ERICKSON'S CONJECTURE ON THE RATE OF ESCAPE OF d-DIMENSIONAL RANDOM WALK

BY

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ABSTRACT. We prove a strengthened form of a conjecture of Erickson to the effect that any genuinely d-dimensional random walk $S_n$, $d > 3$, goes to infinity at least as fast as a simple random walk or Brownian motion in dimension $d$. More precisely, if $S_n^*$ is a simple random walk and $B_t$ a Brownian motion in dimension $d$, and $\psi : [1, \infty) \to (0, \infty)$ a function for which $t^{1/2} \psi(t) \downarrow 0$, then $\psi(n)^{-1} |S_n^*| \to \infty$ w.p.1, or equivalently, $\psi(t)^{-1} |B_t| \to \infty$ w.p.1, iff $\int \frac{\psi(t)^2 - 2 - d/2}{t} < \infty$; if this is the case, then also $\psi(n)^{-1} |S_n| \to \infty$ w.p.1 for any random walk $S_n$ of dimension $d$.

1. Introduction. Let $X_1, X_2, \ldots$ be independent identically distributed random $d$-dimensional vectors and $S_n = \sum_{i=1}^n X_i$. Assume throughout that $d > 3$ and that the distribution function $F$ of $X_1$ satisfies

$$\sup \text{supp}(F) \text{ is not contained in any hyperplane.} \quad (1.1)$$

The celebrated Chung-Fuchs recurrence criterion [1, Theorem 6] implies that $|S_n| \to \infty$ w.p.1 irrespective of $F$. In [4] Erickson made the much stronger conjecture that there should even exist a uniform escape rate for $S_n$, viz. that $n^{-\alpha} |S_n| \to \infty$ w.p.1 for all $\alpha < \frac{1}{2}$ and all $F$ satisfying (1.1). Erickson proved his conjecture in special cases and proved in all cases that $n^{-\alpha} |S_n| \to \infty$ w.p.1 for $\alpha < 1/2 - 1/d$. Our principal result is the following theorem which contains Erickson’s conjecture and which makes precise the intuitive idea that a simple random walk $S_n^*$ (which corresponds to the distribution $F^*$ which puts mass $(2d)^{-1}$ at each of the points $(0, 0, \ldots, \pm 1, 0, \ldots, 0)$) goes to infinity slower than any other random walk.

**Theorem.** Let $d > 3$, $S_n$ and $S_n^*$ as above, and let $\{B_t\}_{t \geq 0}$ be a $d$-dimensional Brownian motion ($EB_t = 0$, $EB_t(i)B_t(j) = t \delta_{i,j}$). Assume that the function $\psi : [1, \infty) \to (0, \infty)$ satisfies

$$t^{-1/2} \psi(t) \downarrow 0. \quad (1.2)$$

Then $\psi(n)^{-1} |S_n^*| \to \infty$ w.p.1 and $\psi(t)^{-1} |B_t| \to \infty$ w.p.1 are both equivalent to

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1For any vector $e \in \mathbb{R}^d$ we denote its components by $e(1), \ldots, e(d)$, and $|e| = (\sum_{i=1}^d e^2(i))^{1/2}$. "w.p.1" stands for "with probability one".

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\[ \int_1^\infty \psi(t)^{d-2} t^{-d/2} dt < \infty. \] (1.3)

If (1.2) and (1.3) hold, then also \( \psi(n)^{-1} |S_n| \to \infty \) w.p.1 whenever \( F \) satisfies (1.1).

As we shall see this theorem is almost immediate from

**Proposition 1.** If \( d > 3 \) and \( F \) satisfies (1.1) then there exist constants \( 0 < \Gamma_1(F), \Gamma_2(F) < \infty \) such that for all \( k \geq 1 \) and all \( A > 0 \),

\[
P \left( |S_n| < A \text{ for some } 2^k < n < 2^{k+1} \right) < \Gamma_1(F) \left( ((A + 1)2^{-k/2})^{d-2} + \exp \left( -\Gamma_2(F)k \right) \right). \quad (1.4)
\]

It is worthwhile contrasting the theorem with the observation (see [4, end of §5]) that even if (1.1)–(1.3) hold, there may exist a deterministic sequence of vectors \( a_n \) such that

\[
\lim \inf_{n \to \infty} \psi(n)^{-1} |S_n - a_n| = 0 \quad \text{w.p.1.} \quad (1.5)
\]

Thus, the theorem is not merely a matter of estimating concentration functions for \( S_n \). Nevertheless the result is intimately connected with concentration functions. Erickson's proof for \( \psi(t) = t^{a}, \alpha < 1/2 - 1/d \), uses a concentration function inequality of Esseen [6], and our proof makes heavy use of the following inequalities for concentration functions in dimension two. These estimates too are closely related to those of Esseen [5, Theorem 3], but are more generally applicable. We note that Corollary 1 shows that in the identically distributed case the concentration function decreases like \( n^{-1} \).

**Proposition 2.** Let \( Z_1, Z_2, \ldots, Z_n \) be independent random two-vectors with distribution functions \( G_1, G_2, \ldots, G_n \). Let \( \rho, \rho_1, \rho_2, \ldots, \rho_n \) be strictly positive numbers such that \( \rho_i < \rho \), and let \( A_1, A_2, \ldots, A_n \) be sets in \( \mathbb{R}^2 \times \mathbb{R}^2 \) such that \( A_i \subset \{ u \in \mathbb{R}^2 : |u| > \rho_i \}^2 \). Define the symmetrization \( G_i^\ast \) of \( G_i \) by

\[
G_i^\ast (B) = \int_{\mathbb{R}^2} G_i(B + u) \, dG_i(u) \quad (1.6)
\]

and put\(^3\)

\[
q_i = \int_{|u| > \rho_i} dG_i^\ast (u), \quad \sigma_i^2 = \inf_{|\theta| = 1} \int_{|u| < \rho_i} (\theta, u)^2 \, dG_i^\ast (u),
\]

\[
C = \sum_{i=1}^n \left\{ \sigma_i^2 + \frac{\rho_i^2}{q_i} \int_{u,v \in A_i} dG_i^\ast (u) \, dG_i^\ast (v) \right\}. \quad (1.7)
\]


\(^4\)For a distribution function \( F \) and Borel set \( A \), \( F(A) \) denotes the mass assigned to \( A \) by the Borel measure induced by \( F \).

\(^5\)(\(\langle u,v \rangle\)) denotes the usual inner product of two vectors of the same dimension.
Finally, denote the angle between two vectors $u$ and $v$, determined in such a way that it lies in $[0, \pi]$, by $\varphi(u,v)$. Then there exists a universal constant $K_1 < \infty$ such that

$$
\sup_{z \in \mathbb{R}^2} P \left( \left| \sum_{i=1}^{n} Z_i + z \right| < \rho \right) < \frac{K_1 \rho^2}{C} \exp \left\{ \frac{1}{C} \sum_{i=1}^{n} \frac{\rho_i^2}{C q_i} \int_{u,v \in A_i} dG^t_i(u) dG^t_i(v) \right\} \cdot \log \left( 1 + \frac{\rho_i}{|v| |\sin \varphi(u,v)|} \right). \quad (1.8)
$$

**Corollary 1.** Let $Z_1, \ldots, Z_n$ be independent random two-vectors, all with the same distribution function $G$, and define $G^t$ as in (1.6). If $\rho > 0$ and

$$
q = \int_{|v| > \rho} dG^t(u), \quad \sigma^2 = \inf_{|v| = 1} \int_{|u| < \rho} \langle \theta, u \rangle^2 dG^t(u),
$$

and if there exist sets $B_1, B_2 \subset \{ z \in \mathbb{R}^2 : |z| > \rho \}$ and a constant $c > 0$ such that

$$
|v| |\sin \varphi(u,v)| > c \quad \text{whenever } u \in B_1, v \in B_2, \quad (1.9)
$$

then

$$
\sup_{z \in \mathbb{R}^2} P \left( \left| \sum_{i=1}^{n} Z_i + z \right| < \rho \right) < \frac{K_1}{n} \rho^2 (1 + \rho c^{-1}) \left\{ \sigma^2 + \frac{\rho^2}{q} \int_{B_1} dG^t(u) \int_{B_2} dG^t(v) \right\}^{-1}. \quad (1.10)
$$

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**2. Reduction to a two dimensional problem.** In this section we shall reduce the proof of Proposition 1 to a two dimensional problem. This problem will then be settled together with Proposition 2, by means of estimates on two dimensional concentration functions in §3. Throughout $K_1, K_2, \ldots$ will be strictly positive constants which depend on the dimension $d$ only and whose numerical value is immaterial for our purposes. They will not necessarily have the same value at each appearance. $\Gamma_1, \Gamma_2, \ldots$ and $k_1, k_2, \ldots$ will be constants which depend on $F$ and $d$ or some other parameters; where necessary these parameters will be indicated explicitly, except that we shall not indicate dependence on $d$. For any random variable $Y$ we write $Y^t = Y - Y'$ where
\( Y' \) is independent of \( Y \) and has the same distribution as \( Y \). If \( Y \) has distribution function \( G \), then \( Y' \) has the distribution function \( G' \) given by
\[
G'(B) = \int G(B + y) \, dG(y).
\]

The first reduction shows that we may assume that \( F \) has certain smoothness properties. It is of a purely technical nature and has little to do with the basic idea of the proof of Proposition 1. The reader should skip the proof of Lemma 1 at first reading.

**Lemma 1.** It suffices to prove Proposition 1 in the case where
\[
F(dx) > a \, dx \quad \text{on } C_0
\]
for some \( a > 0 \) and some closed cube \( C_0 \subseteq \mathbb{R}^d \) (which does not reduce to a point).

**Proof.** Let \( F \) satisfy (1.1). We shall construct an \( F_1 \) which satisfies (2.1) (and a fortiori (1.1)) such that Proposition 1 holds for the original \( S_n \) as soon as it holds for a random walk, \( S_n(F_1) \) say, whose increments have distribution function \( F_1 \). For this purpose we first find a large cube
\[
C_1 = \{ z \in \mathbb{R}^d : -L < z(i) < L, 1 \leq i \leq d \}
\]
such that \( F(C_1) > 0 \) and such that the intersection of \( \text{supp}(F) \) with \( C_1 \) is not contained in any hyperplane. This is possible by (1.1). Define the distribution functions \( G \) and \( H \) by
\[
\begin{align*}
p_0 &= \frac{1}{2} F(C_1), \\
G(S) &= (2p_0)^{-1} F(S \cap C_1)
\end{align*}
\]
and
\[
H(S) = (1 - p_0)^{-1} \left\{ \frac{1}{2} F(S \cap C_1) + F(S \cap C_1^c) \right\}.
\]
Then we can write \( F = p_0 G + (1 - p_0) H \), and if \( I_1, I_2, \ldots, U_1, U_2, \ldots, V_1, V_2, \ldots \) are totally independent random variables with
\[
\begin{align*}
P \{ I_j = 0 \} &= 1 - P \{ I_j = 1 \} = p_0, \\
P \{ U_j \in S \} &= G(S), \quad P \{ V_j \in S \} = H(S),
\end{align*}
\]
then the joint distribution of \( \{ X_n \}_{n \geq 1} \) is the same as of \( \{ U_n(1 - I_n) + V_n I_n \}_{n \geq 1} \) (compare [9, p. 1184]). We shall therefore assume that \( U, V \) and \( I \) are defined on our original probability space and that \( X_n = U_n(1 - I_n) + V_n I_n \). Let \( \sigma_1 < \sigma_2 < \ldots \) be the successive (random) indices for which \( I_n = 1 \). Then, with \( \sigma_0 = 0 \), the \( \sigma_{i+1} - \sigma_i \) are independent and all with the distribution
\[
P \{ \sigma_{i+1} - \sigma_i = r \} = p_0^{r-1} (1 - p_0), \quad r > 1.
\]

---

\(^5\)For any set \( S \subseteq \mathbb{R}^d \), \( S^c \) denotes its complement, i.e., \( \mathbb{R}^d \setminus S \).
Even more, if $\mathcal{G}_n$ is the $\sigma$-field generated by \( \{ U_j, V_j, I_j : 1 < j < n \} \), then for each \( \{ \mathcal{G}_n \} \) stopping time \( T \) and

\[ \sigma^*(T) = \text{smallest } \sigma_i \text{ which exceeds } T, \]

one has

\[ P\{ \sigma^*(T) = T + r | \mathcal{G}_T \} = p_0^{r-1}(1 - p_0), \quad r > 1. \]

It is also not hard to see that the \( \{ S_{n+1} - S_n \}_{i \geq 0} \) are independent and all with the same distribution function\(^6\)

\[ F_2 = \sum_{r=1}^{\infty} P\{ \sigma_i = r \} G^{(r-1)} \ast H = \sum_{r=1}^{\infty} p_0^{r-1}(1 - p_0)G^{(r-1)} \ast H, \]

and that also, conditional on $\mathcal{G}_T$, $S_{\sigma^*(T)} - S_T$ has the distribution $F_2$ on the set \( \{ T < \infty \} \). Now take

\[ T = \inf \{ n \in \mathbb{N}^+ : 2^k + 1 \} \text{ with } |S_n| < A \]

(= $\infty$ if no such \( T \) exists). Then, since $|S_T| < A$,

\[ P \left\{ |S_0| < 2A \text{ for some } l \in \left[ \frac{1}{2} (1 - p_0)2^k2(1 - p_0)2^{k+2} \right] \right\} \]

\[ > P \left\{ T < \infty, |S_{\sigma^*(T)} - S_T| < A \right\} \]

\[ - P \left\{ T < \infty, \sigma^*(T) = \sigma_i \text{ with } l \in \left[ \frac{1}{2} (1 - p_0)2^k2(1 - p_0)2^{k+2} \right] \right\} \]

\[ > P \left\{ T < \infty \right\} F_2 \left( \left\{ z \in \mathbb{R}^d : |z| < A \right\} \right) \]

\[ - P \left\{ T < \infty, \sigma^*(T) - T > 2^k \right\} \]

\[ - P \left\{ \exists l \notin \left[ \frac{1}{2} (1 - p_0)2^k2(1 - p_0)2^{k+2} \right] \text{ with } \sigma_i \in \left[ 2^k, 2^{k+2} \right] \right\}. \tag{2.4} \]

Now for some $A_0 = A_0(F) < \infty$ and all $A > A_0$,

\[ F_2 \left( \left\{ z \in \mathbb{R}^d : |z| < A \right\} \right) > \frac{1}{2}, \quad P \left\{ T < \infty, \sigma^*(T) - T > 2^k \right\} < p_0^{2k}. \]

Also, with

\[ l_1 = \left[ \frac{1}{2} (1 - p_0)2^k \right], \quad l_2 = \left[ 2(1 - p_0)2^{k+2} \right], \]

the last term in (2.4) is at most

\[ P \left\{ \sigma_{l_1} > 2^k \text{ or } \sigma_{l_2} < 2^{k+2} \right\} < \Gamma_3 \exp -\Gamma_4 2^k, \]

for some $\Gamma_3, \Gamma_4 < \infty$ depending on $p_0$ only (by (2.3) and standard exponential

\(^6G \ast H \) denotes the convolution of $G$ and $H$, and $G^{(s)}$ denotes the $s$-fold convolution of $G$ with itself.
estimates; compare [9, formulae (5.40) – (5.42)]. Thus (2.4) yields for $A > A_0$

$$P \{ |S_n| < A \text{ for some } 2^k < n < 2^{k+1} \} = P \{ T < \infty \}
< 2P \{ |S_n| < 2A \text{ for some } l \in \left[ \frac{1}{2}(1 - p_0)2^k, 2(1 - p_0)2^{k+2} \right] \}
+ 2p_0^k + 2\Gamma_3 \exp - \Gamma_2 \cdot (2.5)$$

It is clear from (2.5) that if Proposition 1 holds for the random walk $S_n(F_2) = S_{A_k}$, whose increments $S_{n+1} - S_n$ all have the distribution $F_2$, then Proposition 1 also holds for the original $S_n$. Actually (2.5) was derived only for $A > A_0$. However, for $A < A_0$ or even $A < 2^{k/8}$, (1.4) is immediate from Esseen’s estimate [6, Corollary to Theorem 6.2]

$$P \{ |S_n| < A \} < K_2(A + 1)^d \sup_z P \{ |S_n + z| < 1 \}
< \Gamma_3(F)(A + 1)^d n^{-d/2}, \quad (2.6)$$

for some $\Gamma_3(F) < \infty$.

$F_2$ itself does not have to satisfy (2.1). However, set

$$G_2 = \sum_{r=1}^{\infty} p_0^{r-1} (1 - p_0) G^{(r-1)}, \quad (2.7)$$

so that $F_2 = G_2 \ast H$, and assume that we can find a distribution function $G_1$ with a continuous density such that

$$\int_{R^d} x^v dG_1(x) = \int_{R^d} x^v dG_2(x) \quad \text{for } \|v\| < 16. \quad (2.8)$$

(Here we use the standard multi-index notation; $x^v = \prod_{i=1}^d x(i)^{v(i)}$, for positive integers $v(i)$ and $\|v\| = \sum_{i=1}^d |v(i)|$. We claim that then $F_1 \equiv G_1 \ast H$ has the required properties. Before constructing $G_1$ we shall prove this claim. (2.1) is obvious for $F_1$; indeed $G_1$ has a continuous density and hence $F_1$ has a density which is lower semicontinuous. Thus we merely have to prove that the validity of Proposition 1 for $S_n(F_1)$ implies the validity of Proposition 1 for $S_n(F_2)$ (since we already showed that Proposition 1 then also holds for our original $S_n$). Now fix $k$ and $A > 2^{k/8}$; we already saw above that (1.4) follows from (2.6) if $A < 2^{k/8}$ so that these are the only values of interest. Let $N, \chi_1, \chi_2, \ldots$ be independent random $d$-vectors, also independent of $(S_n)_{n \geq 0}$ and such that $N$ has a normal distribution with mean zero and covariance matrix $k^{-1}A^2$ times the identity matrix, and such that each $\chi_i$ has distribution $F_1$. Then, for any $n,$
\[ P \{ |S_\alpha| < 2A \} < P \{ |S_\alpha(j)| < 2A, 1 < j < d \} \]
\[ < P \{ |S_\alpha(j) + N(j)| < 3A, 1 < j < d \} + \sum_{j=1}^{d} P \{ |N(j)| > A \} \]
\[ < P \{ |S_\alpha(j) + N(j)| < 3A, 1 < j < d \} + K_3 \exp -\frac{A^2}{2A^2k^{-1}} \]
\[ < P \{ |S_\alpha(j) + N(j)| < 3A, 1 < j < d \} + K_3 \exp -\frac{k}{2} . \quad (2.9) \]

Similarly
\[ P \left\{ \left| \sum_{j=1}^{n} X(j) + N(j) \right| < 3A, 1 < j < d \right\} \]
\[ < P \left\{ \left| \sum_{j=1}^{n} X(j) \right| < 4d^{1/2}A \right\} + K_3 \exp -\frac{k}{2} . \quad (2.10) \]

If \( \varphi_1, \varphi_2, \psi \) are the characteristic functions of, respectively, \( G_1, G_2 \) and \( H \), then the characteristic functions of \( S_\alpha + N \) and \( \sum X_i + N \) are
\[ \exp \left( -\frac{A^2}{2k} |\theta|^2 \right) \varphi_2(\theta)^n \psi(\theta)^n \quad \text{and} \quad \exp \left( -\frac{A^2}{2k} |\theta|^2 \right) \varphi_1(\theta)^n \psi(\theta)^n . \]

Thus, by the inversion formula,
\[ \left| P \{ |S_\alpha(j) + N(j)| < 3A, 1 < j < d \} \right| \]
\[ < \pi^{-d} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d\theta_1 \cdots d\theta_d \frac{\sin 3\theta A}{\theta} \left| \varphi_2(\theta)^n - \varphi_1(\theta)^n \exp \left( -\frac{A^2}{2k} |\theta|^2 \right) |\psi(\theta)|^n \right| \]
\[ \leq \left( \frac{3A}{\pi} \right)^d \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\varphi(\theta) - \varphi(\theta)| \exp \left( -\frac{A^2}{2k} |\theta|^2 \right) d\theta_1 \cdots d\theta_d . \]

But, by virtue of (2.8),
\begin{align*}
|\varphi_1(\theta) - \varphi_2(\theta)| &= \left| \int e^{i\langle \theta, x \rangle} \, d(G_1(x) - G_2(x)) \right| \\
&= \left| \int \left( e^{i\langle \theta, x \rangle} - \sum_{r=0}^{15} \frac{i^r\langle \theta, x \rangle^r}{r!} \right) \, d(G_1(x) - G_2(x)) \right| \\
&\leq \frac{1}{16!} \int \langle \theta, x \rangle^{16} \, d(G_1(x) + G_2(x)) = \frac{2}{16!} \int \langle \theta, x \rangle^{16} \, dG_2(x) \\
&\leq \frac{2}{16!} |\theta|^{16} \sum_{r=1}^{\infty} p_0^{r-1} (1 - p_0)((r - 1)L)^{16} \\
&\text{(see (2.7) and recall that } G \text{ is concentrated on } C_i) \text{. Consequently, for } A > 2k/8 \text{ and } n < 2k+2, \\
\left| P \left\{ |S_n(j) + N(j)| < 3A, 1 < j < d \right\} \right| \\
&= P \left\{ \left| \sum_{i=1}^{n} \chi_i(j) + N(j) \right| < 3A, 1 < j < d \right\} \\
&< \Gamma_6(F)nA^d \int \cdots \int |\theta|^{16} \exp \left( -\frac{4^2}{2k} |\theta|^2 \right) \, d\theta_1 \cdots d\theta_d \\
&< \Gamma_7(F)nA^{d-d-16k(d+16)/2} < \Gamma_7(F)k^{(d+16)/2}2^{-k+2}. \\
\end{align*}

Combined with (2.9) and (2.10) this yields

\begin{equation}
P \left\{ |S_n| < 2A \right\} < P \left\{ \left| \sum_{i=1}^{n} \chi_i \right| < 4d^{1/2}A \right\} + \Gamma_8(F)e^{-k/2}. \tag{2.11} \end{equation}

Interchanging the subscripts 1 and 2 we prove similarly

\begin{equation}
P \left\{ |S_n| < A \right\} > P \left\{ \left| \sum_{i=1}^{n} \chi_i \right| < \frac{1}{2}d^{-1/2}A \right\} - \Gamma_8(F)e^{-k/2}, \tag{2.12} \end{equation}

for all \( A > 2^{k/8}, n < 2^{k+2} \). Finally, to prove our claim we appeal to the proof of Theorem 3 in [9]. Just as in the estimates on pp. 1179–1181 of [9],

\begin{equation}
P \left\{ |S_n| < A \text{ for some } 2^k < n < 2^{k+1} \right\} \\
< \left( \sum_{n=0}^{2^k} P \left\{ |S_n| < A \right\} \right)^{-1} \sum_{n=2^k}^{2^{k+2}-1} P \left\{ |S_n| < 2A \right\}, \tag{2.13} \end{equation}

and also
\[ P \left\{ \left| \sum_{i=1}^{n} x_i \right| < 4d^{1/2}A \text{ for some } 2^k < n < 2^{k+2} \right\} \]

\[ > \left[ \sum_{n=0}^{2^{k+2}} P \left( \left| \sum_{i=1}^{n} x_i \right| < 8d^{1/2}A \right) \right]^{-1} \sum_{n=2^k}^{2^{k+2}-1} P \left( \left| \sum_{i=1}^{n} x_i \right| < 4d^{1/2}A \right) \]

\[ > K_4 \left[ \sum_{n=0}^{2^k} P \left( \left| \sum_{i=1}^{n} x_i \right| < \frac{1}{2} d^{-1/2}A \right) \right]^{-1} \sum_{n=2^k}^{2^{k+2}-1} P \left( \left| \sum_{i=1}^{n} x_i \right| < 4d^{1/2}A \right). \quad (2.14) \]

Since

\[ \sum_{n=0}^{2^k} P \{ |S_{n_a}| < A \} \to \infty \quad \text{as } k \to \infty \text{ and } A > 2^{k/8}, \]

it follows from (2.11)-(2.14) that for \( k > k_1(F), \)

\[ P \{ |S_{n_a}| < A \text{ for some } 2^k < n < 2^{k+1} \} \]

\[ < 2K_4^{-1} \left[ P \left( \left| \sum_{i=1}^{n} x_i \right| < 4d^{1/2}A \text{ for some } 2^k < n < 2^{k+1} \right) \right] \]

\[ + P \left( \left| \sum_{i=1}^{n} x_i \right| < 4d^{1/2}A \text{ for some } 2^{k+1} < n < 2^{k+2} \right) \]

\[ + \Gamma_9(F) \exp(-k/2). \]

Since \( \Sigma_i x_i \) can be taken for \( S_n(F_i) \) (its increments \( x_n \) have distribution \( F_i \)), we see from this that if Proposition 1 holds for \( S_n(F_i) \), then it also holds for \( S_n(F_2) \) and for the original \( S_n \). (Note that we can always obtain (1.4) for \( k < k_1 \) by increasing \( \Gamma_1 \).)

To complete the proof of the lemma we show finally that there exists a \( G_1 \) with a continuous density and satisfying (2.8). Let there be \( M \) positive integer \( d \)-vectors \( v \) with \( 1 < \|v\| < 16 \) and consider the set

\[ \mathcal{M} = \left\{ (x_v)_{1 < \|v\| < 16} : |z_v| < \Delta_v + 1, z_v = \int x^v \ dR \text{ for some distribution function } R \text{ on } \mathbb{R}^d \text{ and all } 1 < \|v\| < 16 \right\}, \]

where the constants \( \Delta_v < \infty \) are chosen large enough such that

\[ \mu_v = \int x^v \ dG_2(x) \in [-\Delta_v, \Delta_v] , \quad 1 < \|v\| < 16. \]

Then \( \mathcal{M} \) is a bounded convex subset of \( \mathbb{R}^M \) and \( \mu = (\mu_v)_{1 < \|v\| < 16} \in \mathcal{M} \).
Assume first that \( \mu \in \partial \mathcal{M} \). Then there exists a supporting hyperplane of \( \mathcal{M} \) at \( \mu \) (see \([3, \text{Theorem 8}]\)), i.e., there exist constants \( c_0, c_\nu, 1 < \|\nu\| < 16, \) not all zero, such that

\[
\sum_{1 < \|\nu\| < 16} c_\nu z_\nu + c_0 > 0 \quad \text{for all } z \in \mathcal{M}, \tag{2.15}
\]

and

\[
\sum_{1 < \|\nu\| < 16} c_\nu \mu_\nu + c_0 = 0. \tag{2.16}
\]

Then, if

\[
P(x) = \sum_{\nu = 1}^{M} c_\nu x^\nu + c_0.
\]

(2.15) implies that \( P(x) dR(x) > 0 \) for all distribution functions \( R \) on \( \mathbb{R}^d \) with

\[
\int x^\nu dR(x) < \Delta_\nu + 1, \quad 1 < \|\nu\| < 16.
\]

It is not hard to see from this and (2.16) that \( P(x) = 0 \) for \( x \in \text{supp}(G_2) \), and, hence, for \( x \in \text{supp}(G^{s_2}) = \{0\} \) as well as for

\[
x \in \text{supp}(G^{(s)}) = \{ x_1 + \cdots + x_r : x_i \in \text{supp} \ G \} \quad \text{for some } s > 1.
\]

\( P(0) = 0 \) shows that \( c_0 = 0 \). Moreover, if \( y_1, \ldots, y_{d+1} \) are \((d + 1)\) points in \( \text{supp}(G) = \text{supp}(F) \cap C_1 \) which do not lie in any hyperplane, then also, for any integers \( k_1 > 0, \ldots, k_{d+1} > 0, \Sigma k_i y_i \in \text{supp}(G_2) \) and hence

\[
\sum_{1 < \|\nu\| < 16} c_\nu (k_1 + \cdots + k_{d+1})^{-16}(k_1 y_1 + \cdots + k_{d+1} y_{d+1})^\nu = 0.
\]

If we let \( k_i \to \infty \) such that

\[
k_i (k_1 + \cdots + k_{d+1})^{-1} \lambda_i > 0 \quad \text{with} \quad \sum_{i = 1}^{d+1} \lambda_i = 1,
\]

we obtain

\[
\sum_{\|\nu\| = 16} c_\nu (\lambda_1 y_1 + \cdots + \lambda_{d+1} y_{d+1})^\nu = 0
\]

for all \( \lambda_i > 0 \) with \( \sum_{i = 1}^{d+1} \lambda_i = 1 \), i.e.,

\[
\sum_{\|\nu\| = 16} c_\nu x^\nu = 0
\]

for all \( x \) in the closed convex hull of \( y_1, \ldots, y_{d+1} \). Since these \((d + 1)\) points do not lie in any hyperplane, their convex hull has a nonempty interior (see \([3, \text{Theorem 4}]\)) and we conclude \( c_\nu = 0 \) for \( \|\nu\| = 16 \). But then also
and continuing in this way we find that all \( c_\nu = 0 \). This was excluded, so that 
\( \mu \in \partial \mathcal{M} \) is impossible. But then \( \mu \) lies in the interior of \( \mathcal{M} \) and for each of 
the \( 2^M \) choices of \( \alpha_\nu \) with \( \alpha_\nu = +1 \) or \( -1 \), we can find a point \( z(\alpha) \in \mathcal{M} \) 
such that
\[
\text{sgn}(z_\nu(\alpha) - \mu_\nu) = \text{sgn} \alpha_\nu
\]  
and
\[
z_\nu(\alpha) = \int x^* \, dR_\alpha(x)
\]
for some \( R_\alpha \) with a continuous density. Indeed any point of \( \mathcal{M} \) can be approximated by the moments of continuous densities on \( \mathbb{R}^d \). It is a question 
of simple algebra only to deduce from (2.17) that some convex combination 
of the \( z(\alpha) \) equals \( \mu \). Say
\[
\mu_\nu = \sum_\alpha \gamma(\alpha)z_\nu(\alpha), \quad 1 < ||\nu|| < 16,
\]
with \( \gamma(\alpha) > 0 \), \( \sum_\alpha \gamma(\alpha) = 1 \). Then clearly \( G_1 = \sum_\alpha \gamma(\alpha)R_\alpha \) 
has the moments \( \mu_\nu \) for \( ||\nu|| < 16 \) and has a continuous density as required. □

From now on we assume that (2.1) holds. We introduce the following 
quantities: \( A_k \) will be any fixed positive number not less than \( 2^k/8 \). \( \omega \) will 
stand for a generic unit vector in \( \mathbb{R}^d \) and for any such vector we set
\[
t(\omega) = t(\omega,k) = \min\{n: P\{|S_n,\omega| > A_k\} > (8d + 8)^{-1}\}.
\]
As we shall see in the next lemma \( t(\omega) \) is bounded on \( ||\omega|| = 1 \), and we can 
therefore pick an \( \omega_d = \omega_d^k \) which maximizes \( t(\cdot,k) \), i.e. for which
\[
t(\omega_d^k) = \max_{||\omega|| = 1} t(\omega,k).
\]
After that we can successively pick \( \omega_{d-1}^k, \ldots, \omega_1^k \) such that \( \langle \omega_j^k, \omega_j^k \rangle = \delta_{ij} \) 
and
\[
t(\omega_{d-1}^k) = \max\{t(\omega,k): \langle \omega, \omega_j^k \rangle = 0, d - l < j < d\}.
\]
We define
\[
T = T(k) = t(\omega_d^k,k)
\]
and note that by our construction \( \omega_1, \ldots, \omega_d \) form an orthonormal basis for 
\( \mathbb{R}^d \),
\[
T(k) < t(\omega_j^k,k), \quad j > 3,
\]
and if
\[
\mathcal{H} = \mathcal{H}^k = \text{span of } \{\omega_1^k, \omega_2^k\} = \{z: z = \alpha \omega_1^k + \beta \omega_2^k\},
\]
then
\[
t(\omega, k) < T(k) \quad \text{for } \omega \in \mathcal{K}.
\] (2.20)

Next we take \( M(\omega^2, k) = 1 \) and for all \( \omega \neq \omega_2 \) we take \( M(\omega) = M(\omega, k) \) as the maximum over \( 1 < n < T(k) \) of any \( 1 - (8d + 8)^{-1} \) quantile of \( A_k^{-1}\langle S_n, \omega \rangle \), i.e.,
\[
P\{ A_k^{-1}\langle S_n, \omega \rangle < M(\omega) \} > 1 - (8d + 8)^{-1}, \quad 1 < n < T(k), \quad (2.21)
\]
and for some \( 1 < N(\omega) = N(\omega, k) < T(k) \),
\[
P\{ A_k^{-1}\langle S_N, \omega \rangle > M(\omega) \} > (8d + 8)^{-1}. \quad (2.22)
\]

By definition of \( T \), (2.22) also holds for \( \omega = \omega_2 \) and \( N(\omega_2, k) = T(k) \). Also, because \( M(\omega, k) \) must be at least as big as a \( 1 - (8d + 8)^{-1} \) quantile of \( A_k^{-1}\langle S_{t(\omega, k), \omega} \rangle \) if \( t(\omega, k) < T \), we may and shall choose \( M(\omega, k) \) such that
\[
M(\omega, k) > 1 \quad \text{for } \omega \in \mathcal{K}. \quad (2.23)
\]

Observe also that \( t(\omega_j) > T(k) \) for \( j > 2 \) and therefore
\[
P\{ \langle S_n, \omega \rangle < 2A_k \} > P\{ \langle S_{n-1}, \omega \rangle < A_k \} P\{ X < A_k \} > 1 - (4d + 4)^{-1} \quad (2.24)
\]
whenever
\[
k > k_2(F), \quad n < T(k) \quad \text{and} \quad \omega = \omega_j \quad \text{with } 2 < j < d \quad (2.25)
\]
for a suitable \( k_2(F) < \infty \).

**Lemma 2.** There exists a \( \Gamma_{10} = \Gamma_{10}(F) < \infty \) such that for all \( |\omega| = 1, k > 1 \),
\[
t(\omega, k) < \Gamma_{10}A_k^2. \quad (2.26)
\]
Moreover there exist a universal \( K_0 < \infty \) and a \( k_3 = k_3(F) < \infty \) such that for all \( |\omega| = 1, k > k_3 \),
\[
T(k)P\{ \langle X_1, \omega \rangle > M(\omega, k)A_k \} < K_0 \quad (2.27)
\]
and\(^7\)
\[
T(k)\sigma^2\{ \langle X_1, \omega \rangle I[\langle X_1, \omega \rangle < M(\omega, k)A_k] \} < K_0 \{ M(\omega, k)A_k \}^2. \quad (2.28)
\]
\[
T(k) \to \infty \quad \text{and} \quad A_k \inf_{|\omega| = 1} M(\omega, k) \to \infty \quad \text{as } k \to \infty, \quad (2.29)
\]
and there exists a universal \( 0 < \eta < 1 \) (see (2.44)) such that for all \( |\omega| = 1, k > 1 \),
\[
P\{ \langle S_{T, \omega} \rangle > \frac{1}{2} M(\omega, k)A_k \} > \eta. \quad (2.30)
\]

\(^7\)For a random variable \( Y \), \( \sigma^2\{ Y \} \) denotes its variance. \( I[ ] \) denotes the indicator function of the event between square brackets.
Proof. For any real random variable $Y$ its concentration function $Q(Y,L)$ is defined by

$$Q(Y,L) = \sup_y P\{y < Y < y + L\}.$$  

It is easy to see and well known (see [7, Chapters 1.1 and 2.1]) that this sup is taken on for some $y$, that for $0 < L_1 < L_2$,

$$Q(Y,L_2) < (L_1^{-1}L_2 + 1)Q(Y,L_1), \quad (2.31)$$

and that for $Y_1$ and $Y_2$ independent,

$$Q(Y_1 + Y_2,L) < \min\{Q(Y_1,L),Q(Y_2,L)\}. \quad (2.32)$$

If $Y_1, Y_2, \ldots$ are independent and identically distributed with distribution function $G$, then Theorem 3.1 of Esseen [6] gives

$$Q\left(\sum_{i=1}^n Y_i,L\right) < K_5n^{-1/2}\left\{L^{-2}\int_{|x|<2L} x^2 dG^*(x) + \int_{|x|>2L} dG^*(x)\right\}^{-1/2} \quad (2.33)$$

for some universal constant $K_5 < \infty$, independent of $G$, $n$ and $L$. We shall now apply this with $Y_i = \langle x_i, \omega \rangle$ for an arbitrary unit vector $\omega$. First choose $L$ again such that $\text{supp}(F) \cap C_1$ is not contained in any hyperplane, where $C_1$ is as in (2.2). Then

$$\Gamma_{11}^2 \equiv L^{-2} \inf_{|\omega| = 1} \int \langle x, \omega \rangle^2 dF^*(x) > 0,$$

and thus by (2.31) and (2.33),

$$P\{|\langle S_n, \omega \rangle| < A_k\} < 2\{L^{-1}A_k + 1\}Q(\langle S_n,\omega \rangle,L)$$

$$< 2\{L^{-1}A_k + 1\}K_5\Gamma_{11}^{-1}n^{-1/2} < 1 - (8d + 8)^{-1}$$

as soon as

$$n > \left(\frac{8d + 8}{8d + 7}\right)^2\{2L^{-1}A_k + 2\}^2K_5^2\Gamma_{11}^{-2}.$$

This implies (2.26). The same argument shows for any $L' > L$,

$$P\{|\langle S_n, \omega \rangle| < \frac{1}{2}L'\} < Q(\langle S_n,\omega \rangle,L')$$

$$< (L^{-1}L' + 1)K_5\Gamma_{11}^{-1}n^{-1/2} < 1 - (8d + 8)^{-1}$$

as soon as $n > (2K_5)^2(L^{-1}L' + 1)^2\Gamma_{11}^{-2}$. Thus also $M(\omega)A_k > \frac{1}{2}L'$ whenever $T(k) > (2K_5)^2(L^{-1}L' + 1)^2\Gamma_{11}^{-2}$. However, we must have $T(k) \to \infty$ as $k \to \infty$ because for each fixed $t$,

$$\sup_{\omega} P\{|\langle S_n, \omega \rangle| > A_k\} < P\{|S_i| > 2^{k/8}\} \to 0, \quad k \to \infty.$$

Thus also for each fixed $L'$, $A_k \inf_{\omega} M(\omega,k) > \frac{1}{2}L'$ eventually, and (2.29) holds.

We turn to (2.27) and (2.28). We have for each $\omega$, 

\[ P \left\{ |\langle S_{T-1}, \omega \rangle| < M(\omega)A_k \right\} \geq 1 - (8d + 8)^{-1} > \frac{1}{2} \quad (2.34) \]

(see (2.21) for \( \omega \neq \omega_2 \), and the definition of \( T(k) \) and \( M(\omega_2, k) = 1 \) for \( \omega = \omega_2 \)). On the other hand, by (2.31) with \( L_1 = \frac{1}{8} M(\omega)A_k \) and (2.33), the left-hand side of (2.34) is bounded above by

\[ 17K_5 (T - 1)^{-1/2} P \left\{ |\langle X_{1, \omega}^t, \omega \rangle| > \frac{1}{4} M(\omega)A_k \right\}^{-1/2}, \]

so that

\[ (T - 1) P \left\{ |\langle X_{1, \omega}^t, \omega \rangle| > \frac{1}{4} M(\omega)A_k \right\} \leq 2^{12} K_5^2. \quad (2.35) \]

Now (2.32) for \( Y_1 = Y, Y_2 = Y' \) shows that

\[ Q(Y', L) < Q(Y, L) \]

and this, together with (2.35), yields

\[ Q \left( \langle X_1, \omega \rangle, \frac{1}{2} M(\omega)A_k \right) > Q \left( \langle X_1, \omega \rangle, \frac{1}{2} M(\omega)A_k \right) \]

\[ > P \left\{ |\langle X_{1, \omega}^t, \omega \rangle| \leq \frac{1}{4} M(\omega)A_k \right\} \geq 1 - (T - 1)^{-1/2} K_5^2. \quad (2.36) \]

Finally, take \( T_{12} = T_{12}(F) \) so large that

\[ P \left\{ |X_1| > T_{12} \right\} \leq \frac{1}{2}, \quad (2.37) \]

and \( k_3 = k_3(F) \) so large that for all \( k > k_3 \),

\[ \frac{1}{2} A_k \inf_{\omega} M(\omega, k) > T_{12} \]

as well as

\[ (T(k) - 1)^{-1/2} K_5^2 < \frac{1}{4} \quad \text{and} \quad T(k) > 2. \]

Then (2.36) shows that there exists some interval \([a, a + \frac{1}{2} M(\omega)A_k]\) which contains \( \langle X_1, \omega \rangle \) with a probability at least \( 1 - (T(k) - 1)^{-1/2} K_5^2 > \frac{3}{4} \). By (2.37), for \( k > k_3 \) this interval must intersect \([T_{12}, T_{12} + T_{12}] \subset [-\frac{1}{2} M(\omega)A_k, \frac{1}{2} M(\omega)A_k]. \)]

Thus we proved for \( k > k_3 \),

\[ P \left\{ |\langle X_1, \omega \rangle| \leq M(\omega)A_k \right\} \geq P \left\{ |\langle X_1, \omega \rangle| \in [a, a + \frac{1}{2} M(\omega)A_k] \right\} \]

\[ > 1 - (T(k) - 1)^{-1/2} K_5^2. \]

This proves (2.27) for \( k > k_3(F) \).

To obtain (2.28) we again apply (2.33) with \( Y_i = \langle X_i, \omega \rangle \) and take \( L = M(\omega)A_k \). Combined with (2.31) and (2.34) this gives

\[ 3K_5 (T - 1)^{-1/2} M(\omega)A_k \left\{ \int |\langle x, \omega \rangle| \leq 2 M(\omega)A_k \right\}^{-1/2} \]

\[ > 1 - (8d + 8)^{-1} > \frac{1}{2}, \]

and therefore,
\[ T(k) \int_{|\langle x, \omega \rangle| < 2M(\omega)A_k} \langle x, \omega \rangle^2 \ dF^* (x) < 2^7 \{ K_3 M(\omega) A_k \}^2. \]

Now let \( Y \) be any random variable and \( c > 0 \) a constant. Denote the distribution function of \( Y \) by \( G \) and put

\[ d = \left\{ \int_{|y| < c} dG(y) \right\}^{-1} \int_{|y| < c} y \ dG(y) = E \{ Y \mid Y \leq c \}. \]

Then \( |d| < c \) and

\[
\int_{|y| < 2c} y^2 \ dG^2(y) > \int \left( \left( \int \left( (y_1 - d)^2 + 2(y_1 - d)(y_2 - d) + (y_2 - d)^2 \right) dG(y_1) \right) dG(y_2) \right)
\]

\[
= \int \left( \int \left( (y_1 - d)^2 + 2(y_1 - d)(y_2 - d) + (y_2 - d)^2 \right) dG(y_1) \right) dG(y_2)
\]

\[
= 2 \int_{|y_1| < c} dG(y_2) \int_{|y_1| < c} (y_1 - d)^2 dG(y_1).
\]

Consequently,

\[
\sigma^2 \{ YI[|Y| < c] \} < E \left\{ \left( YI[|Y| < c] - d \right)^2 \right\}
\]

\[
= \int_{|y| < c} (y - d)^2 dG(y) + d^2 \int_{|y| > c} dG(y)
\]

\[
< \frac{1}{2} \left( P \{ |Y| < c \} \right)^{-1} \int_{|y| < 2c} y^2 dG^2(y) + P \{ |Y| > c \} c^2.
\]

When this is applied to \( Y = \langle X_1, \omega \rangle, c = M(\omega)A_k \), we obtain from the above inequality:

\[
T(k)\sigma^2 \{ \langle X_1, \omega \rangle I[\langle X_1, \omega \rangle < M(\omega)A_k] \}
\]

\[
< \left( 2^6 K_3^2 \left( P \{ |\langle X_1, \omega \rangle| < M(\omega)A_k \} \right) \right)^{-1}
\]

\[
+ T(k) \left( P \{ |\langle X_1, \omega \rangle| > M(\omega)A_k \} \right) \{ M(\omega)A_k \}^2.
\]

Together with (2.27) and (2.29) this implies (2.28).

Lastly we prove (2.30). For \( \omega = \omega_2 \) (2.30) is immediate since we already observed that (2.22) holds for \( \omega = \omega_2 \) with \( N \) replaced by \( T \). We therefore fix \( \omega \neq \omega_2 \) for the remainder of the proof. Since, for any \( n < T, \langle S_T, \omega \rangle \) is the sum of the independent random variables \( \langle S_{T_n - 1}, \omega \rangle \) and \( \langle S_T - \langle T_n - 1 \rangle, \omega \rangle \), we have
\[
P \{ |\langle S_{T}, \omega \rangle| < \frac{1}{2} M(\omega) A_k \}
\]
\[< 33 Q \left( \langle S_{T}, \omega \rangle, \frac{1}{32} M(\omega) A_k \right) \text{ (by (2.31))} \]
\[< 33 Q \left( \langle S_{[T_n^{-1}], \omega} \rangle, \frac{1}{32} M(\omega) A_k \right) \text{ (by (2.32))} \]
\[< 33 K_2 \left[ T/n \right]^{-1/2} \left( P \left\{ |\langle S_{n}, \omega \rangle| > \frac{1}{16} M(\omega) A_k \right\} \right)^{-1/2} \text{ (by (2.33))} \]
\[< 33 K_2 \left[ T/n \right]^{-1/2} \left\{ 1 - Q \left( \langle S_{n}, \omega \rangle, \frac{1}{8} M(\omega) A_k \right) \right\}^{-1/2} \text{ (as in (2.36)). (2.38)} \]

Now in order to prove (2.30) we only have to consider those \( \omega \) for which
\[P \left\{ |\langle S_{T}, \omega \rangle| > \frac{1}{2} M(\omega) A_k \right\} < \frac{1}{2}. \]

For such an \( \omega \) (2.38) implies for all \( n < T \),
\[Q \left( \langle S_{n}, \omega \rangle, \frac{1}{8} M(\omega) A_k \right) > 1 - 2^{14} K_2^3 n T^{-1}. \tag{2.39} \]

Now apply (2.39) with \( n = N' \), where
\[N' = N'(\omega, k) = \min \left\{ n : P \left\{ |\langle S_{n}, \omega \rangle| > \frac{1}{2} M(\omega) A_k \right\} > (8d + 8)^{-1} \right\}. \]

Then, by (2.22), \( N'(\omega, k) < N(\omega, k) < T(k) \). Also, by the very definition of \( N' \),
\[P \left\{ \langle S_{N'}, \omega \rangle > \frac{1}{2} M(\omega) A_k \right\} > (16)^{1-(d+1)} \tag{2.40} \]
or the inequality holds with \( \langle S_{N'}, \omega \rangle \) replaced by \( -\langle S_{N'}, \omega \rangle \). For the sake of definiteness assume (2.40). Then any interval containing \( \langle S_{N'}, \omega \rangle \) with a probability larger than \( (16d + 15)(16d + 16)^{-1} \) must contain points to the right of \( \frac{1}{2} M(\omega) A_k \). By (2.39) there exists such an interval of length \( \frac{1}{8} M(\omega) A_k \) which contains \( \langle S_{N'}, \omega \rangle \) with a probability at least
\[1 - 2^{14} K_2^3 N'T^{-1} > (16d + 15)/ (16d + 16) \]
whenever \( T/N' > 2^{18} K_2^2 (d+1) + 4 \). In this case, therefore,
\[P \left\{ \frac{1}{8} M(\omega) A_k < \langle S_{N'}, \omega \rangle \right\} > 1 - 2^{14} K_2^3 N'T^{-1} \tag{2.41} \]
and
\[P \left\{ \langle S_{T}, \omega \rangle > \frac{1}{2} M(\omega) A_k \right\} \]
\[> P \left\{ \langle S_{(j+1)N'} - S_{N', \omega} \rangle > \frac{3}{8} M(\omega) A_k, \right\} \]
\[0 < j < \left[ \frac{T}{N'} \right], \langle S_{T-[T(N')^{-1}], N', \omega} \rangle > -\frac{1}{2} M(\omega) A_k \right\} \]
\[> \left( 1 - 2^{14} K_2^3 \frac{N'}{T} \right)^{T/N'} \min_{i < N'} P \left\{ |\langle S_i, \omega \rangle| < \frac{1}{2} M(\omega) A_k \right\} \]
\[> \frac{8d + 7}{8d + 8} \exp - 2^{15} K_2^2 \text{ (use the definition of } N'). \tag{2.42} \]
If $2 < T/N' < 2^{18} K_2^2 (d + 1) + 4$, we use (2.40) instead of (2.41) and obtain, as in (2.42),

$$P \left\{ \langle S_{T, \omega} \rangle > \frac{1}{2} M (\omega) A_k \right\}$$

$$> P \left\{ \langle S_{(j+1)N'} - S_{jN'}, \omega \rangle > \frac{1}{2} M (\omega) A_k, \right\}$$

$$0 < j < \left[ \frac{T}{N'} \right], \langle S_{T - \lfloor T(N')^{-1} \rfloor N', \omega} \rangle > \frac{1}{2} M (\omega) A_k \right\}$$

$$> (16d + 16)^{-1} \left( \frac{T}{N'} \right) (8d + 7)(8d + 8)^{-1}$$

$$> (8d + 7)(8d + 8)^{-1} \exp - \left( 2^{18} K_2^2 (d + 1) + 4 \right) \log(16d + 16). \quad (2.43)$$

Lastly, if $1 < T(N')^{-1} < 2$, then $N' < N < T < 2N'$ and a fortiori $T - N < N'$. But then

$$P \left\{ \langle S_{T, \omega} \rangle > \frac{1}{2} M (\omega) A_k \right\}$$

$$> P \left\{ \left| \langle S_{u, \omega} \rangle \right| > \frac{1}{2} M (\omega) A_k \right\}$$

$$> \frac{1}{8d + 8} \min_{l < N'} P \left\{ \left| \langle S_{u, \omega} \rangle \right| < \frac{1}{2} M (\omega) A_k \right\} \quad \text{(by (2.22))}$$

$$> (8d + 7)(8d + 8)^{-2} \quad \text{(by definition of } N').$$

Thus in all cases (2.30) holds with

$$\eta = \frac{8d + 7}{8d + 8} \exp - \left( 2^{18} K_2^2 (d + 1) + 4 \right) \log(16d + 16). \quad \square \quad (2.44)$$

We can now give the reduction of Proposition 1 to a two dimensional problem. For the remainder of the proof we shall use the abbreviation

$$m(k) = M (\omega^k, k). \quad (2.45)$$

Next we define

$$J_n = J_n^k = I \left[ \left| \langle X_{n, \omega^k} \rangle \right| > m(k) A_k \text{ or } \left| \langle X_{n, \omega^k} \rangle \right| > A_k \right],$$

$$v_r = v_r^k = \inf \left\{ n: \sum_{i=1}^n (1 - J_i) = rT(k) \right\},$$

$$Y_r = Y_r^k = \sum_{n = v_{r-1} < n < v_r} X_n. \quad (2.46)$$

$Y_r$ is the sum over a bunch of $X_n$'s, exactly $T(k)$ of which satisfy $\left| \langle X_{n, \omega^k} \rangle \right| < m(k) A_k$ and $\left| \langle X_{n, \omega^k} \rangle \right| < A_k$. It is easy to see from this that the $Y_r, r > 1$, are independent and identically distributed. Moreover, we can write

$$Y_r = \sum_{v_{r-1} < n < v_r} X_n J_n + \sum_{v_{r-1} < n < v_r} X_n (1 - J_n),$$
and by definition the second term contains exactly $T(k)$ summands with $1 - J_n \neq 0$. Thus if $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are independent, each $\alpha_i$ ($\beta_i$) with the conditional distribution of $X_1$, given $J_1 = 1$ (respectively $J_1 = 0$), then

$$\sum_{n_{r-1} \leq n \leq n_r} X_n (1 - J_n)$$

has the same distribution as

$$\sum_{n=1}^{T(k)} \beta_n.$$  \hspace{1cm} (2.48)

Moreover, the sum (2.47) is independent of $\sum_{n_{r-1} \leq n \leq n_r} X_n J_n$, which has the same distribution as

$$\sum_{n=1}^{\Lambda} \alpha_n$$

where $\Lambda = \Lambda(k)$ is independent of all $\alpha_i, \beta_i$ and has the negative binomial distribution

$$P\{A = l\} = P\{\nu_r - \nu_{r-1} = T(k) + l\}$$

$$= \binom{T(k) - 1 + l}{l} (P\{J_1 = 1\})^l (1 - P\{J_1 = 1\})^{T(k)},$$

$$l > 0. \hspace{1cm} (2.50)$$

**Lemma 3.** There exist constants $\Gamma_{13}(F) - \Gamma_{15}(F) < \infty$ such that for $2^{k/8} < A_k < 2^{k/2}$ and any choice of the unit vector $\omega_0 = \omega^k_0 \in \mathcal{K}^k$ and $k > 1$ one has

$$P\{|S_n| < A_k \text{ for some } 2^k < n < 2^{k+1}\} < \Gamma_{13}\{A_k 2^{-k/2}\}^{d-2}$$

$$\sum_{2^k-1 < r T(k) < (K_0 + \Gamma_{10} + 2) 2^k} P\left\{\left| \sum_{n=1}^{r} Y_n, \omega^k_j \right| \right\}$$

$$< 4M(\omega^k_j, A_k) \text{ for } 0 < j < 2$$

$$+ \Gamma_{14} \exp - \Gamma_{15} k. \hspace{1cm} (2.51)$$

**Proof.** Let $\mathcal{G}_n$ be the $\sigma$-field generated by $X_1, \ldots, X_n$, and consider for some fixed $k$ the $\{\mathcal{G}_n\}$ stopping times

$$\tau = \tau(k) = \min\{n: 2^k < n < 2^{k+1}, |S_n| < A_k\}$$

($= \infty$ if no such $n$ exists),

$$\sigma = \sigma(k) = \inf\{n_r: n_r > \tau\}.$$
Then the set
\[ \{ \tau < \infty, J_l = 0 \text{ for exactly } s \text{ indices } l \text{ with } \}
\]
\[ l < \tau \text{ but } l \text{ greater than the last } v_r < \tau \} \quad (2.52) \]
belongs to \( \mathcal{G}_r \). Now fix an \( \omega \in \mathcal{C} \cup \{ \omega_3, \ldots, \omega_d \} \) and set
\[ M^*(\omega,k) = \begin{cases} M(\omega,k) & \text{if } \omega \in \mathcal{C}, \\ 1 & \text{if } \omega \in \{ \omega_3, \ldots, \omega_d \}. \end{cases} \]
Then \( M^*(\omega,k) > 1 \) by (2.23), and on the set (2.52),
\[ P \left\{ \left| \langle S_s, \omega \rangle \right| > 4 M^*(\omega,k) A_k \mathcal{G}_r \right\} \]
\[ < P \left\{ \left| \langle S_s - S_{T(k)}, \omega \rangle \right| > 3 M^*(\omega,k) A_k \mathcal{G}_r \right\} \]
\[ = P \left\{ \left| \langle S_s, \omega \rangle \right| > 3 M^*(\omega,k) A_k \right\}, \]
where
\[ \gamma = \gamma(s,k) = \inf \left\{ n : \sum_{i=1}^{n} (1 - J_i) = T(k) - s \right\} \]
(note that only \( 0 < s < T(k) \) can occur).

By definition \( \gamma(s,k) > T(k) - s \), and for each \( L \) and \( k > k_2(F) \),
\[ P \left\{ \left| \langle S_s, \omega \rangle \right| > 3 M^*(\omega,k) A_k \right\} \]
\[ < P \left\{ \left| \langle S_{T(k)-s}, \omega \rangle \right| > 2 M^*(\omega,k) A_k \right\} \]
\[ + P \left\{ \left| \langle S_s - S_{T(k)-s}, \omega \rangle \right| > M^*(\omega,k) A_k \right\} \]
\[ < (4d + 4)^{-1} + P \{ \gamma - (T(k) - s) > L \} \]
\[ + P \left\{ \max_{j < L} \left| \langle S_{T(k)-s+j} - S_{T(k)-s}, \omega \rangle \right| > M^*(\omega,k) A_k \right\} \]
(see (2.21) and (2.24)).

Now for all \( s < T(k) \),
\[ P \left\{ \gamma(s) - (T(k) - s) > L \right\} = P \left\{ \sum_{i=1}^{T(k)-s+L} (1 - J_i) < T(k) - s \right\} \]
\[ < P \left\{ \sum_{i=1}^{T(k)+L} J_i > L \right\} < \frac{T(k) + L}{L} P \{ J_i = 1 \}. \]
Moreover, by (2.46) and (2.27), for \( k > k_2(F) \),
\[ P \{ J^k_i = 1 \} < P \left\{ \left| \langle X_i, \omega^k_i \rangle \right| > M(\omega^k_i,k) A_k \text{ for } i = 1 \text{ or } i = 2 \right\} \]
\[ < 2K_0 T(k)^{-1}. \quad (2.53) \]
Since $T(k) \to \infty$ with $k$ (see (2.29)) we can fix an $L$, independent of $s < T(k)$, such that for all $k > k_3$,

$$P \{ \gamma(s) - (T(k) - s) > L \} < 2k_0(T(k) + L)(LT(k))^{-1} < (8d + 8)^{-1}.$$  

With $L$ fixed in this way

$$P \left\{ \max_{j \leq L} \left| \langle S_{T(k) - s + j} - S_{T(k) - s}, \omega \rangle \right| > M^*(\omega, k)A_k \right\} \leq \frac{1}{8d + 8}$$

as soon as $\inf_{\omega} M^*(\omega, k)A_k$ is large enough. Thus by (2.29) we can find $k_4(F) < \infty$, such that for $k > k_4(F)$ one has on the set (2.52),

$$P \{ |\langle S_\omega, \omega \rangle > 4M^*(\omega, k)A_k | \beta_r \} < (2d + 2)^{-1},$$

uniformly in $\omega$ and $s < T(k)$. Now with $\omega_j, 1 < j < d$, as right after Lemma 1 and $\omega_0 \in \mathcal{C}$ we have

$$P \{ \tau < \infty, |\langle S_{\omega_j}, \omega_j \rangle > 4M^*(\omega, k)A_k \text{ for some } 0 < j < d \} \leq \sum_{j=0}^{d} P \{ |\langle S_\omega, \omega_j \rangle > 4M^*(\omega, k)A_k | \tau < \infty \} \leq \frac{1}{2} P \{ \tau < \infty \}.$$

After taking complements with respect to $\{ \tau < \infty \}$ and using the fact that $\sigma$ equals some $\nu_r$, we obtain

$$P \{ |S_n| < A_k \text{ for some } 2^k < n < 2^{k+1} \} = P \{ \tau < \infty \} \leq 2P \{ \tau < \infty, |\langle S_\omega, \omega_j \rangle | < 4M^*(\omega, k)A_k \text{ for all } 0 < j < d \} \leq 2P \{ \exists \nu_r \in \left[ 2^k, (K_0 + \Gamma_{10} + 2)2^k \right] \text{ with } \}$$

$$|\langle S_r, \omega_j \rangle | < 4M^*(\omega, k)A_k \text{ for all } 0 < j < d \} + 2P \{ \tau < \infty, \sigma > (K_0 + \Gamma_{10} + 2)2^k \}. \quad (2.54)$$

The last term in (2.54) is easily seen to be exponentially small. Indeed, $\tau < 2^{k+1}$ whenever $\tau < \infty$ so that

$$P \{ \tau < \infty, \sigma > (K_0 + \Gamma_{10} + 2)2^k \}$$

$$< E \{ P \{ \sigma - \tau > (K_0 + \Gamma_{10})2^k | \beta_r \}; \tau < \infty \},$$

and on the set (2.52),

$$P \{ \sigma - \tau > (K_0 + \Gamma_{10})2^k | \beta_r \} = P \{ \gamma(s, k) > (K_0 + \Gamma_{10})2^k \}$$

$$< P \{ \gamma(s, k) - (T(k) - s) > K_02^k \} = P \left\{ \sum_{i} T(k) - s + K_02^k \right\}$$

$$\{ J_i > K_02^k \} \quad (2.55)$$
(recall $T(k) - s < T(k) < \Gamma_0^{10} 2^k \leq \Gamma_1 2^k$ by (2.26) and $A_k < 2^{k/2}$). Since $T(k) - s < \Gamma_0^{10} 2^k$ and $P \{ J_i = 1 \} \to 0$ as $k \to \infty$ (see (2.53)), standard exponential estimates show that the last member of (2.55) has an upper bound of the form $\Gamma_0^{10}(F) \exp - \Gamma_1(F) 2^k$ uniformly in $s < T(k)$ (see [10, Chapter 9, problems 12–16]).

The first term in the last member of (2.54) is more troublesome. Recall that $F$ is now assumed to satisfy (2.1). By taking $C_0$ smaller if necessary we may assume that

$$C_0 = \{ z \in \mathbb{R}^d : |\langle x, \omega \rangle - c_j| < \lambda, 1 \leq j \leq d \}$$

for some $c_j$ and $\lambda > 0$. Then we can write $F = a(2\lambda)^d G_3 + (1 - a(2\lambda)^d)H_3$, where $G_3$ is the uniform distribution on $C_0$ and $H_3$ is some other distribution function on $\mathbb{R}^d$. Correspondingly, we may assume that $X_n = E_n \theta_n + (1 - E_n)\psi_n$ where the $E_n, \theta_n, \psi_n$ are independent,

$$P \{ E_n = 1 \} = 1 - P \{ E_n = 0 \} = a(2\lambda)^d, \quad n \geq 1,$$

each $\theta_n$ has distribution $G_3$ and each $\psi_n$ has distribution $H_3$. Now we want to estimate for $3 < j < d$ the conditional probability of

$$\left| \langle S_n, \omega_j \rangle \right| < 4M^*(\omega_j, k)A_k = 4A_k \quad (2.56)$$

given $\nu_r = s$, $E_n = 1$ for $n = n_1, n_2, \ldots, n_p$ ($1 < n_1 < \cdots < n_p < s$), but not for any other $n < s$, and given the values of $X_n$, $n \not\in \{n_1, \ldots, n_p\}$, as well as $\langle X_n, \omega_j \rangle$ for $0 < j < 2$ and $n \in \{n_1, \ldots, n_p\}$. Given all these data we know that for $j > 3$,

$$\langle S_n, \omega_j \rangle = \langle S_s, \omega_j \rangle = \sum_{n \in \{n_1, \ldots, n_p\}} \langle \theta_n, \omega_j \rangle + D_j,$$

where

$$D_j = \sum_{l \not\in \{n_1, \ldots, n_p\}, 1 \leq l \leq s} \langle X_{l}, \omega_j \rangle$$

is some known constant. Moreover, we know the values of $\langle \theta_n, \omega_j \rangle$ for $n \in \{n_1, \ldots, n_p\}$ and $0 < j < 2$. (Note that also the event $\{ \nu_r = s \}$ is determined only by $J_1, \ldots, J_s$ and hence by $\langle X_1, \omega_1 \rangle, \langle X_s, \omega_2 \rangle, 1 < j < s$.) However, $\theta_n$ has a uniform distribution on $C_0$, and therefore the conditional distribution of $\langle \theta_n, \omega_3 \rangle, \ldots, \langle \theta_n, \omega_2 \rangle$, given $\langle \theta_n, \omega_1 \rangle = x_1$ and $\langle \theta_n, \omega_2 \rangle = x_2$, is the same for all $x_1, x_2$, to wit the uniform distribution on the $(d - 2)$ dimensional cube

$$C_2 = \{ x = (x(3), \ldots, x(d)) : |x(i) - c_i| < \lambda, 3 \leq i \leq d \}.$$

This remains true even if we condition on $\langle \theta_n, \omega_0 \rangle$ as well, because $\langle \theta_n, \omega_0 \rangle$ is a function of $\langle \theta_n, \omega_1 \rangle$ and $\langle \theta_n, \omega_2 \rangle$ (since $\omega_0$ lies in $\mathcal{K}$, the plane spanned by $\omega_1$,
and $\omega_j$). Thus, if $\delta_1, \delta_2, \ldots$ are independent random variables, each one uniformly distributed on $C_2$, then

\[
P \left( \left| \langle S_n, \omega_j \rangle \right| < 4M^* \delta_j A_k \text{ for } 0 < j < d \right) = s,
\]

\[
\sum_{n=1}^{r} E_n = \rho_j \left( \left| \langle S_n, \omega_j \rangle \right| < 4M^* \delta_j A_k \text{ for } 0 < j < 2 \right)
\]

\[
< \sup_{D_j} P \left( \left| \sum_{j=1}^{r} \delta_j (j) + D_j \right| < 4M^* \delta_j A_k \text{ for } 3 < j < d \right). \tag{2.57}
\]

Now the corollary to Theorem 6.2 of [6] applies to the independent identically distributed $(d - 2)$ vectors $\{\delta_j\}_{j>1}$ whose distribution is not concentrated on any $(d - 3)$ dimensional hyperplane and therefore the last member of (2.57) is bounded by

\[
\Gamma_1(F) \rho^{-(d-2)/2} \prod_{j=3}^{d} \left\{ 4\lambda^{-1} A_k + 1 \right\} \leq \Gamma_9(F) \left\{ A_k \rho^{-1/2} \right\}^{d-2}.
\]

It follows that for fixed $r$,

\[
P \left\{ \nu_r \in \left[ 2^k, (K_0 + \Gamma_{10} + 2)2^k \right) \right. \]

\[
< \left. P \left( \sum_{n=1}^{r} E_n < \frac{1}{2} 2^k P \{ E_1 = 1 \} \right) \right.
\]

\[
+ \Gamma_9(F) A_k^{d-2} \left( 2^{k-1} P \{ E_1 = 1 \} \right)^{-(d-2)/2}
\]

\[
\cdot P \left( \nu_r \in \left[ 2^k, (K_0 + \Gamma_{10} + 2)2^k \right), \sum_{n=1}^{r} E_n > 2^{k-1} P \{ E_1 = 1 \}, \right. \]

\[
\left. \left| \langle S_n, \omega_j \rangle \right| < 4M^* \delta_j A_k = 4M \delta_j A_k \text{ for } 0 < j < 2 \right). \tag{2.58}
\]

Since $P \{ E_1 = 1 \} = a(2\lambda)^d$ is independent of $k$, the first term in the right-hand side of (2.58) is at most $\Gamma_20(F) \exp - \Gamma_21(F) 2^k$ (see [10, Chapter 9, problems 12–16]) and we obtain
\[ P \{ \exists \nu_n \in \left[ 2^k, (K_0 + \Gamma_{10} + 2)2^k \right] \text{ with } |\langle S_{\nu_n}, \omega_j \rangle| < 4M^*(\omega_j, k)A_k \text{ for all } 0 < j < d \} \]

\[ < \sum_{2^k-1 < rT(k) < (K_0 + \Gamma_{10} + 2)2^k} \left[ \left( \Gamma_{20} \exp - \Gamma_{21}2^k \right) + \left( \Gamma_{22}(F)\{A_k2^{-k/2}\}^{d-2} \right) \cdot P \left\{ |\langle S_{\nu_n}, \omega_j \rangle| < 4M(\omega_j, k)A_k \text{ for } 0 < j < 2 \right\} \right] + P \left\{ \nu_n \in \left[ 2^k, (K_0 + \Gamma_{10} + 2)2^k \right] \text{ for some } r < 2^{k-1}(T(k))^{-1} \text{ or } r > (K_0 + \Gamma_{10} + 2)2^k(T(k))^{-1} \right\} \]  

Since \( \nu_n \) increases with \( r \) and \( \nu_n > rT(k) \) (by definition of \( \nu_n \)) the last term in the right-hand side of (2.59) is at most

\[ P \left\{ \nu_n > 2^k \text{ for } n = \left[ 2^k-1(T(k))^{-1} \right] \right\} \]

and we leave it to the reader to show that this term again is bounded by \( \Gamma_{23} \exp - \Gamma_{24}2^k \). Thus, since \( S_{\nu_n} = \sum_{n=1}^{\nu_n} Y_n \), the last member of (2.54) is indeed bounded by the right-hand side of (2.51) for \( k > \max(k_2, k_3, k_4) \) and suitable \( \Gamma_{13} - \Gamma_{15} \). Obviously we can then insure the validity of (2.51) for \( k < \max(k_2, k_3, k_4) \) by increasing \( \Gamma_{14} \). □

3. Completion of the proof. Lemma 3 shows that Proposition 1 will follow once we show that there exist \( k_5(F) < \infty \) and \( K_2 < \infty \) such that for a suitable choice of \( \omega_0^k \in \mathcal{C}^k \),

\[ \sum_{r=s}^{2s} P \left\{ \frac{r}{s} \left\| \sum_{n=1}^{r} \langle Y_n^{k}, \omega_j^k \rangle \right\| < 4M(\omega_j, k)A_k \text{ for } 0 < j < 2 \right\} < K_2, \text{ for all } k > k_5, s > 1 \text{ and } A_k \in [2^{k/8}, 2^{k/2}] \]  

(3.1)

Note that we can always obtain (1.4) for \( k < k_5 \) by increasing \( \Gamma_1 \); moreover, (1.4) is vacuously true for \( A > 2^{k/2} \), and, as observed before, (1.4) follows from (2.6) when \( A < 2^{k/8} \). Note also that the last statement of our theorem will be immediate from Proposition 1, because (1.2)–(1.4) imply for all \( b > 0 \),

\[ P \{ \psi(n)^{-1}|S_n| < b \text{ for some } n > 2^k \} \]

\[ < \sum_{l=k}^{\infty} P \{ |S_n| < 2^{1/2}b\psi(2^l) \text{ for some } 2^l < n < 2^{l+1} \} \]

\[ < \Gamma_1(F) \left( \sum_{l=k}^{\infty} \left\{ 2^{1/2}b\psi(2^l) + 1 \right\}^{d-2} t^{-d-2} + \exp - t\Gamma_2(F) \right) \]

\[ < K_3 \Gamma_1(F) b^{d-2} \int_{2k-1}^{\infty} \psi(t)^{d-2} t^{-d/2} dt + K_3 \Gamma_1(F)2^{-k(d-2)/2} \]

\[ + \Gamma_1(F) \{ 1 - \exp - \Gamma_2(F) \}^{-1} \exp - k\Gamma_2(F) \to 0 \quad (k \to \infty). \]
Moreover, it is well known that if (1.2) holds, then $\psi(n)^{-1}|S_n^*| \to \infty$ w.p.1 or $\psi(t)^{-1}|B(t)| \to \infty$ w.p.1 if and only if (1.3) holds (see [2, Theorems 5,6] or [8, §4.2.15]; for $S_n^*$ one can also use the multidimensional analogue of Remark 2, p. 1182 in [9]). Thus to prove our theorem it suffices to prove (1.4), and this in turn has been reduced to proving (3.1).

This will be done by means of certain inequalities on two dimensional concentration functions which are contained in Proposition 2 and some corollaries. We therefore begin with their proofs; the notation is as in Proposition 2.

**Proof of Proposition 2.** By [6, formula (6.9)]

$$\sup_{z \in \mathbb{R}^2} P \left( \left| \sum_{i=1}^n Z_i + z \right| < \rho \right)$$

$$< K_4 \rho^2 \int_{|\theta| < \rho^{-1}} d\theta \exp - \left[ \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^2} (1 - \cos \langle \theta, u \rangle) \, dG_i^*(u) \right]$$

$$< K_4 \rho^2 \int_{|\theta| < \rho^{-1}} d\theta \exp - \left[ \frac{1}{2} \sum_{i=1}^n \left( \int_{|u| < \rho_i} (1 - \cos \langle \theta, u \rangle) \, dG_i^*(u) \right)

+ \frac{1}{2q_i} \int_{|u|, |v| > \rho_i} \left[ (1 - \cos \langle \theta, u \rangle)

+ (1 - \cos \langle \theta, v \rangle) \right] \, dG_i^*(u) \, dG_i^*(v) \right]$$

$$< K_4 \rho^2 \int_{|\theta| < \rho^{-1}} \exp \left\{ - K_5 \sum_{i=1}^n \int_{|u| < \rho_i} \langle \theta, u \rangle^2 \, dG_i^*(u) - \sum_{i=1}^n \frac{1}{4q_i} \right. 

\cdot \int_{u, v \in A_i} \left[ (1 - \cos \langle \theta, u \rangle) + (1 - \cos \langle \theta, v \rangle) \right] 

\left. dG_i^*(u) \, dG_i^*(v) \right\} \, d\theta. \quad (3.2)$$

Now take $C$ as in (1.7) and apply Hölder’s inequality, as in [5, p. 212] or [6, p. 294]. It then follows that the last member of (3.2) is bounded by

$$K_4 \rho^2 \left\{ \int_{|\theta| < \rho^{-1}} d\theta \exp - K_5 C|\theta|^2 \right\} \cdot \exp \sum_i \frac{\rho_i^2}{C d_i} \int_{u, v \in A_i} dG_i^*(u) \, dG_i^*(v) \cdot \log \int_{|\theta| < \rho^{-1}} d\theta \exp - \frac{C}{4\rho_i^2} (2 - \cos \langle \theta, u \rangle - \cos \langle \theta, v \rangle).$$
Clearly
\[ \int_{|\theta|<\rho^{-1}} d\theta \exp - K_5 C|\theta|^2 < \int_{\mathbb{R}^2} d\theta \exp - K_5 C|\theta|^2 = \frac{\pi}{K_5 C}. \]

It is also clear that
\[ \int_{|\theta|<\rho^{-1}} d\theta \exp - \frac{C}{4\rho_i^2} (2 - \cos \langle \theta, u \rangle - \cos \langle \theta, v \rangle) \]
is invariant under a rotation of the pair of vectors \( u, v \) and therefore equals
\( \varphi = \varphi(u,v); \) recall also \( |u|, |v| > \rho > \rho_i \) for \( u, v \in \mathcal{A}_i \)
\[ \int_{|\theta|<\rho^{-1}} d\theta \exp - \frac{C}{4\rho_i^2} \left[ (1 - \cos \theta_i |u|) \right. \]
\[ \left. + (1 - \cos(\theta_i |v| \cos \varphi + \theta_2 |v| \sin \varphi)) \right] \]
\[ \leq \int_{|\theta_i|<\rho_i^{-1}} d\theta_1 \exp - \frac{C}{4\rho_i^2} \left[ 1 - \cos \theta_i |u| \right] \]
\[ \cdot \sup_{b \in \mathbb{R}} \int_{|\theta_2|<\rho^{-1}} d\theta_2 \exp - \frac{C}{4\rho_i^2} \left[ 1 - \cos (b + \theta_2 |v| \sin \varphi) \right] \]
\[ \leq K_6 \frac{\rho_i}{|u|C^{1/2}} \left( |u|\rho_i^{-1} + 1 \right) \cdot \frac{\rho_i}{|v| \sin \varphi C^{1/2}} \left( |v| |\sin \varphi|\rho_i^{-1} + 1 \right) \]
(1.8) now follows by putting these estimates together. □

Corollary 1 is immediate from Proposition 1 by specialization (take all \( \rho_i = \rho \) and \( \mathcal{A}_i = B_1 \times B_2 \)). More important for our proof, though, is the more technical

Corollary 2. Let \( Z_1, Z_2, \ldots \) be independent identically distributed two-vectors and assume that there exist constants \( d_1 > 0, -\infty < d_2 < \infty, d_3 > 0 \) such that
\[ p_1 \equiv P \{ Z_i^1 (2) > d_1, Z_i^1 (1) < d_2 Z_i^1 (2) \} > 0, \] (3.3)
as well as
\[ p_2 \equiv P \{ Z_i^2 (2) > d_1, Z_i^2 (1) > d_2 Z_i^2 (2) + d_3 \} > 0. \] (3.4)
Then for all \( \rho > 0 \) and all \( n > 1 \),
for some constant $\Gamma_3 = \Gamma_3(d_i, p_i) < \infty$ depending on $p_1, p_2, d_1 - d_3$ only. In fact $\Gamma_3 < \Gamma_4(D) < \infty$ for all $p_i, d_i$ and $D < \infty$ with
\[
|d_2| < D, \quad d_1d_3^{-1} < D, \quad p_1 > D^{-1}, \quad p_2 > D^{-1}. \tag{3.6}
\]
Moreover, for all $s > 1$ and $\rho > 0$,
\[
E^\# \left\{ r \in [s,2s]: \left| \sum_1^r Z_i \right| < \rho \right\} < \Gamma_3 \left( \left( \frac{\rho}{d_1} \right)^2 + 1 \right) (1 + \log 2). \tag{3.7}
\]
If instead of (3.3) and (3.4) we assume that for some $d_i > 0$, $-\infty < d_2 < \infty$, and $d_3 > 0$,
\[
p_3 \equiv P \left\{ Z_1(2) > d_1, Z_1(1) < d_2 Z_1(2) \right\} > 0 \tag{3.8}
\]
as well as
\[
p_4 \equiv P \left\{ Z_1(2) > d_1, Z_1(1) > d_2 Z_1(2) + d_3 \right\} > 0, \tag{3.9}
\]
then for each $s > 1, \rho > 0$,
\[
E^\# \left\{ r \in [s,2s]: \left| \sum_1^r Z_i \right| < \rho \right\} < \Gamma_5 \left( \left( \frac{\rho}{d_1} \right)^2 + 1 \right), \tag{3.10}
\]
now with $\Gamma_5 = \Gamma_5(p_i, d_i) < \infty$ depending only on $p_3, p_4$ and $d_1 - d_3$. Moreover, $\Gamma_5 < \Gamma_6(D) < \infty$ for all $p_i, d_i$ and $D < \infty$ with
\[
|d_2| < D, \quad d_1d_3^{-1} < D, \quad p_3 > D^{-1}, \quad p_4 > D^{-1}. \tag{3.11}
\]
Lastly, assume that (3.8) and (3.9) hold and that $W_1, W_2, \ldots$ are independent random vectors each with the distribution of $\sum_1^\Lambda Z_i$, where $\Lambda$ is independent of the $Z_i$, has the distribution given by (2.50), and that for some $d_4, d_5$,
\[
0 < d_4 < T(k)P \{ J_1 = 1 \} < d_5 < \infty, \quad P \{ J_1 = 1 \} \leq \frac{1}{2}. \tag{3.12}
\]
Under these conditions one has for all $\rho > 0$,
\[
\sup_z P \left\{ \left| \sum_1^n W_r + z \right| < \rho \right\} < \Gamma_7 \frac{1}{n} \left( \left( \frac{\rho}{d_1} \right)^2 + 1 \right) \tag{3.13}
\]
for some $\Gamma_7 = \Gamma_7(p_i, d_i)$ depending on $p_3, p_4$ and $d_1 - d_5$ only. Moreover, $\Gamma_7 < \Gamma_8(D) < \infty$ for all $p_i, d_i$ and $D < \infty$ which satisfy (3.11) and $d_4 > D^{-1}$, $d_5 < D$. Estimates (3.10) and (3.13) remain valid if (3.8) and (3.9) hold with $Z_1$ replaced by $-Z_1$ in one or both of these formulae.

**Proof.** We first prove (3.5) with $\rho = d_1$ from (3.3) and (3.4). This is an immediate application of Corollary 1 with

---

8\#\{ \} denotes the number of elements in the set between braces.
and

\[ B_1 = \{ z \in \mathbb{R}^2: z(2) > d_1, z(1) < d_2z(2) \} \]

and

\[ B_2 = \{ z \in \mathbb{R}^2: z(2) > d_1, z(1) > d_2z(2) + d_3 \} \quad (\text{see figure}). \]

Clearly, \( B_1, B_2 \subseteq \{ z \in \mathbb{R}^2: |z| > d_1 \} \) and

\[ \int_{B_i} dG^z(u) = p_i < q = \int_{|u| > d_i} dG^z(u), \quad i = 1, 2. \]

Moreover, for \( u \in B_1, v \in B_2, \)

\[ |\sin \varphi(u,v)| = \left( |u| |v| \right)^{-1} |u(2)v(1) - u(1)v(2)|, \]

and, consequently,

\[ |v| |\sin \varphi(u,v)| \geq \frac{u(2)}{|u(1)| + u(2)} |v(1) - \frac{u(1)}{u(2)} v(2)| \]

\[ > \frac{u(2)}{|u(1)| + u(2)} \left\{ d_2v(2) + d_3 - \frac{u(1)}{u(2)} v(2) \right\}. \quad (3.14) \]

Now \( u(1) < d_2u(2) \) for \( u \in B_1 \) so that the last member of (3.14) is positive. If \( |u(1)| < (1 + 2|d_2|)u(2) \), then it is even bounded below by \( \frac{1}{2}(1 + |d_2|)^{-1}d_3 \); and if \( |u(1)| > (1 + 2|d_2|)u(2) \), then necessarily \( u(1) < -2|d_2|u(2) \) and the last member of (3.14) is bounded below by

\[ \frac{u(2)}{u(1)| + u(2)} \left( \frac{1}{2} \frac{|u(1)|}{u(2)} v(2) \right) > \frac{1}{4} \frac{1}{1 + |d_2|} d_1. \]

Thus in all cases

\[ |v| |\sin \varphi(u,v)| > c \equiv \frac{1}{4} \frac{1}{1 + |d_2|} \min\{d_1, d_3\}, \quad (3.15) \]
and (3.5) follows from (1.10) with
\[ \Gamma_3 \equiv K_1 (1 + d_1 e^{-1}) q(p_1 p_2)^{-1} \]
\[ \leq K_1 (p_1 p_2)^{-1} \{ 1 + 4(1 + |d_2|)(1 + d_1 d_3^{-1}) \}. \] (3.16)
Recall that we took \( \rho = d_1 \) so far. However, once we have (3.5) for \( \rho = d_1 \) it follows for each \( \rho > 0 \) because any disc of radius \( \rho \) can be covered by at most \( K_7(\rho^2 d_1^{-2} + 1) \) discs of radius \( d_1 \). It is immediate from (3.16) that \( \Gamma_3 \) is bounded above whenever \( p_1, d_1 \) satisfy (3.6). Also (3.7) is immediate from (3.5) because \( \Sigma_{i=1}^2 a_i r_i^{-1} \leq 1 + \log 2 \).

Next we prove (3.13) by showing that (3.3) and (3.4) hold for suitable values of \( p_1, p_2 \) and \( Z \) replaced by \( W \). For this purpose take \( B_1 \) as above. Assume further that \( z_1, z_2, z_1' \in B_1 \) satisfy
\[ z_1(1) - d_2 z_1(2) < \min \{ z_2(1) - d_2 z_2(2), z_1(1) - d_2 z_1'(2) \} \] (3.17)
as well as
\[ z_2(2) > z_1'(2). \] (3.18)
Then
\[ z_1(1) + z_2(1) - z_1'(1) - d_2 (z_1(2) + z_2(2) - z_1'(2)) \]
\[ < 2^2 (z_1 - d_2 z_1(2) < 0, \] (3.19)
and
\[ z_1(2) + z_2(2) - z_1'(2) > z_1(2) > d_1. \] (3.20)
(3.19) and (3.20) just say that \( z_1 + z_2 - z_1' \in B_1 \), so that \( z_1 + z_2 - z_1' \in B_1 \) as soon as (3.17) and (3.18) hold. In agreement with the notational convention from before Lemma 1 we now take \( \Lambda' \) an independent copy of \( \Lambda \) and put
\[ W_1 = \sum_{i=1}^\Lambda Z_i, \quad W_1' = \sum_{i=1}^{\Lambda'} Z_i', \quad W_1 = W_1 - W_1'. \]
By comparing the different orders of the quantities in (3.17) and (3.18) we now obtain
\[ P \{ W_1 \in B_1 \} > P \{ \Lambda = 2, \Lambda' = 1 \} \]
\[ \cdot \int_{z_1, z_2, z_1' \in B_1} P \{ Z_1 \in dz_1, Z_2 \in dz_2, Z_1' \in dz_1' \} \]
\[ > \frac{1}{6} P \{ \Lambda = 2 \} P \{ \Lambda' = 1 \} \]
\[ \cdot \int_{z_1, z_2, z_1' \in B_1} P \{ Z_1 \in dz_1 \} P \{ Z_2 \in dz_2 \} P \{ Z_1' \in dz_1' \} \]
\[ = \frac{1}{6} P \{ \Lambda = 2 \} P \{ \Lambda' = 1 \} (P \{ Z_1 \in B_1 \})^3. \] (3.21)
Now, by (2.50) and (3.12),
\[ P\{\Lambda = 2\} > \frac{1}{2!} (T(k)P\{J_1 = 1\})^2 \exp - 2d_5 > \frac{1}{2} d_4^2 \exp - 2d_5. \]
Similarly,
\[ P\{\Lambda = 1\} > d_4 \exp - 2d_5, \]
and finally, \( P\{Z_1 \in B_1\} = p_3 \) (by (3.8)). Thus
\[ P\{W_1'(2) > d_1, W_1'(1) < d_2 W_1'(2)\} \]
\[ = P\{W_1' \in B_1\} > K_8 d_4^2 p_3^3 \exp - 4d_5. \quad (3.22) \]
Next consider \( P\{W_1' \in B_2\). Assume \( z_1, z_2, z_1' \in B_2 \) are such that
\[ z_1(1) - d_2 z_1(2) > \max\{z_2(1) - d_2 z_2(2), z_1'(1) - d_2 z_1'(2)\}, \quad (3.23) \]
and (3.18) holds; then we get
\[ z_1(1) + z_2(1) - z_1'(1) - d_2 (z_1(2) + z_2(2) - z_1'(2)) \]
\[ > z_2(1) - d_2 z_2(2) > d_3, \]
and as before, \( z_1(2) + z_2(2) - z_1'(2) > d_1 \). Thus, as in (3.21) and (3.22),
\[ P\{W_1' \in B_2\} > P\{\Lambda = 2, \Lambda' = 1\} \]
\[ \cdot \int_{z_1, z_2, z_1' \in B_2, (3.18) \text{ and } (3.23) \text{ hold}} P\{Z_1 \in dz_1, Z_2 \in dz_2, Z_1' \in dz_1'\} \]
\[ > K_8 d_4^3 \exp - 4d_5 (P\{Z_1 \in B_2\})^3 \]
\[ > K_8 d_4^3 p_3^3 \exp - 4d_5. \quad (3.24) \]
(3.22) and (3.24) show that (3.3) and (3.4) hold for \( W_1' \) instead of \( Z_1' \) and
\[ p_1 = K_8 d_4^3 p_3^3 \exp - 4d_5, \quad p_2 = K_8 d_4^3 p_3^3 \exp - 4d_5. \quad (3.25) \]
(3.13) is therefore immediate from (3.5). Also the bound \( \Gamma_7 < \Gamma_8(D) \), whenever (3.11) holds together with \( d_4 > D^{-1}, d_5 < D \), is immediate from (3.25) and the fact that \( \Gamma_5 < \Gamma_4(D) \) on (3.6). Since we may everywhere in this argument replace \( Z_1 \) by \(-Z_1\) without influencing the distribution of \( W_1' \), no change is needed if \( Z_1 \) is replaced by \(-Z_1\) in (3.8) or (3.9).
Lastly, we prove (3.10). For this purpose let \( \Lambda_1, \Lambda_2, \ldots \) be a sequence of independent random variables, also independent of the \( Z_i \), and
\[ P\{\Lambda_l = 1\} = \left(\frac{1}{2}\right)^{l+1}, \quad l > 0. \quad (3.26) \]
(3.26) corresponds to \( T(k) = 1, P\{J_1 = 1\} = \frac{1}{2} \) in (2.50). Thus (3.12) is satisfied with \( d_4 = d_5 = \frac{1}{2} \) in this case. Set \( \tau_0 = 0 \), and for \( r > 1 \),
\[ \tau_r = \sum_{i=0}^{r} \Lambda_i, \quad W_r = \sum_{i=\tau_{r-1}+1}^{r} Z_i. \quad (3.27) \]
Then the $W_i$ are independent, identically distributed and by what we just proved (3.13) holds with $\Gamma_7$ depending only on $p_3, p_4, d_1 - d_3$ (since now $d_4 = d_5 = \frac{1}{2}$) and $\Gamma_7 < \Gamma_8(D)$ whenever (3.11) holds. Moreover,

$$E \# \left\{ r \in [s, 2s] : \left| \sum_{i=1}^{r} Z_i \right| < \rho \right\} < (s + 1) P \{ \tau_{[s/2]} > s \} \quad \text{or} \quad \tau_{4s} < 2s$$

$$+ E \# \left\{ r \in [\tau_{[s/2]}, \tau_{4s}] : \left| \sum_{i=1}^{r} Z_i \right| < \rho \right\}. \quad (3.28)$$

By (3.26),

$$E \Lambda_i = 1, \quad (3.29)$$

and by standard exponential bounds for the geometric distribution (compare [9, formulae (5.40)–(5.42)]),

$$P \{ |\tau_n - n| > \frac{1}{2} n \} = P \left\{ \sum_{i=1}^{n} \Lambda_i < \frac{1}{2} n \text{ or } \sum_{i=1}^{n} \Lambda_i > \frac{3}{2} n \right\}$$

$$< K_n \exp - K_{10} n. \quad (3.30)$$

Thus, the first term in the right-hand side of (3.28) is at most

$$2(s + 1) K_n \exp - K_{10} \frac{s}{2} < K_11.$$

The second term in the right-hand side of (3.28) equals

$$\sum_{[s/2] < j < 4s} E \# \left\{ r \in [\tau_j, \tau_{j+1}] : \left| \sum_{i=1}^{r} Z_i \right| < \rho \right\}$$

$$= \sum_{[s/2] < j < 4s} \sum_{m=0}^{\infty} P \left\{ \left| \sum_{i=1}^{\tau_{j+1} - \tau_j} Z_i \right| < \rho, \tau_{j+1} - \tau_j > m \right\}$$

$$= \sum_{[s/2] < j < 4s} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} P \left\{ \tau_{j+1} - \tau_j > m, \sum_{l=\tau_j+1}^{\tau_{j+1}} Z_l \in dz \right\}$$

$$\cdot P \left\{ \left| \sum_{i=1}^{\tau_j} Z_i + z \right| < \rho \right\}$$

$$< \sum_{[s/2] < j < 4s} \sum_{m=0}^{\infty} P \{ \tau_{j+1} - \tau_j > m \} \sup_z P \left\{ \left| \sum_{i=1}^{j} W_i + z \right| < \rho \right\}$$

$$< \sum_{[s/2] < j < 4s} E \{ \tau_{j+1} - \tau_j \} \Gamma_7 \frac{1}{j} \left\{ \left( \frac{\rho}{d_i} \right)^2 + 1 \right\} \quad \text{(by (3.13))}$$

$$< \Gamma_7 K_{12} \left\{ \left( \frac{\rho}{d_i} \right)^2 + 1 \right\} \quad \text{(by (3.29)).} \quad (3.31)$$
The last estimate works only for \( s > 2 \), but for \( s = 1 \) (3.10) needs no proof anyway. From (3.28), (3.31) and the estimate for the first term in the right-hand side of (3.28), we finally see that (3.28) is, for \( s > 2 \), at most

\[
K_{11} + \Gamma_7 K_{12} \left( \left( \frac{\rho}{a_1} \right)^2 + 1 \right) \leq K_{13} (\Gamma_7 + 1) \left( \left( \frac{\rho}{a_1} \right)^2 + 1 \right). \tag{3.32}
\]

We now return to the proof of the bound in (3.1). The notation will be as in § 2 for the remainder of this section. We distinguish two cases:

\[
T(k)P \{ J^k = 1 \} < 2^{-5} \eta \tag{3.33}
\]

and

\[
2^{-5} \eta < T(k)P \{ J^k = 1 \} < 2K_0. \tag{3.34}
\]

(\( \eta \) and \( K_0 \) are as in Lemma 2. (3.33) and (3.34) are the only possibilities as we saw in (2.53)). For the remainder of the proof we are only interested in \( \omega \) components with \( \omega \in \mathcal{K} \), the plane spanned by \( \omega_1, \omega_2 \). We shall therefore think of all random variables as being two dimensional and specified by their \( \omega_1 \) and \( \omega_2 \) component. We remind the reader of the notation \( m(k) = M(\omega^k, k) \) and of the choice \( M(\omega_1^k, k) = 1 \). For any random vector \( X \) we define another random vector \( \tilde{X} \in \mathcal{K} \) by scaling of the \( \omega_1 \) and \( \omega_2 \) component as follows:

\[
\tilde{X} = \tilde{X}^k = A_k^{-1} \left\{ \frac{1}{m(k)} \langle X, \omega_1^k \rangle \omega_1^k + \langle X, \omega_2^k \rangle \omega_2^k \right\}. \tag{3.35}
\]

For most quantities we shall no longer indicate its dependence on \( k \) explicitly.

**Lemma 4.** There exist \( k_6 = k_6(F) < \infty \) and \( K_{14} < \infty \) such that the left-hand side of (3.1) is bounded by \( K_{14} \) for all \( s > 1 \) and all \( k > k_6(F) \) for which (3.33) holds.

**Proof.** By dropping the condition on the \( \omega_0 \) component and using the above notation we see that the left-hand side of (3.1) is bounded above by

\[
\sum_{r=2}^{2s} P \left\{ \left| \sum_{n=1}^{r} \langle \tilde{Y}_n, \omega_j \rangle \right| < 4 \text{ for } j = 1,2 \right\}. \tag{3.36}
\]

Of course,

\[
\tilde{Y}_r = \sum_{n=r-1+1}^{r} \{ \langle \tilde{X}_n(1 - J_n), \omega_1 \rangle \omega_1 + \langle \tilde{X}_n(1 - J_n), \omega_2 \rangle \omega_2 \}
+ \sum_{n=r-1+1}^{r} \{ \langle \tilde{X}_n J_n, \omega_1 \rangle \omega_1 + \langle \tilde{X}_n J_n, \omega_2 \rangle \omega_2 \}. \tag{3.37}
\]
By the observations made immediately following the introduction of $Y_r$ in §2 (see (2.47)) the two sums in the right-hand side of (3.37) are independent, and if we call the first sum $\tilde{Z}_r$, i.e.,

$$\tilde{Z}_r = \sum_{n=r-1+1}^{n=r} \{ \langle \chi_n (1 - J_n), \omega_1 \rangle \omega_1 + \langle \chi_n (1 - J_n), \omega_2 \rangle \omega_2 \},$$

then the two dimensional analogue of (2.32) gives

$$\sup_z P \left( \left| \sum_{n=1}^{r} \langle \tilde{Y}_n, \omega_j \rangle + z_j \right| < 4, j = 1, 2 \right) \leq \sup_z P \left( \left| \sum_{n=1}^{r} \langle \tilde{Z}_n, \omega_j \rangle + z_j \right| < 4, j = 1, 2 \right). \quad (3.38)$$

Moreover, as observed in (2.47) and (2.48), each $\tilde{Z}_r$ has the distribution of $\Sigma_{n=1}^{r} \beta_n$, where the $\beta_n$ are independent, and each with the distribution of $X_1$, given $J_1 = 0$. Thus the variance of each $\langle \tilde{Z}_n, \omega \rangle$ equals

$$\sigma^2(\omega, k) = T(k) \sigma^2 \{ \langle \tilde{X}_1, \omega \rangle | J_1 = 0 \}.$$

Note also that

$$|\tilde{X}_1| < A_k^{-1} (m_k^{-1} |\langle X_1, \omega_1 \rangle| + |\langle X_2, \omega_2 \rangle|) < 2$$

whenever $J_1 = 0$. Put

$$\Delta_k = \inf \{ \sigma^2(\omega, k) : |\omega| = 1, \omega \in \mathcal{C} \}$$

and apply the corollary to Theorem 6.2 of Esseen [6] to the independent and identically distributed random two-vectors $\tilde{\beta}_1, \ldots, \tilde{\beta}_r T(k)$. Esseen's $\chi_1(4)$ can now be taken at least

$$2 \inf_{|\omega| = 1} \sigma^2 \{ \langle \tilde{\beta}_1, \omega \rangle \} = 2 T(k)^{-1} \Delta_k,$$

and therefore the right-hand side of (3.38) is bounded by

$$K_{15} \left( r T(k) T(k)^{-1} \Delta_k \right)^{-1} = K_{15} (r \Delta_k)^{-1}.$$

Consequently, the left-hand side of (3.1) is at most

$$\sum_{r=s}^{2s} K_{15} (r \Delta_k)^{-1} \leq \frac{K_{15}}{\Delta_k} (1 + \log 2).$$

Thus we have a bound of the desired form for (3.1) as soon as $\Delta_k$ exceeds some strictly positive $\epsilon_1^2$. We choose $\epsilon_1$ as follows:

$$K_{16} = 32 k_0^{1/2} \eta^{-1/2} + 4, \quad K_{17} = \frac{1}{16} \left( 10 k_0^{1/2} \eta^{-1/2} + 2 \right)^{-1}, \quad (3.39)$$

$$\epsilon_1 = \frac{1}{32} \eta^{1/2} \frac{K_{17}^2}{K_{17} + 8 K_{16}}. \quad (3.40)$$
(K₀ and η are as in Lemma 2.) For the remainder of the lemma we may now assume

\[ \Delta_k < \varepsilon_1^2. \]  

(3.41)

By definition of \( \Delta_k \) this means that there exists a unit vector \( \Omega_1 \in \mathcal{C} \) with \( \sigma^2(\Omega_1,k) < 2\varepsilon_1^2 \). By Chebyshev's inequality we then have, with \( \mu = E\langle \tilde{Z}_1, \Omega_1 \rangle \) and \( T = T(k) \),

\[
P\left\{ \left| \langle \tilde{S}_T, \Omega_1 \rangle - \mu \right| > 8\varepsilon_1\eta^{-1/2} \right\}
< P\left\{ \tilde{S}_T \neq \tilde{Z}_1 \right\} + P\left\{ \left| \langle \tilde{Z}_1, \Omega_1 \rangle - \mu \right| > 8\varepsilon_1\eta^{-1/2} \right\}
< P\left\{ J_n = 1 \text{ for some } n < T \right\} + \sigma^2(\Omega_1,k)/64\varepsilon_1^2\eta^{-1}
< \eta/32 + \eta/32 \quad \text{(by (3.33) and (3.41))} = \eta/16. \quad (3.42)

We first show that (3.42) necessitates

\[ |\mu| > K_{17} - 8\varepsilon_1\eta^{-1/2} > \frac{1}{2}K_{17} \]  

(3.43)

for \( k > k_2(F) + k_3(F) \). Indeed if (3.43) fails, then (3.42) shows

\[ P\left\{ \left| \langle \tilde{S}_T, \Omega_1 \rangle \right| > K_{17} \right\} < \eta/2. \]  

(3.44)

But if

\[ \Omega_1 = (\cos \varphi_0)\omega_1 + (\sin \varphi_0)\omega_2, \]

and we define

\[ \Omega_2 = \left(m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right)^{-1/2}\left(\frac{\cos \varphi_0}{m(k)}\omega_1 + (\sin \varphi_0)\omega_2\right), \]

then (3.44) says

\[ P\left\{ \left| \langle S_T, \Omega_2 \rangle \right| > K_{17} \left\{ m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right\}^{-1/2}A_{17} \right\} < \eta/2. \]  

(3.45)

At the same time \( \Omega_2 \) is a unit vector in \( \mathcal{C} \), and thus by (2.23), \( M(\Omega_2,k) > 1 \), and by (2.30),

\[ P\left\{ \left| \langle S_T, \Omega_2 \rangle \right| > \frac{1}{2}A_{17} \right\} > P\left\{ \left| \langle S_T, \Omega_2 \rangle \right| > \frac{1}{2}M(\Omega_2)A_{17} \right\} > \eta. \]  

(3.46)

(3.45) and (3.46) together imply

\[ K_{17} \left\{ m(k)^{-2}\cos^2\varphi_0 + \sin^2\varphi_0\right\}^{-1/2} > \frac{1}{2} \]

and

\[ \sin^2\varphi_0 < 4K_{17}^2 < \frac{3}{4}, \quad |\cos \varphi_0| > \frac{1}{2}. \]  

(3.47)

However, we also have from (2.30):

\[ P\left\{ \left| \langle S_T, \omega_1 \rangle \right| > \frac{1}{2}m(k)A_{17} \right\} > \eta; \]  

(3.48)
and from (2.24):

$$P\left\{ \left| \langle S_{T,\omega_2} \rangle \right| < 2A_k \right\} > \frac{3}{4} \quad \text{for } k > k_2(F). \quad (3.49)$$

(3.44) and (3.48) together give

$$\frac{\eta}{2} < P\left\{ \left| \langle S_{T,\omega_1} \rangle \right| > \frac{1}{2} m(k)A_k \right\} \quad (3.50)$$

(3.49) and (3.50) together show that any interval which contains $\langle S_{T,\omega_2} \rangle$ with a probability $(1 - \frac{1}{4} \eta)$ must contain both $2A_k$ and $(16K_{17})^{-1}A_k$ or both $-2A_k$ and $-(16K_{17})^{-1}A_k$. Thus any such interval must have length at least

$$\left( (16K_{17})^{-1} - 2 \right) A_k > 10K_0^{1/2} \eta^{-1/2}A_k \quad (\text{see (3.39)}).$$

For $k > k_3(F)$, however, this contradicts

$$P\left\{ \left| \langle S_{T,\omega_2} \rangle - \sum_{n=1}^{T(k)} E\left\{ \left| \langle X_n,\omega_2 \rangle I[\left| \langle X_n,\omega_2 \rangle \right| < A_k] \right\} \right| > 4K_0^{1/2} \eta^{-1/2}A_k \right\}$$

$$< \sum_{n=1}^{T(k)} P\left\{ \left| \langle X_n,\omega_2 \rangle \right| > A_k \right\}$$

$$+ \frac{\eta}{16} \frac{T(k)}{K_0 A_k^2} \sigma^2 \left\{ \langle X_1,\omega_2 \rangle I[\left| \langle X_1,\omega_2 \rangle \right| < A_k] \right\}$$

$$< T(k) P\{ J_1 = 1 \} + \frac{K_0 \eta}{16} \quad (\text{by (2.28) and } M(\omega_2) = 1)$$

$$< \eta/8 \quad (\text{by (3.33)}). \quad (3.51)$$

Thus, the assumption that (3.43) fails indeed leads to a contradiction when $k > k_2 + k_3$, and for the remainder we may use (3.43) when $k > k_6(F) \equiv k_2(F) + k_3(F)$.

Now bring in the unit vector

$$\Omega_3 = (-\sin \varphi_0)\omega_1 + (\cos \varphi_0)\omega_2$$

orthogonal to $\Omega_1$ in $\mathfrak{C}$. If

$$Q\left( \langle \bar{S}_{T,\Omega_3} \rangle, K_{17} \right) > 1 - \eta/4, \quad (3.52)$$
then there exists a constant $f$ such that

$$\mathbb{P}\{||\langle S_T, \Omega_3 \rangle - f|| < \frac{1}{2} K_{17}\} > 1 - \eta/4,$$

which together with (3.42) and (3.40) would imply

$$\mathbb{P}\{||\langle S_T - \mu, \Omega_1 - f, \Omega_3 \rangle < \frac{1}{2} K_{17} + 8 \epsilon_1 \eta^{-1/2} < K_{17}\} > 1 - \eta/2,$$

and a fortiori for $\Omega_4$ a unit vector in $\mathcal{H}$ orthogonal to $\mu, \Omega_1 + f, \Omega_3$,

$$\mathbb{P}\{||\langle \tilde{S}_T, \Omega_4 \rangle - \langle \mu, \Omega_1 + f, \Omega_3, \Omega_4 \rangle \rangle < K_{17}\} > 1 - \eta/2.$$

Apart from the change from $\Omega_1$ to $\Omega_4$ this again is (3.44), which we showed to be impossible. Thus (3.52) must fail and no interval of length $K_{17}$ can contain $\langle S_T, \Omega_3 \rangle$ with probability $1 - \eta/4$. Thus, if $\nu$ is a median of $\langle S_T, \Omega_3 \rangle$, then

$$\mathbb{P}\{||\langle \tilde{S}_T, \Omega_3 \rangle - \nu|| > \frac{1}{2} K_{17}\} > \eta/4,$$

and at least one of the inequalities

$$\mathbb{P}\{\langle \tilde{S}_T, \Omega_3 \rangle > \nu + \frac{1}{2} K_{17}\} > \eta/8 \tag{3.53}$$

or

$$\mathbb{P}\{\langle \tilde{S}_T, \Omega_3 \rangle < \nu - \frac{1}{2} K_{17}\} > \eta/8 \tag{3.54}$$

must hold. For the sake of definiteness assume (3.53) holds. Since $\nu$ is a median of $\langle \tilde{S}_T, \Omega_3 \rangle$ we also have

$$\mathbb{P}\{\langle \tilde{S}_T, \Omega_3 \rangle < \nu\} > \frac{1}{2}. \tag{3.55}$$

We are almost ready to apply Corollary 2; we merely need an estimate on $\nu$, or, more generally, the distribution of $|\tilde{S}_T|$. Just as in (3.51) we have for any $x > 0$ and $k > k_6(F)$,

$$\mathbb{P}\{||\tilde{S}_T, \omega_1 || - T(k)E\{<X_1, \omega_1 || I[|<X_1, \omega_1 || < m(k)A_k]| > x\}$$

$$< T(k)\mathbb{P}\{||<X_1, \omega_1 || > m(k)A_k\}$$

$$+ \frac{T(k)}{x^2} \sigma^2 \{<X_1, \omega_1 || I[|<X_1, \omega_1 || < m(k)A_k]|\}$$

$$< T(k)\mathbb{P}\{J_1 = 1\}$$

$$+ \frac{T(k)}{x^2} \{m(k)A_k\}^{-2} \sigma^2 \{<X_1, \omega_1 || I[|<X_1, \omega_1 || < m(k)A_k]|\}$$

$$< \eta/32 + x^{-2}K_0 \quad \text{(see (3.33) and (2.28))}. \tag{3.56}$$

But also by (2.21),
\begin{align}
P \left\{ |\langle \tilde{S}_T, \omega_1 \rangle| < 2 \right\} &= P \left\{ |\langle S_T, \omega_1 \rangle| < 2m(k)A_k \right\} \\
&> 1 - (8d + 8)^{-1} > \frac{3}{4}.
\end{align}

Together with (3.56) this shows (take $x = 8K_0^{1/2} \eta^{-1/2}$ in (3.56))

\[ T(k)E \left\{ |\langle \tilde{X}_t, \omega_1 \rangle| \right\} \leq |\langle X_t, \omega_1 \rangle| < m(k)A_k \right\} | < 8K_0^{1/2} \eta^{-1/2} + 2,
\]

and therefore (again take $x = 8K_0^{1/2} \eta^{-1/2}$ in (3.56))

\[ P \left\{ |\langle \tilde{S}_T, \omega_1 \rangle| > 16K_0^{1/2} \eta^{-1/2} + 2 \right\} < \eta/16.
\]

The same inequality holds when $\omega_1$ is replaced by $\omega_2$ (use (3.49) to get (3.57)) and, therefore,

\[ P \left\{ |\langle \tilde{S}_T, \Omega_3 \rangle| > 32K_0^{1/2} \eta^{-1/2} + 4 \right\} < \eta/8 < \frac{1}{2}.
\]

In particular, we must have

\[ |\nu| < K_16 = (32K_0^{1/2} \eta^{-1/2} + 4) \quad \text{(see (3.39))}.
\]

We can now complete the proof of the lemma by means of an application of Corollary 2. By (3.43) either $\mu > 0$ or $\mu < 0$. For the sake of argument take $\mu > 0$. Then (3.42) and (3.43) give

\[ P \left\{ K_{17}/4 < \mu/2 < \mu - 8\varepsilon_1 \eta^{-1/2} < \langle \tilde{S}_T, \Omega_1 \rangle < \mu + 8\varepsilon_1 \eta^{-1/2} \right\} > 1 - \eta/16.
\]

(3.59) and (3.55) yield

\[ P \left\{ \langle \tilde{S}_T, \Omega_1 \rangle > \frac{1}{4}K_{17}, \langle \tilde{S}_T, \Omega_3 \rangle < \langle \nu + \frac{1}{8}K_{17}, \mu^{-1} \langle \tilde{S}_T, \Omega_1 \rangle \right\} > P \left\{ \mu - 8\varepsilon_1 \eta^{-1/2} < \langle \tilde{S}_T, \Omega_1 \rangle < \mu + 8\varepsilon_1 \eta^{-1/2}, \langle \tilde{S}_T, \Omega_3 \rangle < \nu \right\} > P \left\{ \langle \tilde{S}_T, \Omega_3 \rangle < \nu \right\} - \frac{\eta}{16} > \frac{1}{2} - \frac{\eta}{16} > \frac{1}{4}.
\]

In exactly the same way one obtains from (3.59) and (3.53):

\[ P \left\{ \langle \tilde{S}_T, \Omega_1 \rangle > \frac{1}{4}K_{17}, \langle \tilde{S}_T, \Omega_3 \rangle > \langle \nu + \frac{1}{8}K_{17}, \mu^{-1} \langle \tilde{S}_T, \Omega_1 \rangle + \frac{1}{8}K_{17} \right\} > P \left\{ \mu - 8\varepsilon_1 \eta^{-1/2} < \langle \tilde{S}_T, \Omega_1 \rangle < \mu + 8\varepsilon_1 \eta^{-1/2}, \langle \tilde{S}_T, \Omega_3 \rangle > \nu + \frac{1}{2}K_{17} \right\} > \eta/8 - \eta/16 = \eta/16.
\]

Lastly, we observe that

\[ P \left\{ Y_1 \neq \tilde{S}_T \right\} = P \left\{ \sum_{1}^{r_1} X_n \neq \sum_{1}^{T} X_n \right\} = P \left\{ \nu_1 \neq T \right\} < P \left\{ J_n = 1 \text{ for some } n < T \right\} < \frac{1}{32} \eta \quad \text{(by (3.33))},
\]
so that with
\[ d_1 = \frac{1}{4} K_{17}, \quad d_2 = \left( \nu + \frac{1}{8} K_{17} \right) \mu^{-1}, \quad d_3 = \frac{1}{8} K_{17}, \]
(3.60) and (3.61) give
\[ P\left\{ \langle \tilde{Y}_1, \Omega_1 \rangle > d_1, \langle \tilde{Y}_1, \Omega_3 \rangle < d_2 \langle \tilde{Y}_1, \Omega_1 \rangle \right\} > \frac{1}{4} - \eta / 32 > \frac{1}{8}, \]
and
\[ P\left\{ \langle \tilde{Y}_1, \Omega_1 \rangle > d_1, \langle \tilde{Y}_1, \Omega_3 \rangle > d_2 \langle \tilde{Y}_1, \Omega_1 \rangle + d_3 \right\} > \eta / 16 - \eta / 32 = 2^{-5}\eta. \]
These inequalities are just (3.8) and (3.9) for \( Z_1(1) = \langle \tilde{Y}_1, \Omega_3 \rangle, \ Z_1(2) = \langle \tilde{Y}_1, \Omega_1 \rangle \) and \( p_3 = \frac{1}{8}, \ p_4 = 2^{-5}\eta. \) Moreover, \( d_3 d_1^{-1}, \ p_3, \ p_4 \) all have strictly positive lower bounds which are independent of \( F, \) whereas (see (3.43) and (3.58))
\[ |d_2| < \left( K_{16} + \frac{1}{8} K_{17} \right) |\mu|^{-1} < 2 \left( K_{16} + \frac{1}{8} K_{17} \right) K_{17}^{-1}. \]
It follows there from (3.10) and (3.11) that for some \( K_{18} < \infty \) and all \( s > 1, \)
(3.36) is bounded by
\[ E \# \left\{ r \in [s, 2s] : \sum_{n=1}^{r} \left\{ \langle \tilde{Y}_n, \Omega_1 \rangle \Omega_1 + \langle \tilde{Y}_n, \Omega_3 \rangle \Omega_3 \right\} \leq 8 \right\} \]
(because \( \Omega_1, \Omega_3 \) is an orthonormal basis of \( \mathcal{H} \))
\[ < K_{18} \left\{ (32K_{17}^{-1})^2 + 1 \right\}. \]
This gives the desired bound for (3.36) and (3.1) under (3.33).

**Lemma 5.** There exist a \( k_7 = k_7(F) < \infty \) and \( K_{19} < \infty \) such that the left-hand side of (3.1) is bounded by \( K_{19} \) for all \( k > k_7(F) \) for which (3.34) holds.

**Proof.** Set
\[ \tilde{W}_r = \sum_{n=r-1+1}^{r} \{ \langle \tilde{X}_n, \Omega_1 \rangle \Omega_1 + \langle \tilde{X}_n, \Omega_2 \rangle \Omega_2 \}. \]
As mentioned in (3.37) and the lines following it, \( \tilde{Y}_r = \tilde{Z}_r + \tilde{W}_r \) and all \( \tilde{Z}_r, \tilde{W}_r, \ r > 1, \ s > 1, \) are independent. Just as in (3.38) the two dimensional analogue of (2.32) therefore gives for any unit vectors \( \Omega_i \in \mathcal{K} \) and \( x_i > 0, \ i \in I \) (some finite set of positive integers)
\[ \sup_{z} P \left\{ \left| \sum_{n=1}^{r} \tilde{Y}_n \Omega_i \right| + z_i < x_i, \ i \in I \right\}, \]
\[ < \sup_{z} P \left\{ \left| \sum_{n=1}^{r} \tilde{W}_n \Omega_i \right| + z_i < x_i, \ i \in I \right\}. \]
The distribution of
\[ W_r = \sum_{n=\nu_r-1+1}^{\nu_r} X_n J_n \]
was given in (2.49) and (2.50). As there we take \( \alpha_1, \alpha_2, \ldots \) independent and each with the conditional distribution of \( X_1 \) given \( J_1 = 1 \). Then \( \tilde{\alpha}_i \) has the conditional distribution of \( X_1 \) given \( J_1 = 1 \) and \( \tilde{W}_r \) the distribution of \( \sum_{i=1}^{\nu_r} \tilde{\alpha}_i \). Moreover, by (2.46),
\[ P \{ |\langle \tilde{\alpha}_i, \omega_1 \rangle| > 1 \text{ or } |\langle \tilde{\alpha}_i, \omega_2 \rangle| > 1 \} \]
\[ = P \{ |\langle X_1, \omega_1 \rangle| > m(k) A_k \text{ or } |\langle X_1, \omega_2 \rangle| > A_k \} \cdot J_1 = 1 \} = 1, \]
and a fortiori
\[ P \{ |\tilde{\alpha}_i| > 1 \} = 1. \quad (3.64) \]
Now consider the unit vectors
\[ u_l = \left( \cos(2l + 1) \frac{\pi}{16} \right) \omega_1 + \left( \sin(2l + 1) \frac{\pi}{16} \right) \omega_2, \quad l = 0, \ldots, 7, \]
all of which lie in \( \mathcal{C} \cap \{ z \in \mathbb{R}^2 : z(2) > 0 \} \), and are such that every vector in \( \mathcal{C} \) makes an angle at most \( \pi/16 \) with some \( u_l \) or \( -u_l \). Thus one can choose a \( 0 < l < 7 \) such that
\[ P \left\{ \varphi(\tilde{\alpha}_i, u_l) < \frac{\pi}{16} \text{ or } \varphi(\tilde{\alpha}_i, -u_l) < \frac{\pi}{16} \right\} > \frac{1}{8}. \quad (3.65) \]
With \( l \) fixed in this way one can also find two orthogonal unit vectors \( \Omega_1, \Omega_2 \) in \( \mathcal{C} \) such that \( \varphi(\Omega_1, u_l) < \pi/16 \),
\[ P \left\{ \varphi(\tilde{\alpha}_i, \Omega_1) < \pi/8 \text{ or } \varphi(\tilde{\alpha}_i, -\Omega_1) < \pi/8 \right\} > \frac{1}{16} \quad (3.66) \]
as well as
\[ P \left\{ \varphi(\tilde{\alpha}_i, \Omega_1) < \pi/8 \text{ or } \varphi(\tilde{\alpha}_i, -\Omega_1) < \pi/8 \right\} > \frac{1}{16} \quad (3.67) \]
Note that (3.66) merely says that there is a probability at least \( \frac{1}{16} \) for \( \tilde{\alpha}_i \) to lie in the first or third quadrant with respect to the \( \Omega_1, \Omega_2 \) axes and even within \( \pi/8 \) from the positive or negative \( \Omega_1 \) axis. (3.67) says the same thing with first and third quadrant replaced by second and fourth quadrant; such \( \Omega_1, \Omega_2 \) are easily obtained by continuity considerations, for if one chooses \( \Omega_1 \) first along one boundary line of the set in braces in (3.65) then there is probability at least \( \frac{1}{8} \) that \( \tilde{\alpha}_i \) lies in the first or third quadrant, and when \( \Omega_1 \) is rotated to the other boundary line then one ends up with probability at least \( \frac{1}{8} \) that \( \tilde{\alpha}_i \) lies in the second or fourth quadrant. Note that \( \Omega_1, \Omega_2 \) really depend on \( k \), since \( \tilde{\alpha}_i \) does. The same holds for \( \varphi_1 \) and \( \omega_0 \) below, but this will not influence the
sequel. Assume

\[ \Omega_1 = (-\sin \varphi_1)\omega_1 + (\cos \varphi_1)\omega_2, \quad \Omega_2 = -\Omega_2 = (\cos \varphi_1)\omega_1 + (\sin \varphi_1)\omega_2 \]

and fix \( \omega_0 \), a unit vector in \( \mathcal{C} \), \( K_{20} < \infty \) and \( \varepsilon_2 > 0 \) as follows:

\[ \omega_0 = \left\{ m(k)^{-2} \cos^2 \varphi_1 + \sin^2 \varphi_1 \right\}^{-1/2} \]

\[ \cdot \left\{ m(k)^{-1} (\cos \varphi_1)\omega_1 + (\sin \varphi_1)\omega_2 \right\}, \]

(3.68)

\[ L_0 = 32K_0\eta^{-1} + 1, \quad x_0 = 24K_0\eta^{-1} + 1, \]

(3.69)

\[ K_{20} = \min\left\{ (64L_0)^{-1}, 2^{-7} (12x_0 + 3)^{-1} \right\}, \quad \varepsilon_2 = (32L_0)^{-1}\eta. \]

(3.70)

For brevity we shall write

\[ \Theta = \Theta(k) = \left\{ m(k)^{-2} \cos^2 \varphi_1 + \sin^2 \varphi_1 \right\}^{1/2}. \]

We consider three separate cases now. Which case we are in depends on which of the inequalities (3.71)–(3.73) holds:

\[ P \left\{ \left| \mathbf{a}_1\mathbf{\beta}_2 \right| \right| > \frac{1}{2} \right\} > \varepsilon_2, \quad (3.71) \]

\[ P \left\{ K_{20} M (\omega_0, k) \Theta(k) < \left| \mathbf{a}_1\mathbf{\beta}_2 \right| < \frac{1}{2} \right\} > \varepsilon_2, \quad (3.72) \]

\[ P \left\{ \left| \mathbf{a}_1\mathbf{\beta}_2 \right| > K_{20} M (\omega_0, k) \Theta(k) \right\} < 2\varepsilon_2. \quad (3.73) \]

First assume (3.71); without loss of generality we may then assume (if necessary replace \( \mathbf{\beta}_2 \) by \(-\mathbf{\beta}_2 \) and/or \( \mathbf{\varphi}_1 \) by \(-\mathbf{\varphi}_1 \)) that

\[ P \left\{ \mathbf{\varphi}_1 \mathbf{\omega}_2 > \frac{1}{2}, \mathbf{\varphi}_1 \mathbf{\omega}_1 > 0 \right\} > \frac{1}{4} \varepsilon_2. \quad (3.74) \]

In this case we take

\[ \Omega_3 = \frac{1}{\sqrt{2}} \{ -\Omega_1 + \Omega_2 \}, \quad \Omega_4 = \frac{1}{\sqrt{2}} \{ \Omega_1 + \Omega_2 \}. \]

\( \Omega_3 \) and \( \Omega_4 \) are orthogonal unit vectors, bisecting the second (resp. first) quadrant with respect to \( \Omega_1, \Omega_2 \). Obviously \( \mathbf{\varphi}_1 \mathbf{\omega}_2 > \frac{1}{2}, \mathbf{\varphi}_1 \mathbf{\omega}_1 > 0 \) entails

\[ \langle \mathbf{\varphi}_1, \Omega_3 \rangle = -\langle \mathbf{\varphi}_1, \Omega_4 \rangle + \sqrt{2} \langle \mathbf{\varphi}_1, \Omega_2 \rangle > -\langle \mathbf{\varphi}_1, \Omega_4 \rangle + 1/\sqrt{2} \]

as well as

\[ \langle \mathbf{\varphi}_1, \Omega_4 \rangle > 2^{-3/2}. \]

Thus (3.74) implies

\[ P \left\{ \langle \mathbf{\varphi}_1, \Omega_4 \rangle > 2^{-3/2}, \langle \mathbf{\varphi}_1, \Omega_3 \rangle > -\langle \mathbf{\varphi}_1, \Omega_4 \rangle + 2^{-1/2} \right\} > \frac{1}{4} \varepsilon_2. \quad (3.75) \]

But also, from (3.64) and (3.67), one has

\[ P \left\{ \langle \mathbf{\varphi}_1, \Omega_4 \rangle > \cos \frac{3\pi}{8}, \langle \mathbf{\varphi}_1, \Omega_3 \rangle < -\langle \mathbf{\varphi}_1, \Omega_4 \rangle \right\} > \frac{1}{32} \]

or the same inequality with \( \mathbf{\varphi}_1 \) replaced by \(-\mathbf{\varphi}_1 \). Thus if we put
\[ d_1 = \frac{1}{3}, \quad d_2 = -1, \quad d_3 = 2^{-1/2}, \quad p_3 = \frac{1}{32}, \quad p_4 = \frac{1}{4} \varepsilon_2, \]

then (3.9) holds with \( Z_1(1) = \langle \tilde{a}_1, \Omega_3 \rangle, \) \( Z_1(2) = \langle \tilde{a}_1, \Omega_4 \rangle, \) and (3.8) holds with the same replacement or with \(-Z_1(1) = \langle \tilde{a}_1, \Omega_3 \rangle, -Z_1(2) = \langle \tilde{a}_1, \Omega_4 \rangle.\) Moreover, by (3.34) and (2.29) we also have (3.12) with \( d_4 = 2^{-\gamma}, d_5 = 2K_0 \) as soon as \( k > k_7 \) for suitable \( k_7 = k_{17} < \infty. \) Thus, by (3.63) and (3.13),

\[
\sup_z P \left\{ \left| \sum_{n=1}^{r} \langle \tilde{Y}_n, \omega_j \rangle + z_j \right| < 4, j = 1,2 \right\} \leq \sup_z P \left\{ \left| \sum_{n=1}^{r} \tilde{W}_n + z \right| < 8 \right\} < K_{21} r^{-1}
\]

for some \( K_{21} < \infty \) depending on \( \varepsilon_2, \gamma \) and \( K_0 \) only. Thus \( K_{21} \) does not depend on \( F, \) and in the case where (3.71) holds we obtain the bound

\[
\sum_{r=s}^{2s} K_{21} r^{-1} < K_{21} (1 + \log 2)
\]

for (3.36) and the left-hand side of (3.1).

Next we consider the case where (3.72) holds. Again we assume that the signs of \( \tilde{\beta}_1, \tilde{\beta}_2 \) have been chosen such that

\[
P \left\{ K_{20} M(\omega_0, k) \Theta(k) < \langle \tilde{a}_1, \Omega_2 \rangle < \frac{1}{2}, \langle \tilde{a}_1, \Omega_1 \rangle > 0 \right\} > \frac{1}{4} \varepsilon_2 \quad (3.76)
\]

(compare (3.74)). Since \( |\tilde{a}_1| > 1 \) w.p.1 (see (3.64)), \( |\langle \tilde{a}_1, \Omega_2 \rangle| < \frac{1}{2} \) implies \( |\langle \tilde{a}_1, \Omega_1 \rangle| > \frac{1}{2} \). Thus, if we define for any random vector \( X, \)

\[
\tilde{X} = \langle \tilde{X}, \Omega_1 \rangle \Omega_1 + \{ M(\omega_0, k) \Theta(k) \}^{-1} \langle \tilde{X}, \Omega_2 \rangle \Omega_2
\]

then (3.76) implies

\[
P \left\{ \langle \tilde{a}_1, \Omega_1 \rangle = \langle \tilde{a}_1, \Omega_1 \rangle > \frac{1}{2}, \langle \tilde{a}_1, \Omega_2 \rangle > K_{20} \right\} > \frac{1}{4} \varepsilon_2. \quad (3.77)
\]

Also, (3.67) together with (3.64) gives

\[
P \left\{ |\langle \tilde{a}_1, \Omega_1 \rangle| > \cos \frac{\pi}{8}, \langle \tilde{a}_1, \Omega_1 \rangle \langle \tilde{a}_1, \Omega_2 \rangle < 0 \right\}
\]

\[
> P \{ |\tilde{a}_1| > 1, \varphi(\tilde{a}_1, \Omega_1) < \pi / 8 \}
\]

or \( \varphi(\tilde{a}_1, -\Omega_1) < \pi / 8, \langle \tilde{a}_1, \Omega_1 \rangle \langle \tilde{a}_1, \Omega_2 \rangle < 0 \} > 1/16,
\]

so that

\[
P \left\{ \langle \tilde{a}_1, \Omega_1 \rangle > \cos \frac{\pi}{8}, \langle \tilde{a}_1, \Omega_2 \rangle < 0 \right\} > \frac{1}{32} \quad (3.78)
\]

or
If we take
\[
d_1 = \frac{1}{2}, \quad d_2 = 0, \quad d_3 = K_{20}, \quad p_3 = \frac{1}{32}, \quad p_4 = \frac{1}{4} e_2,
\]
then (3.77) is again (3.9) with \( Z_1(1) = \langle \alpha_1, \Omega_2 \rangle, Z_1(2) = \langle \alpha_1, \Omega_1 \rangle; \) (3.78) becomes (3.8) and (3.79) is (3.8) with \( Z \) replaced by \(-Z\). As before we also have (3.12) with \( d_4 = 2^{-1/2} \) and \( d_5 = 2K_0 \) for \( k > k_7 \) from (3.34) and (2.29). Thus, by (3.13) and (3.63),
\[
\sup_{Z} P \left\{ \left\lvert \sum_{n=1}^{r} \langle \tilde{Y}_n, \Omega_j \rangle + z \right\rvert < 8, j = 1, 2 \right\} < \sup_{Z} P \left\{ \left\lvert \sum_{n=1}^{r} \langle \tilde{W}_n, \Omega_j \rangle + z \right\rvert < 8, j = 1, 2 \right\}
\]
\[
< \sup_{Z} P \left\{ \left\lvert \sum_{n=1}^{r} \tilde{W}_n + z \right\rvert < 16 \right\} < K_{22} r^{-1}.
\]
Again \( K_{22} < \infty \) depends neither on \( F \) nor \( k \). However,
\[
\sum_{n=1}^{r} \langle \tilde{Y}_n, \Omega_1 \rangle = \sum_{n=1}^{r} \langle \tilde{Y}_n, \Omega_1 \rangle
\]
\[
= A_k^{-1} \left\{ - \sum_{n=1}^{r} \langle Y_n, \omega_1 \rangle \frac{1}{m(k)} \sin \varphi_1 + \sum_{n=1}^{r} \langle Y_n, \omega_2 \rangle \cos \varphi_1 \right\}
\]
and
\[
\left\lvert \sum_{n=1}^{r} \langle \tilde{Y}_n, \Omega_2 \rangle \right\rvert = \left\lvert M(\omega_0, k) \Theta(k) \right\rvert^{-1} \left\lvert \sum_{n=1}^{r} \langle \tilde{Y}_n, \Omega_2 \rangle \right\rvert
\]
\[
= \left\lvert A_k M(\omega_0, k) \Theta(k) \right\rvert^{-1}
\]
\[
\cdot \left\lvert \sum_{n=1}^{r} \langle Y_n, \omega_1 \rangle \frac{1}{m(k)} \cos \varphi_1 + \sum_{n=1}^{r} \langle Y_n, \omega_2 \rangle \sin \varphi_1 \right\rvert
\]
\[
= A_k^{-1} M(\omega_0, k)^{-1} \left\lvert \sum_{n=1}^{r} \langle Y_n, \omega_0 \rangle \right\rvert.
\]
It follows that the condition
\[
\left\lvert \sum_{n=1}^{r} \langle Y_n, \omega_j \rangle \right\rvert < 4M(\omega_j, k)A_k
\]
for \( j = 0 \) is equivalent to
and condition (3.83) for \( j = 1 \) and 2 implies

\[
\left| \sum_{n=1}^{r} \langle \bar{Y}_n, \Omega_2 \rangle \right| < 4|\sin \varphi_1| + 4|\cos \varphi_1| < 8
\]

(recall \( M(\omega_1, k) = m(k), M(\omega_2, k) = 1 \)). Thus, (3.1) is bounded by

\[
\sum_{r=1}^{2s} \sup_{z} P \left\{ \left| \sum_{n=1}^{r} \langle \bar{Y}_n, \Omega_j \rangle + z \right| < 8, j = 1, 2 \right\}
\]

\[
< \sum_{r=1}^{2s} K_{22} r^{-1} \quad (\text{see (3.80)}) \leq K_{22} (1 + \log 2).
\]

This settles the case where (3.72) holds and we are left with the case where

(3.73) holds. Without loss of generality we may also assume that (3.71) fails.

We begin with an estimate on the distribution of \( \langle \bar{S}_T, \Omega_2 \rangle \) in this case.

Analogously to (3.82),

\[
\left| \langle \bar{S}_T, \Omega_2 \rangle \right| = \Theta(k) A_k^{-1} \left| \langle S_T, \omega_0 \rangle \right|,
\]

so that by (2.30),

\[
P \left\{ \left| \langle \bar{S}_T, \Omega_2 \rangle \right| > \frac{1}{2} \Theta(k) M(\omega_0, k) \right\} > \eta. \tag{3.84}
\]

Obviously,

\[
\bar{S}_T = \sum_{n=1}^{T} \bar{X}_n J_n + \sum_{n=1}^{T} \bar{X}_n (1 - J_n).
\]

Moreover, for any \( L > 1 \),

\[
P \left\{ \left| \sum_{n=1}^{T} \langle \bar{X}_n J_n, \Omega_2 \rangle \right| > \frac{1}{4} \Theta(k) M(\omega_0, k) \right\}
\]

\[
< P \left\{ \sum_{n=1}^{T} J_n > L \right\} + P \left\{ \sum_{i=1}^{L} |\bar{a}_i, \Omega_2 \rangle > \frac{1}{4} \Theta(k) M(\omega_0, k) \right\}; \tag{3.85}
\]

indeed once we know that \( J_n = 1 \) exactly for \( n \in \{n_1, \ldots, n_L\} \) and no other \( n \in [1, T] \), the conditional distribution of

\[
\sum_{n=1}^{T} \bar{X}_n J_n \tag{3.86}
\]

is simply the distribution of

\[
\sum_{i=1}^{L} \bar{a}_i \tag{3.87}
\]
The right-hand side of (3.85) is bounded by
\[
\frac{T}{L} EJ_1 + LP \left\{ \left| \langle \tilde{a}_1, \Omega_2 \rangle \right| \geq \frac{1}{4L} \Theta(k) M(\omega_0, k) \right\}.
\]

Now take \( L = L_0 \) (see (3.69)). Then it follows from (2.53), (3.70) and (3.73) that (3.85) is at most
\[
\left( \frac{T}{L_0} \right) P \{ J_1 = 1 \} + L_0 P \left\{ \left| \langle \tilde{a}_1, \Omega_2 \rangle \right| > K_20 \Theta(\omega_0, k) \right\}
\]
\[
< \frac{2}{L_0} \eta_0 + L_02e_2 < \eta/8.
\]

When this bound for (3.85) is combined with (3.84) we find
\[
P \left\{ \left\| \sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n), \Omega_2 \rangle \right\| \geq \frac{1}{4} \Theta(k) M(\omega_0, k) \right\} > \frac{7}{8} \eta.
\]

Next we need an estimate for the distribution of
\[
\sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n), \omega_j \rangle, \quad j = 1, 2,
\]
which is a slight variation on (3.56). E.g., take \( j = 1 \) and write temporarily
\[
I_n = I[|\langle X_n, \omega_1 \rangle| < m(k) A_k] = I[|\langle \tilde{X}_n, \omega_1 \rangle| < 1].
\]

Then, as in (3.56), for \( k > k_3(F) \),
\[
P \left\{ \left\| \sum_{n=1}^{T} \langle \tilde{X}_n I_n, \omega_1 \rangle - TE \langle \tilde{X}_1, \omega_1 \rangle I_1 \right\| > \xi \right\}
\]
\[
< (T(k)/\xi^2) \sigma^2 \{ \langle \tilde{X}_1, \omega_1 \rangle I[|\langle X_1, \omega_1 \rangle| < m(k) A_k] \} < K_0/\xi^2. (3.89)
\]

But also
\[
I_n - (1 - J_n) = I[|\langle \tilde{X}_n, \omega_1 \rangle| < 1, |\langle \tilde{X}_n, \omega_1 \rangle| > 1],
\]
so that
\[
\left\| \sum_{n=1}^{T} \langle \tilde{X}_n(1 - J_n), \omega_1 \rangle - \sum_{n=1}^{T} \langle \tilde{X}_n I_n, \omega_1 \rangle \right\|
\]
\[
< \sum_{n=1}^{T} |\langle \tilde{X}_n, \omega_1 \rangle| |1 - J_n - I_n|
\]
\[
< \# \{ n < T: |\langle \tilde{X}_n, \omega_1 \rangle| < 1, |\langle \tilde{X}_n, \omega_1 \rangle| > 1 \}
\]
\[
< \sum_{n=1}^{T} J_n.
\]

It follows that
\[ P \left\{ \left| \sum_{n=1}^{T} \langle \bar{X}_n (1 - J_n), \omega_1 \rangle - \sum_{n=1}^{T} \langle \bar{X}_n f_n, \omega_1 \rangle \right| > x \right\} \leq x^{-1} T(k) E \langle \bar{X}_1, \omega_1 \rangle \left[ \langle \bar{X}_1, \omega_1 \rangle < 1 \right] > 2x \]

Combined with (3.89) this estimate yields

\[ P \left\{ \left| \sum_{n=1}^{T} \langle \bar{X}_n (1 - J_n), \omega_1 \rangle - T E \langle \bar{X}_1, \omega_1 \rangle I[\langle \bar{X}_1, \omega_1 \rangle < 1] > 2x \right\} \leq K_0 (2x^{-1} + x^{-2}). \]

This inequality remains valid when \( \omega_1 \) is replaced by \( \omega_2 \) throughout and thus, if we define

\[ \rho = \rho(k) = -E \langle \bar{X}_1, \omega_1 \rangle I[\langle \bar{X}_1, \omega_1 \rangle < 1] \sin \varphi_1 \]
\[ + E \langle \bar{X}_1, \omega_2 \rangle I[\langle \bar{X}_1, \omega_2 \rangle < 1] \cos \varphi_1, \]

then

\[ P \left\{ \left| \sum_{n=1}^{T} \langle \bar{X}_n (1 - J_n), \Omega_1 \rangle - T(k) \rho(k) \right| > 4x \right\} < 2K_0 (2x^{-1} + x^{-2}). \]  \hspace{1cm} (3.90)

Now observe that by (2.21) and (2.24),

\[ P \left\{ \langle \bar{S}_T, \Omega_1 \rangle < 3 \right\} > P \left\{ \langle \bar{S}_T, \omega_1 \rangle < 1, \langle \bar{S}_T, \omega_2 \rangle < 2 \right\} \]
\[ > 1 - P \left\{ \langle \bar{S}_T, \omega_1 \rangle > m(k) A_k \right\} - P \left\{ \langle \bar{S}_T, \omega_2 \rangle > 2A_k \right\} \]
\[ > 1 - 3(8d + 8)^{-1} > \frac{1}{2}. \]

Thus, for \( x_0 \) as in (3.69) we have

\[ 2K_0 (2x_0^{-1} + x_0^{-2}) < \eta/4 < 1/4, \]

and

\[ P \left\{ \sum_{n=1}^{T} \langle \bar{X}_n J_n, \Omega_1 \rangle + T(k) \rho(k) \right\} < 4x_0 + 3, \]
\[ \left| \sum_{n=1}^{T} \langle \bar{X}_n (1 - J_n), \Omega_1 \rangle - T(k) \rho(k) \right| < 4x_0 \]

\[ > P \left\{ \langle \bar{S}_T, \Omega_1 \rangle < 3, \right. \left. \sum_{n=1}^{T} \langle \bar{X}_n (1 - J_n), \Omega_1 \rangle - T(k) \rho(k) \right\} < 4x_0 > 1/4. \]

Using the relation between the distributions of (3.86) and (3.87) we obtain
\[ \frac{1}{4} < P \left\{ \left| \sum_{n=1}^{T} \langle \tilde{X}_n, J_n, \Omega_1 \rangle + T(k) \rho(k) \right| < 4x_0 + 3 \right\} \]

\[ < P \left\{ \sum_{l=1}^{T} J_n > L_0 \right\} \]

\[ + \sum_{l=0}^{L_0} P \left\{ \sum_{l=1}^{T} J_n = l \right\} P \left\{ \left| \sum_{l=1}^{T} \langle \tilde{a}_l, \Omega_1 \rangle + T(k) \rho(k) \right| < 4x_0 + 3 \right\} \]

\[ < \frac{\eta}{8} + \max_{l < L_0} P \left\{ \left| \sum_{l=1}^{T} \langle \tilde{a}_l, \Omega_1 \rangle + T(k) \rho(k) \right| < 4x_0 + 3 \right\} \]

(contrast the estimates for (3.85)). Thus, for some \( 0 < l_0 < L_0 \),

\[ P \left\{ \sum_{l=1}^{l_0} \langle \tilde{a}_l, \Omega_1 \rangle + T(k) \rho(k) < 4x_0 + 3 \right\} > \frac{1}{8} \quad (3.91) \]

Similarly we get from (3.88) and (3.90),

\[ \frac{5}{8} \eta < P \left\{ \sum_{n=1}^{T} \langle X_n (1 - J_n), \Omega_2 \rangle > \frac{1}{4} \Theta(k) M(\omega_0, k), \right\} \]

\[ \left| \sum_{n=1}^{T} \langle X_n (1 - J_n), \Omega_1 \rangle - T(k) \rho(k) \right| < 4x_0 \right\} \]

\[ < P \left\{ \sum_{l=1}^{T} J_n > L_0 \right\} \]

\[ + \max_{l < L_0} P \left\{ \sum_{l=1}^{T-l} \langle \tilde{\beta}_n, \Omega_2 \rangle > \frac{1}{4} \Theta(k) M(\omega_0, k), \right\} \]

\[ \left| \sum_{n=1}^{T-l} \langle \tilde{\beta}_n, \Omega_1 \rangle - T(k) \rho(k) \right| < 4x_0 \right\} \]

where we used the same notation as in (2.47) and (2.48). Again there exists a \( 0 < l_1 < L_0 \) such that

\[ P \left\{ \sum_{n=1}^{T-l_1} \langle \tilde{\beta}_n, \Omega_2 \rangle > \frac{1}{4} \Theta(k) M(\omega_0, k), \right\} \]

\[ \left| \sum_{n=1}^{T-l_1} \langle \tilde{\beta}_n, \Omega_1 \rangle - T(k) \rho(k) \right| < 4x_0 \right\} > \frac{\eta}{2} \quad (3.92) \]
But, by (2.29) and (2.53) we can choose \( k_F(F) < \infty \) such that for \( k > k_7 \),
\[
P \left\{ \left| \sum_{T-i+1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle \right| > \frac{1}{8} \Theta(k) M(\omega_0, k) \right\}
\]
\[
= P \left\{ \left| \sum_{j=1}^{i_1} \langle \beta_j, \omega_0 \rangle \right| > \frac{1}{8} M(\omega_0, k) A_k \right\}
\]
\[
< l_1 P \left\{ |X_1| > (8l_1)^{-1} M(\omega_0, k) A_k|J_1 = 0 \right\}
\]
\[
< l_1 (P \{ J_1 = 0 \})^{-1} P \left\{ |X_1| > (8l_1)^{-1} M(\omega_0, k) A_k \right\} < \eta/8.
\]
A similar estimate gives for \( k > k_7 \),
\[
P \left\{ \left| \sum_{T-i+1}^{T} \langle \tilde{\beta}_n, \Omega_1 \rangle \right| > 4x_0 \right\} < \frac{\eta}{8},
\]
and we therefore conclude from (3.92) that
\[
P \left\{ \left| \sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle \right| > \frac{1}{8} \Theta(k) M(\omega_0, k),
\right. \]
\[
\left. \left| \sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_1 \rangle - T(k)p(k) \right| < 8x_0 \right\} > \frac{\eta}{4}. \tag{3.93}
\]
However, with \( \alpha_i, \beta_j \) and \( \Lambda \) independent, \( Y_i \) has the distribution of \( \sum_{n=1}^{T} \beta_n + \sum_{n=1}^{T} \alpha_i \) (see (2.46)-(2.50)); (3.91) and (3.93) together therefore give
\[
P \left\{ \langle \tilde{Y}_1, \Omega_2 \rangle > \frac{1}{16} \Theta(k) M(\omega_0, k), \langle \tilde{Y}_1, \Omega_1 \rangle < 12x_0 + 3 \right\}
\]
\[
> P \{ \Lambda = l_0 \} P \left\{ \left| \sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_2 \rangle \right| > \frac{1}{8} \Theta(k) M(\omega_0, k),
\right. \]
\[
\left. \left| \sum_{n=1}^{T} \langle \tilde{\beta}_n, \Omega_1 \rangle - T(k)p(k) \right| < 8x_0 \right\}
\]
\[
\cdot P \left\{ \left| \sum_{i=1}^{l_0} \langle \tilde{\alpha}_i, \Omega_2 \rangle \right| < \frac{1}{16} \Theta(k) M(\omega_0, k),
\right. \]
\[
\left. \left| \sum_{i=1}^{l_0} \langle \tilde{\alpha}_i, \Omega_1 \rangle + T(k)p(k) \right| < 4x_0 + 3 \right\}
\]
\[
> P \{ \Lambda = l_0 \} \cdot \frac{\eta}{4} \left\{ \frac{1}{8} - l_0 P \{ |\langle \tilde{\alpha}_1, \Omega_2 \rangle| > K_20 \Theta(k) M(\omega_0, k) \} \right\}
\]
\[
> \frac{1}{60} \eta P \{ \Lambda = l_0 \} \quad \text{(see (3.70) and (3.73)).} \tag{3.94}
\]
Moreover, by (2.50) and (3.34) for any \( l \),

\[
\Pr \{ \Lambda = l \} > \left( l! \right)^{-1} (TP \{ J_1 = 1 \})^l (1 - 2K_0T^{-1})^T \\
> \left( l! \right)^{-1} (2^{\frac{-\eta}{2}})^l \exp - 4K_0,
\]
as soon as \( K_0T(k)^{-1} < \frac{1}{4}, \) i.e., for \( k > k_7(F) \) if \( k_7(F) \) is chosen large enough (see (2.29)). Thus, in terms of \( \bar{Y}_1 \), (3.94) yields for \( k > k_7 \) and some \( K_{23} > 0 \), independent of \( F \) and \( k \),

\[
\Pr \left\{ \nu_1 = T + l_0, \left| \left\langle \bar{Y}_1, \Omega_2 \right\rangle \right| > \frac{1}{16} \right\}
\]

\[
\left| \left\langle \bar{Y}_1, \Omega_1 \right\rangle \right| < 16(12x_0 + 3)\left| \langle \bar{Y}_1, \Omega_2 \rangle \right| > K_{23}.
\]

We can therefore find a number \( x_1 = x_1(F,k), |x_1| < 16(12x_0 + 3) \) such that

\[
\Pr \left\{ \nu_1 = T + l_0, \left\langle \bar{Y}_1, \Omega_2 \right\rangle > \frac{1}{16}, \left\langle \bar{Y}_1, \Omega_1 \right\rangle < x_1 \langle \bar{Y}_1, \Omega_2 \rangle \right\} > \frac{1}{4}K_{23}, \quad (3.95)
\]
as well as

\[
\Pr \left\{ \nu_1 = T + l_0, \left\langle -\bar{Y}_1, \Omega_2 \right\rangle > \frac{1}{16}, \left\langle -\bar{Y}_1, \Omega_1 \right\rangle > x_1 \langle -\bar{Y}_1, \Omega_2 \rangle \right\} > \frac{1}{4}K_{23}, \quad (3.96)
\]
or both of these inequalities hold with \( \langle \bar{Y}_1, \Omega_2 \rangle \) replaced by \( -\langle \bar{Y}_1, \Omega_2 \rangle \). By changing the sign of \( \Omega_2 \), if necessary, we may restrict ourselves to the case where (3.95) and (3.96) hold. Lastly, we observe that (3.64), (3.70), the inequality \(|x_1|K_{20} < 16(12x_0 + 3)K_{20} < \frac{1}{4}, \) \( (3.73) \) and the fact that \( (3.71) \) fails imply

\[
\Pr \left\{ \left| \left\langle \tilde{a}_{0+1}, \Omega_2 \right\rangle \right| < K_{20}, \left| \left\langle \tilde{a}_{0+1}, \Omega_1 \right\rangle \right| > x_1 \langle \tilde{a}_{0+1}, \Omega_2 \rangle + \frac{1}{4} \right\}
\]

\[
> \Pr \left\{ \left| \left\langle \tilde{a}_{0+1}, \Omega_2 \right\rangle \right| < \min \left\{ \frac{1}{2}, K_{20} \Theta(k)M(\omega_0,k) \right\}, \left| \left\langle \tilde{a}_{0+1}, \Omega_1 \right\rangle \right| > \frac{1}{2} \right\}
\]

\[
> \Pr \left\{ \left| \left\langle \tilde{a}_{0+1}, \Omega_2 \right\rangle \right| < \min \left\{ \frac{1}{2}, K_{20} \Theta(k)M(\omega_0,k) \right\}, |\tilde{a}_{0+1}| > 1 \right\}
\]

\[
> 1 - e_2 - 2e_2 > \frac{1}{2}.
\]

Let us assume that

\[
\Pr \left\{ \left| \left\langle \tilde{a}_{0+1}, \Omega_2 \right\rangle \right| < K_{20}, \left\langle \tilde{a}_{0+1}, \Omega_1 \right\rangle > x_1 \langle \tilde{a}_{0+1}, \Omega_2 \rangle \right\}

> \frac{1}{4} > x_1 \langle \tilde{a}_{0+1}, \Omega_2 \rangle + \frac{1}{4} \right\} > \frac{1}{4}.
\]

(If this fails then we have

\[
\Pr \left\{ \left| \left\langle \tilde{a}_{0+1}, \Omega_2 \right\rangle \right| < K_{20}, \left\langle \tilde{a}_{0+1}, \Omega_1 \right\rangle < -x_1 \langle \tilde{a}_{0+1}, \Omega_2 \rangle \right\}

> \frac{1}{4} > x_1 \langle \tilde{a}_{0+1}, \Omega_2 \rangle - \frac{1}{4} \right\} > \frac{1}{4};
this can be treated analogously.) Then we have from (2.50), (3.96) and (3.34),

\[ P \left( \langle \bar{Y}_1, \Omega_2 \rangle > \frac{1}{16} - K_{20} > \frac{1}{32}, \langle \bar{Y}_1, \Omega_1 \rangle > x_1 \langle \bar{Y}_1, \Omega_2 \rangle + \frac{1}{4} \right) \]

> \( P \{ \Lambda = l_0 + 1 \} \)

\[ P \left( \left\{ \sum_{1 \leq i < l_0} \tilde{\alpha}_i + \sum_{n=1}^{T} \tilde{\beta}_n \Omega_2 \right\} > \frac{1}{16} , \right. \]

\[ \left. \left\{ \sum_{1 \leq i < l_0} \tilde{\alpha}_i + \sum_{n=1}^{T} \tilde{\beta}_n \Omega_1 \right\} > x_1 \left\{ \sum_{1 \leq i < l_0} \tilde{\alpha}_i + \sum_{n=1}^{T} \beta_n \Omega_2 \right\} , \right. \]

\[ \left| \langle \bar{\alpha}_{l_0 + 1}, \Omega_2 \rangle \right| < K_{20}, \langle \bar{\alpha}_{l_0 + 1}, \Omega_1 \rangle > x_1 \langle \tilde{\alpha}_{l_0 + 1}, \Omega_2 \rangle + \frac{1}{4} \right) \]

\[ \left( \langle \bar{Y}_1, \Omega_1 \rangle \right) > x_1 \left( \langle \bar{Y}_1, \Omega_2 \rangle \right) \]

\[ > \frac{1}{16} K_{22} T(k) P \{ J_1 = 1 \} (l_0 + 1)^{-1} > 2^{-3} \eta K_{23} (L_0 + 1)^{-1} = K_{24} \] (3.97)
say, with \( K_{24} > 0 \) independent of \( F \) and \( k \). We can now complete the proof by a final application of Corollary 2 as before. (3.95) and (3.97) again are versions of (3.8) (resp. (3.9)) with \( Z_1(1) = \langle \bar{Y}_1, \Omega_1 \rangle, Z_1(2) = \langle \bar{Y}_1, \Omega_2 \rangle, d_1 = 1/32, d_2 = x_1, d_3 = \frac{1}{4}, p_3 = \frac{1}{4} K_{23}, p_4 = K_{24} \). Thus, by (3.10) we have for some \( K_{25} < \infty \),

\[ \sum_{r=2}^{2s} P \left( \left| \sum_{1 \leq i < r} \bar{Y}_n \right| < 12 \right) < K_{25}, \quad s > 1. \] (3.98)

As in (3.81)-(3.83) the left-hand side of (3.98) is, however, an upper bound for the left-hand side of (3.1). Thus (3.1) holds also in this last case. □

Lemmas 4 and 5 prove (3.1) with \( k_5 = \max(k_6, k_7), K_2 = \max(K_{14}, K_{19}) \). As observed in the beginning of this section, this proves Proposition 1 and our theorem.

**References**


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