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$(E^3/X) \times E^1 \approx E^4$ (X, A CELL-LIKE SET): AN ALTERNATIVE PROOF[1]

BY

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ABSTRACT. The author gives an alternative proof that a cell-like closed-0-dimensional decomposition of E^3 is an E^4 factor. The argument is essentially 2-dimensional. The 3- and 4-dimensional topology employed is truly minimal.

1. Introduction. W. T. Eaton and C. Pixley [2] and R. D. Edwards and R. T. Miller [3] have given nice proofs of the theorem of the title. We venture an alternative proof based on a simple idea for shrinking a decomposition (§2), on the neighborhood lemma of D. R. McMillan [4] polished à la Eaton-Pixley (§3), and on an essentially 2-dimensional argument (§§4, 5, and 6). The 3- and 4-dimensional topology employed is truly minimal. [1], [2], or [3] supply further motivation and background material.

Setting. Let X denote a cell-like set in E^3. Let G denote the decomposition of $E^4 = E^3 \times E^1$ having as its nondegenerate elements the sets $X \times \{t\}$, $t \in E^1$. Let $\pi\colon E^4 \to E^4/G$ denote the projection.

THEOREM. *The spaces E^4 and E^4/G are homeomorphic.*

PROOF. Suppose given disjoint compact PL 3-manifolds D and E in E^4 and a neighborhood N of the saturation relation $\pi^{-1}\pi\colon E^4 \to E^4$. (See [1] for a discussion of relations and their neighborhoods.) By the Shrinking Lemma of §2, the theorem follows provided we can prove the existence of a homeomorphism $h\colon E^4 \to E^4$ in N such that $\pi hD \cap \pi hE = \varnothing$.

There exist an open set U in E^3 and points $a_0 < a_1 < \cdots < a_n$ in E^1 such that $X \subset U$, $D \cup E \subset E^3 \times (a_0, a_n)$, and such that any homeomorphism $h\colon E^3 \times E^1 \to E^3 \times E^1$, fixed outside $U \times (a_0, a_n)$ and changing no E^1 coordinate by as much as $2 \cdot \max_i(a_i - a_{i-1})$, lies in N.

It is well known (and a simple consequence of the Neighborhood Lemma of §3), that each $X \times \{t\}$ ($t \in E^1$) is PL cellular in $E^3 \times E^1$. Thus there exist

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[2]These results were obtained while the author was visiting at Utah State University.

disjoint PL 4-balls $B_1, \ldots, B_{n-1}$ such that $X \times \{a_i\} \subset \text{Int } B_i \subset B_i \subset U \times (a_{i-1}, a_{i+1})$ $(i = 1, \ldots, n-1)$. Thus, by a PL homeomorphism of E^4 fixed outside of $B_1 \cup \cdots \cup B_n$, it is possible to adjust D and E so that they miss $X \times \{a_1\}, \ldots, X \times \{a_{n-1}\}$.

The proof will clearly be complete once we show, for each of the intervals (a_{i-1}, a_i), the existence of a PL homeomorphism h_i: $E^3 \times E^1 \to E^3 \times E^1$, having compact support in $U \times (a_{i-1}, a_i)$, such that no element $X \times \{t\}$ with $t \in (a_{i-1}, a_i)$ hits both h_iD and h_iE. This homeomorphism will be constructed in §6. We recommend that the reader turn immediately to §6 and refer to the other sections as needed for terminology and lemmas.

2. Shrinking monotone decompositions of manifolds.

SHRINKING LEMMA. *Let M denote a Cat n-manifold* (*Cat* = DIFF, TOP or PL), *and let π: $M \to M/G$ denote the projection map of a monotone upper semicontinuous decomposition G of M. Then M and M/G are homeomorphic provided the following is satisfied*:

(*) *Given disjoint $(n-1)$-dimensional compact CAT-submanifolds D and E of M, each neighborhood N in $M \times M$ of the saturation relation $\pi^{-1}\pi$: $M \to M$ contains a CAT homeomorphism h: $M \to M$ such that $\pi hD \cap \pi hE = \emptyset$.*

PROOF. We treat only the case of compact M. The noncompact case follows from exactly the same argument applied to the one-point compactification M^+ of M, all homeomorphisms of M^+ chosen to fix the point at infinity.

Suppose a positive number ε and a neighborhood N of $\pi^{-1}\pi$ in $M \times M$ given. By Bing's Shrinking Criterion [**1**, Appendix I], it suffices to show the existence of a homeomorphism h: $M \to M$ in N such that each element $g \in G$ has image $h(g)$ under h of diameter less than ε.

Let $(D_1, E_1), \ldots, (D_k, E_k)$ denote finitely many pairs of $(n-1)$-dimensional compact CAT-submanifolds of M, D_i and E_i disjoint for each i, such that any continuum in M having diameter at least ε intersects both D_i and E_i for some i. By [**1**, Theorem A12], there exist neighborhoods $N_1, \ldots, N_k$ of $\pi^{-1}\pi$ in $M \times M$ such that $N_1^{-1} \cdot \cdots \cdot N_k^{-1} \subset N$.

By (*), there is a CAT homeomorphism h_1: $M \to M$ in N_1 such that $\pi h_1 D_1 \cap \pi h_1 E_1 = \emptyset$. Cutting N_2 down in size if necessary we find from [**1**, Theorem A12] that we may assume that

$$(\pi \circ N_2 \circ h_1 D_1) \cap (\pi \circ N_2 \circ h_1 E_1) = \emptyset.$$

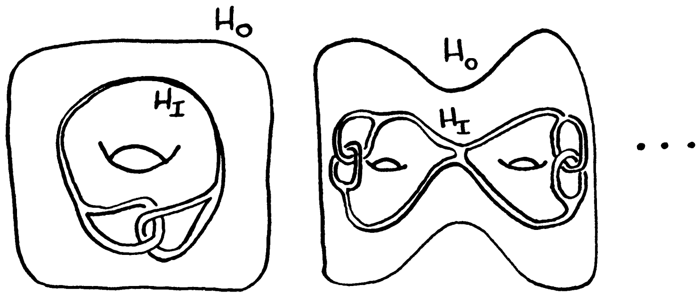
By (*), there is a CAT homeomorphism h_2: $M \to M$ in N_2 such that $\pi h_2 h_1 D_2 \cap \pi h_2 h_1 E_2 = \emptyset$. By the choice of N_2, $\pi h_2 h_1 D_1 \cap \pi h_2 h_1 E_1 = \emptyset$ as well. Proceeding inductively, we find CAT homeomorphisms $h_1, \ldots, h_k$ in $N_1, \ldots, N_k$, respectively, such that, for each i,

$$\pi h_k \circ \cdots \circ h_1 D_i \cap \pi h_k \circ \cdots \circ h_1 E_i = \emptyset.$$

Then $h = h_1^{-1} \circ \cdots \circ h_k^{-1}$: $M \to M$ is a homeomorphism satisfying the requirements of the second paragraph of this proof.

3. Neighborhoods of cell-like sets in E^3.

DEFINITION. A *split-handle pair* (H_O, H_I) consists of an outer handlebody H_O and an inner handlebody H_I, H_I contained on H_O in the simple fashion pictured in Figure 1.



One split handle Two split handles . . . etc.

FIGURE 1

Note the simple linking and the lack of knotting of H_I in H_O.

NEIGHBORHOOD LEMMA. *If $X \subset U \subset E^3$, X cell-like, U open, then there is a split-handle pair (H_O, H_I) and a* PL *embedding f: $H_O \to U$ such that* $X \subset \mathrm{Int}(fH_I)$. (*The pair (fH_O, fH_I) is called a split-handle neighborhood of X in U.*)

PROOF. By [4], there is a PL bouquet B of n loops in U (some $n > 1$) such that X lies in some regular neighborhood of B in U and such that B is contractible in U.

Let D_O be a PL wedge of n disks; by [5, Theorem 3] there is a PL map g: $D_O \to U$ which takes Bd D_O homeomorphically onto B and such that the only singularities of gD_O are disjoint simple arcs $A_1, A_2, \ldots, A_k$ where two sheets of $g(D_O)$ cross exactly in the manner indicated by Figure 2.

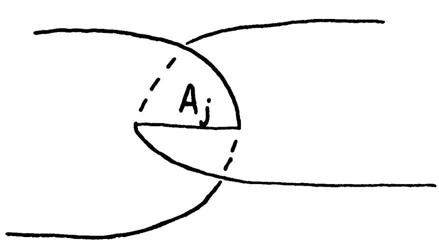


FIGURE 2

It is therefore clearly possible to remove from D_O the interiors of k disjoint pairs of disks in Int D_O to form a disk-with-holes D_I such that, near each arc A_j of singularity, $g(D_I)$ looks exactly like Figure 3.

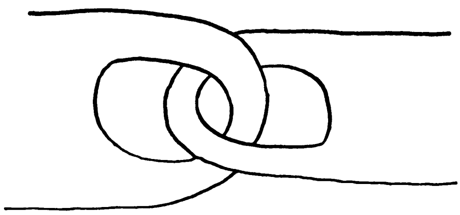


FIGURE 3

Let H_O' and H_I' be regular neighborhoods of $g(D_O)$ and $g(D_I)$ in U, respectively, $H_I' \subset \text{Int } H_O'$. Since $B \subset g(D_I)$, there is a PL homeomorphism h: $U \to U$ such that $X \subset \text{Int } h(H_I')$. Then (hH_O', hH_I') is a split-handle neighborhood of X in U.

4. Graphs on a cylinder over a bouquet.

Setting. Let $B = J_1 \cup \cdots \cup J_n$ be a bouquet of n PL loops $J_1, \ldots, J_n$ wedged at wedge point $*$. Let $C = B \times E^1$ denote the PL cylinder over B with subcylinders $C_1 = J_1 \times E^1, \ldots, C_n = J_n \times E^1$.

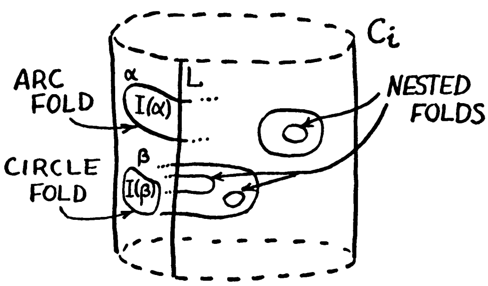


FIGURE 4

Graphs. A graph in C is a PL 1-complex G in C such that, for each i, $G \cap C_i$ is a compact 1-manifold transverse to the line $L = \{ * \} \times E^1$ in C_i. A *fold* in G (see Figure 4) is a component α of $G \cap (C - L)$ such that some component $I(\alpha)$ of $C - (\alpha \cup L)$ has compact closure in C; the component $I(\alpha)$ is uniquely determined by α and is called the *interior* of α. A *nested fold* is a fold contained in the interior of another fold. A fold α is an *arc-fold* if α is an open arc and a *circle-fold* if α is a simple closed curve.

UNNESTING LEMMA. *Any graph G in C can be changed into a graph G' in C which has no nested folds by a finite sequence of moves of the following two types*:

Type 1. *For some i, perform a* PL *isotopy of $G \cap C_i$ in C_i; accept unknown changes in $G \cap (C_j - L)$ for each $j \neq i$.*

Type 2. *Let A be a* PL *arc in C irreducibly joining $G - L$ and $L - G$; let N be a small regular neighborhood of A in C intersecting G in a small arc β containing $A \cap G$; replace* Int β *by that component of* $\mathrm{Fr}_C N$ *which intersects L.*

PROOF. *Step* 1. Perform a finite sequence of moves of Type 1, each reducing the cardinality of $G \cap L$, until no further move of Type 1 will reduce the cardinality of $G \cap L$. The altered G can have no arc-folds. Thus G clearly satisfies the following Inductive Hypothesis.

Inductive hypothesis. If a component K of G has a fold, then

(1) K separates C into exactly two components and is the frontier of each in C;

(2) one component $I(K)$ of $C - K$ has compact closure in C and is called the interior of K; and

(3) if α is a fold in K, then $I(\alpha) \subset I(K)$.

Complexity. To each graph G satisfying this hypothesis assign the complexity sequence $(G(1), G(2), G(3), \dots)$ where $G(n)$ is the number of those nested folds α in G such that $\alpha \subset I(K)$ for exactly n folded components K of G. Define $(G'(1), G'(2), G'(3), \dots) < (G(1), G(2), G(3), \dots)$ if $G'(n) < G(n)$ where n is the last index k such that $G'(k) \neq G(k)$. Note that there does not exist an infinite, strictly decreasing sequence of complexity sequences.

Step 2. Reduce $(G(1), G(2), G(3), \dots)$ to the zero sequence $(0, 0, 0, \dots)$ by a finite sequence of moves of Type 2 as follows. We carefully reintroduce arc folds in a controlled fashion in the process. Suppose $(G(1), G(2), \dots) > (0, 0, \dots)$. Choose a fold α in G that is not nested but such that $I(\alpha)$ contains a nested fold. We consider two cases.

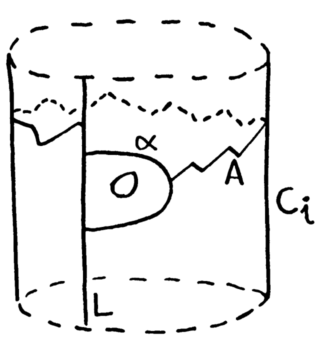


FIGURE 5

Case 1. (See Figure 5.) If α is an arc-fold, there is an arc A in C irreducibly joining $G - L$ and $L - G$ with one endpoint on α such that $A \cup L \cup \alpha$

separates the two ends of that subcylinder C_i of C which contains α. Use A to perform a move of Type 2.

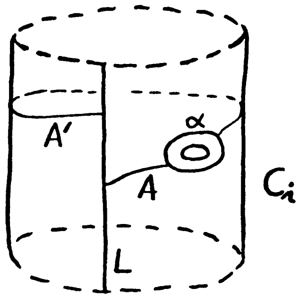


FIGURE 6

Case 2. (See Figure 6.) If α is a circle-fold, there are disjoint arcs A and A' in C, each irreducibly joining $G - L$ and $L - G$, each having one endpoint on α, such that $A \cup \alpha \cup A' \cup L$ separates the two ends of that subcylinder C_i of C which contains α. Use A and A' to perform two moves of Type 2.

In either case it is easy to check that the new graph G' obtained still satisfies the inductive hypothesis but has smaller complexity. Thus, complexity $(0, 0, 0, \dots)$ will be reached after finitely many steps. But a graph with complexity $(0, 0, 0, \dots)$ has no nested folds.

5. Straightening, splitting, and flattening unnested graphs. Let B, C, G, etc. be as in the previous section, G having no nested folds. The figures illustrating this section show one of the subcylinders C_i of C cut apart along L and laid flat. The reader is to consider the resulting two copies of L as a single line, however.

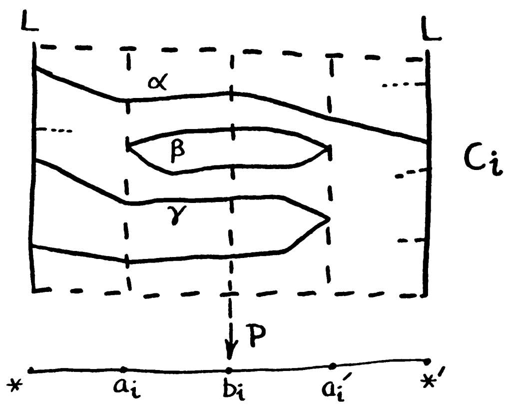


FIGURE 7

Straightening G. (See Figure 7.) Fix $i \in \{1, \dots, n\}$. In J_i choose points a_i, b_i, a_i' as in the figure. Let p: $C \to B$ denote the projection map. By an isotopy of C_i fixing L adjust G so that p acts on G in the simplest conceivable manner: nonfold compnonents α map 1-1 under p, circle-fold components β map 2-1 onto $a_i a_i'$ under p, arc-fold components γ map 2-1 onto $* a_i'$ or onto $a_i *'$ under p, all as pictured in the figure. Repeat for the other indices in $\{1, \ldots, n\}$.

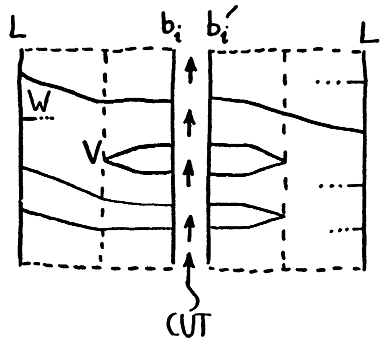
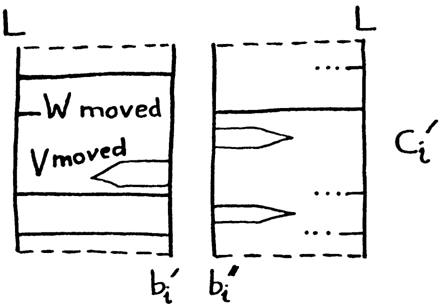


FIGURE 8

Splitting C and G. (See Figure 8.) Split C along the lines $\{b_i\} \times E^1$ $(i = 1, \ldots, n)$ to form a set C' of the form $B' \times E^1$, B' a $2n$-od. Let G' be the resulting graph in C'.



(It is impossible to give a more accurate rendering of the flattened G' without knowing, for example, whether W is "above" or "below" V in C_i')

FIGURE 9

Flattening G' in C'. (See Figure 9.) It is an easy matter to show that there is a PL isotopy of C' such that no two components of the image of G' under the isotopy intersect the same horizontal level $B' \times \{t\}$ of C', $t \in E^1$.

6. The homeomorphism h_i. Let X, U, D, E, and (a_{i-1}, a_i) be exactly as at the end of §1. By the Neighborhood Lemma, X has a split-handle neighborhood (fH_O, fH_I) in U such that $(fH_O \times \{a_{i-1}, a_i\}) \cap (D \cup E) = \varnothing$. Since all further changes take place in $(\operatorname{Int} fH_O) \times (a_{i-1}, a_i)$, we assume without loss that $f =$ identity.

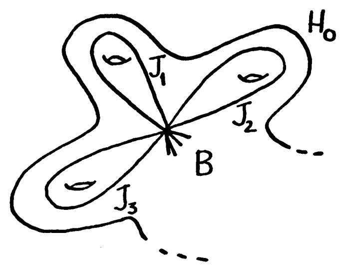


FIGURE 10

Let $B = J_1 \cup \cdots \cup J_n$ be a bouquet of n PL loops, with wedge point $*$, forming a spine for H_O as in Figure 10. Let B' be a PL spine for H_I coinciding with B except for small linked handles, as in Figure 11. Let q: $B' \to B$ be the natural projection, as pictured in Figure 12.

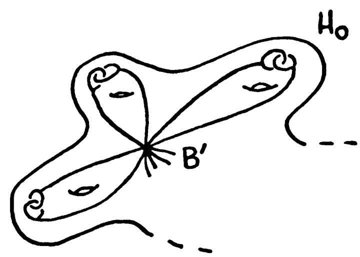


FIGURE 11

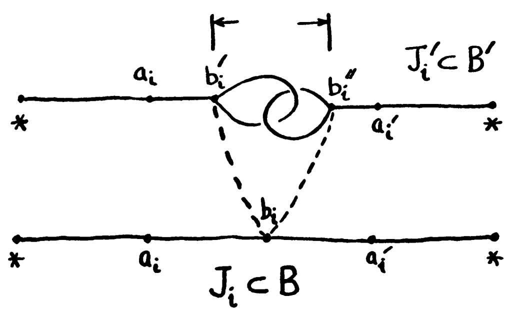


FIGURE 12

By a small move, put $D \cup E$ and $C = B \times (a_{i-1}, a_i)$ in general position. Then $G = (D \cup E) \cap C$ is a graph in C as in §4. It is easy to see that moves of $D \cup E$ in $(\operatorname{Int} fH_O) \times (a_{i-1}, a_i)$ allow one to perform moves of Types 1 and 2 on G as in §4. Thus we may assume G has no nested folds. Further isotopies of $D \cup E$ in $H_O \times (a_{i-1}, a_i)$ straighten G as in §5. Let $G' = q^{-1}G$. Then G' is, except for the addition of small loops attached at the points of $G' \cap (\{b_i', b_i''\} \times (a_{i-1}, a_i))$ exactly the splitting G' of G described in §5. Thus there is a flattening of G' in $C' = B' \times (a_{i-1}, a_i)$ as in §5 which may be realized by a homeomorphism α of space. There is a regular neighborhood M of the (flattened G') $= \alpha G'$ in C' such that no two components of M intersect the same horizontal level $B' \times \{t\}$ of C'. There is a horizontal homeomorphism β of E^4 which on $B' \times (a_{i-1}, a_i) = C'$ so nearly approximates $q \times \mathrm{id}$ that $(D \cup E) \cap \beta C' \subset \beta C' \subset \beta\alpha^{-1}M$ and such that no component of $\beta\alpha^{-1}M$ intersects both D and E. Then $\alpha\beta^{-1}(D \cup E)$ is such that no horizontal level $B' \times \{t\}$ of C' intersects both $\alpha\beta^{-1}D$ and $\alpha\beta^{-1}E$. A final standard horizontal push fixing B' but otherwise moving points away from B' results in an adjusted D and E that do not hit the same horizontal level $H_I \times \{t\}$, $t \in (a_{i-1}, a_i)$. This final push completes the construction of h_i.

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