CONDITIONALLY COMPACT SEMITOPOLOGICAL ONE-PARAMETER INVERSE SEMIGROUPS OF PARTIAL ISOMETRIES

M. O. BERTMAN

Abstract. The algebraic structure of one-parameter inverse semigroups has been completely described. Furthermore, if $B$ is the bicyclic semigroup and if $B$ is contained in any semitopological semigroup, the relative topology on $B$ is discrete. We show that if $F$ is an inverse semigroup generated by an element and its inverse, and $F$ is contained in a compact semitopological semigroup, then the relative topology is discrete; in fact, if $F$ is any one-parameter inverse semigroup contained in a compact semitopological semigroup, then the multiplication on $F$ is jointly continuous if and only if the inversion is continuous on $F$, and we describe $F$ in that case. We also show that if $(J_t)$ is a one-parameter semigroup of bounded linear operators on a (separable) Hilbert space, then $(J_t) \cup (J_t^*)$ generates a one-parameter inverse semigroup $T$ with $J_t^{-1} = J_t^*$ if and only if $(J_t)$ is a one-parameter semigroup of partial isometries, and we describe the weak operator closure of $T$ in that case.

1. Introduction. An inverse semigroup is a semigroup in which each element $x$ has a unique inverse $x^{-1}$ with the properties that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If $G$ is any subgroup of the positive real numbers under ordinary multiplication, let $P = G \cap [1, \infty)$. Let $F_P = \{(x, y, z) \in P^3: y \geq x$ and $y \geq z\}$ together with the following multiplication:

$$(x, y, z)(r, s, t) = \left( \frac{xyz}{y \wedge zr}, \frac{yzs}{(y \wedge zr)(zr \wedge s)}, \frac{zst}{zr \wedge s} \right)$$

where $x \wedge y = \min(x, y)$. Any inverse semigroup generated by a homomorphic image of $P$ is a homomorphic image of $F_P$ [6] and is called a one-parameter inverse semigroup. We mention at this point one homomorphic image of $F_P$. Let $B_P = P \times P$ together with this multiplication:

$$(x, y)(z, w) = \left( \frac{xz}{y \wedge z}, \frac{yw}{y \wedge z} \right).$$

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If $P = \{1, x, x^2, \ldots \}$ where $x > 1$, then $B_P$ is the *bicyclic semigroup* whose properties are discussed in [4].

Suppose that $Y$ is a semigroup with a Hausdorff topology in which the maps $x \mapsto xy$ and $x \mapsto yx$ of $Y \rightarrow Y$ are continuous for each $y$ in $Y$. Then the multiplication on $Y$ is said to be *separately continuous* and $Y$ is called a *semitopological semigroup*. If the map $(x, y) \mapsto xy$ of $Y \times Y \rightarrow Y$ is continuous, then the multiplication on $Y$ is *jointly continuous* (or just *continuous*) and $Y$ is called a *topological semigroup*.

Let $X$ be a one-parameter inverse semigroup generated by the image of $P$ under a continuous homomorphism $f: P \rightarrow X$. Let $j: P \rightarrow F_P$ be the map $x \mapsto (x, x, x)$. Then $j$ is an isomorphism of $P$ into $F_P$ and we can identify $P$ with its image in $F_P$. Then $x^{-1} = (1, x, 1)$ and hence $(x, y, z) = xy^{-1}z$. Furthermore, if $\tilde{f}: F_P \rightarrow X$ is defined by

$$\tilde{f}(xy^{-1}z) = f(x)f(y)^{-1}f(z),$$

then $\tilde{f}$ is a homomorphism and the following diagram commutes [6], [1]:

$$\begin{array}{ccc}
F_P & \xrightarrow{\tilde{f}} & X \\
\downarrow{j} & & \\
P & \xrightarrow{f} & X
\end{array}$$

If $F_P$ is endowed with the relative product topology which it inherits from $R^3$, then the multiplication on $F_P$ is jointly continuous. We shall show that if $X$ is embedded densely in a compact semitopological semigroup and that if the inversion on $f(P) \cup f(P)^{-1}$ is continuous, then in fact $\tilde{f}$ is a *continuous* homomorphism of $F_P$ onto $X$.

One-parameter inverse semigroups occur naturally in $B(H)$, the semigroup of bounded linear operators on Hilbert space. In the weak operator topology, the closed unit ball is a compact semitopological semigroup, and the map $T \rightarrow T^*$ is continuous where $T^*$ is the adjoint of the operator $T$. A partial isometry is an operator which is isometric on the orthogonal complement of its kernel; it is known [8, p. 99] that $T$ is a partial isometry if and only if $TT^*T = T$. We will show that if $J = \{J_t: t \in [0, \infty)\}$ is a one-parameter semigroup of partial isometries, then $J \cup J^*$ generates a one-parameter inverse semigroup.

This example shows that it is not unreasonable to ask that the map $x \mapsto x^{-1}$ be continuous on $X$, and in that case we are able to give a description of $X$. 
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1.1. Remark. The closure of an inverse semigroup in a compact topological semigroup is an inverse semigroup, but Brown and Moran [3, Proposition 1] give an example of a unitary group whose weak closure is not an inverse semigroup.

2. The one-parameter inverse semigroups. In this section we summarize the possible congruence relations on $F_P$ (and hence its homomorphic images), as given in [6, 3.10]. We then point out that every one-parameter semigroup $J$ of partial isometries generates a one-parameter inverse semigroup $X$, and that each of the possibilities for $X$ can be realized as an inverse semigroup generated by a partial isometry.

If $1 < t \in P$, let $I_t = \{xy^{-1}z : y > t\} \subseteq F_P$ and $I_t^0 = \{xy^{-1}z \in F_P ; y > t\}$. $I_t$ and $I_t^0$ are the only ideals of $F_P$. Let $\alpha$ and $\beta$ be the congruences on $F_P$ defined so that $xy^{-1}z \alpha rs^{-1}t$ if and only if $x = r$ and $yt = sz$ and $xy^{-1}z \beta rs^{-1}t$ if and only if $z = t$ and $yr = xs$. Then $F_P/\alpha \approx B_P \approx F_P/\beta$. For each subgroup $N$ of $G$, define $\sigma_N$ to be the group congruence on $F_P$ defined so that $xy^{-1}z \sigma_N rs^{-1}t$ if and only if $xzsrty \in N$. Then if $I$ is an ideal of $F_P$ and $\delta = \alpha, \beta,$ or $\sigma_N$, define the relation $I\delta$ by $xy^{-1}z I\delta rs^{-1}t$ if and only if either

(a) $xy^{-1}z$ and $rs^{-1}t$ are in $I$ and $xy^{-1}z \delta rs^{-1}t$ or
(b) $xy^{-1}z = rs^{-1}t \notin I$.

2.1. Proposition (Eberhart and Selden). Every congruence on $F_P$ is $I\delta$ for some ideal $I$ and congruence $\delta = \alpha, \beta,$ or $\sigma_N$. Hence the only possible homomorphic images of $F_P$ are $(F_P \setminus I) \cup B_P$, $(F_P \setminus I) \cup (G/N)$ or $F_P$ [6, Theorem 3.10].

2.2. Proposition. Let $T$ be a semigroup with an involution $*$, i.e., a mapping $u \rightarrow u^*$ such that $(uv)^* = v^*u^*$ and $(u^*)^* = u$, and let $R = \{R_t : t \in P\}$ be a one-parameter subsemigroup of $T$. Then $\langle R, R^* \rangle$, the semigroup generated by $R \cup R^*$, is an inverse semigroup if and only if

(i) $R_tR_t^*R_t = R_t$ for each $t$ in $P$ and
(ii) $R_sR_t^*R_tR_s = R_sR_t^*R_tR_s$ for each $s$ and $t$ in $P$.

Proof. The conditions are clearly necessary. Now if $R$ is as above, and $s > t$, we have

$R_sR_t^* = R_{s-t}R_tR_t^* = R_{s-t}R_t^*R_{s-t} = R_sR_t^*R_{s-t} = R_sR_t^*R_{s-t}$.

Similarly, $R_t^*R_s = R_{s-t}R_t^*R_s$, and it follows that any "word" from $\langle R, R^* \rangle$ can be expressed as a triple $R_xR_yR_z$ where $y > x$ and $y > z$. The arguments
of [6, pp. 56–57] can be used to show that the map \( h: F_p \to \langle R, R^* \rangle \) defined by \( h(xy^{-1}z) = R_xR_yR_z \) is a homomorphism, and since \( F_p \) is an inverse semigroup, so is its image \( \langle R, R^* \rangle \).

Now let \( J = \{ J_t \}_{t=0}^\infty \) be a one-parameter semigroup of partial isometries on a (separable) Hilbert space. By [7, Theorem B],

\[
J_t = U_t \oplus K_t \oplus S_t \oplus T_t,
\]

where \( U_t \) is unitary, \( K_t \) is purely isometric, \( S_t \) is purely coisometric, and \( T_t \) is a nilpotent and hence, by [12], the direct integral of truncated shifts. Recall that a truncated shift is an operator which is unitarily equivalent to \( R_t \), where for \( f \) in \( L^2(K, a) \) (if \( K \) is a separable Hilbert space and \( a > 0 \), \( L^2(K, a) \) is the Hilbert space of measurable \( K \)-valued functions on \([0, a]\) with square-integrable \( K \)-norm):

\[
R_t f(x) = \begin{cases} 0 & \text{if } t < x < a - t \\ f(x - t) & \text{if } t < x \leq a \\ f(x + t) & \text{if } 0 < x < a - t \\ 0 & \text{if } x \geq a \\ f(x) & \text{if } x = 0 \\ 0 & \text{if } x < 0 
\end{cases}
\]

and \( R_t = 0 \) if \( t > a \). This \( a \) is called the index of \( R_t \). Since

\[
R_t^* f(x) = \begin{cases} 0 & \text{if } a - t < x \leq a \\ f(x) & \text{if } 0 < x < a - t \\ 0 & \text{if } x \geq a \\ f(x) & \text{if } x = 0 \\ 0 & \text{if } x < 0 
\end{cases}
\]

and \( R_t^* = 0 \) if \( t > a \), we have the following.

2.3. Proposition. If \( R = (R_t) \) is a one-parameter semigroup of truncated shifts of index \( a \), then \( \langle R, R^* \rangle \) is isomorphic to \( F_p / I_a \).

Proof. Since, for \( x, y, \) and \( z \) in \( P \), with \( y < a \),

\[
R_x R_y R_z f(t) = \begin{cases} 0 & \text{if } t < x \text{ or } t > a - y + x \\ f(t - x + y - z) & \text{otherwise,} 
\end{cases}
\]

it is easy to see that \( R \) satisfies the conditions of 2.2 and hence \( \langle R, R^* \rangle \) is an inverse semigroup. Furthermore, one sees that \( R_x R_y^* R_z = 0 \) if and only if \( y > a \) and hence by 2.1, \( \langle R, R^* \rangle \approx F_p / I_a \).

It now follows that if \( T = \{ T_t \} \) is a one-parameter semigroup of nilpotent partial isometries of index \( a \) (and hence a direct integral of truncated shifts of index less than or equal to \( a \)), then \( \langle T, T^* \rangle \) is isomorphic to \( (F_p \setminus I_a) \cup \{ 0 \} \).

Proposition 2.2 also gives the following.

2.4. Proposition. If \( J = \{ J_t \} = \{ U_t \oplus K_t \oplus S_t \oplus T_t \} \) is a one-parameter semigroup of partial isometries, then \( \langle J, J^* \rangle \) is a one-parameter inverse semigroup. Furthermore, if \( \{ U_t \} \neq I \) is a semigroup of unitary operators, \( \{ K_t \} \) of nonunitary isometries, \( \{ S_t \} \) of nonunitary coisometries, and \( \{ T_t \} \) of nilpotents of index \( a \), then

1. \( \langle \{ U_t \oplus S_t \}, \{ U_t^* \oplus S_t^* \} \rangle \approx (F_p \setminus I_a) \cup R^+; \)
2. \( \langle \{ K_t \oplus T_t \}, \{ K_t^* \oplus T_t^* \} \rangle \approx (F_p \setminus I_a) \cup B_p; \)
3. Conditionally compact one-parameter inverse semigroups. We now assume that $X$ is a one-parameter inverse semigroup densely embedded in a compact semitopological semigroup $Y$. In case $P = \{1, x, x^2, \ldots \}$ where $x > 1$, we show that $X$ has the discrete topology as a subspace of $Y$. Furthermore, if $P$ is dense in $[1, \infty)$, and the inversion $x \mapsto x^{-1}$ on $P \cup P^{-1}$ is continuous, then the multiplication is continuous on $X$. We then assume, from 3.2.7 on, that inversion is continuous on $X$, and describe $X$.

Since semigroups of the type $B_P$ are somewhat more accessible than those of the more general type, and some information is known about them in case $P = \{1, x, x^2, \ldots \}$ for $x > 1$, we devote a short section to these semigroups.

3.1. Bicyclic Semigroups. As we have mentioned before, if $P = \{1, x, x^2, \ldots \}$ where $x > 1$, then $B_P$ is the familiar bicyclic semigroup, $B$, so-called because it is generated by two mutually inverse elements $p$ and $q$ subject to the generating relation $pq = 1$; we summarize the results of [2] on this subject:

(a) $B$ is a discrete subspace of $\overline{B}$ in the relative topology. (Originally proved in [5], as was (b).)

(b) $\overline{B} \setminus B$ is a closed two-sided ideal of $\overline{B}$.

(c) The minimal idempotent $e$ of the monothetic semigroup $\Gamma = \{q, q^2, \ldots \}$ is the minimal idempotent of $\overline{B}$, and the minimal ideal $K$ of $\Gamma$, which is a compact topological group, with identity $e$, in the relative topology from $\overline{B}$, is the minimal ideal of $\overline{B}$. Furthermore $K = e\overline{B} = \overline{B}e$.

(d) The decreasing sequence $q^n p^n$ of idempotents converges to an idempotent $w$; $wB$ is a cyclic group with identity $w\overline{B} = wB = B \setminus B$.

(e) If the map $x \mapsto x^*$ is a continuous involution, then $e$ and $w$ are selfadjoint, and for each $x \in \overline{B}$, $(ex)^* = ex^* = (ex)^{-1}$ in $K$.

(f) If $B = \langle T, T^* \rangle$ where $T$ is an isometry, then $w$ is the Wold idempotent which decomposes $H$ into $H_1 \oplus H_2$ where $T|H_1$ is unitary and $T|H_2$ is a unilateral shift. The minimal idempotent $e$ decomposes $H$ into $H_3 \oplus H_4$, where $T|H_3$ is unitary, and $H_4$ is the closed linear span of the eigenvalues of $T$ (and $T^*$) corresponding to its unimodular eigenvalues.

These results show that we have no more (nor less) information about the closure of $B$ than we have about the closure of a group, which examples in [3] and [12] show may be arbitrarily pathological.

If $P$ is a dense subsemigroup of $[1, \infty)$, the existence of the idempotent $w$ follows exactly as in [2], as does the fact that $wB_P$ is a group. That $B_P \setminus B_P = wB_P$ turns out to be equivalent to the continuity of the inversion on $B_P$ as we see for $F_P$ in 3.2.6, and from 3.2.7 on, we shall assume this condition.
3.2. $F_p$. Suppose $X = F_p$. Then $X$ may be pictured as below:

![Diagram](image)

The idempotents of $F_p$ are precisely those of the form $x(yx)^{-1}y$ where $x$ and $y$ are elements of $P$. [6, 2.13]. Now the idempotents $\{xx^{-1} : x \in P\}$ and $\{x^{-1}x : x \in P\}$ form two decreasing nets as $x \to \infty$, and hence there exist two idempotents $E$ and $F$ such that $xx^{-1} \to E$ and $x^{-1}x \to F$ as $x \to \infty$. Since $xx^{-1}$ is decreasing, $E \neq xx^{-1}$ for any $x$ in $P$, and similarly, $F \neq x^{-1}x$ for any $x$ in $P$. Furthermore, $xx^{-1}E = E = Exx^{-1}$ for every $x$ in $P$ and $x^{-1}xF = F = Fx^{-1}x$ for every $x$ in $P$.

**Proposition 3.2.1.** $E$ and $F$ commute with every element $xy^{-1}z$ of $F_p$, and hence with every element of $\overline{F_p}$.

**Proof.** As $x \to \infty$, $xx^{-1} \to E$, so if $r \in P$, $rxx^{-1} = r(r^{-1}r)(xx^{-1}) = r(xx^{-1})(r^{-1}r)$ (since idempotents commute) = $(rx)(rx)^{-1}r \to Er$; but $rxx^{-1} \to rE$, so $Er = rE$. On the other hand, for each $r$ in $P$, $xx^{-1}r^{-1} = xx^{-1}rr^{-1} = r^{-1}xx^{-1}r^{-1} = r^{-1}(rx)(xx^{-1}) \to r^{-1}E$; but $xx^{-1}r^{-1} \to Er^{-1}$, so $Er^{-1} = r^{-1}E$ for each $r$ in $P$. Hence if $xy^{-1}z \in F_p$, $Exy^{-1}z = xEy^{-1}z = xy^{-1}Ez = xy^{-1}zE$. Similarly, $F$ commutes with each element of $F_p$ and hence with each element of $\overline{F_p}$.

**Proposition 3.2.2.** If $\{z_a\}$ and $\{y_a\}$ are unbounded increasing nets from $P$, then $\{x_a^{-1}y_a^{-1}y_a^{-1}\}$ converges to $EF$.

**Proof.** Let $(x_{a_k}^{-1}y_{a_k}^{-1}y_{a_k}^{-1})$ be a subnet which converges, say to $Z$. Then $EFx_{a_k}^{-1}y_{a_k}^{-1}y_{a_k}^{-1} = Ex_{a_k}^{-1}Fy_{a_k}^{-1}y_{a_k}^{-1} = EF$, but this net converges to $EFZ$, so $EF = EFZ$. Now let $x \in P$, and assume $x_{a_k} > x$. Then $xx^{-1}x_{a_k}^{-1}y_{a_k}^{-1}y_{a_k}^{-1}y_{a_k}^{-1} \to Z$, so $xx^{-1}Z = Z$ for each $x$ in $P$. Similarly $Zx^{-1}x = Z$ for each $x$ in $P$, and
hence, letting $x$ approach $\infty$, $EZ = Z = ZF$, so $EFZ = EZF = EZ = Z$, so $EF = EFZ = Z$. It now follows that $x_a^{-1}y_a^{-1}y_a \to EF$.

**Proposition 3.2.3.** $EF_p \cap F_p = \emptyset$; $FF_p \cap F_p = \emptyset$ and $EFF_p \cap F_p = \emptyset$.

**Proof.** Suppose $E \in F_p$; then $E = xx^{-1}y^{-1}y$ for some $x$ and $y$ in $P$. But if $k > x$, $E = kk^{-1}E = kk^{-1}xx^{-1}y^{-1}y = kk^{-1}y^{-1}y \neq xx^{-1}y^{-1}y = E$. Thus $E$ is not in $F_p$, and a similar maneuver shows that $F$ and $EF$ are not in $F_p$. Now we show that $EF_p \cap F_p = \emptyset$. Suppose that there exists $x$ in $P$ such that $Ex^{-1} \in F_p$. Then $E = Exx^{-1} = xEx^{-1} \in F_p$. So there is no such $x$. Now if $xy^{-1}z$ is any element of $F_p$ such that $Exy^{-1}z \in F_p$, then

$$E(y/x)^{-1} = Exx^{-1}(y/x)^{-1} = Exy^{-1}$$

$$= xy^{-1}Exz^{-1} = (Exy^{-1})z^{-1} \in F_p,$$

and this contradiction proves that there is no such $xy^{-1}z$. The proof for $F$ and $EF$ is similar.

**Proposition 3.2.4.** If $(x_a y_a^{-1}z_a)$ is a net from $FP$ such that $(y_a)$ is unbounded, then if $u \in EFP \cup FF_p$.

**Proof.** Note that if $e$ is any idempotent and $x$ any element of $FP$, then $ex = x$ if and only if $x \in eFP$. Now if $(x_a)$ is unbounded, for any $k$ in $P$, $x_a$ is eventually larger than $k$, so $kk^{-1}x_a y_a^{-1}z_a = x_a y_a^{-1}z_a \to u$, so $kk^{-1}u = u$, and hence $Eu = u$. If $u$ were in $F_p$, $u$ would be in $EF_p$, so $u \notin F_p$. Similarly, if $(z_a)$ is unbounded, $u$ would be in $FP \setminus F_p$. If $(x_a)$ and $(z_a)$ are both bounded nets and $(y_a)$ is unbounded, then for every $x > \sup\{x_a\}$,

$$x^{-1}xx_a y_a^{-1}z_a = x^{-1}xx_a x_a^{-1}(y_a/x_a)^{-1}z_a$$

$$= x_a x_a^{-1}x^{-1}x(y_a/x_a)^{-1}z_a$$

$$= x_a x_a^{-1}(y_a/x_a)^{-1}z_a = x_a y_a^{-1}z_a \to u.$$

But $x^{-1}xx_a y_a^{-1}z_a \to x^{-1}xu$, so $x^{-1}xu = u$, and again $u \in FP \setminus F_p$.

**Proposition 3.2.5.** If $P = \{1, x, x^2, x^3, \ldots \}$, then $F_p$ has the discrete topology as a subspace of any compact semitopological semigroup.

**Proof.** Suppose $x_a y_a^{-1}z_a \to xy^{-1}z$. Then by 3.2.4, $(y_a)$, $(x_a)$ and $(z_a)$ are bounded nets in $P$. Hence we can find $x_0$, $y_0$, $z_0$ and subnets $x_{a_1}$, $y_{a_1}$, $z_{a_1}$ such that $x_{a_1} \to x_0$, $y_{a_1} \to y_0$ and $z_{a_1} \to z_0$. Thus eventually $x_{a_1} = x_0$, $y_{a_1} = y_0$, and $z_{a_1} = z_0$, and hence eventually $x_{a_1} y_{a_1}^{-1} z_{a_1} = x_0 y_0^{-1} z_0$. Thus since

$$x_{a_1} y_{a_1}^{-1} z_{a_1} \to xy^{-1}z, \quad x_0 y_0^{-1} z_0 = xy^{-1}z.$$
multiplication on $F_P$ and the inversion map on $F_P$ are continuous functions. Moreover, since every convergent net which is bounded in its second coordinate converges to a point of $F_P$, and every convergent net which is unbounded in its second coordinate converges to a point of $\overline{F_E P}$ or $\overline{F_F P}$, we see that $\overline{F_E P} \cup \overline{F_F P}$ is an ideal of $\overline{F_P}$ and $\overline{F_P} \setminus F_P = \overline{E_F P} \setminus \overline{F_F P}$. We next show that all these related conditions are equivalent.

**Proposition 3.2.6.** If $F_P$ is contained densely in a compact semitopological semigroup, and $P \subseteq F_P$ is a continuous embedding, then the following are equivalent:

(a) Inversion is continuous on $P \cup P^{-1}$;
(b) $P^{-1} \subseteq F_P$ is a continuous embedding;
(c) The topology on $F_P$ is the product topology;
that is, $x_a y_a^{-1} z_a \rightarrow x y z$ if and only if $x_a \rightarrow x, y_a \rightarrow y$, and $z_a \rightarrow z$;
(d) $\{y_a\}$ is a bounded net from $P$, and if $x_a y_a^{-1} z_a \rightarrow u, u \in F_P$;
(e) $\overline{F_P} \setminus F_P = \overline{EF_P} \cup \overline{FF_P}$.

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are easy. We show (a) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a), and (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (d).

(a) $\Rightarrow$ (c). Suppose inversion is continuous in $P \cup P^{-1}$. We note that if $x_a y_a^{-1} z_a \rightarrow u$ where $x_a \rightarrow x$ and $z_a \rightarrow z$, and if $x > 1$, and $1 < r < x$, then since $r < x_a$ eventually, $r r^{-1} x_a y_a^{-1} z_a = x_a y_a^{-1} z_a$ eventually, so $r r^{-1} u = u$. (Similarly $u t^{-1} t = u$ if $t < z$.) Now if $x_a \rightarrow x$, we show that $x_a x_a^{-1} \rightarrow xx^{-1}$.

We can assume that $x_a x_a^{-1} \rightarrow u$ for some $u \in \overline{F_P}$. If $r > x$, then eventually $r > x_a$, so $r^{-1} r x_a x_a^{-1} = r^{-1} r$, so $r^{-1} = r^{-1} u$; letting $r \rightarrow x$, $r^{-1} \rightarrow r^{-1}$ by assumption and thus $x^{-1} = x^{-1} u$, and $xx^{-1} = xx^{-1} u$. If $x_a > x$ frequently, then

$$xx^{-1} x_a x_a^{-1} = x_a x_a^{-1}$$

frequently, so $xx^{-1} u = u$. On the other hand, if $x_a < x$ eventually, then since $xx_a^{-1} \rightarrow xx^{-1}$ we have

$$xx_a^{-1} u = (x / x_a) x_a x_a^{-1} u = x / x_a u$$

by the above, so $xx^{-1} u = u$, so in either case, $u = xx^{-1} u = xx^{-1}$. Similarly $x^{-1} x_a \rightarrow x^{-1} x$. Now if $x_a \rightarrow x, y_a \rightarrow y, z_a \rightarrow z$, we can assume $x_a y_a^{-1} z_a \rightarrow u$.

Let $r > x$ and $t > z$; then

$$rr^{-1} x_a y_a^{-1} z_a t^{-1} t = r \left( x_a y_a^{-1} z_a \right)^{-1} t,$$

so

$$rr^{-1} u t^{-1} t = r \left( y / x z \right)^{-1} t = rr^{-1} (y t / x z) t.$$
Letting $r \to x$, we have $xx^{-1}ut^{-1}t = xx^{-1}(yt/xz)^{-1}z = x(y/z)^{-1}t = x(y/z)^{-1}t$, and letting $t \to z$,

$$u = xx^{-1}uz^{-1}z = x(y/z)^{-1}z^{-1}z = xy^{-1}z.$$

It follows that $x\alpha x^{-1}z \to xy^{-1}z$ if and only if $x \to x, y \to y, and z \to z$.

(d) $\Rightarrow$ (a). Let $x \to x$ in $P$, and suppose $x^{-1} \to rs^{-1}t$ in $F_p$. We show $r < x$, and $t < x$. For suppose $r > x$. Then $r > x_\alpha$ eventually, so since

$$rr^{-1}\left(\frac{r}{x_\alpha}\right) = rx_\alpha^{-1} \to rs^{-1}t = r^2\left(\frac{r^2s}{r^2 + s}\right)^{r^2 s}$$

we have

$$rr^{-1}/x = r^2\left(\frac{r^2s}{r^2 + s}\right)^{-1}$$

and hence $r = 1$, and this contradiction shows that $r < x$; similarly $t < x$.

Now suppose that $s < x$. Then since $s < x_\alpha$ eventually, and $x_\alpha^{-1} = x_\alpha^{-1}ss^{-1} \to rs^{-1}ss^{-1} = rt(st)^{-1}t$, we have $t = 1$. Similarly, we can show $r = 1$. Thus $x_\alpha^{-1} \to s^{-1}$ where $s < x$. Now if $x_\alpha < x$ frequently, the fact that

$$(x/x)x^{-1} = x_\alpha^{-1}xx^{-1} \to s^{-1}xx^{-1} = (x/s)x^{-1}$$

implies that $x = s$. If $x_\alpha > x$ frequently, then

$$s^{-1} = \lim_{\alpha} x_\alpha^{-1} = \lim_{\alpha} (x_\alpha^{-1}xx^{-1}) = s^{-1}xx^{-1} = (x/s)x^{-1},$$

so $x = s$. Hence if $x_\alpha^{-1} \to rs^{-1}t$, $r < x$, $t < x$, and $s > x$. Suppose $s > x$.

Then $s > x_\alpha$ eventually, so since $(s/x_\alpha)s^{-1} = x_\alpha^{-1}ss^{-1} \to rs^{-1}ss^{-1} = rt(st)^{-1}t, t = 1$, and $s/x = r$. Similarly, $r = 1$, so $s = x$. Therefore, if $x_\alpha \to x$, then $x_\alpha^{-1} \to x^{-1}$.

(d) $\Rightarrow$ (e). Let $u \in F_p \setminus F_p$. Then $u = \lim_{\alpha} x_\alpha y_\alpha^{-1}z_\alpha$, where $\{y_\alpha\}$ is unbounded. As in 3.2.4, either $Eu = u$ or $Fu = u$. Hence $u \in \overline{F_p} \cup \overline{F_p}$. If $v \in \overline{F_p} \cup \overline{F_p}$, then $Ev = v$ or $Fv = v$, so by 3.2.3, $v \in \overline{F_p} \setminus F_p$.

(e) $\Rightarrow$ (f). Suppose $F_p \setminus F_p = \overline{F_p} \cup \overline{F_p}$. If $u \in \overline{F_p} \setminus F_p$, then either $Eu = u$ or $Fu = u$, so if $v \in \overline{F_p}$, $uv = Ev \in F_p$, or else $uv = Fuw \in \overline{F_p}$, so $uv \in \overline{F_p} \setminus F_p$.

(f) $\Rightarrow$ (d). We first show that if $\{y_\alpha\}$ is bounded and $y_\alpha^{-1} \to u$, then $u \in F_p$.

Let $r > y_\alpha$ for each $\alpha$. We can assume $y_\alpha$ converges, say to $y$ in $P$, and hence $r^{-1}ru = r^{-1}r/y$, and since $r^{-1}ru \in F_p, u \in F_p$. Now let $x_\alpha y_\alpha^{-1}z_\alpha \to v$. Let $r > x_\alpha$ and $t > z_\alpha$ for each $\alpha$. Then

$$rr^{-1}x_\alpha y_\alpha^{-1}z_\alpha^{-1}t^{-1} = r\left(\frac{y_\alpha t}{x_\alpha z_\alpha}\right)^{-1}t;$$

we can assume $r_\alpha t^{-1}/x_\alpha z_\alpha$ converges to $q$, hence $q \in F_p$; so $rr^{-1}vt^{-1}t = qrt \in F_p$ and hence $v \in F_p$. 
3.3. For the remainder of this paper, we will assume \( X \) is a homomorphic image of \( F_p \) embedded densely in a compact semitopological semigroup, and that the inversion on \( X \) is a continuous mapping.

**Proposition 3.3.1.** If \( X = F_p/\delta \) and if \( \delta^\delta \) is the natural map of \( F_p \) to \( X \), then, under the assumption that inversion is continuous on \( X \), \( \delta^\delta \) is a continuous homomorphism from \( F_p \) with the product topology onto \( X \).

**Proof.** It follows as in 3.2.6 that \( \delta^\delta \) is continuous on \( F_p \setminus I \) where \( \delta = I_a \).
Now since \( \delta^\delta \) must be continuous on \( P \), we know \( I = I_a \) for some \( a > 1 \) and if \( \sigma = \sigma_N \) for some subgroup \( N \) of \( G \), \( N \) must be a closed subgroup of \( G \). We can assume \( P \) is dense in \([1, \infty)\). We prove the proposition for \( F_p/I_a \sigma_N \) and point out that the proofs for the remaining cases are similar.

Suppose \( X = F_p/I_a \sigma_N \). Then we note that \( X \approx (F_p/I_a) \cup B_p \), where the multiplication is as follows: If both \( u \) and \( v \) are in \( F_p \) then \( uv \) is as usual. If either \( u \) or \( v \) is in \( B_p \), then \( uv = \alpha^\delta(u)\alpha^\delta(v) \). For the sake of simplifying the notation, we equate \( xy^{-1}z \) with \((x, y/z)\) if \( y > a \).

Let \( x_a \rightarrow a \) from \([1, a)\), and suppose \( x_a x_a^{-1} \rightarrow e \). Since for each \( r < a \), \( x_a x_a^{-1} rr^{-1} = x_a x_a^{-1} \) eventually, \( err^{-1} = e \). Now \( x_a x_a^{-1}(a, a) = (a, a) \), so \( e(a, a) = (a, a) \). But \( e(a/x_a)^{-1} \rightarrow e \) as \( x_a \rightarrow a \), and

\[
e(a/x_a)^{-1} = e x_a x_a^{-1}(a/x_a)^{-1} = e(x_a, a) \rightarrow e(a, a) = (a, a),
\]

so \( e = (a, a) \). Similarly, if \( x_a^{-1} x_a \rightarrow f, f(1, 1) = (1, 1) \), so since

\[
f(a/x_a) = f x_a^{-1} x_a(a/x_a) = f x_a^{-1}(a, 1) = f(a/x_a, 1) \rightarrow f(1, 1)
\]

and \( f(a/x_a) \rightarrow f \), we have \( f = f(1, 1) = (1, 1) \). Now suppose that \( x_a \rightarrow x \), and \( y_a \rightarrow b > a \); we assume \( x_a y_a^{-1} \rightarrow u \). Then \( y_a > a \) eventually, so \( x_a y_a^{-1} = (x_a, y_a) \) eventually, and \( x_a y_a^{-1} \rightarrow (x, b) \). If \( y_a \rightarrow a \), we can assume \( y_a < a \) eventually. As above, if \( r < a \), \( x_a y_a^{-1} = x_a y_a^{-1} rr^{-1} \rightarrow ur^{-1} \) so \( u = ur^{-1} \) and hence, letting \( r \rightarrow a \), \( u = u(a, a) \). But \( x_a y_a^{-1}(a, a) = (x_a/a, y_a) \rightarrow (x, a) \) so \( u = (x, a) \). If \( z_a \rightarrow z \), and \( y_a \rightarrow b > a \), then since \( y_a^{-1} z_a = (1, y_a/z_a) \) eventually, \( y_a^{-1} z_a \rightarrow (1, b/z) \). If \( y_a \rightarrow a \), assume \( y_a^{-1} z_a \rightarrow u \), and \( y_a < a \) eventually.

Then if \( r < a, r^{-1} u = u \), and, letting \( r \rightarrow a, (1, 1) u = u \) by the above; but \( (1, 1)(y_a^{-1} z_a) = (1, y_a/z_a) \rightarrow (1, a/z) \) so \( u = (1, a/z) \). Now we show in general that if \( x_a \rightarrow x \), \( y_a \rightarrow y \), and \( z_a \rightarrow z \), then \( x_a y_a^{-1} z_a \rightarrow xy^{-1}z \). We can assume \( x_a y_a^{-1} z_a \rightarrow u \). If \( y > a \), then \( x_a y_a^{-1} z_a = (x_a, y_a/z_a) \) eventually and we are done. If \( y = a \) and \( x > 1 \), suppose \( x < a \), and let \( x < r < a \). Then eventually \( rr^{-1} x_a y_a^{-1} z_a = r(y_a/x_a) z_a \rightarrow (r, xy/zz) \) so \( r^{-1} u = (r, ry/zz) \), and, letting \( r \rightarrow x, xx^{-1} u = (x, y/z) \) but as before \( xx^{-1} u = u \), so \( u = (x, y/z) \). Finally, if \( x = a \), then \( u = (a, a) u = (a, a/z) \). Thus if \( \delta^\delta(x_a) \rightarrow \delta^\delta(x), \delta^\delta(y_a) \rightarrow \delta^\delta(y), \) and \( \delta^\delta(z_a) \rightarrow \delta^\delta(z) \), then \( \delta^\delta(x_a y_a^{-1} z_a) \rightarrow \delta^\delta(xy^{-1}z) \), and it follows that \( \delta^\delta \) is a continuous homomorphism of \( F_p \) onto \( X \). The proof is exactly analogous if \( X = F_p/I_a \sigma_N \).
Now recall that for each $X$, the decreasing nets $\{\delta^q(x)\delta^q(x)^{-1}: x \in P\}$ and $\{\delta^q(x)^{-1}\delta^q(x): x \in P\}$ converge to idempotents $E$ and $F$ respectively as $x \to \infty$. Furthermore, in each case, $E$ and $F$ commute with $X$, and 3.3.1 allows us to conclude that in fact, $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$. We point out that if $X = F_p/I_\alpha{\sigma}_\alpha$, $E = F = \delta^q(a^{-1})$ is the identity element of the group which is the minimal ideal of $X$. If $X = F_p/I_\alpha{\sigma}_\alpha$, $F = (1, 1) = \delta^q(a^{-1})$ but $E \notin X$. In this case, $\overline{X} \setminus X = \overline{EX}$. Similarly, if $X = F_p/I_\alpha{\beta}$, then $\overline{X} \setminus X = \overline{FX}$.

**Proposition 3.3.3.** If $X$ is any homomorphic image of $F_p$, then $EX \approx B_p$ or $EX$ is a commutative group; $EX$ is a group if and only if $E = EF$, and these remarks are true if $E$ and $F$ are interchanged. Furthermore, $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$.

Proof. Since $E$ commutes with each element of $X$, the map $u \to Eu$ is a homomorphism of $X$ to $EX$. Hence $EX = F_p/I_\alpha{\sigma}_\alpha$ for some $b \in P$ and congruence $\sigma$ on $FP$. Since $Exx^{-1} = E$ for each $x$ in $(I_\alpha{\sigma}_\alpha)(P)$, $b = 1$, and hence $EX = F_p/\beta$ or $F_p/\sigma_\alpha$. In the first case, $EX \approx B_p$ with $E$ as identity and

$$E\delta^q(xy^{-1}z) = E(y/x)^{-1}z \approx (y/x, z),$$

and the idempotents $Ex^{-1}x$ converge to $EF$ as $x \to \infty$. It is easy to see that $EFX$ is a commutative group in this case. In the second case, $EX$ is a group and since $Ex^{-1}x = E$, $E = EF$. The discussion in the previous paragraph together with 3.2.6 implies that $\overline{X} = X \cup \overline{EX} \cup \overline{FX}$.

The idempotents $E$ and $F$ are thus maximal among idempotents of $\overline{X} \setminus X$, if $\overline{X} \setminus X \neq \emptyset$. We now turn to a discussion of minimal idempotents of $\overline{X}$. The one-parameter semigroups $\delta^q(P)$ and $\delta^q(P^{-1})$ possess as kernels (minimal ideals) commutative compact topological groups $K$ and $K'$, respectively, with identities $e_0$ and $f_0$. If $X = B_p$, it follows as in [2] that $e_0 = f_0$ and hence $K = K'$ is the minimal ideal for $\overline{X}$; furthermore $K \subseteq EX$ and $K = \overline{EX}$ if and only if $e_0 = E = w$. These facts are easy to prove if $X$ is any of the other images of $F_p$; we discuss the case where $X = F_p$.

If $\{x_a\}$ is a net from $P$ which converges to $e_0$, by 3.2.6, $\{x_a\}$ is unbounded. If $x \in P$, $x < x_a$ frequently, so $xx^{-1}x_a = x_a$ frequently and hence $xx^{-1}e_0 = e_0 = Ee_0 = e_0E$. Similarly $e_0F = e_0 = Fe_0$. Thus $EFx_0 = e_0$ so $e_0 \in \overline{EFX}$, and hence $e_0$ commutes with each element of $\overline{X}$. Now if $y \in P$, $x_0y^{-1} \to e_0y^{-1}$. But since $x_0 > y$ frequently, $x_0y^{-1} = x_0x_0^{-1}x_0/y$, and $e_0x_0^{-1}y \to e_0y^{-1}$. Thus since $e_0x_0^{-1} = e_0x_0^{-1}x_0/y = e_0x_0/y$, $e_0y^{-1}$ is in $\overline{K} = K$. Similarly $y^{-1}e_0$ is in $\overline{K}$. Thus $e_0X = Xe_0 \subseteq K$, and $e_0\overline{X} = \overline{Xe_0} \subseteq K$. Now if $e$ is any idempotent of $\overline{X}$, $(e_0e)(e_0e) = e_0e_0ee = e_0e$, and since $e_0e \in K$, $e_0e = e$. Hence $e_0$ is the minimal idempotent for $\overline{X}$. It follows that $e_0 = f_0$ and as in [2, Proposition 2] that $K$ is the kernel for $\overline{X}$ and that $K = e_0\overline{X}$. We summarize this discussion in the following proposition.
Proposition 3.3.4. If \( X = F_P/\delta \), then the kernel of \( \overline{X} \) is \( K \), the compact topological group which is the kernel of \( \delta^\ast(P) \). If \( e_0 \) is the identity element of \( K \), then \( K = e_0\overline{X} \).

We now derive some corollaries and give some examples from the space of bounded linear operators on complex Hilbert space. This one is proved as in [2, Proposition 5].

Corollary 3.3.5. If \( \overline{X} \) is a compact semitopological semigroup with a continuous involution \( t \to t^* \), all idempotents of \( X \), \( EX \) and \( FX \) are selfadjoint, as are \( EF \) and \( e_0 \). If \( x \in K \), then \( x^* \in K \), and furthermore, \( x^*x = e = xx^* \) for every \( x \) in \( K \).

The following corollary was proved in [1, Theorem 4].

Corollary 3.3.6. If \( \overline{X} \) is a compact topological semigroup, then since \( EX \neq B_P \neq FX \), \( E = F \) and hence \( \overline{X} = X \cup \overline{EX} \).

Example 3.3.7. If \( U \) is unitary, and \( T \) and \( S \) are unilateral shifts, let \( J = U \oplus T \oplus S^* \). Then \( J^n = U^n \oplus T^n \oplus (S^*)^n \); let \( X = \langle J, J^* \rangle \). We have \( E = I_1 \oplus O_2 \oplus I_3 \), \( F = I_1 \oplus I_2 \oplus O_3 \), and \( EF = I_1 \oplus O_2 \oplus O_3 \). \( EX \) is the bicyclic semigroup generated by \( U \oplus O_2 \oplus S^* = p \) and \( U^* \oplus O_2 \oplus S = q \) with \( 1 = pq = E \), and \( FX \) is the bicyclic semigroup \( \langle U \oplus T \oplus O_3, U^* \oplus T^* \oplus O_3 \rangle \). It is easy to see that the structure of \( EFX \) is totally dependent on \( U \).

Remark 3.3.9. If \( \{J_t\} = J \) is a one-parameter semigroup of partial isometries on a finite-dimensional complex space, then \( J_t = U_t \oplus T_t \), where \( U_t \) is unitary and \( T_t \) is a truncated shift.

Proof. The uniform and weak topologies coincide, so \( \overline{X} = \langle J \cup J^* \rangle \) is a compact topological semigroup. If \( J_t = U_t \oplus S_t \oplus C_t^* \oplus T_t \) where either \( S_t \) or \( C_t \) is a nonunitary isometry, then either \( EX \) or \( FX \) would be bicyclic. Since this cannot happen in a compact topological semigroup, \( S_t \) and \( C_t^* \) are both unitary.

References


DEPARTMENT OF MATHEMATICS, CLARKSON COLLEGE, POTSDAM, NEW YORK 13676