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# ON THE GROWTH OF SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS(1)

BY

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ABSTRACT. In this paper we determine estimates for the growth of both real-valued and complex-valued solutions of algebraic differential equations on an interval $(x_0, +\infty)$. One of the main results of the paper (Theorem 3) confirms E. Borel's conjecture on the growth of real-valued solutions for a broad class of solutions of second-order algebraic differential equations. The conjecture had previously been shown to be false for third-order equations.

1. **Introduction.** In this paper we investigate the growth of solutions of algebraic differential equations having polynomial coefficients (i.e. equations of the form

$$\Omega(z, y, y', \ldots, y^{(m)}) = 0, \tag{1}$$

where $\Omega$ is a polynomial in $z, y, y', \ldots, y^{(m)}$ which is not identically zero).

For first-order equations (1), it was shown by Lindelöf [17] that any real-valued solution on an interval $(x_0, +\infty)$ is majorized by a function of the form $\exp(x^\lambda)$ for some constant $\lambda$. In [7] and [25], Vijayaraghavan and others proved that second-order equations (1) can possess real-valued solutions on $(0, +\infty)$ having arbitrarily rapid growth by showing that for any preassigned increasing function $\psi(x)$ on $(0, +\infty)$, it is possible to construct a real-valued solution $y(x)$ on $(0, +\infty)$ of a second-order equation (1) such that $|y(x)| > \psi(x)$ at a sequence of $x$ tending to $+\infty$. The examples constructed in [7] and [25] are of the form $(2 - \cos(x) - \cos(\beta x))^{-1}$ and $(P(\alpha x) + \bar{P}(\alpha\bar{x}))/2$, respectively, for certain constants $\alpha$ and $\beta$ depending on $\psi(x)$, where $P(u)$ is the Weierstrass $Pe$-function. In addition, it is easy to see from the construction in [25] that for a given $\psi(x)$, there is a suitable $\alpha$ such that the function $y = P(\alpha x)$ is a complex-valued solution on $(0, +\infty)$ of a first-order equation (1) for which $|y(x)| > \psi(x)$ at a sequence of $x$ tending to $+\infty$. The examples, $P(\alpha z)$, $(P(\alpha z) + \bar{P}(\alpha\bar{z}))/2$, and $(2 - \cos(z) - \cos(\beta z))^{-1}$, which were constructed in [7] and [25] are all meromorphic

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functions on the plane of finite order of growth, and thus are quotients of entire functions of finite order. Using the minimum modulus estimate [20, p. 336] for such entire functions (which arises from the Hadamard factorization theorem), it is easy to see that in order to achieve arbitrarily rapid growth on $(0, +\infty)$, the constants $\alpha$ and $\beta$ above must be chosen so that the solutions have poles arbitrarily close to the positive real axis (but, of course, no poles on the positive real axis). One of the main results of Part 1 of this paper (Theorem 1 below) deals with meromorphic functions $y(z)$ on the plane which have no poles on $(0, +\infty)$ (and which need not be real-valued on $(0, +\infty)$ nor of finite order of growth on the plane). If $y(z)$ satisfies an equation (1), then we obtain an estimate for the growth of $|y(x)|$ on $(1, +\infty)$ in terms of the distance function from $x$ to the set of poles of $y$, and possibly the counting function for the number of distinct zeros of $y$ near $(0, +\infty)$. Hence we obtain quantitative information on how close to $(0, +\infty)$ the solution $y(z)$ must have poles in order for $|y(x)|$ to achieve a preassigned growth on $(0, +\infty)$. In fact, Theorem 1 applies to any solution having no poles on $(0, +\infty)$, which is defined and meromorphic in a region which for some $\varepsilon > 0$ contains the semi-infinite strip $|\mathrm{Im}(z)| < \varepsilon$, $\mathrm{Re}(z) > 0$. This result is a corollary of a general result (see §8) on the growth of solutions of equation (1) which are defined and analytic on simply-connected regions of the plane. We remark here that for these results, we require that, if the order of (1) is higher than two, the solution $y(z)$ not be a solution of some equation $\Omega_q = 0$, where $\Omega_q$ denotes the homogeneous part of $\Omega$ of total degree $q$ in the indeterminates $y, y', \ldots, y^{(m)}$. The reason for imposing this condition in the case of equations of order higher than two is that if this condition is violated, then quantities besides the counting functions for the zeros and poles of the solution can enter into the estimate for the growth of such a solution, and this estimate can be extremely complicated. (See [5, pp. 55, 63–65] for a complete discussion.) Of course, even if this condition is violated, then one can always find a polynomial $Q$ of degree $m$ which is not a solution of $\Omega = 0$ (e.g. see [5, p. 57]), and one can try to obtain some information on $y$ by applying our results to the solution $v = y - Q$ of the equation $\Lambda = 0$, where

$$\Lambda = \Omega(z, v + Q, v' + Q', \ldots, v^{(m)} + Q^{(m)}).$$

(It is obvious that $\Lambda_0 = 0$ has no solutions.) Finally, we remark that the basic ideas and techniques used in the development of the estimates in Theorems 1 and 2 below had their origins in the author's papers [2] and [6], but in the present paper these ideas and techniques are improved (see §4) and are made more general. For example, in [2] and [6], estimates were developed for the growth of analytic solutions in regions where the zeros of the solution are "sparse". In the present paper, no such restriction is needed. One reason for this is a result proved in §6 below which provides an estimate for the growth

of an arbitrary meromorphic function in the unit disk in terms of the growth of its logarithmic derivative. (See also §7.)

In Part 2 of the paper, we consider positive, increasing solutions of second-order equations (1) on an interval $(x_0, +\infty)$. In [9], E. Borel indicated a line of reasoning which would show that positive, increasing solutions of $m$th order equations (1) on an interval $(x_0, +\infty)$ are eventually majorized by $\exp_{m+1} x$ (where $\exp_k x$ is the $k$th iterate of the exponential function). However, as pointed out by several authors (e.g. Hardy [13], Fowler [12], and Vijayaraghavan [25]), Borel's proof was clearly incomplete, and, in fact, the author [6, §14, p. 53] constructed a counterexample in the case of third-order equations. For arbitrary second-order equations (1), it is not known whether positive, increasing solutions on $(x_0 + \infty)$ can have arbitrarily rapid growth, and it is this difficult question that we investigate in Part 2. (None of the examples of arbitrarily rapid growing solutions which were constructed by Vijayaraghavan and others in [7] and [25], and which were discussed earlier, are increasing.) Fowler [12] considered the special equation $y'' = R(x,y)/Q(x,y)$, where $R$ and $Q$ are polynomials, and his results show that real-valued solutions on $(x_0, +\infty)$ of this special equation are eventually majorized by a function of the form $\exp(x^\lambda)$. (We mention here that in some of his results, Fowler permitted $R(x,y)$ and $Q(x,y)$ to be polynomials in $y$ with coefficients of the form $x^\alpha f(x^{-\beta})$, where $\alpha$ and $\beta$ are rational with $\beta > 0$, and $f(z)$ is analytic at $z = 0$.) However, for the general second-order equation (1), the only general result which is known for positive, increasing solutions is a result of the author [6, §15, p. 56]. This result states that if $y(x)$ is a positive solution on $(x_0, +\infty)$ with the properties that (i) for every $\alpha \geqslant 0$, $y/x^\alpha \to +\infty$ as $x \to +\infty$, and (ii) $\log y(x)$ is an increasing convex function of $\log x$, then for some constant $\lambda$, either $y(x)$ is eventually majorized by $\exp(x^\lambda)$ or the function $v = xy'/y$ satisfies $v'/v = O(x^\lambda)$ as $x \to +\infty$ outside a possible exceptional set of finite measure. (It is not known whether the possible exceptional set can be removed, but if it can, this would obviously confirm Borel's conjecture for such solutions.) In the present paper (§13), we take a different approach and prove that for a second-order equation (1), there is a constant $A$ (which can be calculated directly from the equation) such that any positive, increasing solution $y(x)$ on $(x_0, +\infty)$ which satisfies (a) for every $\alpha \geqslant 0$, $y/x^\alpha \to +\infty$ as $x \to +\infty$, and (b) $x^{-A}y'/y \to +\infty$ as $x \to +\infty$, then also satisfies $y(x) \leqslant \exp_2 x^\lambda$ for some constant $\lambda$ on an interval $(x_1, +\infty)$. (In the case when (1) has constant coefficients, $A$ can always be taken to be zero.) This confirms Borel's conjecture for positive, increasing solutions satisfying (a) and (b). In addition, we point out in §14 that the same result holds if, instead of polynomial coefficients, we allow the coefficients of (1) to lie in a certain type of function field which was

introduced and investigated by W. Strodt [23]. These fields consist of functions which are defined and analytic in a region of the form $|\arg z| < \theta$, $|z| > r$, and have an asymptotic expansion, as $z \to \infty$ in the sector $|\arg z| < \theta$, in terms of decreasing real (but not necessarily rational) powers of $z$. This allows us to consider coefficients which are more general than those treated by Fowler (see §15).

We conclude with two remarks. For the reader who is interested in the growth of real solutions of algebraic difference equations, algebraic functional equations and differential-difference equations, we refer the reader to the papers of Cooke [10], [11], Lancaster [16], and Shah [21], [22]. Secondly, the author would like to acknowledge valuable conversations with his colleague, Robert P. Kaufman.

## Part 1. Complex solutions

2. We now state one of our main results. The proof will be completed in §11.

THEOREM 1. *Let $\Omega(z, y, y', \ldots, y^{(m)})$ be a polynomial in $z, y, y', \ldots, y^{(m)}$ which is not identically zero. Let $y_0(z)$ be a function which is defined and meromorphic on a region $R$, which for some $\varepsilon > 0$ contains the strip in the right half-plane which is bounded by the lines $x = 0$, $y = \varepsilon$, and $y = -\varepsilon$. Suppose that $y_0(z)$ has no poles on $(0, +\infty)$ and satisfies the equation $\Omega = 0$ at each point of analyticity. For each $x > 0$, let $\delta(x)$ denote the minimum of $\varepsilon$ and the distance from $x$ to the set of poles in $R$ of $y_0(z)$. (If $y_0$ has no poles, set $\delta(x) = \varepsilon$.) For $r > 0$, denote by $\bar{n}_1(r)$ the number of distinct zeros of $y_0(z)$ in $|z| \leqslant r$ which lie in the region bounded by $x = 0$ and the curves $y = \delta(x)$ and $y = -\delta(x)$. Then there exist positive constants $c_1$ and $c_2$, such that if we set*

$$a(x) = c_1 \exp\left(c_2 \int_\varepsilon^x \frac{dt}{\delta(t)}\right) \quad \text{for } x \geqslant \varepsilon, \tag{2}$$

*then the following hold:*

(a) *If $m = 1$, then $|y_0(x)| \leqslant \exp(a(x))$ for $x \geqslant \varepsilon$.*

(b) *If $m > 1$, and if for some nonnegative integer $q$, $y_0(z)$ is not a solution of the equation $\Omega_q = 0$ (where $\Omega_q$ is the homogeneous part of $\Omega$ of total degree $q$ in $y, y', \ldots, y^{(m)}$), then for $x \geqslant \varepsilon$,*

$$|y_0(x)| \leqslant \exp\big(a(x)(\bar{n}_1(a(x)) + \log^+ a(x))\big). \tag{3}$$

(c) *If $m = 2$, and if for some positive integer $q$ for which $\Omega_q$ is not the zero polynomial, the function $y_0(z)$ is a solution of $\Omega_q = 0$, then for $x \geqslant \varepsilon$,*

$$|y_0(x)| \leqslant \exp_2\Big(a(x)\Big(1 + \big(\bar{n}_1(a(x))\big)^2\Big)\Big). \tag{4}$$

3. **Notation.** For $0 < R \leqslant +\infty$, and a meromorphic function $g(z)$ in $|z| < R$, we will use the standard notation for the Nevanlinna functions

$m(r, g)$, $N(r, g)$ and $T(r, g)$ introduced in [18, pp. 6, 12]. We will also use the notation $n(r, g)$ to denote the number of poles (counting multiplicity) of $g$ in $|z| \leqslant r$, and we will denote by $\bar{n}(r, g)$ the number of distinct poles of $g$ in $|z| \leqslant r$. As in [18, p. 70], we will denote by $\bar{N}(r, g)$ the function obtained by replacing $n(r, g)$ by $\bar{n}(r, g)$ in the definition of $N(r, g)$. For an equation (1), we will denote by $\Omega_q$ the homogeneous part of total degree $q$ in $y, y', \ldots, y^{(m)}$. As in [2], we use the abbreviation "n.e. in [0, 1)" (*nearly everywhere* in [0, 1)) to mean everywhere in [0, 1) except for a set $W$ such that $\int_W (1-r)^{-1}\, dr < +\infty$. We shall make frequent use of the following simple fact proved in [3, p. 68]. If $f(r)$ and $h(r)$ are monotone nondecreasing functions on [0, 1) such that n.e. in [0, 1), $f(r) \leqslant h(r)$, then there is a positive constant $b$, with $b < 1$, such that for all $r$ in [0, 1),

$$f(r) \leqslant h(s(r)), \quad \text{where } s(r) = 1 - b(1-r). \tag{5}$$

4. We begin with two results which are improvements of [2, §3] and [2, §4], respectively.

LEMMA A. *Let $\Omega(z, y, y', \ldots, y^{(m)})$ be a polynomial in $y, y', \ldots, y^{(m)}$, whose coefficients are functions of $z$ which are defined and meromorphic in the unit disk. For each $r < 1$, let $\Phi(r)$ be the maximum of the Nevanlinna characteristics of the coefficients. Let $y_0(z)$ be a meromorphic function in the unit disk which is not identically zero and which satisfies the equation $\Omega = 0$, but which for some nonnegative integer $q$ does not satisfy the equation $\Omega_q = 0$. Then n.e. in* [0, 1), *we have as $r \to 1$,*

$$T(r, y_0) = O\big(\bar{N}(r, y_0) + \bar{N}(r, 1/y_0) + \Phi(r) + \lambda(r, y_0)\big), \tag{6}$$

*where*

$$\lambda(r, y_0) = \log\big((1 + T(r, y_0))/(1-r)\big).$$

PROOF. If $\sigma$ is the maximum integer such that $y_0$ is not a solution of $\Omega_\sigma = 0$, then since $y_0$ satisfies $\Omega = 0$, we have

$$\sum_{q=0}^{\sigma} R_q(y_0(z))^q = 0 \tag{7}$$

where $R_q$ is a polynomial in $y_0'/y_0, \ldots, y_0^{(m)}/y_0$, whose coefficients are those of $\Omega_q$, and where $R_\sigma \not\equiv 0$. Using the elementary rules for calculating with the Nevanlinna characteristic [18, p. 14], it follows [14, p. 108] that

$$T(r, y_0) \leqslant \sum_{q=0}^{\sigma} T(r, R_q) + O(1) \quad \text{as } r \to 1. \tag{8}$$

Now by the Nevanlinna theory [19, p. 256], it follows that $m(r, y_0'/y_0) = O(\lambda(r, y_0))$ n.e. in [0, 1) as $r \to 1$. By induction on $j$, it thus follows that n.e. in [0, 1),

$$m(r, y_0^{(j)}/y_0) = O(\lambda(r, y_0)), \tag{9}$$

and

$$T(r, y_0^{(j)}) = O(T(r, y_0) + \lambda(r, y_0)). \tag{10}$$

From (9), we have n.e. in [0, 1),

$$m(r, R_q) = O(\Phi(r) + \lambda(r, y_0)). \tag{11}$$

Now it is easily verified that if we set $w_0 = y_0'/y_0$, then for each $j$, $y_0^{(j)}/y_0$ can be written as a polynomial in $w_0, w_0', \ldots, w_0^{(j-1)}$. Since $N(r, w_0^{(j)}) = O(N(r, w_0))$, it follows that

$$N(r, R_q) = O(N(r, w_0) + \Phi(r)) \quad \text{as } r \to 1. \tag{12}$$

Since $N(r, w_0) = \overline{N}(r, y_0) + \overline{N}(r, 1/y_0)$, the result now follows from (8), (11) and (12).

5. Lemma B. *Let $\Omega(z, y, y', \ldots, y^{(m)})$ be a polynomial in $z, y, y', \ldots, y^{(m)}$. Let $D$ be a simply-connected region which is not the whole plane, and let $g$ be a univalent analytic mapping of the unit disk onto $D$. Let $y_0(z)$ be a meromorphic function on $D$ which is not identically zero and which satisfies $\Omega = 0$, but which for some nonnegative integer $q$, does not satisfy the equation $\Omega_q = 0$. Then if we set $\varphi(\zeta) = y_0(g(\zeta))$ for $|\zeta| < 1$, we have n.e. in* [0, 1) *as $r \to 1$,*

$$T(r, \varphi) = O\left(\overline{N}(r, \varphi) + \overline{N}(r, 1/\varphi) + \lambda(r, \varphi)\right), \tag{13}$$

*where $\lambda(r, \varphi)$ is as defined in Lemma* A.

Proof. Letting $f$ be the inverse of $g$, it easily follows by induction that for $j \geqslant 1$ and $z$ in $D$,

$$y_0^{(j)}(z) = \sum_{k=1}^{j} \varphi^{(k)}(f(z))\Gamma_{kj}(f'(z), \ldots, f^{(j)}(z)), \tag{14}$$

where $\Gamma_{kj}(u_1, \ldots, u_j)$ is a polynomial in $u_1, \ldots, u_j$ with constant coefficients. If $H_1(z), \ldots, H_\sigma(z)$ are the coefficients of $\Omega$, it follows that $\varphi(\zeta)$ is a solution of an algebraic differential equation $\Lambda(\zeta, \varphi, \varphi', \ldots, \varphi^{(m)}) = 0$, where each coefficient $G(\zeta)$ is a polynomial (with constant coefficients) in the variables $H_k(g(\zeta))$, $f^{(j)}(g(\zeta))$, where $1 \leqslant k \leqslant \sigma$ and $1 \leqslant j \leqslant m$. In addition, it is clear from (14) that $\varphi$ is not a solution of $\Lambda_q = 0$. From the Koebe distortion theorem [15, p. 351] and its consequence [15, Theorem 17.4.7, p. 353], there exist positive constants $L_1$, $L_2$ and $c$, such that

$$L_2(1 - r) \leqslant |g'(\zeta)| \leqslant L_1(1 - r)^{-3}, \tag{15}$$

$$|g(\zeta)| \leqslant c(1 - r)^{-2}, \tag{16}$$

on any circle $|\zeta| = r < 1$. Since each $H_k(z)$ is a polynomial, clearly

$$|H_k(g(\zeta))| \leqslant c_k(1-r)^{-2\alpha_k} \quad \text{on } |\zeta| = r,$$

where $c_k$ is a positive constant, and $\alpha_k$ is the degree of $H_k(z)$. From (15),

$$|f'(g(\zeta))| \leqslant L_2^{-1}(1-r)^{-1} \quad \text{on } |\zeta| = r < 1.$$

From the Cauchy formula for the derivative, applied to $u_j(\zeta) = f^{(j)}(g(\zeta))$ (using the contour $|z| = (1 + |\zeta|)/2$), and (15), it easily follows by induction on $j$, that there exist constants $k_j > 0$ and $a_j > 0$ such that

$$|f^{(j)}(g(\zeta))| \leqslant k_j(1-r)^{-a_j} \quad \text{on } |\zeta| = r < 1.$$

It thus follows (using [18, p. 140, (3)]) that for each coefficient $G(\zeta)$ in $\Lambda$, we have

$$T(r, G) = O\big(\log(1/(1-r))\big) \quad \text{as } r \to 1,$$

and hence the result follows from Lemma A.

6. To handle the case of second-order equations completely, we need the following result.

LEMMA C. *Let $y(z)$ be any meromorphic function on the unit disk which is not identically zero, and let $w = y'/y$. Then there exist positive constants $b$, $K$ and $K_1$, with $b < 1$, such that if $s(r) = 1 - b(1-r)$, then for all $r$ in* (0, 1),

$$T(r, y) \leqslant K\Big((1-r)^{-1}N(s(r), y) + \exp\big(K_1\Psi(s(r))\big)\Big), \tag{17}$$

*where $\Psi(r) = (1-r)^{-1}\Phi(r)$, and where,*

$$\Phi(r) = \log(1/(1-r))(T(r, w) + 1) + N(r, w)\log^+ N(r, w). \tag{18}$$

PROOF. Clearly we can assume $w \not\equiv 0$. Let $\{a_n\}$ and $\{b_m\}$ be the sequences of zeros and poles, respectively, of $w$ in the disk (each arranged in order of increasing moduli). Let $0 < r < R < 1$, and let $z = re^{i\theta}$ be any point on $|z| = r$ which is not a zero or pole of $w$. Then from the Poisson-Jensen formula [18, p. 3] it follows that

$$\log|w(z)| \leqslant ((R+r)/(R-r))m(R, w) + J(R, z), \tag{19}$$

where

$$J(R, z) = \sum_{|b_m| < R} \log\left|\frac{R^2 - \bar{b}_m z}{R(z - b_m)}\right|. \tag{20}$$

Let $r_m = |b_m|$. Clearly if $r_m < R$, then $|R^2 - \bar{b}_m z| \leqslant 2R$ since $R < 1$. Hence if $r$ is not equal to any $r_m$, then

$$J(R, z) \leqslant \sum_{r_m < R} \log(2/|r - r_m|). \tag{21}$$

Now from the definition of $N(R, w)$, it easily follows that there is a constant $c_1 \geqslant 0$, such that for $\frac{1}{2} \leqslant r < R < 1$,

$$n(r, w) \leqslant (R - r)^{-1}(N(R, w) + c_1). \tag{22}$$

For the moment, assume that the sequence $\{b_m\}$ is not empty. If this sequence is infinite, let $m_0$ be an index such that $r_m \geqslant \frac{1}{2}$ for $m \geqslant m_0$. If the sequence $\{b_m\}$ is finite, let $m_0$ be the last index. Now clearly, $m \leqslant n(r_m, w)$, so applying (22) with $r = r_m$ and $R = \alpha(r_m)$, where $\alpha(r) = (3r + 1)/4$, it easily follows that if we set

$$\varepsilon_m = \left( \frac{1 - r_m}{N(\alpha(r_m), w) + c_1} \right)^2 \quad \text{for } m > m_0, \tag{23}$$

then $\sum_{m>m_0} \varepsilon_m$ converges and $0 < \varepsilon_m < 16$ for $m > m_0$. Now set

$$\alpha_m = \varepsilon_m(1 - r_m)/18 \quad \text{for } m > m_0. \tag{24}$$

Then it is easily verified that if $E_1$ is the union of all intervals $[r_m - \alpha_m, r_m + \alpha_m]$ for $m > m_0$, together with the set $\{|a_n|: n \geqslant 1\}$, then $E_1$ lies in $[0, 1)$ and

$$\int_{E_1} \frac{dx}{1 - x} \quad \text{is finite.} \tag{25}$$

Referring to (21), let $r \geqslant \alpha(r_{m_0})$, $r \not\in E_1$, and take $R = \alpha(r)$. Then for $m > m_0$, $|r - r_m| \geqslant \alpha_m$, while for $m \leqslant m_0$, $|r - r_m| \geqslant (1 - r_{m_0})/4$. Thus from (21),

$$J(\alpha(r), z) \leqslant c_2 + \sum \log(2/\alpha_m), \tag{26}$$

where $c_2$ is a constant independent of $r$, and the sum is over all $m > m_0$ for which $r_m < \alpha(r)$. If we set $\beta(r) = (1 + r)/2$, then clearly $\alpha(r_m) \leqslant \beta(r)$ if $r_m < \alpha(r)$. Noting that the sum in (26) has at most $n(\alpha(r), w)$ terms, it follows easily from (19) (with $R = \alpha(r)$), (23), (24) and (26) that there are positive constants $c_3$ and $R_1$, with $\frac{1}{2} \leqslant R_1 < 1$, such that for $R_1 \leqslant r < 1$ and $r \not\in E_1$, we have on $|z| = r$,

$$\log|w(z)| \leqslant (8/(1 - r))m(\alpha(r), w) + U(r), \tag{27}$$

where

$$U(r) = c_3 n(\alpha(r), w)\big(\log^+ N(\beta(r), w) + \log(1/(1 - \alpha(r)))\big). \tag{28}$$

(Of course, (27) is valid even if the sequence $\{b_m\}$ is empty by (9).) If we let $E$ be the union of $E_1$ and $[0, R_1]$, then clearly $E$ satisfies (25), and if $r \not\in E$, (27) is valid on $|z| = r$. Now let $V(r)$ denote the right side of (27). Choose $\varepsilon$, $0 < \varepsilon < 1$, such that $y$ has no zeros or poles on $0 < |z| \leqslant \varepsilon$. Then clearly for $\frac{1}{2} \leqslant r < 1$,

$$|N(r, y)| \leqslant (n(r, y)/\varepsilon) + n(0, y)\log 2, \tag{29}$$

$$|N(r, 1/y)| \leqslant (n(r, 1/y)/\varepsilon) + n(0, 1/y)\log 2. \tag{30}$$

By Jensen's formula [19, p. 168], there is a constant $\lambda > 0$ such that on $(0, 1)$,

$$T(r, 1/y) = T(r, y) + h(r) \quad \text{where } |h(r)| \leqslant \lambda. \tag{31}$$

Set

$$c_4 = n(0, y)\log 2 + n(0, 1/y)\log 2 + \lambda,$$

and let

$$B(r) = \frac{2}{\varepsilon}\, n(r, y) + \left(2\pi + \frac{1}{\varepsilon}\right)\exp(V(r)) + c_4. \tag{32}$$

We now assert that if $r$ belongs to $[0, 1)$ and $r \notin E$, then

$$\log^+|y(z)| \leqslant B(r) \quad \text{on } |z| = r. \tag{33}$$

Suppose that (33) fails for a point $z_0 = re^{i\theta_0}$, where $r \notin E$. Hence $\log|y(z_0)| > B(r)$. Let $z_1 = re^{i\theta}$ (where $\theta_0 < \theta < \theta_0 + 2\pi$) be any point on $|z| = r$ distinct from $z_0$, and let $\Gamma$ be the arc $\zeta = re^{-i\varphi}$, where $-\theta \leqslant \varphi \leqslant -\theta_0$. Since $r \notin E$, clearly $y$ is analytic and nowhere zero on some simply-connected neighborhood of $\Gamma$, so there exists an analytic branch $g$ of $\log y$ on this neighborhood. Since $g' = y'/y = w$, we have $g(z_0) - g(z_1) = \int_\Gamma w(\zeta)\, d\zeta$. It follows by taking exponentials, and using (27), that $\log|y(z_1)| > B(r) - 2\pi(\exp V(r))$. Clearly, this holds for $z_1 = z_0$ also, so that (using (32)) it follows that

$$m(r, y) > B(r) - 2\pi\big(\exp(V(r))\big) \quad \text{and} \quad m(r, 1/y) = 0. \tag{34}$$

But by the argument principle and (27) we have $n(r, 1/y) \leqslant n(r, y) + \exp(V(r))$. Since $m(r, 1/y) = 0$, it then easily follows from (29)–(32) that we obtain an inequality which is in direct contradiction to the first inequality in (34), thus proving (33). Hence n.e. in $[0, 1)$, we have $m(r, y) \leqslant B(r)$ and thus $T(r, y) \leqslant N(r, y) + B(r)$. Now estimating $B(r)$ using (22) (applied to $y$ and to $w$) and then using (5), it follows by a routine calculation that (17) holds.

7. **Remark.** We observe that the estimate (17) for $T(r, y)$ involves not only the growth of $y'/y$, but also the counting function for the poles of $y$. (We can replace this counting function for the poles by the counting function for the zeros of $y$ by applying Lemma C to $1/y$ instead of $y$ in view of (31).) We remark here that it is not possible to obtain an estimate for $T(r, y)$ in terms of the growth of $y'/y$ alone, because it is possible [4, p. 334] to construct meromorphic functions in the disk which have arbitrarily rapid growth in the disk, but whose logarithmic derivatives are of finite order in the disk (and, in fact, of bounded characteristic using a similar construction).

8. THEOREM 2. *Let $\Omega(z, y, y', \ldots, y^{(m)})$ be a polynomial in $z, y, \ldots, y^{(m)}$ which is not identically zero. Let $D$ be a simply-connected region which is not the whole plane. Let $f(z)$ be any univalent analytic map of $D$ onto the unit disk, and set $L(z) = 1/(1 - |f(z)|)$. Let $y(z)$ be an analytic function on $D$ which satisfies the equation $\Omega = 0$, and let $\bar{n}_1(r)$ denote the number of distinct zeros of*

*$y(z)$ in $D$ which lie in $|z| < r$. Then there exist positive constants $K$ and $K_1$ such that the following hold on $D$.*

(a) *If $m = 1$, then $|y(z)| \leqslant K_1 \exp(L(z))^K$.*

(b) *If $m \geqslant 1$, and for some nonnegative integer $q$, the function $y(z)$ is not a solution of the equation $\Omega_q = 0$, then,*

$$|y(z)| \leqslant \exp\Big(KL(z)\Big(\bar{n}_1\Big(K(L(z))^2\Big) + \log^+ KL(z)\Big)\Big). \tag{35}$$

(c) *If $m = 2$, and if for some positive integer $q$ for which $\Omega_q$ is not the zero polynomial, $y(z)$ is a solution of the equation $\Omega_q = 0$, then*

$$|y(z)| \leqslant \exp_2\Big(K_1 (L(z))^K \Big(1 + \Big(\bar{n}_1\Big(K_1 (L(z))^2\Big)\Big)^2\Big)\Big). \tag{36}$$

PROOF. Part (a) follows immediately from [1, §2, p. 574]. Let $g$ be the inverse of $f$, and set $\varphi(\zeta) = y(g(\zeta))$ for $|\zeta| < 1$. In view of (16), it easily follows that for $0 \leqslant r < 1$,

$$\bar{n}(r, 1/\varphi) \leqslant \bar{n}_1\big(c(1-r)^{-2}\big), \tag{37}$$

and hence, for $0 < r < 1$,

$$\bar{N}(r, 1/\varphi) \leqslant (1/\varepsilon)\bar{n}_1\big(c(1-r)^{-2}\big), \tag{38}$$

where $\varepsilon$ is a constant, $0 < \varepsilon < 1$, such that $\varphi$ has no zeros on $0 < |\zeta| \leqslant \varepsilon$.

Now assume the hypothesis of part (b). From Lemma B and the fact that $\varphi$ is analytic on $|\zeta| < 1$, it easily follows, using (38), that n.e. in $[0, 1)$,

$$T(r, \varphi) = O\Big(\bar{n}_1\big(c(1-r)^{-2}\big) + \log(1/(1-r))\Big). \tag{39}$$

Using the relation between the maximum modulus of $\varphi$ and $T(r, \varphi)$ (see [18, p. 140, (13)]), and then (5), it easily follows (noting that $y(z) = \varphi(f(z))$) that (35) is valid on $D$, proving part (b).

Now assume the hypothesis of part (c). Then clearly $h = y'/y$ satisfies a first-order equation

$$\sum H_{kj}(z) v^k (v')^j = 0, \tag{40}$$

where the $H_{kj}$ are polynomials. Set $u(\zeta) = h(g(\zeta))$ for $|\zeta| < 1$. Then $u(\zeta)$ satisfies the equation

$$\sum F_{kj}(\zeta)(u(\zeta))^k (u'(\zeta))^j = 0, \tag{41}$$

where $F_{kj}(\zeta) = H_{kj}(g(\zeta))/(g'(\zeta))^j$. Set $p = \max\{k + j\colon F_{kj} \not\equiv 0\}$, and $m = \max\{j\colon F_{p-j,j} \not\equiv 0\}$. Using (15) and (16), it is proved in [1, pp. 575–576] that there exist positive constants $K_2$, $K_3$, $q$, $\sigma$ and $r_0$, with $q \geqslant 1$ and $r_0 < 1$, such that on $|\zeta| = r < 1$, $|F_{kj}(\zeta)| \leqslant K_2(1-r)^{-q}$ for all $(k, j)$, while $|F_{p-m,m}(\zeta)| \geqslant K_3(1-r)^{\sigma}$ if $r > r_0$. It then follows from [3, §3, p. 63] that there are positive constants $K_4$ and $b_1 < 1$ such that if $s_1(r) = 1 - b_1(1-r)$, then on $[0, 1)$,

$$T(r, u) \leqslant K_4\big((1-r)^{-\lambda} + n(s_1(r), u) + N(s_1(r), u)\big), \tag{42}$$

where $\lambda = q + \sigma > 1$. Now if $\varphi(\zeta) = y(g(\zeta))$, then clearly, $\varphi'(\zeta)/\varphi(\zeta) = g'(\zeta)u(\zeta)$. Hence $N(r, u) = \bar{N}(r, 1/\varphi)$ since $\varphi$ and $1/g'$ are analytic. Estimating $n(s_1(r), u)$ in (42), in terms of $N((1 + s_1(r))/2, u)$, using (22), and estimating $T(r, g')$ using (15), it easily follows that on [0, 1),

$$T(r, \varphi'/\varphi) \leqslant K_5(1-r)^{-1}\big((1-r)^{-\lambda+1} + \bar{N}(s_2(r), 1/\varphi)\big), \tag{43}$$

where $K_5 > 0$ and $s_2(r) = 1 - b_2(1-r)$, where $0 < b_2 < 1$. Now using (38), Lemma C and the analyticity of $\varphi$, it follows by routine calculation, that on [0, 1),

$$T(r, \varphi) \leqslant K_7 \exp\big(K_6(\Psi(s_3(r)))\big), \tag{44}$$

where

$$\Psi(r) \leqslant (1-r)^{-K}\Big(1 + \big(\bar{n}_1(c_1(1-r)^{-2})\big)^2\Big),$$

and where $K$, $K_6$, $K_7$ and $c_1$ are positive constants and $s_3(r) = 1 - b_3(1-r)$ for some positive $b_3 < 1$. Converting (44) to an estimate on the maximum modulus of $\varphi$, and noting that $y(z) = \varphi(f(z))$, we obtain (36), proving part (c).

9. Lemma D. *Let $D$ be a simply-connected region which is not the whole plane. Let $z_0$ be a point of $D$ and let $f$ be the univalent analytic mapping of $D$ onto the unit disk satisfying $f(z_0) = 0$, $f'(z_0) > 0$. For each $z$ in $D$, let $\Delta(z)$ denote the distance from $z$ to the boundary of $D$. Then for all $z$ in $D$,*

$$\left|\frac{1+f(z)}{1-f(z)}\right| \leqslant \exp\left(2\int_{z_0}^{z} \frac{|d\zeta|}{\Delta(\zeta)}\right), \tag{45}$$

*where the contour of integration is any path from $z_0$ to $z$ lying in $D$.*

Proof. If we denote the inverse of $f$ by $g$, then we assert that

$$|f'(g(\zeta))| \leqslant (1 - |\zeta|^2)/\Delta(g(\zeta)) \quad \text{on } |\zeta| < 1. \tag{46}$$

To prove (46), let $\sigma = \Delta(g(\zeta))$. Then for $|w| < 1$, the point $g(\zeta) + \sigma w$ lies in $D$, so that $H(w) = f(g(\zeta) + w\sigma)$ is an analytic function from the disk into the disk and $H(0) = \zeta$. Let $F(u) = (u - \zeta)/(1 - \bar{\zeta}u)$, and $\psi(w) = F(H(w))$. Then $\psi$ maps the disk into the disk and $\psi(0) = 0$, so by Schwarz's lemma, $|\psi'(0)| \leqslant 1$ from which (46) follows.

Let $h(z)$ denote the analytic branch on $D$ of $\frac{1}{2}\log((1 + f(z))/(1 - f(z)))$ which vanishes at $z_0$. Then from (46), $|h'(z)| \leqslant 1/\Delta(z)$ on $D$. Since $(1 + f(z))/(1 - f(z)) = \exp(2h(z))$ on $D$, clearly (45) follows immediately.

10. Lemma E. *Let $y = \delta(x)$ be a function on $(0, +\infty)$ such that for some*

*$\varepsilon > 0, 0 < \delta(x) \leqslant \varepsilon$ for each $x$, and which satisfies a Lipschitz condition of the form $|\delta(x_1) - \delta(x_2)| \leqslant |x_1 - x_2|$ for all positive $x_1, x_2$. Let $D_\delta$ be the region bounded by the $y$-axis and the curves $y = \delta(x)$ and $y = -\delta(x)$ for $x > 0$. For each $z$ in $D_\delta$, let $\Delta(z)$ denote the distance from $z$ to the boundary of $D_\delta$. Then*

(a) $\Delta(x) \geqslant \delta(x)/\sqrt{2}$ *for* $x \geqslant \varepsilon$.

(b) *If $f$ is the univalent analytic mapping of $D_\delta$ onto the unit disk such that $f(\varepsilon) = 0, f'(\varepsilon) > 0$, then for $x > \varepsilon, f(x)$ is a positive increasing function and*

$$\frac{1}{1 - f(x)} \leqslant \exp\left(2\sqrt{2} \int_\varepsilon^x \frac{dt}{\delta(t)}\right) \quad \text{for } x \geqslant \varepsilon. \tag{47}$$

PROOF. For part (a), let $x \geqslant \varepsilon$, and let $\lambda_1$ and $\lambda_2$ denote the lines of slope 1 and $-1$, respectively, through $(x, \delta(x))$. In view of the Lipschitz condition on $\delta$, it is easy to see that the open triangle bounded by $\lambda_1, \lambda_2$ and the real axis lies in $D_\delta$. By the symmetry of the region $D_\delta$ with respect to the real axis, part (a) now follows easily.

To prove part (b), we observe first that since $D_\delta$ is symmetric with respect to the real axis, it follows by the uniqueness part of the Riemann mapping theorem [20, p. 229] that $f(z) \equiv \bar{f}(\bar{z})$. Thus $f$ is real on $(0, +\infty)$, and since $f'$ is nowhere zero, it follows that $f'(x) > 0$ on $(0, +\infty)$. Since $f(\varepsilon) = 0$, clearly $f$ is positive and increasing for $x > \varepsilon$, and $f(x) < 1$ by the maximum modulus principle. Part (b) now follows immediately from part (a) and Lemma D using the segment $[\varepsilon, x]$ as the contour of integration.

11. PROOF OF THEOREM 1. Assume the hypothesis and notation of Theorem 1. It is easy to see that the function $\delta(x)$ defined in Theorem 1 satisfies the hypothesis of Lemma E, and since the solution $y_0(z)$ is obviously defined and analytic on $D_\delta$, the conclusions of Theorem 1 now follow immediately from Theorem 2 (applied to $D_\delta$) and Lemma E.

12. **Remark.** If in Theorem 1 the solution $y_0(z)$ is a meromorphic function on the plane whose growth is known, then one can crudely estimate the quantity $\bar{n}_1(r)$ since $\bar{n}_1(r) \leqslant \bar{n}(r, 1/y_0)$, and the latter quantity can be estimated from the growth. (E.g., if $y_0$ is of finite order less than $\lambda$, then by [19, p. 221], $n(r, 1/y_0) \leqslant \lambda r^\lambda$ for all sufficiently large $r$.) Of course, $\bar{n}_1(r)$ can be much smaller than $\bar{n}(r, 1/y_0)$, as in the case of $y_0(z) = \exp_2 z - 1$, which satisfies part (b) of Theorem 1 for $m = 2$ in the strip $x > 0, -1 < y < 1$, and has no zeros in this strip. (Here $\delta(x) \equiv 1$.)

## PART 2. REAL SOLUTIONS

13A. **Preliminaries.** Suppose we are given a second-order algebraic differential equation

$$\Omega(x, y, y', y'') = \sum f_{ijk}(x) y^i (y')^j (y'')^k = 0, \tag{48}$$

where we assume that the coefficients $f_{ijk}$ are polynomials. (More general classes of coefficients will be discussed in §14.)

Set $p = \max\{i + j + k: f_{ijk} \not\equiv 0\}$, and let $I$ be the set of all $(i, j, k)$ with $f_{ijk} \not\equiv 0$ and $i + j + k = p$. Set $\Delta = \{j + 2k$: for some $i$, $(i, j, k) \in I\}$, say $\Delta$ consists of $q_1 < q_2 < \cdots < q_r$, where $r \geqslant 1$. For $1 \leqslant m \leqslant r$, let $J_m$ be the subset of $I$ consisting of those $(i, j, k)$ such that $j + 2k = q_m$, and let $\delta_m$ be the maximum degree of all coefficients $f_{ijk}$ where $(i, j, k)$ belongs to $J_m$. If $r > 1$, set

$$A_1 = \max\{(\delta_m - \delta_r)/(q_r - q_m): 1 \leqslant m < r\}. \tag{49}$$

For $\varepsilon > 0$, set

$$Q_m(x, \varepsilon) = \sum \left\{f_{ijk}(x)(1 + \varepsilon)^k: (i, j, k) \text{ belongs to } J_m\right\}. \tag{50}$$

It is easy to see that there exists $\varepsilon_0 > 0$ with the property that if $0 < \varepsilon < \varepsilon_0$, then the degree in $x$ of the polynomial $Q_m(x, \varepsilon)$ is $\delta_m$ for all $m$, $1 \leqslant m \leqslant r$.

Set $\beta = \min\{k$: for some $(i, j), f_{ijk} \not\equiv 0\}$ and define $\lambda = \max\{i + j: f_{ij\beta} \not\equiv 0\}$. Let $j_1 < j_2 < \cdots < j_s$ be the set of all $j$ for which $f_{\lambda-j,j,\beta} \not\equiv 0$, and let $d_k$ denote the degree of this polynomial for $j = j_k$. If $s > 1$, set

$$A_2 = \max\{(d_k - d_s)/(j_s - j_k): 1 \leqslant k < s\}. \tag{51}$$

With this notation we now prove

13B. THEOREM 3. *Let $A$ be a real number defined as follows. If $r > 1$ and $s > 1$, set $A = \max\{A_1, A_2\}$. If $r > 1$ and $s = 1$, set $A = A_1$. If $r = 1$ and $s > 1$, set $A = A_2$. If $r = 1$ and $s = 1$, let $A$ be any real number. Then if $y(x)$ is any positive, increasing solution of* (48) *on an interval $(x_0, +\infty)$, having a continuous second derivative and satisfying the conditions that* (i) *for every $\alpha \geqslant 0$, $y(x)/x^\alpha \to +\infty$ as $x \to +\infty$, and* (ii) *$x^{-A}y'/y \to +\infty$ as $x \to +\infty$, then there exist positive constants $c$ and $x_1$ such that $y(x) \leqslant \exp_2(x^c)$ for $x \geqslant x_1$.*

PROOF. We will first prove that for any $\varepsilon_1 > 0$, there exists an $x_1 = x_1(\varepsilon_1)$ such that

$$y' < y^{1+\varepsilon_1} \quad \text{on } (x_1, +\infty). \tag{52}$$

Choose a fixed number $\varepsilon < \varepsilon_1$, with $0 < \varepsilon < \varepsilon_0$ (where $\varepsilon_0$ is as in §13A) such that

$$\varepsilon < (p - (i + j + k))/2(j + 2k), \tag{53}$$

if $i + j + k < p$ (where $p$ is as in §13A) and $j + 2k > 0$. To prove (52) it obviously suffices to prove $y' < y^{1+\varepsilon}$ on an interval $(x_1, +\infty)$.

We assume the contrary. Since the integral $\int_{x_0+1}^{+\infty}(y'/y^{1+\varepsilon})\,dx$ converges, it follows that $y' < y^{1+\varepsilon}$ holds except on a set of finite measure, so by our assumption and Rolle's theorem (applied to $y'/y^{1+\varepsilon}$) it easily follows that on

a sequence $\{\zeta_n\} \to +\infty$, we have

$$yy'' = (1+\varepsilon)(y')^2 \quad \text{and} \quad y' < y^{1+\varepsilon}. \tag{54}$$

Now by isolating the terms in (48) where $i+j+k=p$, and dividing by $y^p$, we obtain

$$\sum_{i+j+k=p} f_{ijk}(y'/y)^j(y''/y)^k = -\Phi, \tag{55}$$

where $\Phi = \sum_{i+j+k<p} h_{ijk}$, and where

$$h_{ijk} = f_{ijk}\left(\frac{y'}{y}\right)^j\left(\frac{y''}{y}\right)^k y^{i+j+k-p} \quad \text{for } i+j+k<p. \tag{56}$$

From (53), (54) and (i) of the hypothesis, it follows that for all sufficiently large $n$,

$$|h_{ijk}| \leqslant |f_{ijk}|(1+\varepsilon_0)^k y^{(i+j+k-p)/2} \quad \text{at } \zeta_n, \tag{57}$$

and, hence, from (i), it follows that for any $\alpha \geqslant 0$,

$$\Phi(\zeta_n) = o(\zeta_n^{-\alpha}) \quad \text{as } n \to \infty. \tag{58}$$

We consider the polynomial

$$G(x, v) = \sum_{m=1}^{r} Q_m(x, \varepsilon)v^{q_m}$$

(using the notation of §13A). Since $G(x, v)$ is of degree $q_r$ and has polynomial coefficients, it follows now (e.g. from the factorization theorem of W. Strodt [23, §62]) that there exist $d$ distinct functions $B_1(z), \ldots, B_d(z)$, each defined and meromorphic in a region $|\arg z| < \pi$, $|z| > K$, with the following properties: (A) If $B_j(x) \not\equiv 0$, then there exist a complex constant $c_j$ and a real constant $\alpha_j$ such that $B_j(x)/c_jx^{\alpha_j} \to 1$ as $x \to +\infty$; (B) There exists a real number $b$ such that $x^b(B_i - B_j) \to \infty$ if $i \neq j$; (C) There exist positive integers $m_1, \ldots, m_d$ such that $m_1 + \cdots + m_d = q_r$, and

$$G(x, v) = Q_r(x, \varepsilon)(v - B_1(x))^{m_1} \cdots (v - B_d(x))^{m_d}, \tag{59}$$

for all functions $v = v(x)$ on $(K, +\infty)$. From (B), it follows that on some interval $[x_2, +\infty)$,

$$|x^bB_i(x) - x^bB_j(x)| > 2 \quad \text{if } i \neq j. \tag{60}$$

Let us denote the left side of (55) by $\Lambda(x)$. Then in view of (54) and §13A, clearly $\Lambda(\zeta_n) = G(\zeta_n, y'(\zeta_n)/y(\zeta_n))$. If we set $u_n = \zeta_n^b y'(\zeta_n)/y(\zeta_n)$, then since $Q_r(x, \varepsilon)$ is a nontrivial polynomial in $x$, it easily follows from (55), (58) and (59) that for all $\alpha \geqslant 0$,

$$(u_n - \zeta_n^bB_1(\zeta_n))^{m_1} \cdots (u_n - \zeta_n^bB_d(\zeta_n))^{m_d} = o(\zeta_n^{-\alpha}) \tag{61}$$

as $n \to \infty$. If $a_n$ denotes the left side of (61), then for some $n_0$, $|a_n| < (1/2)^q$ for $n \geqslant n_0$. Hence for each $n \geqslant n_0$, it is clearly impossible for $|u_n - \zeta_n^b B_j(\zeta_n)| > \frac{1}{2}$ for each $j = 1, \ldots, d$. Thus clearly there is an index $t$, $1 \leqslant t \leqslant d$, such that $|u_n - \zeta_n^b B_t(\zeta_n)| \leqslant \frac{1}{2}$ holds for infinitely many $n$, say $n_1 < n_2 < \ldots$. In view of (60), if $k$ is sufficiently large, then $|u_n - \zeta_n^b B_j(\zeta_n)| \geqslant 1$ for $n = n_k$ and all $j \neq t$, and hence, from (61), when $n = n_k$, it follows that for any $\alpha \geqslant 0$, $u_n - \zeta_n^b B_t(\zeta_n) = o(\zeta_n^{-\alpha})$ as $k \to \infty$. Thus clearly, for any $\alpha \geqslant 0$, when $n = n_k$,

$$y'(\zeta_n)/y(\zeta_n) - B_t(\zeta_n) = o(\zeta_n^{-\alpha}) \quad \text{as } k \to \infty. \tag{62}$$

If $B_t \equiv 0$, this is impossible, since by hypothesis, $x^{-A}y'/y \to +\infty$ as $x \to +\infty$ for some constant $A$. If $B_t \not\equiv 0$, then clearly $r > 1$, and the algorithm in [23, §28, p. 236] (or [24, §2.85, p. 28]) shows that the constant $\alpha_t$ in Property (A), satisfies $\alpha_t \leqslant A_1$ where $A_1$ is defined by (49). Thus $\zeta_n^{-A}B_t(\zeta_n)$ tends to a finite limit as $n \to \infty$, which by (62) would contradict the hypothesis that $x^{-A}y'/y \to +\infty$ as $x \to +\infty$. This proves (52).

We now assert that for any $\varepsilon_1 > 0$, there is an $x_3$ such that

$$yy'' \leqslant (1 + \varepsilon_1)(y')^2 \quad \text{on } [x_3, +\infty). \tag{63}$$

To prove (63), choose $\varepsilon < \varepsilon_1$ as in the previous part (i.e. $0 < \varepsilon < \varepsilon_0$ and satisfying (53)), and we will show that

$$yy'' < (1 + \varepsilon)(y')^2 \quad \text{on an interval } [x_3, +\infty). \tag{64}$$

First, it is clear that it is impossible for $yy'' > (1 + \varepsilon)(y')^2$ to hold on an interval $[x_4, +\infty)$, for this would imply that $y'/y^{1+\varepsilon}$ is increasing on $[x_4, +\infty)$ which would contradict $y'/y^{1+\varepsilon} \to 0$ as $x \to +\infty$ from (52). Thus if (64) were false, then in view of (52), there would be a sequence $\{\zeta_n\} \to +\infty$ at which (54) is valid, and this would lead to the same contradiction as in the proof of (52). Thus (63) holds.

We now assert that

$$y'' \geqslant 0 \quad \text{on an interval } [x_5, +\infty). \tag{65}$$

To prove (65), we note first that by hypothesis (i), it is impossible that $y'' < 0$ on an interval $[x_6, +\infty)$. Hence if we assume that (65) is false, then the set where $y'' > 0$ certainly contains a union of nonempty disjoint open intervals $(a_n, \zeta_n)$, where $\{\zeta_n\} \to +\infty$ as $n \to \infty$. Of course, $y''(\zeta_n) = 0$. Now by definition of $\beta$ (see §13A)), we may write (48) in the form

$$(y'')^\beta \sum_{k \geqslant \beta} f_{ijk} y^i (y')^j (y'')^{k-\beta} = 0,$$

and hence

$$\sum_{k \geqslant \beta} f_{ijk} y^i (y')^j (y'')^{k-\beta} = 0$$

on each interval $(a_n, \zeta_n)$. By continuity, this relation must hold at $\zeta_n$ also, and since $y''(\zeta_n) = 0$, we thus have $\Sigma f_{ij\beta} y^i (y')^j = 0$ at each $\zeta_n$. Isolating those terms where $i + j = \lambda$ (see §13A), and dividing by $y^\lambda$, we obtain

$$\sum_{k=1}^{s} f_{\lambda - j_k, j_k, \beta} (y'/y)^{j_k} = - \sum_{i+j<\lambda} f_{ij\beta} (y'/y)^j y^{i+j-\lambda} \tag{66}$$

at $\zeta_n$. If we choose $\varepsilon > 0$ such that $\varepsilon < (\lambda - (i + j))/2j$ if $i + j < \lambda$ and $j > 0$, and apply (52) and (i) of the hypothesis, it easily follows that at $\zeta_n$, the right side of (66) is $o(\zeta_n^{-\alpha})$ for any $\alpha > 0$. We consider the polynomial

$$G(x, v) = \sum_{k=1}^{s} f_{\lambda - j_k, j_k, \beta} v^{j_k},$$

and by the Strodt factorization theorem as before, $G(x, v)$ factors as in (59), where the roots $B_j$ have Properties (A) and (B). From (66) it follows as before that (62) is valid for some index $t$ and a sequence $n_1 < n_2 < \ldots$, and this is impossible as before by the definitions of $A$ and $A_2$. This proves (65).

Returning now to the original equation in the form (55), and choosing $\varepsilon > 0$ satisfying (53), it is clear from (63) (using $\varepsilon_1 = 1$), (65), (52) (applied to $\varepsilon$), and (i) of the hypothesis that for any $\alpha \geqslant 0$, $\Phi(x) = o(x^{-\alpha})$ as $x \to +\infty$. If we set $v_0 = y'/y$, then (55) can be written

$$\sum g_{mn} v_0^m (v_0')^n = -\Phi(x), \tag{67}$$

where the $g_{mn}$ are polynomials. By [5, p. 57] not all $g_{mn}$ can be identically zero, and it is not possible by (67) that $g_{00}$ be the only nontrivial coefficient. Thus if we set $h_{mn} = g_{mn}$ if $(m, n) \neq (0, 0)$, and $h_{00} = g_{00} + \Phi(x)$, then $v_0$ satisfies the first-order algebraic differential equation

$$\sum h_{mn} v^m (v')^n = 0, \tag{68}$$

$h_{mn}$ is a polynomial if $(m, n) \neq (0, 0)$, $h_{00}$ differs from a polynomial by a function $o(x^\alpha)$ for all $\alpha$ as $x \to +\infty$, and some $h_{mn}$ with $m + n > 0$ is nontrivial. As mentioned in §1, it was shown by Lindelöf [17] (see also [8, pp. 95–97]), that if all $h_{mn}$ are polynomials, then a real-valued solution $v_0(x)$ of (68) on an interval $[x_0, +\infty)$ would satisfy

$$|v_0(x)| \leqslant \exp\left(x^{k+1}/(k+1)\right) \quad \text{on } [x_1, +\infty), \tag{69}$$

where $k$ can be taken to be $d + \varepsilon$ for any $\varepsilon > 0$, where $d$ is the maximum of the degrees of the coefficients. It is not difficult to verify that Lindelöf's proof is valid without any changes for our equation (68), and so (69) is valid for our $v_0$ (where, in the calculation of $k$, the "degree" of $h_{00}$ is taken to be the degree of $g_{00}$). Since $v_0 = y'/y$, the conclusion of Theorem 3 now follows immediately, where we may take $c = d + 1 + \varepsilon$, for any $\varepsilon > 0$.

14. REMARK. We now show that Theorem 3 holds for more general classes of coefficients than polynomials.

DEFINITION [23, §§61, 66–68]. Let $\theta$ be a real number, $0 < \theta \leqslant \pi$, and for each $r \geqslant 0$, let $D(r)$ consist of all points $z$ in the sector $|\arg z| < \theta$ satisfying $|z| \geqslant r$. The set of all $D(r)$ for $r \geqslant 0$ is denoted $F(\theta)$ and is clearly a filter base which converges to $\infty$. Let $L$ be a set of functions, each meromorphic in an element of $F(\theta)$, with the following properties: (i) $L$ is a field (where, as usual, we identify two elements of $L$ if they agree on an element of $F(\theta)$); (ii) $L$ contains all functions of the form $Kz^{\alpha}$, where $K$ is a complex number and $\alpha$ is a real number; and (iii) for every element $f$ in $L$ except zero, there exist a nonzero complex number $c$ and a real number $\alpha$ such that $f/cz^{\alpha} \to 1$ as $z \to \infty$ in $|\arg z| < \theta$. Then we will call $L$ a *Strodt field* (briefly, an SF) over $F(\theta)$. If $f/cz^{\alpha} \to 1$, we will denote $\alpha$ by $\delta_0(f)$. For a function $g$ satisfying $g = o(x^{-N})$ for each $N > 0$, as $x \to +\infty$, we will write $\delta_0(g) = -\infty$.

The simplest example of an SF over any $F(\theta)$ is the field generated by all the functions $Kz^{\alpha}$ where $K$ is complex and $\alpha$ is real. This field contains the field of rational functions (and hence the ring of polynomials). A more extensive SF (see [23, §71.3, p. 247]) is the set of all functions having, in an element of $F(\theta)$, a representation $u_0G(u_1, \ldots, u_s)$, where $s$ is a positive integer, $G$ is analytic at $(0, 0, \ldots, 0)$, and $u_j = c_jz^{\alpha_j}$ where $\alpha_j < 0$ for $j = 1, \ldots, s$. This SF clearly contains the coefficients treated by Fowler in [12] (see §1).

It is not difficult to verify that the statement of Theorem 3 remains true if the coefficients of equation (48) are assumed to belong to any SF over some $F(\theta)$, provided that in the definitions of $A_1$ and $A_2$ (see (49) and (51)), we replace $\delta_m$ by the maximum of all $\delta_0(f_{ijk})$, where $i + j + k = p$ and $j + 2k = q_m$, and we replace $d_k$ by $\delta_0(f_{\lambda-j,j,\beta})$ for $j = j_k$. The Strodt factorization theorem [23, §62] used in the proof states, in part, that for any algebraic polynomial with coefficients in an SF over $F(\theta)$, there exists an SF over $F(\theta)$ in which the polynomial factors completely. When (48) has coefficients in an SF, Lindelöf's proof of (69) for real-valued solutions of (68) is easily seen to be valid when we take $k$ to be $2(\max|\delta_0(h_{mn})|) + \varepsilon$ for any $\varepsilon > 0$.

## BIBLIOGRAPHY

1. S. Bank, *On majorants for solutions of algebraic differential equations in regions of the complex plane*, Pacific J. Math. **31** (1969), 573–582.

2. _____, *On analytic and meromorphic functions satisfying nth order algebraic differential equations in regions of the plane*, J. Reine Angew. Math. **253** (1972), 87–97.

3. _____, *A general theorem concerning the growth of solutions of first-order algebraic differential equations*, Compositio Math. **25** (1972), 61–70.

4. _____, *On solutions of algebraic differential equations whose coefficients are analytic functions in the unit disk*, Ann. Mat. Pura Appl. **92** (1972), 323–335.

5. _____, *On determining the growth of meromorphic solutions of algebraic differential equations having arbitrary entire coefficients*, Nagoya Math. J. **49** (1973), 53–65.

6. _____, *Some results on analytic and meromorphic solutions of algebraic differential equations*, Advances in Math. **15** (1975), 41–62.

7. N. Basu, S. Bose and T. Vijayaraghavan, *A simple example for a theorem of Vijayaraghavan*, J. London Math. Soc. **12** (1937), 250–252.

8. R. Bellman, *Stability theory of differential equations*, Dover, New York, 1953.

9. E. Borel, *Mémoire sur les séries divergentes*, Ann. Sci. École Norm. Sup. **16** (1899), 9–136.

10. K. Cooke, *The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order*, Pacific J. Math. **4** (1954), 483–501.

11. _____, *The rate of increase of real continuous solutions of certain algebraic functional equations*, Trans. Amer. Math. Soc. **92** (1959), 106–124.

12. R. Fowler, *Some results on the form near infinity of real continuous solutions of a certain type of second-order differential equation*, Proc. London Math. Soc. (2) **13** (1914), 341–371.

13. G. H. Hardy, *Some results concerning the behavior at infinity of a real and continuous solution of an algebraic differential equation of the first order*, Proc. London Math. Soc. (2) **10** (1912), 451–468.

14. S. Hellerstein and L. Rubel, *Subfields that are algebraically closed in the field of all meromorphic functions*, J. Analyse Math. **12** (1964), 105–111.

15. E. Hille, *Analytic function theory*, Vol. 2, Ginn, Boston, 1962.

16. O. Lancaster, *Some results concerning the behavior at infinity of real continuous solutions of algebraic difference equations*, Bull. Amer. Math. Soc. **46** (1940), 169–177.

17. E. Lindelöf, *Sur la croissance des intégrales des équations différentielles algébriques du premier ordre*, Bull. Soc. Math. France **27** (1899), 205–215.

18. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthiers-Villars, Paris, 1929.

19. _____, *Eindeutige Analytische Funktionen*, 2nd ed., Springer-Verlag, Berlin, 1953.

20. S. Saks and A. Zygmund, *Analytic functions*, English transl., Monografie Mat., Tom 28, Warsaw, 1952.

21. S. M. Shah, *On real continuous solutions of algebraic difference equations*, Bull. Amer. Math. Soc. **53** (1947), 548–558.

22. _____, *On real continuous solutions of algebraic difference equations*. II, Proc. Nat. Inst. Sci. India **16** (1950), 11–17.

23. W. Strodt, *On the algebraic closure of certain partially ordered fields*, Trans. Amer. Math. Soc. **105** (1962), 229–250.

24. W. Strodt and R. Wright, *Asymptotic behavior of solutions and adjunction fields for nonlinear first-order differential equations*, Mem. Amer. Math. Soc. No. 109 (1971).

25. T. Vijayaraghavan, *Sur la croissance des fonctions définies par les équations différentielles*, C. R. Acad. Sci. Paris **194** (1932), 827–829.

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