

PROJECTIVE MODULES OVER SUBRINGS OF $k[X, Y]$

BY

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ABSTRACT. In this paper we study projective modules over subrings of $k[X, Y]$. Conditions are given for projective modules to decompose into free \oplus rank 1 modules. Our main result is that if k is an algebraically closed field and A a subring of $B = k[X, Y]$ with $A \subset B$ integral and $\text{sing}(A)$ finite, then all f.g. projective A -modules have the form free \oplus rank 1. We also give several examples of subrings of $k[X, Y]$ which have indecomposable projective modules of rank 2.

1. Introduction. Seshadri [14] first showed that all f.g. projective $k[X, Y]$ -modules are free. It is well known that if A is an arbitrary subring of $k[X, Y]$, then not all f.g. projective A -modules are free. For example, when $A = k[X^2, X^3, Y]$, $\text{Pic}(A) = k[Y]$. When A is an affine subring of $k[X, Y]$, f.g. projective A -modules have the form free \oplus rank 2 by Serre's Theorem [3, p. 173]. In this paper we study cases where this decomposition can be improved to free \oplus rank 1. In [1] or [2] the author proved the following theorem.

THEOREM 1.1. *Let A be an affine subring of $k[X, Y]$ generated by monomials; then:*

- (1) *All f.g. projective A -modules stably have the form free \oplus rank 1.*
- (2) *If A is normal, then all f.g. projective A -modules are free.*

In §2 we establish notation and a few preliminary results.

In §3 we study conditions for f. g. projective A -modules to decompose into free \oplus rank 1 modules when A has dim 2. Presumably these results are well known; compare [9].

Our main theorem (4.1) is proved in §4. We show that if k is an algebraically closed field and A an affine subring of $B = k[X, Y]$ with $\text{sing}(A)$ finite and $A \subset B$ integral, then all f.g. projective A -modules have the form free \oplus rank 1. A special application of this theorem is to rings of invariants. This is discussed in §5.

In §6 we give several examples of subrings A of $k[X, Y]$ which have indecomposable projective A -modules of rank 2. These examples show that the projective A -module structure depends on the field k .

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2. Preliminaries. All rings will be commutative with 1. A will denote a ring, M a maximal ideal of A , and k a field. Our K -theory notation follows [3], while K_2 will be Milnor's K_2 [6]. $\text{spec}(A)$ will denote the set of all prime ideals of A with the Zariski topology. For I an ideal of A , $V(I) = \{P \in \text{spec}(A) | P \supset I\}$. $\text{max}(A)$ is the subset of $\text{spec}(A)$ consisting of maximal ideals. The dimension of a subset of $\text{spec}(A)$ will be the usual combinatorial dimension. The dimension of a ring A will always mean Krull dimension and will be denoted by $\dim A$. We will denote the homological dimension of an A -module N by $\text{hd}_A N$. If N is an A -module, then $\text{supp } N = \{P \in \text{spec}(A) | N_P \neq 0\}$. The group of units of a ring A will be denoted by A^* . We will abbreviate short exact sequence by SES. The set of singularities of A will be denoted by $\text{sing}(A) = \{P \in \text{spec}(A) | A_P \text{ is not regular}\}$.

We recall that $\tilde{K}_0(A)$ is the subgroup of $K_0(A)$ generated by elements of the form $[P] - [A^{\text{rank } P}]$, where P is an f.g. projective A -module. There is the natural determinant homomorphism $\det: K_0(A) \rightarrow \text{Pic}(A)$ defined by $\det([P]) = \Lambda^n(P)$ where $n = \text{rank } P$ and $\text{Pic}(A)$ is the group of isomorphism classes of f.g. projective A -modules of rank 1. Clearly \det induces an epimorphism $\det: \tilde{K}_0(A) \rightarrow \text{Pic}(A)$; the kernel of this map will be denoted by $SK_0(A)$.

The following elementary lemma will be stated without proof.

LEMMA 2.1. *The following are equivalent.*

- (1) \det is an isomorphism.
- (2) $SK_0(A) = 0$.
- (3) All f.g. projective A -modules of constant rank stably have the form free \oplus rank 1.
- (4) Let P be a f.g. projective A -module of rank n ; then $\Lambda^n(P) \approx A$ iff P is stably free.

A commutative square of rings

$$\begin{array}{ccc}
 A & \xrightarrow{f_1} & A_1 \\
 f_2 \downarrow & & \downarrow g_1 \\
 A_2 & \xrightarrow{g_2} & B
 \end{array}$$

is said to be cartesian if given $x \in A_1$ and $y \in A_2$ such that $g_1(x) = g_2(y)$, then there exists a unique $z \in A$ such that $f_1(z) = x$ and $f_2(z) = y$.

THEOREM 2.2 [3, p. 481]. *Given a cartesian square of rings with g_1 surjective, we have the following "Mayer-Vietoris" exact sequences*

$$0 \rightarrow A^* \rightarrow A_1^* \oplus A_2^* \rightarrow B^* \xrightarrow{\partial} \text{Pic}(A) \rightarrow \text{Pic}(A_1) \oplus \text{Pic}(A_2) \rightarrow \text{Pic}(B), \quad (1)$$

$$K_1(A) \rightarrow K_1(A_1) \oplus K_1(A_2) \rightarrow K_1(B) \xrightarrow{\partial} \tilde{K}_0(A) \rightarrow \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) \rightarrow \tilde{K}_0(B). \quad (2)$$

Using the natural determinant maps we can connect sequences (1) and (2) to obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow \\
 & & SK_1(A) & \rightarrow & SK_1(A_1) \oplus SK_1(A_2) & \rightarrow & SK_1(B) & \xrightarrow{\partial} & SK_0(A) & \rightarrow & SK_0(A_1) \oplus SK_0(A_2) & \rightarrow & SK_0(B) \\
 & & \downarrow \\
 & & K_1(A) & \rightarrow & K_1(A_1) \oplus K_1(A_2) & \rightarrow & K_1(B) & \xrightarrow{\partial} & \tilde{K}_0(A) & \rightarrow & \tilde{K}_0(A_1) \oplus \tilde{K}_0(A_2) & \rightarrow & \tilde{K}_0(B) \\
 & & \downarrow \\
 0 & \rightarrow & A^* & \rightarrow & A_1^* \oplus A_2^* & \rightarrow & B^* & \xrightarrow{\partial} & \text{Pic}(A) & \rightarrow & \text{Pic}(A_1) \oplus \text{Pic}(A_2) & \rightarrow & \text{Pic}(B) \\
 & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

We have the following immediate lemma.

LEMMA 2.3. *Suppose that $SK_0(A_1) = SK_0(A_2) = 0$; then*

- (1) $SK_0(A) = \partial(SK_1(B))$.
- (2) *If $SK_1(B) = 0$, then $SK_0(A) = 0$.*
- (3) *If h is an isomorphism, then $SK_1(B) \approx SK_0(A)$.*

$SK_0(A) = 0$ just implies that stably all f.g. projective A -modules have the form free \oplus rank 1. To remove this stability requirement we will often use the following cancellation theorem.

THEOREM 2.4 (MURTHY-SWAN [9]). *Let A be an affine ring of $\dim \leq 2$ over an algebraically closed field. If P is a f.g. projective A -module and $A \oplus P \approx A \oplus Q$, then $P \approx Q$.*

$(a_0, \dots, a_n) \in A^{n+1}$ is unimodular if $Aa_0 + \dots + Aa_n = A$. The stable range of A , denoted by $sr(A)$, is $\leq d$ if given any unimodular row (a_0, \dots, a_d) , there exist $c_0, \dots, c_{d-1} \in A$ so that $(a_0 + c_0a_d, \dots, a_{d-1} + c_{d-1}a_d) \in A^d$ is still unimodular. For N an A -module, $x \in N$ is unimodular if $N = Ax \oplus N'$.

3. **Rings of dim 2.** If A is an affine domain, $\text{sing}(A)$ is closed [5, p. 245]. However, $\text{sing}(A)$ need not be closed in general [10]. If A is an affine domain of dim 2, then $\dim \text{sing}(A) \leq 1$ because $0 \notin \text{sing}(A)$. If, in addition, A is

normal, then $\text{sing}(A)$ has $\dim 0$ and, hence, is finite because A_P is a DVR for all ht 1 prime ideals P .

We note that for $M \in \max(A)$, $M \notin \text{sing}(A)$ iff A_M is regular iff $\text{hd}_A M < \infty$. If $M \in \max(A)$, then $\text{Ext}_A^1(M, A) \approx \text{Ext}_A^2(A/M, A)$ is annihilated by M . Thus $\text{Ext}_A^1(M, A) \approx \text{Ext}_{A_M}^1(M_M, A_M)$. If $M \notin \text{sing}(A)$ and $\dim A_M = 2$, then A_M is a regular local ring of $\dim 2$, so $\text{Ext}_A^1(M, A) \approx A/M$.

LEMMA 3.1 (SERRE [13]). *Let A be a noetherian ring and M an f.g. A -module with $\text{hd}_A M < 1$. If $\text{Ext}_A^1(M, A)$ is generated by r elements, then there is a SES $0 \rightarrow A^r \rightarrow P \rightarrow M \rightarrow 0$ where P is a f.g. projective A -module.*

LEMMA 3.2 [9]. *Let M be a maximal ideal of A such that A_M is a regular local ring of $\dim 2$. If there is a SES $0 \rightarrow Q \rightarrow P \rightarrow M \rightarrow 0$ where P is a f.g. projective A -module of rank 2, then $\Lambda^2(P) \approx Q$.*

COROLLARY 3.3. *Let I be an ideal of A of ht 2 with $V(I)$ finite and for all $M \in V(I)$, A_M is a regular local ring of $\dim 2$. Then for any SES $0 \rightarrow Q \rightarrow P \rightarrow I \rightarrow 0$, where P is a f.g. projective A -module of rank 2, $\Lambda^2(P) \approx Q$.*

LEMMA 3.4. *Let A be a commutative noetherian ring and M a maximal ideal such that A_M is a regular local ring of $\dim 2$; then $[A/M] \in SK_0(A)$.*

PROOF. Here we consider $K_0(A)$ as $K_0(H(A))$, where $H(A)$ is the category of f.g. A -modules with finite homological dimension [3, p. 407]. By Lemma 3.1 there is a SES $0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$ where P is a f.g. projective A -module of rank 2. By Lemma 3.2, $\Lambda^2(P) \approx A$. So

$$[A/M] = [A] - [M] = [A] - [P] + [A] = [A^2] - [P],$$

but $\Lambda^2(P) \approx A$, so $[A^2] - [P] \in SK_0(A)$. \square

We note that if A_M is a regular local ring of $\dim 1$, it need not be true that $[A/M] \in SK_0(A)$. For in this case M is actually projective, so $[A/M] \in SK_0(A)$ iff $\Lambda^1(M) = M \approx A$, that is, M is principal. We next show that in many cases the $[A/M]$'s generate $SK_0(A)$.

PROPOSITION 3.5. *Let A be a commutative noetherian ring of $\dim 2$ and $X = \{M \in \max(A) | \text{ht } M < 1\}$. Assume that $Y = X \cup \text{sing}(A) \subset Z$, where $Z \subset \text{spec}(A)$ is a closed set of $\dim < 1$. Then $SK_0(A)$ is generated by $[A/M]$ for $M \in \max(A) \setminus Y$.*

PROOF. Any element of $SK_0(A)$ has the form $[P] - [A^2]$, where P is a f.g. projective A -module of rank 2 with $\Lambda^2(P) \approx A$. It is sufficient to find $f \in P^* = \text{Hom}_A(P, A)$ such that $A/\text{im } f$ is artinian and $\text{supp}(A/\text{im } f) \cap Z = \emptyset$.

For let $I = \text{im } f$ and $V(I) = \{M_1, \dots, M_n\} \subset \max(A)$. $\text{supp}(A/I) \cap Z = \emptyset$, so each $[A/M_i] \in SK_0(A)$ by Lemma 3.4. We have a SES $0 \rightarrow Q \rightarrow P$

$\xrightarrow{f} I \rightarrow 0$ where $Q = \ker f$. Since each $M_i \notin \text{sing}(A)$, $\text{hd}_{A_{M_i}} I_{M_i} \leq 1$ and thus $\text{hd}_A I \leq 1$. So $\text{hd}_A Q = 0$ and Q is a f.g. projective A -module. By Corollary 3.3, $Q \approx \Lambda^2(P) \approx A$. Thus $[P] = [A] + [I]$ in $K_0(A)$.

A/I is artinian and thus has a composition series $A/I = N_0 \supset N_1 \supset \dots \supset N_l = 0$, where each $N_i/N_{i+1} \approx A/M_j$ for some $M_j \in V(I)$. So in $K_0(A)$, $[A/I] = \sum [N_i/N_{i+1}] = \sum [A/M_j]$, and thus

$$[P] - [A^2] = [A] + [I] - [A^2] = -[A/I] = -\sum [A/M_j].$$

So $SK_0(A)$ is generated by $\{[A/M] \mid M \in \max A \setminus Z\}$.

Now we show how to find the desired $f \in P^*$. Let $Q = P^*$, then Q is also a f.g. projective A -module of rank 2. Z is closed, so $Z = V(J)$ with $\dim A/J \leq 1$. By Serre's Theorem [3, p. 173], there is an $x \in Q$ so that \bar{x} is unimodular in $\bar{Q} = Q/JQ$ and $O_Q(x) + J = A$ where $O_Q(x) = \{g(x) \mid g \in Q^*\}$. Thus there is an $a \in J$ with $O_Q(x) + Aa = A$. Then (a, x) is unimodular in $A \oplus Q$. By [15, p. 13], there exists a $y \in Q$ so that $\text{ht } O_Q(x + ay) \geq 2$. Let $f = x + ay \in Q = P^*$, then $A/\text{im } f$ is artinian and $O_Q(f) + J = A$, which implies $\text{supp}(A/\text{im } f) \cap Z = \emptyset$. \square

The following theorem is now easily proved.

THEOREM 3.6. *Let A be a commutative noetherian ring of dim 2 and $X = \{M \in \max(A) \mid \text{ht } M \leq 1\}$. Assume that $Y = X \cup \text{sing}(A) \subset Z$ where $Z \subset \text{spec}(A)$ is a closed set of dim ≤ 1 . Then the following are equivalent.*

- (1) $SK_0(A) = 0$.
- (2) All f.g. projective A modules of constant rank stably have the form free \oplus rank 1.
- (3) $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus Y$.
- (4) $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus Z$.

COROLLARY 3.7. *Let A, X, Y , and Z be as in Theorem 3.6 and assume that the cancellation theorem for f.g. projection A -modules holds. Then the following are equivalent.*

- (1) $SK_0(A) = 0$.
- (2) All f.g. projective A -modules of constant rank have the form free \oplus rank 1.
- (3) $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus Y$.
- (4) $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus Z$.
- (5) Each $M \in \max(A) \setminus Y$ can be generated by 2 elements.

PROOF. Clearly (1)–(4) are equivalent. (3) \Rightarrow (5). By Lemmas 3.1 and 3.2, for each $M \in \max(Y)$ there is a SES $0 \rightarrow A \rightarrow P \rightarrow M \rightarrow 0$ where P is a f.g. projective A -module of rank 2 with $\Lambda^2(P) \approx A$. By (2), $P \approx A \oplus \Lambda^2(P) \approx A^2$, so M can be generated by 2 elements.

(5) \Rightarrow (3). For each $M \in \max(A) \setminus Y$ there is a SES $0 \rightarrow L \rightarrow A^2 \rightarrow M \rightarrow 0$

with $L \approx \Lambda^2(A^2) \approx A$ by Lemma 3.2 again. Thus $[A/M] = [A] - [M] = [A] - ([A^2] - [A]) = 0$ in $K_0(A)$. \square

REMARK 3.8. If A is an affine subring of $B = k[X, Y]$ with $A \subset B$ integral, then all maximal ideals of A have ht 2 and $\text{sing}(A)$ is closed of $\dim < 1$. So by Theorem 3.6, $SK_0(A) = 0$ iff $[A/M] = 0$ for all $M \in \max(A) \setminus \text{sing}(A)$.

4. Main theorem.

THEOREM 4.1. *Let k be an algebraically closed field and A an affine subring of $B = k[X, Y]$ with $A \subset B$ integral and $\text{sing}(A)$ finite. Then $SK_0(A) = 0$, so all f.g. projective A -modules have the form free \oplus rank 1.*

k is algebraically closed, so by Theorem 2.4 the cancellation theorem holds for f.g. projective A -modules. So by Remark 3.8 and Corollary 3.7 it is enough to show that $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus \text{sing}(A)$.

Let $X = \{P \in \text{spec}(B) \mid P \cap A \in \text{sing}(A)\}$; then $X \subset \max(B)$ and is finite since $A \subset B$ is integral and $\text{sing}(A)$ is finite. Next we show that we can always avoid a finite number of maximal ideals of B . Geometrically this just says that given any finite set of points in k^2 , there is a line which does not pass through any of these points.

LEMMA 4.2. *Let k be an algebraically closed field and $B = k[X, Y]$. Let $Z \subset \max(B)$ be finite and $M \in \max(B) \setminus Z$. Then there is a $P = (aX + bY + C) \subset M$ so that $P \not\subset N$ for any $N \in Z$.*

PROOF. k is algebraically closed, so we may assume that $M = (X, Y)$ and $Z = \{M_1, \dots, M_n\}$ where each $M_i = (X - a_i, Y - b_i)$. There are only a finite number of ratios b_i/a_i , and since k is infinite, there exists a $0 \neq a \in k$ so that $a \neq -b_i/a_i$ for $1 \leq i \leq n$. Let $P = (aX + Y) \subset M = (X, Y)$; clearly P is not contained in any M_i . \square

$A \subset B$ is integral, so each $M \in \max(A) \setminus \text{sing}(A)$ can be lifted to a $\bar{M} \in \max(B) \setminus X$. By Lemma 4.2 there is a $P = (aX + bY + C) \subset \bar{M}$ which is not contained in any $N \in X$. Let $Q = P \cap A$; then $0 \neq Q \in \text{spec}(A)$ is not contained in any element of $\text{sing}(A)$. We also note that Q is locally principal, and hence invertible (projective). For $M \in \text{sing}(A)$, $Q_M = A_M$, while for $M \notin \text{sing}(A)$, A_M is a regular local ring and so ht 1 prime ideals are principal because A_M is factorial. We also note that $\text{hd}_A A/Q = 1$. So we have shown

LEMMA 4.3. *Let $M \in \max(A) \setminus \text{sing}(A)$; then M contains an invertible prime ideal which is not contained in any $N \in \text{sing}(A)$.*

We have $k \subset A/Q \subset k[X, Y]/(aX + bY + c) \approx k[T]$, which is an integral extension. As A/Q is a 1-dimensional affine domain over k contained in $k[T]$, by Luroth's Theorem, the integral closure of A/Q is

$\overline{A/Q} = k[g(T)]$. So we may assume that $\overline{A/Q} = k[T]$.

LEMMA 4.4. *Let k be an algebraically closed field and R an affine domain over k of dim 1. If ${}_R\overline{R}$ is f.g., where \overline{R} is the integral closure of R , then the natural map $f: G_0(\overline{R}) \rightarrow G_0(R)$ is surjective.*

PROOF. f is well defined because ${}_R\overline{R}$ is f.g. $G_0(R)$ is generated by $[R]$ and $[R/M]$ for $M \in \max(R)$. For $M \in \max(R)$, let $\overline{M} \in \max(\overline{R})$ lie over M . Then $R/M \hookrightarrow \overline{R}/\overline{M}$ is an R -isomorphism because k is algebraically closed. Since $\text{rank}_R \overline{R} = 1$, also $[R] \in \text{im } f$. So f is surjective. \square

LEMMA 4.5. *Let R be a domain with $G_0(R) \approx \mathbf{Z}$; then $[R/I] = 0$ in $G_0(R)$ for all $0 \neq I \in \text{spec}(R)$.*

PROOF. Since R is a domain, we have the SES $0 \rightarrow \ker rk \rightarrow G_0(R) \xrightarrow{rk} \mathbf{Z} \rightarrow 0$. $G_0(R) \approx \mathbf{Z}$, so $\ker rk = 0$. But clearly each R/I has rank 0, so $[R/I] = 0$ in $G_0(R)$. \square

In our case $R = A/Q$ and $\overline{R} = \overline{A/Q} = k[T]$. $G_0(k[T]) \approx \mathbf{Z}$ and $G_0(A/Q) \approx \mathbf{Z} \oplus \tilde{G}_0(A/Q)$ since A/Q is a domain. By Lemma 4.4, $\tilde{G}_0(A/Q) = 0$, so $G_0(A/Q) = \mathbf{Z}$ on $[A/Q]$. So for each $M \in \max(A) \setminus \text{sing}(A)$, M contains an invertible prime ideal Q which misses $\text{sing}(A)$. So $G_0(A/Q) = \mathbf{Z}$ and thus $[A/M] = 0$ in $G_0(A/Q)$ by Lemma 4.5.

If N is a f.g. A/Q -module, then N is also a f.g. A -module and clearly $\text{supp}(N) \subset V(Q)$. Since $\text{hd}_A A/Q = 1$ and $V(Q) \cap \text{sing}(A) = \emptyset$, each f.g. A/Q -module also has finite homological dimension when considered as an A -module. We thus have a natural homomorphism $\psi: G_0(A/Q) \rightarrow K_0(A) = K_0(H(A))$ defined by $\psi([A/Q]) = [A/Q]$. But $[A/M] = 0$ in $G_0(A/Q)$ and, hence, $[A/M] = \psi([A/M]) = 0$ in $K_0(A)$ also. So we have shown that $[A/M] = 0$ in $K_0(A)$ for all $M \in \max(A) \setminus \text{sing}(A)$, and thus the proof of Theorem 4.1 is complete.

If A is an affine normal domain of dim 2, then $\text{sing}(A)$ is finite; so the following corollary is immediate.

COROLLARY 4.6. *Let k be an algebraically closed field and A an affine normal subring of $B = k[X, Y]$ with $A \subset B$ integral. Then all f.g. projective A -modules have the form free \oplus rank 1.*

REMARK 4.7. We note that Theorem 4.1 includes more than just the normal subrings. For example, $A = k[X^2, X^3, XY, Y^2, Y^3]$ is a subring of $k[X, Y]$ whose only singularity is the origin. Also note that $\text{Pic}(A) \neq 0$.

REMARK 4.8. We do not know any examples of rings of the type in Corollary 4.6 for which $\text{Pic}(A) \neq 0$.

REMARK 4.9. In §6 we show that Theorem 4.1 cannot be improved to $\dim \text{sing}(A) < 1$. In fact, $A = \mathbf{C}[X, Y(X^2 - Y), Y^2(X^2 - Y)]$ has indecomposable projective modules of rank 2.

We do not know if it necessary for k to be algebraically closed. However, we do have the following partial result.

PROPOSITION 4.10. *Let A be an affine subring of $B = k[X, Y]$ with $A \subset B$ integral and $\text{sing}(A)$ finite. Then $SK_0(A)$ is torsion.*

PROOF. Let k' be a finite algebraic extension of k with $[k' : k] = n < \infty$. Then $K_0(A) \xrightarrow{f} K_0(k' \otimes_k A) \rightarrow K_0(A)$ is just multiplication by n , so $\ker f$ is n -torsion. The algebraic closure \bar{k} of k is the direct limit of finite algebraic extensions of k , say $\bar{k} = \text{proj lim } k'$. So

$$\bar{A} = \bar{k} \otimes_k A = (\text{proj lim } k') \otimes_k A \approx \text{proj lim}(k' \otimes_k A),$$

and thus $K_0(\bar{A}) = \text{proj lim } K_0(k' \otimes_k A)$. Hence $L = \ker(K_0(A) \rightarrow K_0(\bar{A}))$ is also torsion. But $SK_0(\bar{A}) = 0$ by Theorem 4.1, so $SK_0(A) \subset L$ and, hence, is torsion. \square

REMARK 4.11. We are also not sure if Theorem 4.1 is valid without assuming $A \subset B$ integral. For example, let $A = \mathbb{C}[X(X^2 + Y^3)^3, Y(X^2 + Y^3)^2, X^2 + Y^3] \subset \mathbb{C}[X, Y]$. Then $A \approx \mathbb{C}[X, Y, Z]/(Z^2 - (X^3 + Y^7))$, so A is factorial [12, p. 31]. Thus A is normal and $\text{Pic}(A) = 0$. Clearly $\mathbb{C}(X, Y)$ is the quotient field of A , but $A \subset \mathbb{C}[X, Y]$ is not integral because A is normal. It is not known if all f.g. projective A -modules are free.

One special case when $\text{Pic}(A) = 0$ is when A is graded.

LEMMA 4.12 [7]. *Let $A = A_0 \oplus A_1 \oplus \dots$ be an affine normal domain with A_0 a field; then $\text{Pic}(A) = 0$.*

As an immediate corollary of Lemma 4.12 and Corollary 4.6, we have

COROLLARY 4.13. *Let k be an algebraically closed field and A an affine normal graded subring of $B = k[X, Y]$ with $A \subset B$ integral; then all f.g. projective A -modules are free.*

5. Rings of invariants. A special case of the rings of Corollary 4.6 are rings of invariants of $k[X, Y]$. Let A be a ring and G a subgroup of $\text{Aut}(A)$; then $A^G = \{a \in A \mid \theta(a) = a \ \forall \theta \in G\}$ is a subring of A . Let $B = k[X, Y]$ and $G \subset \text{Aut}_k(B)$ be a finite subgroup; then $A = B^G$ is an affine normal subring of B with $A \subset B$ integral. Proposition 5.1 thus follows from Corollaries 4.6 and 4.13.

PROPOSITION 5.1. *Let k be an algebraically closed field and $G \subset \text{Aut}_k(k[X, Y])$ a finite subgroup. Then all f.g. projective $A = k[X, Y]^G$ -modules have the form free \oplus rank 1. If, in addition, each $\theta \in G$ is a graded automorphism, then all f.g. projective A -modules are free.*

Instead of considering rings of invariants, we can also consider the kernel

of a group of derivations. Let k be a field of char $k = p \neq 0$ and G a finite group of k -derivations of $B = k[X, Y]$. Then $k[X, Y]^p \subset \ker D$ for each $D \in G$; $A = {}^G k[X, Y] = \{f \in k[X, Y] \mid D(f) = 0 \forall D \in G\}$ is a subring of B with $k[X, Y]^p \subset {}^G B \subset B$, and hence $A \subset B$ is integral. If all $D \in G$ are graded, then $A = {}^G B$ will also be graded. So we have the following proposition.

PROPOSITION 5.2. *Let k be an algebraically closed field with char $k = p \neq 0$ and G a finite group of k -derivations of $B = k[X, Y]$. Then all f.g. projective $A = {}^G B$ -modules have the form free \oplus rank 1. Moreover, if all $D \in G$ are graded, then all f.g. projective A -modules are free.*

Again, we do not know if it is necessary for k to be algebraically closed. However, if A is generated by monomials over k , then by Theorem 1.1, all f.g. projective modules are free, regardless of k .

6. Constructing indecomposable projective modules. For arbitrary subrings A of $B = k[X, Y]$ not all f.g. projective A -modules have the form free \oplus rank 1. In fact, this depends on the field k . We give several examples of subrings A of B with $A \subset B$ integral, but $SK_0(A) \neq 0$.

EXAMPLE 6.1. Let $A = k[X, Y(X^2 - Y), Y^2(X^2 - Y)] \subset B = k[X, Y]$. $A \subset B$ is integral, and B is the integral closure of A . We note that $I = Y(X^2 - Y)B$ is contained in the conductor of B/A . We then have the following cartesian square.

$$\begin{array}{ccc}
 k[X, Y(X^2 - Y), Y^2(X^2 - Y)] = A & \longrightarrow & B = k[X, Y] \\
 \downarrow & & \downarrow \\
 K[X] = A/I & \longrightarrow & B/I = k[X, Y]/(Y(X^2 - Y))
 \end{array}$$

By Theorem 2.2, $\text{Pic}(A) = 0$ and $SK_0(A) \approx SK_1(B/I)$ by Lemma 2.3. Using the Mayer-Vietoris K -theory exact sequence for K_2 [6], it is easy to see that $SK_1(B/I) \approx K_2(k[T]/(T^2))/K_2(k)$. If k is a field of char $k \neq 2$ or a perfect field, then $K_2(k[T]/(T^2)) \approx K_2(k) \oplus \Omega_Z^1$, so $SK_0(A) \approx \Omega_Z^1$ [16].

If k is a finite field, then $\Omega_Z^1 = 0$. However, by [15, p. 45], $\text{sr}(A) < 2$ when k is a finite field, so all f.g. projective A -modules are free. But if $k = \mathbb{C}$, then $\Omega_{\mathbb{C}/\mathbb{Z}}^1 \neq 0$. Thus $\text{Pic}(A) = 0$ while $SK_0(A) \neq 0$, so there exist indecomposable projective A -modules of rank 2. For let $0 \neq [P] - [A^2] \in SK_0(A)$. If P were decomposable, then necessarily $P \approx A^2$ since $\text{Pic}(A) = 0$. Note that P is not even stably decomposable. This ring was used by Pedrini [11] as an example of a ring for which $NK_0(A) \neq 0$, but $\text{Pic}(A) = \text{Pic}(A[T]) = 0$.

REMARK 6.2. We note that

$$A = k[X, Y(X^2 - Y), Y^2(X^2 - Y)] \\ \approx k[U, V, W]/(V^3 + W^2 - U^2VW).$$

Thus A is graded if we assign weights of 1, 4, and 6 to $U, V,$ and $W,$ respectively. If $k = \mathbb{C},$ then $\{(U - \alpha, V, W)/(V^3 + W^2 - U^2VW)\} \subset \text{sing}(A),$ so $\dim \text{sing}(A) = 1.$ Thus the hypothesis of Theorem 4.1 cannot be improved to $\dim \text{sing}(A) < 1.$

REMARK 6.3. $A = \mathbb{C}[X, Y(X^2 - Y), Y^2(X^2 - Y)]$ is an example of an affine domain over an algebraically closed field which has an indecomposable projective module of rank 2. This answers a question raised by Murthy [7].

It is not hard to give more examples. They depend upon calculating $SK_1(k[X, Y]/(f))$ for $f \in k[X, Y].$ For example, Theorem 1.1(1) really follows because $SK_1(k[X, Y]/(X^iY^j)) = 0.$

LEMMA 6.4. *Let k be algebraic over a finite field and $f \in k[X, Y];$ then $SK_1(k[X, Y]/(f)) = 0.$*

PROOF. By [15, p. 45], $\text{sr}(k[X, Y]) \leq 2.$ Assuming this, a proof may be found in [4, p. 29]. \square

PROPOSITION 6.5. *Let k be algebraic over a finite field and A an affine subring of $B = k[X, Y]$ with integral closure $B;$ then $SK_0(A) = 0.$ If $\text{Pic}(A) = 0,$ then all f.g. projective A -modules are free.*

PROOF. Since B is the integral closure of $A,$ the conductor I of B/A contains a $0 \neq f \in B.$ By Lemma 6.4, $SK_1(B/fB) = 0,$ so $SK_0(A) = 0$ by Lemma 2.3. Since $\text{sr}(A) \leq 2$ all stably free A -modules are free. So if $\text{Pic}(A) = 0,$ all f.g. projective A -modules are free. \square

We give one final criterion for $SK_0(A) = 0.$

PROPOSITION 6.6. *Let A be an affine subring of $B = k[X, Y]$ with integral closure $B.$ Let I be the conductor of $B/A.$ If $V(I) \subset \text{spec}(B)$ is finite, then $SK_0(A) = 0.$*

PROOF. If $V(I)$ is finite, then B/I is semilocal, and so $SK_1(B/I) = 0.$ Thus $SK_0(A) = 0$ by Lemma 2.3. \square

We conclude this paper with two more examples like Example 6.1.

EXAMPLE 6.7. Let $A = k[X, Y^2, Y(X^3 - Y^2)] \subset B = k[X, Y].$ Clearly B is the integral closure of A and $f = X^3 - Y^2$ is in the conductor of $B/A.$ From the cartesian square

$$\begin{array}{ccc} k[X, Y^2, Y(X^3 - Y^2)] = A & \hookrightarrow & B = k[X, Y] \\ \downarrow & & \downarrow \\ k[S^2] = A/fB & \hookrightarrow & B/fB = k[S^2, S^3] \end{array}$$

and Theorem 2.2 it is easy to calculate $\text{Pic}(A) = 0$ and $SK_0(A) \approx SK_1(B/fB) \approx SK_1(k[S^2, S^3])$.

If k is algebraic over a finite field, then $SK_1(B/fB) = 0$ by Lemma 6.4, so $SK_0(A) = 0$. In this case all f.g. projective A -modules are free because $\text{sr}(A) \leq 2$.

However, if k is algebraically closed and has infinite transcendence degree over \mathbf{Q} , then $SK_1(B/fB) \approx SK_1(k[S^2, S^3]) \neq 0$ [8]. So in this case A has indecomposable projective modules of rank 2.

EXAMPLE 6.8. Let $A = k[X, Y^2, Y(X^2 + Y^2 - 1)] \subset B = k[X, Y]$. Then B is the integral closure of A and $f = X^2 + Y^2 - 1$ is in the conductor of B/A . From the cartesian square

$$\begin{array}{ccc} k[X, Y^2, Y(X^2 + Y^2 - 1)] = A & \xrightarrow{c} & B = k[X, Y] \\ \downarrow & & \downarrow \\ k[X] = A/fB & \xrightarrow{c} & B/fB = k[X, Y]/(X^2 + Y^2 - 1) \end{array}$$

and Lemma 2.3 we see that $SK_0(A) \approx SK_1(B/fB)$. We consider three cases for k .

(1) $\text{char } k = 2$. Then $B/fB = k[X, Y]/(X + Y + 1)^2$, so $SK_1(B/fB) \approx SK_1(k[X, Y]/(X + Y + 1)) = 0$. Thus $SK_0(A) = 0$. By Theorem 2.2, $\text{Pic}(A) \neq 0$.

(2) $k = \mathbf{C}$ (or $\sqrt{-1} = i \in k$). Letting $U = X + iY$ and $V = X - iY$, then $B/fB = \mathbf{C}[X, Y]/(X^2 + Y^2 - 1) \approx \mathbf{C}[U, V]/(UV - 1) \approx \mathbf{C}[U, 1/U]$. Thus $SK_0(A) = 0$, while $\text{Pic}(A) \approx \mathbf{Z}$. By Theorem 2.4 every f.g. projective A -module has the form free \oplus rank 1.

(3) $k = \mathbf{R}$. By [6, p. 128], $SK_1(\mathbf{R}[X, Y]/(X^2 + Y^2 - 1)) \approx \mathbf{Z}/2\mathbf{Z}$, so $SK_0(A) \approx \mathbf{Z}/2\mathbf{Z}$. $(B/fB)^* = \mathbf{R}^*$, so $\text{Pic}(A) = 0$ from Theorem 2.2. Let $0 \neq [P] - [A^2] \in SK_0(A)$; then P is a f.g. projective A -module of rank 2 which is not stably free. Thus P is indecomposable. However, $2([P] - [A^2]) = [P \oplus P] - [A^4] = 0$, so $P \oplus P \approx A^4$.

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