F satisfy the hypotheses of* [2, *Part* III, *Theorem* 15.1].

In this case we assume that $R = \pi_0(E)$ is a subring of the rational numbers whence $\mathfrak{B}^*(F)^E = \mathfrak{B}^*(F;\ R)$. In both cases let Y_0 be an Adams resolution of $M\mathfrak{B} \wedge DF$ which realizes a differential algebra C such that $Y_0 \wedge Y_0$ realizes $C \otimes C$. Then there is a filtered absolute DGMA $\mathcal{C}\mathfrak{B}_E^*(F;\ R)$ which defines the spectral sequence $(*)$. Furthermore, there is a map of DGMAs α: $\mathcal{C}\mathfrak{B}_E^*(F;\ R) \to \mathcal{C}\mathfrak{B}^*(F;\ R)$ such that α_* is an isomorphism.

PROOF. We are given the following Adams resolution:

$$(**) \qquad M\mathfrak{B} \wedge DF \underset{\iota}{\overset{r}{\rightleftarrows}} Y_0 \supset Y_1 \supset \cdots \supset Y_{p+1} \supset \cdots.$$

The filtered spectrum Y_0 is defined by applying the mapping telescope construction to a suitable sequence of fibrations. ι, r is the canonical inclusion, retraction, respectively. We always use $(**)$ evaluated on a subspace U of $\mathbf{R}^\infty$, and our constructions are performed only on that part of $(**)$ which lies in the stable range. Define $\mathcal{C}\mathfrak{B}_E^*(F; R)$ exactly as $\mathcal{C}\mathfrak{B}_*(DF^+; R)$ in Definition 4.2 except that all pairs

$$(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M})), \quad \sigma(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M})), \quad \partial\sigma(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M}))$$

are used as the generating set. $\mathfrak{M}$ is a manifold with $\mathfrak{B} \times DF^+$-structure, and $\hat{\mathcal{P}}(\mathfrak{M})$: $D^* \to Y_0(U)$ such that $r \circ \hat{\mathcal{P}}(\mathfrak{M}) = \mathcal{P}(\mathfrak{M})$. Let α: $\mathcal{C}\mathfrak{B}_E^*(F; R) \to \mathcal{C}\mathfrak{B}^*(F; R)$ be the map of DGMs induced by sending $(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M}))$ to $\mathfrak{M}$. Clearly α_* is an isomorphism. $\mathcal{C}\mathfrak{B}_E^*(F; R)$ is filtered by letting $(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M})) \in F^p\mathcal{C}\mathfrak{B}_E^*(F; R)$ if $\hat{\mathcal{P}}(\mathfrak{M})(D^*) \subset Y_p(U)$. Standard transversality arguments show that

$$H_n[F^p\mathcal{C}\mathfrak{B}_E^*(F; R)] \cong \pi_n(Y_p) \underset{Z}{\otimes} R$$

and

$$H_n[F^p\mathcal{C}\mathfrak{B}_E^*(F; R)/F^{p+q}\mathcal{C}\mathfrak{B}_E^*(F; R)] \cong \pi_n(Y_p, Y_{p+q}) \underset{Z}{\otimes} R.$$

The above isomorphisms are induced by sending a cycle $(\mathfrak{M}, \hat{\mathcal{P}}(\mathfrak{M}))$ to $(-1)^\varepsilon \hat{\mathcal{P}}(\mathfrak{M}) \circ \varphi$. Hence $\mathcal{C}\mathfrak{B}_E^*(F; R)$ is a filtered DGM, except that $\bigcap_{p=0}^\infty F^p\mathcal{C}\mathfrak{B}_E^*(F; R)$ may be nonzero. $\mathcal{C}\mathfrak{B}_E^*(F; R)$ defines the spectral sequence of an exact couple which is isomorphic to the one that defines the Adams spectral sequence $(*)$. In this spectral sequence $E_1 = C$ with product μ_1. By our hypotheses on C and Y_0 we can realize μ_1 by a map μ: $Y_0 \wedge Y_0 \to Y_0$ such that $\mu(Y_p \wedge Y_q) \subset Y_{p+q}$ and $r\mu(\iota \wedge \iota) = (M\omega) \wedge (D\Delta)$. Thus $\mu_1 = \bar{\mu}_*$ where

$$\bar{\mu}: \left(\bigcup_{i=0}^{p} Y_i \wedge Y_{p-i}, \bigcup_{i=0}^{p+1} Y_i \wedge Y_{p-i+1} \right) \to (Y_p, Y_{p+1}).$$

The product on $\mathcal{C}\mathfrak{B}_E^*(F; R)$ is defined exactly as the product on $\mathcal{C}\mathfrak{B}^*(F; R)$ in Theorem 5.3(c). This definition is based upon $(\mathfrak{M}', \hat{\mathcal{P}}(\mathfrak{M}')) \cdot (\mathfrak{M}'', \hat{\mathcal{P}}(\mathfrak{M}'')) = (\mathfrak{M}' \times \mathfrak{M}'', \mu(\hat{\mathcal{P}}(\mathfrak{M}') \wedge \hat{\mathcal{P}}(\mathfrak{M}'')))$ if $V' \perp V''$. A straightforward verification shows that this product meets all of our requirements.

Observe that $\mathcal{C}\mathfrak{B}_E^*$ is natural with respect to triples $(f, Df, \hat{f})$ where (f, Df) is a morphism in $\mathcal{S}_p$ and $\hat{f}$: $Y_0' \to Y_0''$ such that $\hat{f}(Y_p') \subset Y_p''$ for $p \geqslant 0$ and $r \circ \hat{f} \circ \iota \cong 1 \wedge Df$.

7. Matric Massey products. We define matric Massey products on $\mathfrak{B}^*(F)$ and derive several of their basic properties. We consider the case $R = Z$ first. The case of arbitrary R is considered at the end of this section. Our definition generalizes J. C. Alexander's triple products [3], and we will show in Theorem 7.3 that it corresponds to G. J. Porter's higher products [17] under the Thom-Pontrjagin isomorphism. Cobordism matric Massey products can be defined in the four contexts corresponding to [14, §1, (3), (4), (5), (6)]. However, all of our applications are to $\mathfrak{B}^*(F)$ so we keep our notation simple by only discussing this one important case.

In this section F will be a finite CW complex. We assume that we are given Spanier-Whitehead duals $((F^+, *), D_tF, u_t)$ in $\mathfrak{S}_{tp}$ for $1 \leqslant t \leqslant n$ and morphisms $(\Delta, D_{s,t}\Delta)$ in $\mathfrak{S}_{sp+tp}$ such that $D_{s,t}\Delta\colon D_sF \wedge D_tF \to D_{s+t}F$ and

$$D_{s,t+u}\Delta \circ (1 \wedge D_{t,u}\Delta) = D_{s+t,u}\Delta \circ (D_{s,t}\Delta \wedge 1).$$

For example, if $F = *$ then $p = 0$, $D_tF = S^0$ and $D_{s,t}\Delta = 1$ will do.

The theory of this section and §8 also applies to the situation $D_tF = DF \wedge \cdots \wedge DF$ (t times) and $D_{s,t}\Delta = 1$. In this case Definition 7.2 defines matric Toda products which by Theorem 7.3 correspond to Porter's higher Toda products under the Thom-Pontrjagin isomorphism. These Toda products satisfy all the theorems of §§7 and 8.

The following lemma gives conditions under which the union of manifolds with $\mathfrak{B} \times DF$-structure is a manifold with $\mathfrak{B} \times DF$-structure. This lemma allows us to do the pasting required to define Massey products. Note the case $E_1 = \cdots = E_t = \varnothing$ below, where only (a), (b), (d) are applicable. This special case is especially useful.

LEMMA 7.1. *For $1 \leqslant i \leqslant t$, let M_i^n be a manifold with $\mathfrak{B} \times DF$-structure and k-dimensional domain $\mathfrak{D}_i$ in V. For $1 \leqslant i \leqslant t$, let E_i^{n-1} be a submanifold of ∂M_i with $\mathfrak{B} \times DF$-structure. Assume all of the following conditions.*

(a) *$\mathfrak{D}_1 \cup \cdots \cup \mathfrak{D}_t = \mathfrak{D}$ is a domain.*

(b) *Let $2 \leqslant r \leqslant t$, $1 \leqslant k \leqslant r$ and $1 \leqslant i_1 < \cdots < i_r \leqslant t$. Then $M_{i_1} \cap \cdots \cap M_{i_r}$ is an $(n - r + 1)$-dimensional submanifold of M_{i_k}, and $E_{i_1} \cap \cdots \cap E_{i_r}$ is an $(n - r)$-dimensional submanifold of E_{i_k}.*

(c) *For $1 \leqslant r \leqslant t$ and $1 \leqslant i_1 < \cdots < i_r \leqslant t$,*

$$\partial(E_{i_1} \cap \cdots \cap E_{i_r}) = \left(\bigcup_{j \in \{1,\ldots,t\} - \{i_1,\ldots,i_r\}} E_j\right) \cap E_{i_1} \cap \cdots \cap E_{i_r}.$$

(d) *For $1 \leqslant r \leqslant t$ and $1 \leqslant i_1 < \cdots < i_r \leqslant t$,*

$$\partial(M_{i_1} \cap \cdots \cap M_{i_r}) = \left[\left(\bigcup_{j \in \{1,\ldots,t\} - \{i_1,\ldots,i_r\}} M_j\right) \cap M_{i_1} \cap \cdots \cap M_{i_r}\right] \cup [E_{i_1} \cap \cdots \cap E_{i_r}].$$

(e) *For* $1 \leqslant i, j \leqslant t$, $E_i \cap M_j = E_i \cap E_j$.
Then $M_1 \cup \cdots \cup M_t$, $\partial(M_1 \cup \cdots \cup M_t) = E_1 \cup \cdots \cup E_t$ *is a manifold with* $\mathfrak{B} \times DF$*-structure and domain* $\mathfrak{D}$, $\partial\mathfrak{D}$, *respectively.*

PROOF. This lemma is proved by induction on t. The case $t = 2$ is easy. Assume that $t \geqslant 3$ and that the lemma is true for $t - 1$. Define $M_i' = M_i$, $E_i' = E_i$ for $1 \leqslant i \leqslant t-2$, $M_{t-1}' = M_{t-1} \cup M_t$ and $E_{t-1}' = E_{t-1} \cup E_t$. Straightforward computations verify (a) and (e) for $M_1', \ldots, M_{t-1}'$, $E_1', \ldots, E_{t-1}'$. The induction hypothesis thus implies the inductive step.

We now define cobordism Massey products. We let $\bar{x}$ denote $\tau^{1+\deg x}x$.

DEFINITION 7.2. Let $(W_1, \ldots, W_n)$ be a multipliable system of matrices with entries in $\mathfrak{B}^*(F)$ [14,1.1] such that W_1 is a row matrix and W_n is a column matrix. The matric Massey product $\langle W_1, \ldots, W_n\rangle$ is defined if there are matrices $A_{i,j}$ $(0 \leqslant i < j \leqslant n, (i,j) \neq (0, n))$ such that:

(a) For $1 \leqslant i \leqslant n$, the entries of $A_{i-1,i}$ are disjoint closed manifolds with $\mathfrak{B} \times D_1F$-structure and domain $\partial\mathfrak{D}_i$ in V_i. $[A_{i-1,i}] = W_i$ and $V_1 \perp \cdots \perp V_n$.

(b) For $1 < j - i$, the entries of $A_{i,j}$ are disjoint manifolds with $\mathfrak{B} \times D_{j-i}F$-structure and domain $\mathfrak{D}_{i,j} = \mathfrak{D}_{i+1} \times \cdots \times \mathfrak{D}_{j-1} \times \partial\mathfrak{D}_j$ in $V_{i+1} + \cdots + V_j$.

(c) For $0 \leqslant i < h < k < j \leqslant n$, $A_{i,h}A_{h,j} \cap A_{i,k}A_{k,j} = A_{i,h}A_{h,k}A_{k,j}$.
Hence by Lemma 7.1, $\tilde{A}_{i,j} = \bigcup_{t=i+1}^{j-1} A_{i,t}A_{t,j}$ is a matrix of closed manifolds with $\mathfrak{B} \times D_{j-i}F$-structure for $0 \leqslant i < j \leqslant n$.

(d) For $j - i \geqslant 2$, $\partial A_{i,j} = \tilde{A}_{i,j}$.

$\{A_{i,j}\}$ is called a defining system for $\langle W_1, \ldots, W_n\rangle$. $\langle W_1, \ldots, W_n\rangle$ consists of all $[\tilde{A}_{0,n}]$ for all defining systems $\{A_{i,j}\}$. The terms "strictly defined", "indeterminacy" and "defined with zero indeterminacy" are defined as usual [14].

Assume that we are given a defining system $\{A_{i,j}\}$ for $\langle W_1, \ldots, W_n\rangle$. We can isotop the entries of the $A_{i,j}$ to construct a new defining system $\{A_{i,j}'\}$ for any domains $\mathfrak{D}_1', \ldots, \mathfrak{D}_n'$ of the appropriate dimensions which satisfy Definition 7.2(a). Clearly $[\tilde{A}_{0,n}] = [\tilde{A}_{0,n}']$. Thus there is no loss of generality in restricting our attention to convenient domains, as we do in the proofs of Theorems 7.3 and 7.5.

We show next that our Massey products coincide with G. J. Porter's associative higher products [17, §2] under the Thom-Pontrjagin isomorphism $\mathcal{P}$. Both points of view are useful. Ours is convenient when working with spectral sequences, while Porter's point of view is useful when defining systems have to be moved around or changed by a cobordism. (See Theorem 7.5.) In the case where the matrices are not all 1×1 matrices, Porter assumes that $M\mathfrak{B}$ has an abelian sum. This condition is never met by the examples of interest here. Hence in the proof of Theorem 7.3 we define a variation of

Porter's construction which places no additional conditions on $M\mathfrak{B}$. This construction reduces to that of [17] in the case of 1×1 matrices.

THEOREM 7.3. *Let $(W_1, \ldots, W_N)$ be a multipliable system of matrices with entries in $\mathfrak{B}^*(F)$. Let $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$ denote Porter's associative higher Massey product. Then $\langle W_1, \ldots, W_N\rangle$ is defined if and only if $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$ is defined. If both are defined then*

$$\mathcal{P}\langle W_1, \ldots, W_N\rangle = \langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle.$$

PROOF. Let W_k be an $m_k \times n_k$ matrix where $m_1 = n_N = 1$. Write $V_k = \mathbf{R}e_k' + \mathbf{R}e_k'' + V_k'$. Let I_j^q denote the jth subinterval of the regular partition of I into q subintervals. Define

$$I_{i,j}(V_k) = I_i^{m_k} e_k' \times I_j^{n_k} e_k'' \times I(V_k')$$

where the cube $I(V_k')$ is defined from a fixed orthonormal basis of V_k'. Represent $\mathcal{P}(W_k)$ by a $m_k \times n_k$ matrix of maps $(\xi_k(i,j))$ where

$$\xi_k(i,j)\colon \big(I_{i,j}(V_k), \partial I_{i,j}(V_k)\big) \to \big(M\mathfrak{B}(U_k(i,j)) \wedge D_1F, *\big).$$

Observe that $U_{p,q}(i,j) = U_p(i, r_p) + \cdots + U_k(r_{k-1}, r_k) + \cdots + U_q(r_{q-1}, j)$ is independent of $r_1, \ldots, r_{q-1}$ and $U_{p,t}(i,h) + U_{t,q}(h,j) = U_{p,q}(i,j)$. Choose unit vectors $\beta_1, \ldots, \beta_{N-1}$ in $\mathbf{R}^\infty$ such that $V_1 \perp \cdots \perp V_N \perp \mathbf{R}\beta_1 \perp \cdots \perp \mathbf{R}_{N-1}$. Define

$$\begin{aligned} I_{p,q}(i,j) = {} & \cup I_{r_p, r_{p+1}}(V_{p+1}) \times \cdots \times I_{r_{q-1}, r_q}(V_q) \\ & \times I(\mathbf{R}\beta_{p+1} + \cdots + \mathbf{R}\beta_{q-1}) \times \{(0, \ldots, 0)\} \end{aligned}$$

where the union is taken over all sequences $(r_p, \ldots, r_q)$ with $r_p = i$ and $r_q = j$. Let $U\partial I_{0,N}(1,1)$ consist of all points in $\partial I_{0,N}(1,1)$ which have some β-coordinate one. Define $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$ to consist of all homotopy classes which have a representative

$$\varphi\colon \big(\partial I_{0,N}(1,1), U\partial I_{0,N}(1,1)\big) \to \big(M\mathfrak{B}(U_{0,N}(1,1)) \wedge D_NF, *\big)$$

such that:

(i) Away from the basepoint, $P_2 \circ \varphi$ is smooth and transverse regular to the zero section in $M\mathfrak{B}(U_{0,N}(1,1))$.

(ii) There are maps

$$\varphi_{p,q}(i,j)\colon \big(I_{p,q}(i,j), \partial I_{p,q}(i,j)\big) \to \big(M\mathfrak{B}(U_{p,q}(i,j)), *\big)$$

with $\varphi_{p,p+1} = \xi_p$ for $1 \leqslant p \leqslant N$. Furthermore, φ restricted to

$$\begin{aligned} & I_{0,1}(r_0, r_1) \times \cdots \times I_{p-1,p}(r_{p-1}, r_p) \times I_{p,q}(i,j) \\ & \qquad \times I_{q,q+1}(r_q, r_{q+1}) \times \cdots \times I_{N-1,N}(r_{N-1}, r_N) \end{aligned}$$

equals $\xi_1 \times \cdots \times \xi_{p-1} \times \varphi_{p,q} \times \xi_{q+1} \times \cdots \times \xi_N$ followed by $M\omega \wedge D\Delta$.

(iii) $\varphi_{p,q}(i,j)$ restricted to $I_{p,t}(i,h) \times I_{t,q}(h,j)$ equals $\varphi_{p,t}(i,h) \times \varphi_{t,q}(h,q)$ followed by $M\omega \wedge D\Delta$.

Assume that we are given such a defining system for $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$. Consider the domains $D_i = \mathring{I}(V_i) \times [0,1)\beta_i$ in V_i, $1 \leqslant i \leqslant N$. If $(i,j) \neq (0,N)$, $\varphi_{i,j}$ defines a matrix $A_{i,j}$ of manifolds with $\mathcal{B} \times D_1F$-structure and domain $\mathcal{D}_{i,j}$. $\varphi_{i,j}|\partial I_{i,j}$ defined $\tilde{A}_{i,j}$, and φ defines $\tilde{A}_{0,N}$. Thus $\langle W_1, \ldots, W_N\rangle$ is defined, and

$$\mathcal{P}\langle W_1, \ldots, W_N\rangle \supset \langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle.$$

Conversely, given a defining system for $W_1, \ldots, W_N$ with the above domains, we obtain a defining system for $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$ by applying $\mathcal{P}'$. If $\mathcal{M}$ has domain D then $\mathcal{P}'(\mathcal{M})$ is the Pontrjagin-Thom construction with domain $\mathcal{C}l_{\mathbf{R}^\infty}D$. Thus, $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$ is defined and $\mathcal{P}\langle W_1, \ldots, W_N\rangle \subset \langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_N)\rangle$.

Most of the properties of algebraic matric Massey products [14,§§1, 2, 3] are also valid for cobordism Massey products. The following three theorems illustrate this point.

THEOREM 7.4. *$\langle W_1, \ldots, W_n\rangle$ is strictly defined in $\mathcal{B}^*(F)$ if and only if each partial defining system $\{A_{p,q} | q - p \leqslant k\}$, $1 \leqslant k \leqslant n-1$, can be completed to a defining system.*

PROOF. Let $\langle W_1, \ldots, W_n\rangle$ be strictly defined and let $\{A_{p,q} | q - p \leqslant k\}$ be a partial defining system. For $1 \leqslant p \leqslant n-k$,

$$\left[\tilde{A}_{p-1,p+k}\right] \in \langle W_p, \ldots, W_{p+k}\rangle = \{0\}.$$

Hence $\tilde{A}_{p-1,p+k}$ is a boundary and $A_{p-1,p+k}$ can be chosen. Thus the nontrivial half of this theorem follows by induction on $n-k$.

THEOREM 7.5. *Let $\langle W_1, \ldots, W_n\rangle$ be defined in $\mathcal{B}^*(F)$. For $1 \leqslant i \leqslant n$ let $W_i = [A_{i-1,i}]$ as in Definition 7.2(a). Then every element of $\langle W_1, \ldots, W_n\rangle$ can be obtained from defining systems $\{A_{i,j}\}$ which begin with these specific $A_{i-1,i}$.*

PROOF. The analogue of this theorem is easily verified for Porter's higher products. Hence this theorem follows from Theorem 7.3 and the discussion following Definition 7.2.

THEOREM 7.6. $\operatorname{Indet}\langle W_1, W_2, W_3\rangle = \{U_1W_3 + W_1U_3 | U_1, U_3$ *is a row, column matrix, respectively, with entries in* $\mathcal{B}^*(F)\}$.

PROOF. This theorem is a translation of the corresponding fact from [3, p. 201] since our triple products coincide with those of J. C. Alexander.

We now consider matric Massey products in $\mathcal{B}^*(F; R)$.

DEFINITION 7.7. Let $(W_1, \ldots, W_n)$ be a multipliable system of matrices with entries in $\mathcal{B}^*(F; R)$. $\{A_{i,j}; \zeta_1, \ldots, \zeta_n\}$ is called a defining system for

$\langle W_1, \ldots, W_n\rangle$ if $\zeta_1, \ldots, \zeta_n$ are positive integers which are invertible in R such that all the conditions of Definition 7.2 are satisfied except that $[A_{i-1,i}] = \zeta_i W_i$ for $1 \leqslant i \leqslant n$. $\langle W_1, \ldots, W_n\rangle$ is defined as the set of all $\zeta_1^{-1} \cdots \zeta_n^{-1}[\tilde{A}_{0,n}]$ for all such defining systems $\{A_{i,j}; \zeta_1, \ldots, \zeta_n\}$.

Observe that $\langle W_1, \ldots, W_n\rangle = \zeta_1^{-1} \cdots \zeta_n^{-1}\langle \zeta_1 W_1, \ldots, \zeta_n W_n\rangle$ where $\langle \zeta_1 W_1, \ldots, \zeta_n W_n\rangle$ is defined in $\mathfrak{B}^*(F)$ by Definition 7.2.

THEOREM 7.8. *Theorems* 7.3, 7.4, 7.5 *and* 7.6 *remain valid when* $\mathfrak{B}^*$ *is replaced by* $\mathfrak{B}^*(\ ; R)$.

PROOF. The proofs of these theorems for $\mathfrak{B}^*$ when interpreted according to Definition 7.7 become proofs for $\mathfrak{B}^*(\ ; R)$.

8. Matric Massey products and the Adams spectral sequence. There are three basic theorems of J. P. May [14] which relate algebraic Massey products and spectral sequences defined by filtered differential graded algebras. We prove these theorems for the cobordism Massey products of §7 and the Adams spectral sequences of Theorem 6.4. Some of these theorems for triple products were proved in [3] and were known independently to D. S. Kahn [unpublished].

In this section F will be a finite CW complex with Spanier-Whitehead duals $D_t F$ as in §7. We work with a fixed Adams spectral sequence as in Theorem 6.4:

$$\text{(A)} \qquad E_2 = \operatorname{Ext}_{E_*(E)}\big(E_*(F), E_*(M\mathfrak{B})\big) \Rightarrow \mathfrak{B}^*(F)^E.$$

Thus (A) is induced by the filtered DGMA $\mathcal{C}\mathfrak{B}_E^*(F; R)$ which is defined by an Adams resolution Y_0. The products on Y_0 and $\mathcal{C}\mathfrak{B}_E^*(F; R)$ are only homotopy associative. In the appendix we show how Stasheff's theory of A_n-structures can be used to generalize the definition of matric Massey products of §7 from $\mathcal{C}\mathfrak{B}^*(F; R)$ to $\mathcal{C}\mathfrak{B}_E^*(F; R)$. We let π: $Z_r^{p,q} \to E_r^{p,q}$, $r \geqslant 1$, be the canonical map. Recall from Theorem 6.2 that the Adams spectral sequence is bigraded so that d_r has bidegree $(r, r-1)$. Our proofs refer only to the case $R = Z$. It is straightforward to generalize them to the case of arbitrary R.

THEOREM 8.1. *Let* $\{A_{i,j}\}$ *be a defining system for* $\langle V_1, \ldots, V_n\rangle$ *in* $E_{r+1} = H(E_r, d_r)$, $r \geqslant 1$. *For* $1 \leqslant i \leqslant n$, *let* V_i *be an infinite cycle which converges to* W_i. *Assume that* $\langle W_1, \ldots, W_n\rangle$ *is strictly defined in* $\mathfrak{B}^*(F)$ *and that condition* (*) *holds.*

(*) *Let* $u \geqslant 0$, $1 < j - i < n$ *and let* (p,q) *be the degree of an entry of* $A_{i,j}$. *Then* $E_{r+u+1}^{p-u,q+u}$ *consists entirely of infinite cycles.*

Then $\widetilde{A_{0,n}}$ *is an infinite cycle which converges to an element of* $\langle W_1, \ldots, W_n\rangle$.

PROOF. The proof of [14, Theorem 4.1] applies with only minor modifications once we define a defining system $\{B_{i,j}\}$ for $\langle W_1, \ldots, W_n\rangle$ in $\mathcal{C}\mathfrak{B}_E^*(F)$. Most of the definition follows from the requirement that $\{\alpha(B_{i,j})\}$ be a defining system for $\langle W_1, \ldots, W_n\rangle$ in $\mathcal{C}\mathfrak{B}^*(F)$. The only remaining problem is how to define the cycles $\tilde{B}_{i,j}$, since the standard method requires an associative product. Theorem 8A.3 of the appendix deals with this problem.

Observe that condition (*) of Theorem 8.1 above is vacuous if $E_{r+1} = E_\infty$, while condition (**) of Theorem 8.2 below is vacuous if $s = r + 1$.

THEOREM 8.2. *Let $\langle V_1, \ldots, V_n\rangle$ be defined in $E_{r+1} = H(E_r, d_r)$, $r \geqslant 1$. Assume that condition (**) holds and that $d_t(V_i) = 0$ for $r < t < s$, $1 \leqslant i \leqslant n$.*

(**) *Let $1 < j - i < n$, $r < t < s$ and let (p, q) be the degree of an entry in an element of a defining system of $\langle V_1, \ldots, V_n\rangle$. Then $E_t^{p+t,q-t+1} = 0$ and $E_{r+s-t}^{p+t,\,q-t+1} = 0$.*

If $\alpha \in \langle V_1, \ldots, V_n\rangle$ then $d_t(\alpha) = 0$ for $r < t < s$. Furthermore, there are matrices Y_k in E_{r+1} which survive to $d_s(V_k)$, $1 \leqslant k \leqslant n$, such that if

$$X_1 = \left(Y_1, \overline{V}_1\right), \qquad X_i = \begin{pmatrix} V_i & 0 \\ Y_i & \overline{V}_i \end{pmatrix} \text{for } 1 < i < n, \qquad X_n = \begin{pmatrix} V_n \\ Y_n \end{pmatrix}$$

then $\langle X_1, \ldots, X_n\rangle$ is defined in $E_{r+1} = H(E_r, d_r)$ and contains an element γ which survives to $-d_s(\alpha)$.

PROOF. We put this theorem into a context where the proof of [14, Theorem 4.3] applies with minor modifications. Let $\{A_{i,j}\}$ be a defining system for $\langle V_1, \ldots, V_n\rangle$ with $[\tilde{A}_{0,n}] = \alpha$. $\{\mathfrak{B}_{i,j}\}$ is a defining system modulo $F^{*+s}\mathcal{C}\mathfrak{B}_E^*(F)$ for $\langle V_1, \ldots, V_n\rangle$ in $\mathcal{C}\mathfrak{B}_E^*(F)$ if the following conditions hold.

(i) $\pi(\mathfrak{B}_{i,j}) = A_{i,j}$ and $\mathfrak{B}_{i,j} = (B_{i,j}, \hat{\mathcal{P}}(B_{i,j}))$.

(ii) The entries of $B_{i-1,i}$ are disjoint manifolds with $\mathfrak{B} \times D_1F$-structure and domain $\mathfrak{D}_{i+}$ in V_i such that $V_1 \perp \ldots \perp V_n$.

(iii) For $n \geqslant j - 1 > 1$, there are domains $\mathfrak{D}_{i,j}$ as in Definition 7.2(b) such that

$$\mathfrak{D}_{i,j+} = (\mathfrak{D}_{i+1+} \times \mathfrak{D}_{i+1,j}) \cup \bigcup_{t=i+2}^{j-2} (\mathfrak{D}_{i,t} \times \mathfrak{D}_{t,j}) \cup (\mathfrak{D}_{i,j-1} \times \mathfrak{D}_{j+})$$

and

$$\mathfrak{D}_{i,j-} = (\mathfrak{D}_{i-1-} \times \mathfrak{D}_{i+1,j}) \cup (\mathfrak{D}_{i,j-1} \times \mathfrak{D}_{j-}).$$

(iv) For $n > j - 1 > 1$, the $B_{i,j}$ have disjoint entries with domain $\mathfrak{D}_{i,j}$.

(v) For $n > j - i > 1$, $\partial\mathfrak{B}_{i,j} = \tilde{\mathfrak{B}}_{i,j} \cup \partial_0\mathfrak{B}_{i,j}$.

(vi) For $n \geqslant j - i > 1$, $\partial\tilde{\mathfrak{B}}_{i,j} = \tau\partial(\partial_0\mathfrak{B}_{i,j})$ and

$$\pi\left(\partial \tilde{\mathfrak{B}}_{i,j}\right) = \sum_{t=i+1}^{j-1} \pi\left(\partial_0 \mathfrak{B}_{i,t} \cdot \mathfrak{B}_{t,j}\right) - \pi\left(\mathfrak{B}_{i,t} \cdot \partial_0 \mathfrak{B}_{t,j}\right).$$

(vii) For $n \geqslant j - i > 1$, $\tilde{\mathfrak{B}}_{i,j} = (\tilde{B}_{i,j}, \hat{\mathcal{P}}(\tilde{B}_{i,j}))$ where $\tilde{B}_{i,j} = \bigcup_{t=i+1}^{j-1} \bar{B}_{i,t} B_{t,j}$ and $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ is defined in Theorem 8A.4 of the appendix. By Theorem 8A.4, $\pi(\tilde{\mathfrak{B}}_{i,j}) = \tilde{A}_{i,j}$.

(viii) The entries of $\partial \mathfrak{B}_{i,i+1}$, $\partial_0 \mathfrak{B}_{i,j}$ $(j > i + 1)$ have filtration degree s larger than the corresponding entries of $\mathfrak{B}_{i,i+1}$, $\mathfrak{B}_{i,j}$, respectively.

We must show that $\mathfrak{D}_{i,j+}$, $\mathfrak{D}_{i,j-}$ and $\mathfrak{D}_{i,j+} \cap \mathfrak{D}_{i,j-}$ are domains. $D^*_{i,j-}$ and $\partial D^*_{i,j-}$ are simply connected manifolds and $D^*_{i,j-}$ has zero reduced homology. By Smale's characterization of high dimensional smooth discs [16] we see that $\mathfrak{D}_{i,j-}$, $\partial \mathfrak{D}_{i,j-} = \mathfrak{D}_{i,j+} \cap \mathfrak{D}_{i,j-}$ and hence $\mathfrak{D}_{i,j+}$ are domains. Let $\mathfrak{D}_{i,j} = (D_{i,j}, \varphi, U)$. By [16, Theorem 9.7] we can give $D_{i,j}$ a second structure of a domain $(D_{i,j}, \varphi', U)$ which induces $\mathfrak{D}_{i,j+}$ and $\mathfrak{D}_{i,j-}$. Moreover $\varphi^{-1} \circ \varphi'$ is a diffeomorphism of the disc. A straightforward computation verifies the hypotheses of Lemma 7.1 for $\bigcup_{t=i+1}^{j-1} \bar{B}_{i,t} B_{t,j}$. Hence $\tilde{B}_{i,j}$ is a manifold which satisfies (vi) and (vii).

The proof of [14, Theorem 4.3] can now be applied with $G_{i,j} = \pi(\partial_0 \mathfrak{B}_{i,j})$. To construct $\mathfrak{B}_{i,j}$ as in [14] we choose c with $\partial(c) = f + h - \sigma\partial(f)$, $h \in F^{p+t}\mathcal{C}\mathfrak{B}^*_E(F)$, $r < t \leqslant s$ and t maximal.

COROLLARY 8.3. *Assume in addition to the hypotheses of Theorem* 8.2 *that there is only one way to choose* $Y_1, \ldots, Y_n$. *Write* $Y_i = d_s(V_i)$, $1 \leqslant i \leqslant n$. *Suppose that all the Massey products have zero indeterminacy and that* $\langle \bar{V}_1, \ldots, \bar{V}_{k-1}, d_s(V_k), V_{k+1}, \ldots, V_n \rangle$ *is strictly defined in* $E_{r+1} = H(E_r, d_r)$ *for* $1 \leqslant k \leqslant n$. *Then*

$$d_s\langle V_1, \ldots, V_n \rangle = -\sum_{k=1}^{n} \langle \bar{V}_1, \ldots, \bar{V}_{k-1}, d_s(V_k), V_{k+1}, \ldots, V_n \rangle.$$

PROOF. This corollary is a consequence of Theorem 8.2 and [14, Proposition 2.10] applied to the algebraic Massey product $\langle X_1, \ldots, X_n \rangle$.

THEOREM 8.4. *Let* $\langle V_1, \ldots, V_n \rangle$ *be defined in* $E_{r+1} = H(E_r, d_r)$, $r \geqslant 1$. *For* $1 \leqslant i \leqslant n$, *assume that* V_i *is a matrix of infinite cycles which converges to* W_i. *Let* $1 \leqslant q \leqslant n - 2$ *be given such that* $\langle W_k, \ldots, W_{k+q} \rangle$ *is strictly defined in* $\mathfrak{B}^*(F)$ *for* $1 \leqslant k \leqslant n - q$. *Let* $s > r$ *be given such that condition* (*) *of Theorem* 8.1 *holds for* $1 < j - i \leqslant q$, *the first conclusion of condition* (**) *of Theorem* 8.2. *holds for* $q < j - i$, *and the second conclusion of condition* (**) *holds for* $q + 1 < j - i$. *If* $\alpha \in \langle V_1, \ldots, V_n \rangle$ *then* $d_t(\alpha) = 0$ *for* $t < s$. *Furthermore, for* $1 \leqslant k \leqslant n - q$ *there are matrices* Y_k *with entries in* E_{r+1} *which survive to an element of* $\langle W_k, \ldots, W_{k+q} \rangle$ *such that if*

$$X_1 = \left(Y_1, \bar{V}_1\right), \qquad X_i = \begin{bmatrix} V_{i+q} & 0 \\ Y_i & \bar{V}_i \end{bmatrix} \text{ for } 1 \leqslant k \leqslant n - q, \quad X_{n-q} = \begin{pmatrix} V_n \\ Y_{n-q} \end{pmatrix}$$

then $\langle X_1, \ldots, X_{n-q}\rangle$ *is defined in* $E_{r+1} = H(E_r, d_r)$ *and contains an element* γ *which survives to* $d_s(\alpha)$.

PROOF. This theorem is proved by combining the proofs of Theorems 8.1 and 8.2 exactly as in the proof of this theorem for algebraic Massey products in [14, Theorem 4.5].

COROLLARY 8.5. *Assume in addition to the hypotheses of Theorem* 8.4 *that there is only one way to choose* $Y_1, \ldots, Y_{n-q}$. *Suppose that all the algebraic Massey products have zero indeterminacy and that* $\langle \bar{V}_1, \ldots, \bar{V}_{k-1}, Y_k, V_{k+q+1}, \ldots, V_n\rangle$ *is strictly defined in* $E_{r+1} = H(E_r, d_r)$ *for* $1 \leqslant k \leqslant n-q$. *Then*

$$d_s\langle V_1, \ldots, V_n\rangle = \sum_{k=1}^{n-q} \langle \bar{V}_1, \ldots, \bar{V}_{k-1}, Y_k, V_{k+q+1}, \ldots, V_n\rangle.$$

PROOF. This corollary is a consequence of Theorem 8.4 and [14, Proposition 2.10] applied to the algebraic Massey product $\langle X_1, \ldots, X_{n-q}\rangle$.

Appendix to §8. To complete the proofs of Theorems 8.1 and 8.2 we need to define the construction $\tilde{\mathfrak{B}}_{i,j}$ in $\mathcal{C}\mathfrak{B}_E^*(F)$ in the cases when all the $\partial B_{i,i+1} = 0$ and in the case when all the $\partial B_{i,i+1} \neq 0$. We do this in Theorems 8A.3 and 8A.4, respectively, by using A_n-structures for the product μ: $Y_0 \wedge Y_0 \to Y_0$.

LEMMA 8A.1 (D. S. KAHN [UNPUBLISHED]). *Let* $X_0 \supset \cdots \supset X_r \supset \cdots$ *be spectra with* $\iota: F_r \to X_{r+1}$ *the fibre of* $X_{r+1} \hookrightarrow X_r$. *Assume that* ι^*: $E^*(X_r) \to E^*(F_r)$ *is one-to-one for* $r \geqslant 0$. *Let* Y_0 *be an Adams resolution and let* f_i: $X_0 \to Y_0$ *for* $i = 0, 1$ *such that* $f_i(X_r) \subset Y_r$ *for* $r \geqslant 0$. *If* $f_0 \simeq f_1$ *and* $f_0^* = f_1^*$: $E^*(Y_r, Y_{r+1}) \to E^*(X_r, X_{r+1})$ *for* $r \geqslant 0$ *then there is a homotopy* H: $f_0 \simeq f_1$ *such that* $H(I \ltimes X_r) \subset Y_r$ *for* $r \geqslant 0$.

PROOF. We use the methods of [1, §3]. Assume that we have compressed a homotopy H: $f_0 \simeq f_1$ so that $H(I \ltimes X_r) \subset Y_r$ for $r \leqslant t$. Then we can deform $H|I \ltimes X_{t+1}$rel $\dot{I} \ltimes X_{t+1}$ through Y_t into Y_{t+1} if the map

$$\lambda: E^*(Y_t, Y_{t+1}) \to E^*\big(I \ltimes X_{t+1}, \dot{I} \ltimes X_{t+1}\big)$$

induced by H is zero. Write $X_t \simeq SA$, $X_{t+1} \simeq SB$ and $F_t = A \cup CB$.

In Figure 2, $f = (f_0, f_1)$, π collapses $I \times F_t$ to a point and $\hat{H}$ is a null-homotopy of $H \circ (1 \ltimes \iota)$ induced by a null-homotopy of $F_t \to^{\iota} X_{t+1} \to X_t$. The diagram of Figure 2 commutes and $\lambda = (H \cup Cf)^*$.

$$\tilde{E}^*\big(\dot{I} \ltimes (X_t \cup CX_{t+1})\big) \cong \tilde{E}^*(X_t \cup CX_{t+1}) \oplus \tilde{E}^*(X_t \cup CX_{t+1}),$$

and under this isomorphism $\pi^*(a, b) = S(a) - S(b)$. Thus

$$(1 \ltimes \iota)^*\lambda(x) = \pi^* f^*(x) = \pi^*\big(f_0^*(x), f_1^*(x)\big) = 0$$

since $f_0^* = f_1^*$. Hence $\lambda = 0$.

$$\begin{tikzcd}
\widetilde{E}^*(Y_t \cup CY_{t+1}) \arrow[r, "(H \cup Cf)^*"] \arrow[dd, "f^*"'] \arrow[ddr, "(\hat{H} \cup Cf)_*"] & \widetilde{E}^*((I \ltimes X_{t+1}) \cup C(\dot{I} \ltimes X_{t+1})) \arrow[d, "1\text{-}1"', "(1 \ltimes \iota)^*"] & \\
& \widetilde{E}^*((I \ltimes F_t) \cup C(\dot{I} \ltimes F_t)) & \\
& \widetilde{E}^*(C(I \ltimes F_t) \cup C(\dot{I} \ltimes F_t)) \arrow[u] \arrow[d, "\cong"] & \\
\widetilde{E}^*(\dot{I} \ltimes (X_t \cup CX_{t+1})) \arrow[d, equal] & & \\
\widetilde{E}^*(\dot{I} \ltimes (SA \cup CSB)) \arrow[r, phantom, "\cong"] & \widetilde{E}^*(S(\dot{I} \ltimes F_t)) \arrow[uuu, bend right, "\pi^*"'] &
\end{tikzcd}$$

FIGURE 2

THEOREM 8A.2. *Let Y_0 be an Adams resolution, and let μ: $Y_0 \wedge Y_0 \to Y_0$ such that (Y_0, μ) realizes an associative differential algebra C and μ is homotopy associative. Then μ admits an A_n-structure. That is, in the notation of* [18], *there are maps*

$$M_n: K_n \ltimes Y_0 \wedge \cdots \wedge Y_0 \to Y_0$$

such that $M_n(K_n \ltimes Y_{r_1} \wedge \cdots \wedge Y_{r_n}) \subset Y_{r_1 + \cdots + r_n - n + 3}$ for $n \geqslant 3$, $M_2 = \mu$ and $M_n | \partial K_n \ltimes Y_0 \wedge \cdots \wedge Y_0$ is given by [18, *Theorem* 5(2)].

PROOF. $M_2 = \mu$ and M_3: $\mu(\mu \wedge 1) \simeq \mu(1 \wedge \mu)$ is given by Lemma 8A.1. Inductively if the M_i, $2 \leqslant i \leqslant n$, have been defined then $M_{n+1} | \partial K_{n+1} \ltimes Y_0 \wedge \cdots \wedge Y_0$ is defined which extends to define M_{n+1} by the argument of [1, Lemma 3.5].

We used Lemma 8A.1 to show that $M_n(K_n \ltimes Y_{r_1} \wedge \cdots \wedge Y_{r_n})$ is contained in $Y_{r_1 + \cdots + r_n - n + 3}$ instead of $Y_{r_1 + \cdots + r_n - n + 2}$. This slight improvement makes the theorems of this section apply to Massey products defined in $E_2 = H(E_1, d_1)$ as well as $E_{r+1} = H(E_r, d_r)$, $r \geqslant 2$.

THEOREM 8A.3. *Make all the assumptions of Theorem* 8.1 *except that $\langle W_1, \ldots, W_n \rangle$ may not be defined. Let $\{(B_{i,j}, \hat{\mathcal{P}}(B_{i,j})) | j - i < q\}$ be given for some $q \geqslant 1$ such that*:

(a) $\pi(B_{i,j}, \hat{\mathcal{P}}(B_{i,j})) = A_{i,j}$.

(b) $\partial B_{i,j} = \tilde{B}_{i,j}$, *and $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ is defined as in the proof of this theorem. Then for $j - i = q$ we can define $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ such that $\pi(\tilde{B}_{i,j}, \hat{\mathcal{P}}(\tilde{B}_{i,j})) = \tilde{A}_{i,j}$.*

PROOF. We replace the partial defining system $\{\mathcal{P}(B_{i,j}) | j - i < q\}$ for the associative higher product $\langle \mathcal{P}(W_1), \ldots, \mathcal{P}(W_n) \rangle$ by a partial defining system $\{G_{i,j} | j - i < q\}$ for a nonassociative higher product. The A_n-structure

of Theorem 8A.2 can be used to lift the $G_{i,j}$ to the Adams resolution Y_0. The liftings $\tilde{G}_{i,j}$ induce the $\hat{\mathcal{P}}(\tilde{B}_{i,j})$.

We begin by defining the domains $T_{i,j}$, $\tilde{T}_{i,j}$ of the maps $G_{i,j}$, $\tilde{G}_{i,j}$. We also construct maps $R_{i,j}$: $T_{i,j} \to D^*_{ij}$ and homotopy equivalences $\tilde{R}_{i,j}$: $\tilde{T}_{i,j} \to \partial D^*_{ij}$. Let $T_{i,i+1} = D^*_{i,i+1}$ and $R_{i,i+1} = 1$. For $j > i + 1$,

$$\tilde{T}_{i,j} = \bigcup_{m=2}^{j-i} \bigcup_{i<s_1<\cdots<s_{m-1}<j} K_m \ltimes T_{i,s_1} \wedge \cdots \wedge T_{s_{m-1},j}/\sim$$

and

$$\tilde{R}_{i,j} = \bigcup_{m=2}^{j-i} \bigcup_{i<s_1<\cdots<s_{m-1}<j} * \ltimes R_{i,s_1} \wedge \cdots \wedge R_{s_{m-1},j}$$

where

$$\begin{aligned}(\partial_k(u, v)(a, b), t_1, \ldots, t_m) \\ \sim (a, t_1, \ldots, t_{k-1}, (b, t_k, \ldots, t_{k+v-1}), t_{k+v}, \ldots, t_m)\end{aligned}$$

for $u + v = k + 1$, $a \in K_u$, $b \in K_v$ and $(t_1, \ldots, t_m) \in T_{i,s_1} \wedge \cdots \wedge T_{s_{m-1},j}$ $\tilde{R}_{i,j}$ is a homotopy equivalence by Whitehead's Theorem since $K_m \simeq I^{m-2}$. For $j > i + 1$ let $T_{i,j}$ be $(I \times \tilde{T}_{i,j}) \cup C(\tilde{T}_{i,j})$ with the base of the cone identified with the base of the cylinder. $\tilde{T}_{i,j}$ is contained in $T_{i,j}$ as the top of the cylinder. $R_{i,j}$ is the composite map

$$T_{i,j} \xrightarrow{\pi} C(\tilde{T}_{i,j}) \xrightarrow{C(\tilde{R}_{i,j})} C\partial D^*_{i,j} \cong D^*_{i,j}$$

where π collapses the cylinder to its base.

For $1 < j - i < q$ we define $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ and a map

$$G_{i,j}: (T_{i,j}, \tilde{T}_{i,j}) \to (Y_{*-r(j-i-1)}, Y_{*-r(j-i-2)}).$$

We use induction on $j - i$ beginning with $G_{i,i+1} = \hat{\mathcal{P}}(B_{i,i+1}) \circ R_{i,i+1}$. For $j - i > 1$, $\tilde{G}_{i,j} = G_{i,j}|\tilde{T}_{i,j}$ is given by

$$\tilde{G}_{i,j}(k, t_1, \ldots, t_m) = M_m \circ (1 \ltimes G_{i,s_1} \wedge \cdots \wedge G_{s_{m-1},j})(k, t_1, \ldots, t_m)$$

where $k \in K_m$ and $(t_1, \ldots, t_m) \in T_{i,s_1} \wedge \cdots \wedge T_{s_{m-1},j}$. Let $\tilde{Q}_{i,j}$ be a homotopy inverse to $\tilde{R}_{i,j}$. Find $\hat{\mathcal{P}}(\tilde{B}_{i,j}) \simeq \tilde{G}_{i,j} \circ \tilde{Q}_{i,j}$ in $Y_{*-r(j-i-2)}$ such that $r \circ \hat{\mathcal{P}}(\tilde{B}_{i,j}) = \mathcal{P}(\tilde{B}_{i,j})$ since r is a fibration. Then

$$\pi(\tilde{B}_{i,j}, \hat{\mathcal{P}}(\tilde{B}_{i,j})) = [\tilde{G}_{i,j}] \circ [\tilde{Q}_{i,j}] = \tilde{A}_{i,j}.$$

The last equality follows from the observations that for $m \geqslant 3$, $r \geqslant 1$:

$$\begin{aligned}\tilde{G}_{i,j}(K_m \ltimes T_{i,s_1} \wedge \cdots \wedge T_{s_{m-1},j}) &\subset Y_{*-r(j-i-m)-m+3} \\ &= Y_{*-r(j-i-2)+(r-1)m+3-2r}\end{aligned}$$

and $(r-1)m+3-2r \geqslant 1$. $\hat{\mathcal{P}}(B_{i,j}) \circ R_{i,j}$ is a null-homotopy of $\hat{\mathcal{P}}(\tilde{B}_{i,j}) \circ \tilde{R}_{i,j}$ in $Y_{*-r(j-i-1)}$. Let $G_{i,j}$ be this null-homotopy followed by a homotopy from $\hat{\mathcal{P}}(\tilde{B}_{i,j}) \circ \tilde{R}_{i,j}$ to $\tilde{G}_{i,j}$ in $Y_{*-r(j-i-2)}$. Then $G_{i,j} \simeq \hat{\mathcal{P}}(B_{i,j}) \circ R_{i,j}$ in $(Y_{*-r(j-i-1)}, Y_{*-r(j-i-2)})$. If $j-i=q$ then the preceding constructions of $\tilde{G}_{i,j}$ and $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ apply.

THEOREM 8A.4. *Make all the assumptions of Theorem* 8.2. *Let* $\{\mathcal{B}_{i,j}, \tilde{\mathcal{B}}_{i,j}, \partial_0\mathcal{B}_{i,j} | j-i<q\}$ *be given for* $q \geqslant 1$ *such that*:

(a) (i)–(viii) *as listed in the proof of Theorem* 8.2 *are valid.*

(b) $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ *is defined as in the proof of this theorem.*

Then for $j-i=q$ *we can define* $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ *such that* $\pi(\tilde{\mathcal{B}}_{i,j}) = \tilde{A}_{i,j}$ *and*

$$\pi\left(\partial\tilde{\mathcal{B}}_{i,j}\right) = \sum_{t=i+1}^{j-1} \pi(\partial_0\mathcal{B}_{i,t} \cdot \mathcal{B}_{t,j}) - \pi(\mathcal{B}_{i,t} \cdot \partial_0\mathcal{B}_{t,j}).$$

PROOF. The proof of this theorem is similar to the proof of Theorem 8A.3. We sketch the details. Let $T'_{i,i+1} = (D_{i,i+1+})^*$ and let $T'_{i,j} = T_{i,j}$ if $j > i+1$. For $j > i+1$ define $\tilde{T}_{i,j}$, $\tilde{R}_{i,j}$ as in 8A.3 but with the $T_{p,q}$ replaced by $T'_{p,q}$. Define $\partial\tilde{T}_{i,j} = \tilde{R}_{i,j}^{-1}(\partial D^*_{i,j+})$. Then $\tilde{R}_{i,j}$: $(\tilde{T}_{i,j}, \partial\tilde{T}_{i,j}) \to (D^*_{i,j+}, \partial D^*_{i,j+})$ is a homotopy equivalence. Let $\partial_0 T_{i,j} = D^*_{i,j-} \cup (I \times \partial D^*_{i,j-})$ where $\partial D^*_{i,j-}$ is identified with the base of the cylinder. $\partial_0 R_{i,j}$: $\partial_0 T_{i,j} \to D^*_{i,j-}$ is given by $1 \cup P_2$. Let $T_{i,j} = C(\tilde{T}_{i,j} \cup \partial_0 T_{i,j}) \cup [I \times (\tilde{T}_{i,j} \cup \partial_0 T_{i,j})]$ where $\partial\tilde{T}_{i,j}$ is identified by $\tilde{R}_{i,j}$ with $\{1\} \times \partial D^*_{i,j+}$ in $\partial_0 T_{i,j}$, and the base of the cone is identified with the base of the cylinder. $R_{i,j}$: $T_{i,j} \to D^*_{i,j}$ is defined as in 8A.3. As in 8A.3, $\tilde{G}_{i,j}$: $\tilde{T}_{i,j} \to Y_{*-r(j-i-2)}$ is defined by the A_m-structures on Y_0. $\tilde{G}_{i,j}$ defines $\hat{\mathcal{P}}(\tilde{B}_{i,j})$ so that $\tilde{G}_{i,j} \cong \hat{\mathcal{P}}(\tilde{B}_{i,j}) \circ \tilde{R}_{i,j}$ from $(\tilde{T}_{i,j}, \partial\tilde{T}_{i,j})$ to $(Y_{*-r(j-i-2)}, Y_{*-r(j-i-1)+s})$. $\partial_0 G_{i,j}$: $\partial_0 T_{i,j} \to Y_{*-r(j-i-1)+s}$ is given by $\hat{\mathcal{P}}(\partial_0 B_{i,j}) \circ \partial_0 R_{i,j}$ union a homotopy from $\hat{\mathcal{P}}(\partial_0 B_{i,j}) \circ \tilde{R}_{i,j} | \partial\tilde{T}_{i,j}$ to $\tilde{G}_{i,j} | \partial\tilde{T}_{i,j}$. Define $G_{i,j}$: $T_{i,j} \to Y_{*-r(j-i-1)}$ as in 8A.3.

9. **Brown-Peterson homology.** Let $Z_{(p)}$ denote the integers localized at p. Recall that the Brown-Peterson spectrum BP is a direct summand of $MU_{(p)}$ for a fixed prime p. (See [2].) We use this fact to construct sub DGMs $\mathcal{C}BP_*$ of $\mathcal{C}BU_*(\ ;\ Z_{(p)})$ whose homology is BP_*. However, $\mathcal{C}BP_*$ is not a chain functor because of the failure of Definition 4.1(2) for CW pairs. Thus there is no hope of generalizing the results of §6 to BP. The results of §§4, 5, 7 and 8, however, are also valid for the case $\mathcal{B} = BP$. Note that in §§6 and 8 the case $E = BP$ was allowed. It is the case $\mathcal{B} = BP$ which was not allowed but is the focal point of our present considerations.

THEOREM 9.1. *There are two functors, both denoted* $\mathcal{C}BP_*$, *from the categories of CW complexes, and CW pairs to the category of DGMs indexed on* J. *In both cases* $\mathcal{C}BP_*(\mathcal{X})$ *is a sub DGM of* $\mathcal{C}BU_*(\mathcal{X};\ Z_{(p)})$, *and there are natural isomorphisms* $H_n[\mathcal{C}BP_*(\ )] \cong BP_n(\ )$. $\mathcal{C}BP_*(\mathcal{X})$ *has a generating set* G

defined by $X \in G_{m,\mathfrak{D}}$ *if* kX *is in the natural generating set of* $\mathcal{C}BU_m(\mathcal{X}; Z_{(p)})_{\mathfrak{D}}$ *for some* $k \in Z^+$.

PROOF. Let $\mathcal{X}$ denote a CW complex, or a CW pair, and let $\mathfrak{D} \in J_n$. For $m \in \{n-1, n\}$ define $\mathcal{C}BP'_m(\mathcal{X})_{\mathfrak{D}}$ as the submonoid of all representative cycles in $\mathcal{C}BU_m(\mathcal{X}; Z_{(p)})_{\mathfrak{D}}$ of $BP_m(\mathcal{X})$. For $m \in \{n-1, n, n+1\}$ let $\mathcal{C}BP''_m(\mathcal{X})_{\mathfrak{D}}$ denote the submonoid of $\mathcal{C}BU_m(\mathcal{X}; Z_{(p)})_{\mathfrak{D}}$ spanned by $\mathcal{C}BP'_m(\mathcal{X})_{\mathfrak{D}}$ and by all X such that $\partial(X) \in \mathcal{C}BP'_{m-1}(\mathcal{X})_{\mathfrak{D}}$ and $X \neq X' + X$ with $\partial(X') = 0$, $X' \neq 0$. Define $\mathcal{C}BP_m(\mathcal{X})_{\mathfrak{D}}$ as the sub DGM of $\mathcal{C}BU_m(\mathcal{X})_{\mathfrak{D}}$ spanned by $\mathcal{C}BP''_m(\mathcal{X})_{\mathfrak{D}}$. The conclusions of the theorem now follow easily.

EXAMPLE 9.2. Let (X, A) be a CW pair with X connected and $BP_n(X, A) \neq 0$. For $\mathfrak{D} \in J_n$ the canonical map

$$H_n[\mathcal{C}BP(X)_{\mathfrak{D}}/\mathcal{C}BP(A)_{\mathfrak{D}}] \to MU_*(X, A; Z_{(p)})$$

is onto and hence $H_n[\mathcal{C}BP(X)_{\mathfrak{D}}/\mathcal{C}BP(A)_{\mathfrak{D}}] \cong BP_n(X, A)$. From another point of view, the following short exact sequence of DGMs does not induce a long exact sequence in homology.

$$0 \to \mathcal{C}BP_*(A) \to \mathcal{C}BP_*(X) \to \mathcal{C}BP_*(X)/\mathcal{C}BP_*(A) \to 0.$$

We can now use Spanier-Whitehead duality to construct a functor $\mathcal{C}BP^*(\ )$ from $\mathcal{S}_q$ to the category of DGMs indexed on $J^{(q)}$ such that $H^n[\mathcal{C}BP^*(\ )]$ is naturally isomorphic to $BP^n(\ )$. We can also use the statement and proof of Theorem 5.3 in the case $\mathfrak{B} = BU$, $R = Z_{(p)}$ to realize the BP_* cross-product, the BP_* smash-product and the BP^* cup-product as pairings of DGMs.

The following definition and theorem generalize the results of §§7, 8 to BP.

DEFINITION 9.3. Let ι: $BP \to MU_{(p)}$, ρ: $MU_{(p)} \to BP$ denote the canonical inclusion, projection, respectively. Let $(W_1, \ldots, W_n)$ be a multipliable system of matrices with entries in $BP^*(F)$. Then $\langle W_1, \ldots, W_n\rangle$ is defined if $\langle \iota(W_1), \ldots, \iota(W_n)\rangle$ is defined in $MU^*(F; Z_{(p)})$. In that case $\langle W_1, \ldots, W_n\rangle = \rho\langle \iota(W_1), \ldots, \iota(W_n)\rangle$.

THEOREM 9.4. (a) *Theorems* 7.3 *and* 7.6 *remain valid if* $\mathfrak{B}^*$ *is replaced by* BP^*.

(b) *All theorems of* §8 *with* $R = \pi_0(E) = Z_{(p)}$ *remain valid if* $\mathfrak{B}^*$ *is replaced by* BP^*.

PROOF. (a) Theorems 7.3 and 7.6 for BP^* are a consequence of these theorems for $MU^*(\ ; Z_{(p)})$ and Definition 9.3.

(b) Observe that the spectral sequence under consideration is a direct summand of the spectral sequence of Theorem 6.4 for $\mathfrak{B} = BU$ and the same E. From this observation, the theorems of §8 for BP^* follow from these theorems for $MU^*(\ ; Z_{(p)})$.

REFERENCES

1. J. F. Adams, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214. MR **20** #2711.

2. _____, *Stable homotopy and generalised homology*, Univ. of Chicago Press, Chicago, 1974.

3. J. C. Alexander, *Cobordism Massey products*, Trans. Amer. Math. Soc. **166** (1972), 197–214. MR **45** #2700.

4. D. W. Anderson, *Chain functors and homology theories*, Lecture Notes in Math., no. 249, Springer-Verlag, Berlin, 1971, pp. 1–12. MR **49** #3895.

5. M. F. Atiyah, *Bordism and cobordism*, Proc. Cambridge. Philos. Soc. **57** (1961), 200 – 208. MR **23** #A4150.

6. R. O. Burdick, P. E. Conner and E. E. Floyd, *Chain theories and their derived homology*, Proc. Amer. Math. Soc. **19** (1968), 1115–1118. MR **38** #1668.

7. J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR **25** #3543.

8. J. M. Cohen, *Stable homotopy*, Lecture Notes in Math., no. 165, Springer-Verlag, Berlin and New York, 1970. MR **42** #8486.

9. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N.J., 1952. MR **14**, 398.

10. S. O. Kochman, *The symplectic cobordism ring* (to appear).

11. A. Kock, L. Kristensen and I. Madsen, *Cochain functors for general cohomology theories*. II, Math. Scand. **20** (1967), 151–176. MR **35** #4908.

12. R. Lashof, *Poincaré duality and cobordism*, Trans. Amer. Math. Soc. **109** (1963), 257–277. MR **27** #6281.

13. S. Mac Lane, *Homology*, Grundlehren math. Wiss., Bd. 114, Springer-Verlag, Berlin; Academic Press, New York, 1963. MR **28** #122.

14. J. P. May, *Matric Massey products*, J. Algebra **12** (1969), 533–568. MR **39** #289.

15. _____, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math., no. 577, Springer-Verlag, Berlin and New York, 1977.

16. J. W. Milnor, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, N.J., 1965. MR **32** #8352.

17. G. J. Porter, *Higher products*, Trans. Amer. Math. Soc. **148** (1970), 315–345. MR **41** #1053.

18. J. D. Stasheff, *Homotopy associativity of H-spaces*. I, II, Trans. Amer. Math. Soc. **108** (1963), 275–312. MR **28** #1623.

19. R. E. Stong, *Notes on cobordism theory*, Princeton Univ. Press, Princeton, N.J.; Univ. Tokyo Press, Tokyo, 1968. MR **40** #2108.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

Current address: Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B9