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ULTRAPOWERS AND LOCAL PROPERTIES OF BANACH SPACES

BY

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ABSTRACT. The present paper is an approach to the local theory of Banach spaces via the ultrapower construction. It includes a detailed study of ultrapowers and their dual spaces as well as a definition of a new notion, the notion of a u-extension of a Banach space. All these tools are used to give a unified definition of many classes of Banach spaces characterized by local properties (such as the $\mathcal{L}_p$-spaces). Many examples are given; also, as an application, it is proved that any $\mathcal{L}_p$-space, $1 < p < \infty$, has an ultrapower which is isomorphic to an L_p-space.

1. Introduction. Ultraproducts and ultrapowers of normed spaces have been known for some time [7], [2] and are now commonly used by Banach space theorists. Nevertheless, they have been mainly considered as technical tools and basic questions concerning their individual properties have been left unanswered until now. The present paper is an attempt to fill this gap.[1] We feel such an approach is useful: the theory of ultrapowers provides a natural framework for the study of local properties of Banach spaces and thus may yield a better understanding of some geometric situations. This paper can be viewed as a continuation of §6 of [15], which included some results connecting ultrapowers and finite representability. Let us also mention that ultrapowers have been used recently by Dacunha-Castelle and Krivine [3] in their work on subspaces of L_1 and by the author in order to solve the so-called problem of envelopes [16].

As was pointed out by the referee, the study of ultrapowers of Banach spaces is closely connected with the "nonstandard hull" construction of nonstandard analysis and especially with work of Luxemburg, Ward Henson, Moore ([17], [18]). Although these two lines of research have been carried on very separately, they are quite close to each other in detail and spirit and it is not difficult to translate a given result from one theory to another.

Before we describe in some detail the content of the paper, we recall some definitions from [2] or [15]. Notations are standard and come from [11] or [15].

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[1] A preliminary version of this paper has been presented at the Seminaire Maurey-Schwartz, 1974–1975.

Let $(E_i)_{i \in I}$ be a set of Banach spaces; $\mathfrak{U}$ an ultrafilter on the set I; let Π_0 be the set of mappings $(x_i)_{i \in I}$ such that $x_i \in E_i$ and for some λ, $\|x_i\| < \lambda$ for any i. Π_0 is endowed with the seminorm $\| \ \|$ defined by $\|(x_i)_{i \in I}\| = \lim_{\mathfrak{U}} \|x_i\|$.

The set N of elements of Π_0 with seminorm 0 is a subspace of Π_0.

DEFINITION 1.1. The *ultraproduct* $\Pi_{i \in I} E_i / \mathfrak{U}$ of the spaces $(E_i)_{i \in I}$ is the quotient space Π_0 / N.

If $x = (x_i)_{i \in I}$ is in Π_0, then the norm of $x + N$ as an element of the ultrapower Π_0 / N is given by the expression $\lim_{\mathfrak{U}} \|x_i\|$.

Let $l_\infty((E_i)_{i \in I})$ be the space Π_0 endowed with the supremum norm. $\Pi_{i \in I} E_i / \mathfrak{U}$ can be viewed as the quotient space $l_\infty((E_i)_{i \in I})/N$, therefore it is a Banach space.

When all the spaces $(E_i)_{i \in I}$ are identical, we use the word *ultrapower* instead of ultraproduct; $E^I / \mathfrak{U}$ denotes the ultrapower; the *canonical embedding* from E into $E^I / \mathfrak{U}$ is the mapping which associates to a given element x the element of $E^I / \mathfrak{U}$ given by the constant function equal to x.

It should be noted that if E is finite dimensional, $E^I / \mathfrak{U}$ is isomorphic to E (because the unit cell of E is compact).

For this reason, in the theorems as well as in the proofs, Banach spaces will be implicitly assumed to be infinite dimensional. The corresponding proofs in the finite dimensional case will be omitted.

DEFINITION 1.2. Let E, F be Banach spaces, $E \subseteq F$; F is a *u-extension* of E if there exist an ultrapower of E, $E^I / \mathfrak{U}$, and a mapping ϕ: $F \to E^I / \mathfrak{U}$ such that:

(i) ϕ is an isometry from F into $E^I / \mathfrak{U}$.

(ii) $\phi \restriction E$ is the canonical embedding from E into $E^I / \mathfrak{U}$.

Definition 1.2 has an "isomorphic" analog:

DEFINITION 1.3. Let E, F be Banach spaces, $E \subseteq F$; F is a *weak u-extension* of E if there exist an ultrapower of E, $E^I / \mathfrak{U}$, and a mapping φ: $F \to E^I / \mathfrak{U}$ such that

(i) ϕ is an isomorphism from F into $E^I / \mathfrak{U}$.

(ii) $\phi \restriction E$ is the canonical embedding from E into $E^I / \mathfrak{U}$.

REMARK. Any weak u-extension of E is isomorphic to a u-extension of E.

Using the above definitions it is possible to enlarge a given class C of Banach spaces in such a way that some of the local properties of the elements of C are preserved. For example, the following observation, already made in [15], is the starting point of our work:

PROPOSITION 1.1. *Let E, F be Banach spaces; the following conditions are equivalent*:

(i) *F is a subspace of some ultrapower of E.*

(ii) *For any finite dimensional subspace A of F and any $\varepsilon > 0$, there exists a subspace B of E, $1 + \varepsilon$-isomorphic to A.*

When (i) *or* (ii) *holds we say F is finitely representable in E.*

DEFINITION 1.4. Let E be a Banach space; we let $S(E)$ be the smallest class of Banach spaces containing E, closed under isometry, ultrapowers and the formation of subspaces.

Similarly we let $Q(E)$ be the smallest class closed under isometry, ultrapowers and the formation of quotient spaces and containing E as an element.

DEFINITION 1.5. Let C be a class of Banach spaces. We let $\mathcal{C}$ denote the class of Banach spaces E such that some element F of C is isometric to a u-extension of E.

Similarly, $\tilde{\mathcal{C}}$ denotes the class of Banach spaces E such that some element F of C is isometric to an ultrapower of E.

We now describe the organization of the paper.

In §2, we study ultrapowers and their dual spaces. Thus, if E is a given space, $(E^I/\mathcal{U})^*$ is isometric to $E^{*I}/\mathcal{U}$ for any ultrafilter $\mathcal{U}$ on a set I if and only if E is super-reflexive. When E is not super-reflexive we still have some information on $(E^I/\mathcal{U})^*$; for example, we show that $(E^I/\mathcal{U})^*$ is a u-extension of $E^{*I}/\mathcal{U}$.

§3 is devoted to u-extensions; we recall an alternative definition of u-extensions and weak-u-extensions, more local in character. We also prove that if E is a complemented subspace of F and F is a subspace of an ultrapower of E, then F is a weak-u-extension of E.

In §4, we apply the preceding results to the study of the classes $S(E)$, $Q(E)$, $\mathcal{C}$, $\tilde{\mathcal{C}}$ defined above:

(1) We prove some duality theorems on the classes $S(E)$ and $Q(E^*)$. As $S(E)$ is the class of Banach spaces finitely representable in E, we have a satisfactory answer to the rather vague question: "what is the dual notion for finite-representability?"

(2) We show that the operation which enables us to go from a class C to the class $\mathcal{C}$ formalizes a procedure which has been used in several places in the theory of Banach spaces. Thus if C is the class of spaces isomorphic to an L_p-space, $\mathcal{C}$ is the class of $\mathfrak{L}_p$-spaces [9]; if C is the class of spaces isomorphic to a Banach lattice, $\mathcal{C}$ is the class of spaces with a local unconditional structure. Using general theorems about u-extensions we prove some closure properties for the class $\mathcal{C}$. We also give the following characterization of $\mathcal{C}$ when C is closed under isomorphism and ultrapowers:

$E \in \mathcal{C}$ if and only if E^{**} is a complemented subspace of a Banach space F which belongs to C and is finitely-representable in E.

In the case of spaces with a local unconditional structure this is an improvement of results of [5].

(3) Concerning the class $\tilde{\mathcal{C}}$ we show the following result:

THEOREM. (i) *If E is an $\mathcal{L}_p$-space, $1 < p < \infty$, there exists an ultrapower of E, $E^I/\mathfrak{U}$, isomorphic to an L_p-space.*

(ii) *If E is complemented in an L_1-space (resp. in a $C(K)$-space), there exists an ultrapower of E, $E^I/\mathfrak{U}$, isomorphic to an L_1-space (resp. a $C(K)$-space).*

This theorem answers an unpublished question of Krivine; the new information it gives on $\mathcal{L}_p$-spaces shows, inter alia, that there is no hope to single out the spaces isomorphic to an L_p-space from the $\mathcal{L}_p$-spaces by a set of formulas in the style of [8] or [15].

The proof of the theorem combines the Pelczyński decomposition method with a result taken from model theory, due originally to Keisler (see [1]) and extended by Shelah [14].

The results which come from model theory are proved in an appendix so that the text itself can be read without any familiarity with mathematical logic.

2. Ultrapowers and their dual spaces.

2.1. *Iteration of ultrapowers.*

PROPOSITION 2.1. *Let E be a Banach space; $\mathfrak{U}$ and $\mathfrak{V}$ be ultrafilters on sets I and J respectively; the ultrapower $(E^I/\mathfrak{U})^J/\mathfrak{V}$ is isometric to $E^{I\times J}/\mathfrak{U}\times\mathfrak{V}$ where $\mathfrak{U}\times\mathfrak{V}$ is defined by: $X \in \mathfrak{U}\times\mathfrak{V}$ if and only if $\{j: \{i: (i,j) \in X\} \in \mathfrak{U}\} \in \mathfrak{V}$.*

PROOF. We recall Definition 1.1 and we let Π_0 be the set of mappings x_{ij} such that $x_{ij} \in E$ and for some λ, $\|x_{ij}\| \leqslant \lambda$ for every pair (i,j). For any element $x = (x_{ij})_{ij\in I\times J}$ of Π_0 we let $\phi_j(x)$ be the element of $E^I/\mathfrak{U}$ given by $(x_{ij})_{i\in I}$; we let $\phi(x)$ be the element of $(E^I/\mathfrak{U})^J/\mathfrak{V}$ given by $(\phi_j(x))_{j\in J}$. Clearly ϕ is linear.

To see that ϕ is onto it is enough to observe that any element x in $(E^I/\mathfrak{U})^J/\mathfrak{V}$ can be given by mapping $(x_j)_{j\in J}$ with $\|x_j\| \leqslant \|x\|$; similarly x_j can be given by $(x_{ij})_{i\in I}$ with $\|x_{ij}\| \leqslant \|x_j\|$. It remains to show that if $\|x\|$ denotes the seminorm of the element x of Π_0, $\|\phi(x)\| = \|x\|$; from this fact it follows that if N denotes the set of elements of Π_0 with seminorm zero, Π_0/N is isometric to $(E^I/\mathfrak{U})^J/\mathfrak{V}$. Now assume $\|x\| = \alpha$ and let $\varepsilon > 0$; let $X = \{j: \{i: |\,\|x_{ij}\| - \alpha| < \varepsilon\} \in \mathfrak{U}\}$; $X \in \mathfrak{V}$; for any j in X we get $|\,\|\phi_j(x)\| - \alpha| \leqslant \varepsilon$ hence $|\,\|\phi(x)\| - \alpha| \leqslant \varepsilon$; as ε is arbitrary we have $\|\phi(x)\| = \|x\|$.

The next result is a consequence of a deep result of Keisler and Shelah; it will be used subsequently in this paper. The proof is given in the appendix.

THEOREM 2.1. *Let E be a Banach space, $\mathfrak{U}$ an ultrafilter on a set I; $\mathfrak{V}$ an ultrafilter on a set J; there exists a set K and an ultrafilter $\mathfrak{W}$ on K such that the spaces $(E^I/\mathfrak{U})^K/\mathfrak{W}$ and $(E^J/\mathfrak{V})^K/\mathfrak{W}$ are isometric.*

Another theorem from model theory will be used in this paper and proved in the appendix:

THEOREM 2.2. *Let E be a Banach space, F a separable subspace of E; there exists a separable subspace L, $F \subseteq L \subseteq E$, such that L and E have isometric ultrapowers. Furthermore if E is a Banach lattice, we may assume L is a sublattice of E.*

2.2. *The dual of $E^I/\mathfrak{U}$.* Let $E^I/\mathfrak{U}$ be an ultrapower of E; for any element f of $E^{*I}/\mathfrak{U}, f = (f_i)_{i \in I}$, let $j(f)$ be the linear functional on $E^I/\mathfrak{U}$ defined by $j(f)((x_i)_{i \in I}) = \lim_{\mathfrak{U}} f_i(x_i)$. It is easy to see that j is well defined and that it is an isometric embedding from $E^{*I}/\mathfrak{U}$ into $(E^I/\mathfrak{U})^*$. A natural question is the following: When is $E^{*I}/\mathfrak{U}$ equal to $(E^I/\mathfrak{U})^*$? (By this, we mean when is j onto?) The answer is given by the next result, which is a restatement of results contained in **[17]**, **[18]** and whose proof will be omitted.

THEOREM 2.3. *Let E be a Banach space; the following conditions are equivalent:*

(i) *E is super-reflexive.*

(ii) *For any ultrafilter $\mathfrak{U}$, $(E^I/\mathfrak{U})^*$ is equal to $E^{*I}/\mathfrak{U}$.*

(iii) *There exists a nontrivial ultrafilter $\mathfrak{U}$ on $\mathbf{N}$ such that $(E^{\mathbf{N}}/\mathfrak{U})^*$ is equal to $E^{*\mathbf{N}}/\mathfrak{U}$.*

REMARKS. (1) When we say $(E^I/\mathfrak{U})^*$ is equal to $E^{*I}/\mathfrak{U}$ we mean (formally) that the above defined j is onto.

(2) A *super-reflexive* Banach space is a Banach space E such that any F finitely representable in E is reflexive. By Proposition 1.1, E is super-reflexive if and only if any ultrapower of E is reflexive.

Even when E is not super-reflexive, the spaces $E^{*I}/\mathfrak{U}$ and $(E^I/\mathfrak{U})^*$ are still locally quite similar.

THEOREM 2.4. *Let E be a Banach space, $\mathfrak{U}$ an ultrafilter on a set I, there exist an ultrafilter $\mathfrak{V}$ on a set J and a mapping ϕ: $(E^I/\mathfrak{U})^* \to (E^{*I}/\mathfrak{U})^J/\mathfrak{V}$ such that:*

1. *ϕ is an isometry from $(E^I/\mathfrak{U})^*$ onto a strongly complemented subspace of $(E^{*I}/\mathfrak{U})^J/\mathfrak{V}$.*

2. *$\phi \restriction E^{*I}/\mathfrak{U}$ is the canonical embedding from $E^{*I}/\mathfrak{U}$ into $(E^{*I}/\mathfrak{U})^J/\mathfrak{V}$.*

REMARK. (1) By strongly complemented subspace, we mean a subspace on which there is a projection of norm 1.

(2) The above theorem should be compared to Theorem 6.4 of **[15]**, where it is proved that there exists an isometry ϕ from E^{**} onto a strongly complemented subspace of some ultrapower $E^I/\mathfrak{U}$, $\phi \restriction E$ being the canonical embedding.

To prove the theorem, we need a lemma which can be compared to the so-called principle of local reflexivity [10]; in order to state the lemma we introduce some notations; we assume

$(g_k)_{1 \leqslant k \leqslant p}$ are given elements in $(E^I/\mathfrak{U})^*$,
$(f_m)_{1 \leqslant m \leqslant q}$ are given elements in $E^{*I}/\mathfrak{U}$,
$(x_n)_{1 \leqslant n \leqslant r}$ are given elements in $E^I/\mathfrak{U}$,
$(\alpha_{mk})_{1 \leqslant m \leqslant q; 1 \leqslant k \leqslant p}$ are given real numbers.
We let

$$\lambda_m = \left\| f_m + \sum_{k=1}^{p} \alpha_{mk} g_k \right\|, \quad 1 \leqslant m \leqslant q,$$

$$\mu_{kn} = g_k(x_n), \qquad 1 \leqslant k \leqslant p;\ 1 \leqslant n \leqslant r.$$

We assume that for any k there is an index m with $g_k = f_m + \sum_{k=1}^{p} \alpha_{mk} g_k$.

LEMMA 2.2. *For any strictly positive real ε, there exists a sequence $h_1, \ldots, h_p$ of elements of $E^{*I}/\mathfrak{U}$ such that*

$$\left\| f_m + \sum_{k=1}^{p} \alpha_{mk} h_k \right\| \leqslant \lambda_m + \varepsilon, \quad 1 \leqslant m \leqslant q,$$

$$|h_k(x_n) - \mu_{kn}| \leqslant \varepsilon, \qquad 1 \leqslant k \leqslant p;\ 1 \leqslant n \leqslant r.$$

PROOF OF THE LEMMA. Assume that f_m is given by $(f_m^i)_{i \in I}$; similarly, x_n is given by $(x_n^i)_{i \in I}$. To prove the lemma it is enough to find elements h_k^i, $i \in I$, $1 \leqslant k \leqslant p$, and a set X in $\mathfrak{U}$ such that for any i in X, the following inequalities hold

$$\left\| f_m^i + \sum_{k=1}^{p} \alpha_{mk} h_k^i \right\| \leqslant \lambda_m + \varepsilon, \qquad |h_k^i(x_n^i) - \mu_{kn}| \leqslant \varepsilon.$$

Then, if h_k is the element of $E^{*I}/\mathfrak{U}$ given by $(h_k^i)_{i \in I}$, the sequence $(h_k)_{1 \leqslant k \leqslant p}$ meets the requirements of the lemma. (Notice that for some m_k, $\|h_k^i\| \leqslant \lambda_{m_k} + \varepsilon$.)

From now on, we consider the space $E^p \times \mathbf{R}$ (endowed with the supremum norm); we notice that $(E^p \times \mathbf{R})^I/\mathfrak{U}$ is canonically isometric to $(E^I/\mathfrak{U})^p \times \mathbf{R}$; if $(A_i)_{i \in I}$ are subsets of $E^p \times \mathbf{R}$ we let $\prod_{i \in I} A_i/\mathfrak{U}$ be the set of elements equal to some element $(x_i)_{i \in I}$ with $x_i \in A_i$. We let:

B_m^i be the set of elements $v_m^i(u)$, $\|u\| < 1$ where $v_m^i = (\alpha_{m1}u, \ldots, \alpha_{mp}u, f_m^i(u) - \lambda_m - \varepsilon)$, $1 \leqslant m \leqslant q$ and $u \in E$.

$y_{kn}^i = (0, \ldots, 0, -x_n^i, 0, \ldots, 0, \mu_{kn} - \varepsilon)$ where $-x_n^i$ is the kth factor, $1 \leqslant k \leqslant p$, $1 \leqslant n \leqslant r$.

$z_{kn}^i = (0, \ldots, 0, x_n^i, 0, \ldots, 0, -\mu_{kn} - \varepsilon)$ where x_n^i is the kth factor, $1 \leqslant k \leqslant p$, $1 \leqslant n \leqslant r$.

$-1 = (0, \ldots, 0, \ldots, 0, -1)$.

Also, if u is a variable ranging over $E^I/\mathfrak{U}$, $u = (u_i)_{i \in I}$, we let

$B_m = \{v_m(u), \|u\| \leqslant 1\}$ where $v_m(u)$ is the element of $(E^p \times \mathbf{R})^I/\mathfrak{U}$ given by $(v_m^i(u_i))_{i\in I}$.

$y_{kn} = (y_{kn}^i)_{i\in I}$

$z_{kn} = (z_{kn}^i)_{i\in I}$.

$-1 = (-1)_{i\in I}$.

It is easy to see that $\Pi_{i\in I}B_m^i/\mathfrak{U} \subseteq B_m$. We let H_i be the convex hull of the sets B_m^i, $1 \leqslant m \leqslant p$, and of the elements y_{kn}^i, z_{kn}^i, $1 \leqslant k \leqslant p$, $1 \leqslant n \leqslant r$ and -1. Similarly K is the convex hull of the sets B_m, $1 \leqslant m \leqslant p$, and of the elements y_{kn}, z_{kn}, $1 \leqslant k \leqslant p$, $1 \leqslant n \leqslant r$, and -1. We claim $\Pi_{i\in I}H_i/\mathfrak{U} \subseteq K$. This follows from the next sublemma.

SUBLEMMA. *Let s be a fixed integer and assume H_i is the convex hull of s convex sets $(C_{il})_{1\leqslant l\leqslant s}$; then $\Pi_{i\in I}\, H_i/\mathfrak{U}$ is included in the convex hull of the sets $\Pi_{i\in I}\, C_{il}/\mathfrak{U}$, $1 \leqslant l \leqslant s$.*

To prove the sublemma, let $t_i = \Sigma\gamma_{il}w_{il}$ with $\Sigma_{l=1}^s\gamma_{il} = 1$ and $w_{il} \in C_{il}$, if $\lim_{\mathfrak{U}} \gamma_{il} = \gamma_l$ and if w_l is the element of $\Pi_{i\in I}C_{il}/\mathfrak{U}$ given by $(w_{il})_{i\in I}$ we get $t = (t_i)_{i\in I} = \Sigma_{l=1}^s\gamma_l w_l$ and $\Sigma_{l=1}^s\gamma_l = 1$, so that t is in the convex hull of the sets $\Pi_{i\in I}C_{il}/\mathfrak{U}$, $1 \leqslant l \leqslant s$.

Now let g be the linear functional on $(E^I/\mathfrak{U})^p \times \mathbf{R}$ such that

$$g(w_1, \ldots, w_p, \xi) = \sum_{k=1}^{p} g_p(w_p) + \xi.$$

From the hypotheses, it follows that $\|g\| \leqslant 1 + \Sigma_{k=1}^p\|g_k\|$ and that g is $\leqslant -\varepsilon$ on K; if $\delta(K)$ is equal to $\inf\{\|v\|: v \in K\}$ we get

$$\delta(K) \geqslant \varepsilon\|g\|^{-1}.$$

From this inequality and the inclusion $\Pi_{i\in I}H_i/\mathfrak{U} \subseteq K$ we get

$$\{i: \delta(H_i) \geqslant \varepsilon\|g\|^{-1}/2\} \in \mathfrak{U}.$$

This, in turn, shows that if H_i' is the set of elements of $E^p \times \mathbf{R}$ whose distance to H_i is strictly smaller than $\varepsilon\|g\|^{-1}/4$ then

$$\{i: 0 \notin H_i'\} \in \mathfrak{U}.$$

H_i' is convex and open, therefore there exists a linear functional h^i on $E^p \times \mathbf{R}$ which is strictly negative on H_i' (hence on H_i); we may assume $h^i(-1) = -1$ so that h^i is given by a sequence $(h_1^i, \ldots, h_n^i, 1)$.

From the fact that h^i is negative on B_m^i it follows that

$$\left\| f_m^i + \sum_{k=1}^{p} \alpha_{mk} h_k^i \right\| \leqslant \lambda_m + \varepsilon;$$

from the fact that h^i is negative on any element y_{kn}^i, z_{kn}^i it follows that

$$\left|h_k^i(x_n^i) - \mu_{kn}\right| \leqslant \varepsilon.$$

This is exactly what remained to be proved.

We now prove Theorem 2.4.

Let J be the set of pairs $j = (F_j, X_j)$ where F_j is a finite subset of $(E^I/\mathcal{U})^*$ and X_j is a finite subset of $E^I/\mathcal{U}$. J is ordered by the relation $j \leqslant j'$ if and only if $F_j \subseteq F_{j'}$ and $X_j \subseteq X_{j'}$. The set of elements $\hat{j} = \{j' : j \leqslant j'\}$ has the finite intersection property and, thus, can be extended to an ultrafilter $\mathcal{V}$ on J. If we let $n(j)$ be the number of elements in F_j we notice that $\lim_{\mathcal{V}}(n(j))^{-1} = 0$.

For any element f of $(E^I/\mathcal{U})^*$ we define the element f_j of $E^{*I}/\mathcal{U}$ in such a way that

if $f \notin F_j$ then f_j is 0,

if $f \in F_j \cap E^{*I}/\mathcal{U}$ then f_j is f,

if $f^1, \ldots, f^n$ are elements of F_j and $\varepsilon_1, \ldots, \varepsilon_n$ is a sequence of $+$ and $-$ then

$$(*) \qquad \left\| \sum_{k=1}^{n} \varepsilon_k f_j^k \right\| \leqslant \left\| \sum_k \varepsilon_k f^k \right\| + (n(j))^{-1},$$

if f is in F_j and x is in X_j

$$(**) \qquad |f_j(x) - f(x)| \leqslant (n(j))^{-1}.$$

The fact that the mapping $f \to f_j$ exists follows from Lemma 2.2 and the observation that $(*)$ and $(**)$ include only a finite number of inequalities.

We let $\phi: (E^I/\mathcal{U})^* \to (E^{*I}/\mathcal{U})^J/\mathcal{V}$ be defined by $\phi(f) = (f_j)_{j \in J}$. We first check ϕ is norm preserving. Let f belong to $(E^I/\mathcal{U})^*$; if $f \in F_j$ we get from $(*)$

$$\|f_j\| \leqslant \|f\| + (n(j))^{-1}$$

so that $\lim_{\mathcal{V}} \|f_j\| \leqslant \|f\|$.

On the other hand let x be an element of norm 1 of $E^I/\mathcal{U}$ such that $f(x) \geqslant \|f\| - \eta$; from $(**)$ we get as long as $x \in X_j$ and $f \in F_j$

$$|f_j(x) - f(x)| \leqslant (n(j))^{-1}$$

so that $\|f_j\| \geqslant f(x) - (n(j))^{-1}$. Finally $\lim_{\mathcal{V}} \|f_j\| \geqslant f(x) \geqslant \|f\| - \eta$, as η is arbitrary, we get

$$\lim_{\mathcal{V}} \|f_j\| \geqslant \|f\|.$$

To prove that ϕ is linear, it is enough to show that

$$\|\phi(f) + \phi(g) - \phi(h)\| \leqslant \|f + g - h\|.$$

If f, g, h are in F_j we get from $(*)$

$$\|f_j + g_j - h_j\| \leqslant \|f + g - h\| + (n(j))^{-1}$$

so that

$$\|\phi(f) + \phi(g) - \phi(h)\| = \lim_{\mathcal{V}} \|f_j + g_j - h_j\| \leqslant \|f + g - h\|.$$

The fact that $\phi \restriction E^{*I}/\mathcal{U}$ is the canonical embedding from $E^{*I}/\mathcal{U}$ into $(E^{*I}/\mathcal{U})^J/\mathcal{V}$ is clear because if $f \in F_j$ and $f \in E^{*I}/\mathcal{U}$ then $f_j = f$.

We now define π: $(E^{*I}/\mathcal{U})^J/\mathcal{V} \to (E^I/\mathcal{U})^*$; if g is the element of $(E^{*I}/\mathcal{U})^J/\mathcal{V}$ given by $(g_j)_{j \in J}$ we let for x in $E^I/\mathcal{U}$,

$$\pi(g)(x) = \lim_{\mathcal{V}} g_j(x).$$

It is easy to see that π is linear and that $\|\pi\| \leqslant 1$. To finish the proof of the theorem, it is enough to prove that, for any f in $(E^I/\mathcal{U})^*$, $\pi(\phi(f)) = f$. Letting $g = \pi(\phi(f))$ we have

$$g(x) = \lim_{\mathcal{V}} f_j(x);$$

but if f is in F_j we have

$$|f(x) - f_j(x)| \leqslant (n(j))^{-1},$$

so that

$$g(x) = \lim_{\mathcal{V}} f_j(x) = f(x).$$

3. *u*-extensions of Banach spaces. The definition of u-extensions and weak-u-extensions has been given in §1 (Definitions 1.2 and 1.3). The aim of this section is to give local characterizations of these notions.

The following result is proved in [**15**, §6].

PROPOSITION 3.1. *Let E, F be Banach spaces, $E \subseteq F$; the following conditions are equivalent:*

(i) *F is a u-extension of E.*

(ii) *For any finite dimensional subspace A of F and for any $\varepsilon > 0$, there exists an application $\phi_A: A \to E$ such that*

$\phi_A \restriction A \cap E$ is the identity,

ϕ_A is a $1 + \varepsilon$-isomorphism from A onto $\phi_A(A)$.

If F is a weak-u-extension of E, there exists an isomorphism ψ from F onto a u-extension of E, F', such that ψ is the identity on E; from this we get:

PROPOSITION 3.2. *Let E, F be Banach spaces, $E \subseteq F$; the following conditions are equivalent:*

(i) *F is a weak-u-extension of E.*

(ii) *There exists a real number λ such that for any finite dimensional subspace A of F, one can find an application $\phi_A: A \to E$ with the following properties:*

$\phi_A \restriction A \cap E$ is the identity,

ϕ_A is a λ-isomorphism from A onto $\phi_A(A)$.

We now give a condition which is sufficient to ensure that a given space F is a weak-u-extension of one of its subspaces E.

THEOREM 3.1. *Let E, F be Banach spaces, $E \subseteq F$; assume E is complemented in F and F is finitely representable in E; then F is a weak-u-extension of E.*

It should be noticed that if F is a u-extension of E, the hypotheses of the above theorem are "almost" satisfied in the following sense:

PROPOSITION 3.3. *Let F be a u-extension of E, then E^{**} is complemented in F^{**} and F^{**} is finitely representable in E^{**}.*

This result is implicit in [15]: it is proved in this paper that there exists an application $\pi: F \to E^{**}$ such that:
$\|\pi\| \leqslant 1$,
$\pi \restriction E$ is the canonical embedding i from E into E^{**}.
But $i^{**}(E^{**})$ is complemented in E^{****} so that if $\tilde{\pi}$ denotes a projection from E^{****} onto $i^{**}(E^{**})$, $\tilde{\pi} \circ \pi^{**}$ is a projection from F^{**} onto E^{**} (we even get $\|\tilde{\pi} \circ \pi^{**}\| \leqslant 1$). F^{**} is finitely representable in F (by "local reflexivity"), therefore, because F is finitely representable in E, F^{**} is finitely representable in E, hence in E^{**}.

Theorem 3.1 is a consequence of the following lemma, where E and F are as in the theorem and where Q denotes a projection from F onto E.

LEMMA 3.1. *Let A be a finite dimensional subspace of $(I - Q)F$; for any integer n and any $\delta > 0$, there exists a sequence of finite dimensional subspaces of E, say $A_1, \ldots, A_n$, such that (1) each space A_k, $1 \leqslant k \leqslant n$, is $1 + \delta$-isomorphic to A,*

(2) if S_k denotes the unit sphere of A_k, then $d(S_k, S_{k'}) \geqslant \|Q\|^{-1}(1 + \delta)^{-1}$ if $k \neq k'$. ($d(S_k, S_{k'})$ is $\inf_{x \in S_k; y \in S_{k'}} \|x - y\|$ as usual.)

To prove the lemma it is enough to show by induction on n that, for any n, there exist an ultrapower of E, $E^{J_n}/\mathcal{V}_n$ and a sequence of subspaces of $E^{J_n}/\mathcal{V}_n$, $B_1, \ldots, B_n$, such that:
B_k, $1 \leqslant k \leqslant n$, is isometric to A,
the mutual distances of the unit spheres of $B_1, \ldots, B_n$ are at least $\|Q\|^{-1}$.

To get the lemma one carries back in E, via a $1 + \delta$-isomorphism, the space spanned by $B_1, \ldots, B_n$.

The property to show is known for $n = 1$ (with $J_n = I$ and $\mathcal{V}_n = \mathcal{U}$). To go from n to $n + 1$, consider the projection Q_n from $F^{J_n}/\mathcal{V}_n$ onto $E^{J_n}/\mathcal{V}_n$ defined by $Q_n((x_j)_{j \in J_n}) = (Q(x_j))_{j \in J_n}$. Let B_{n+1} be the subspace $A^{J_n}/\mathcal{V}_n$ of the ultrapower $F^{J_n}/\mathcal{V}_n$, where $A^{J_n}/\mathcal{V}_n$ is the set of elements $(x_j)_{j \in J_n}$ with $x_j \in A$; B_{n+1} is isometric to A. Now, if $\|x\| = \|y\| = 1$, $x \in B_k$, $k \leqslant n$, and $y \in B_{n+1}$ we have $Q_n(x - y) = x$ because Q_n is 0 on B_{n+1} so that:

$$1 = \|x\| = \|Q_n(x - y)\| \leqslant \|Q_n\| \|x - y\| \leqslant \|Q\| \|x - y\|$$

or

$$\|x - y\| \geqslant \|Q\|^{-1}.$$

To finish the proof we just observe that $B_1, \ldots, B_{n+1}$ are subspaces of $F^{J_n}/\mathcal{V}_n$ which is finitely representable in F, therefore in E, and thus is isometric to a subspace of $E^{J_{n+1}}/\mathcal{V}_{n+1}$ for some ultrafilter $\mathcal{V}_{n+1}$ on a set J_{n+1}.

We now give the proof of the Theorem 3.1.[2]

Let W be a finite dimensional subspace of F. Let A be $(I - Q)W$ and let K be the set of elements of $Q(W)$ whose norm is at most 2. K is compact. We claim that for a given $\varepsilon > 0$ there exists a subspace B of E, $1 + \varepsilon$-isomorphic to A, and such that if S denotes the unit sphere of B, $d(S, K) \geqslant \|Q\|^{-1}(1 - \varepsilon)/2$. If this is not true we may pick an ε'-dense sequence in K: $x_1, \ldots, x_n$, where $\varepsilon' = \varepsilon\|Q\|^{-1}/4$. We let $A_1, \ldots, A_{n+1}$; $S_1, \ldots, S_{n+1}$, given by Lemma 3.1 with $\delta = \varepsilon/2$; as we assume $d(S_k, K) < \|Q\|^{-1}(1 - \varepsilon)/2$ it follows easily that there exist an index m, $1 \leqslant m \leqslant n$, and two different indexes k, k', $1 \leqslant k < k' \leqslant n + 1$ such that

$$d(x_m, S_k) < \frac{\|Q\|^{-1}}{2}\left(1 - \frac{\varepsilon}{2}\right),$$

$$d(x_m, S_{k'}) < \frac{\|Q\|^{-1}}{2}\left(1 - \frac{\varepsilon}{2}\right);$$

we get

$$d(S_k, S_{k'}) < \|Q\|^{-1}\left(1 - \frac{\varepsilon}{2}\right)$$

contradicting the choice of S_k, $S_{k'}$. Let τ be a $1 + \varepsilon$-isomorphism from A onto B. If $w \in W$, $T(w)$ is defined by

$$T(w) = Q(w) + \tau(w - Q(w)).$$

We now show that for some real number λ depending only on $\|Q\|$ and ε, T is a λ-isomorphism from W onto $T(W)$. Also T is the identity on $W \cap E$; in view of Proposition 3.2., the theorem follows. Clearly:

$$\|T(w)\| \leqslant [\|Q\| + (1 + \varepsilon)(1 + \|Q\|)]\|w\|.$$

In the other direction, we have

$$\|T(w)\| \geqslant \frac{\|Q\|^{-1}}{8}\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)\|w\|.$$

To check this, we consider two cases:

Case 1. $\|Q(w)\| \leqslant 2\|y\|$ where $y = \tau(w - Q(w))$; then it follows from $d(K, S) \geqslant \|Q\|^{-1}(1 - \varepsilon)/2$ that

[2] The proof is modelled after the Lindenstrauss-Rosenthal proof that any complemented subspace of an $\mathcal{L}_p$-space which is not an $\mathcal{L}_2$-space is an $\mathcal{L}_p$-space.

$$\|Q(w)+y\| \geq \frac{\|Q\|^{-1}}{2}(1-\varepsilon)\|y\| \geq \frac{\|Q\|^{-1}}{4}(1-\varepsilon)\frac{\|y\|+\|Q(w)\|}{2}$$

$$\geq \frac{\|Q\|^{-1}}{8}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)(\|w-Q(w)\|+\|Q(w)\|).$$

Case 2. $\|Q(w)\| \geq 2\|y\|$; then

$$\|Q(w)+y\| \geq \|Q(w)\|-\|y\| \geq \frac{\|Q(w)\|}{2} \geq \frac{\|Q(w)\|+\|y\|}{3}$$

$$\geq \frac{1}{3(1+\varepsilon)}(\|Q(w)\|+\|w-Q(w)\|).$$

This finishes the proof.

REMARK. Actually, it is not difficult to derive from the above inequalities that there exists an application ψ from F onto a u-extension of E, which is the identity on E and is a $(8\|Q\|(2\|Q\|+1))$-isomorphism.

4. Classes of Banach spaces closed under ultrapowers.

4.1. *The classes* $S(E)$ *and* $Q(E)$. Let E be a Banach space; the classes $S(E)$ and $Q(E)$ have been defined in the introduction. It is easy to prove the following result:

PROPOSITION 4.1. $S(E)$ *is the class of Banach spaces finitely representable in* E.

In order to state a similar result for $Q(E)$ we need a definition.

DEFINITION 4.1. Let E be a Banach space; a *local quotient space* of E is any space F which is isometric to the quotient of an ultrapower of E by a closed subspace.

PROPOSITION 4.2. $Q(E)$ *is the class of local quotient spaces of* E.

PROOF. The only thing to check is the fact that the class of local quotient spaces of E is closed under ultrapowers. So, assume F is the quotient space of $E^I/\mathcal{U}$ by one of its subspaces. Let π: $E^I/\mathcal{U} \to F$ be the canonical mapping from $E^I/\mathcal{U}$ onto its quotient space. Define $\tilde{\pi}$: $(E^I/\mathcal{U})^J/\mathcal{V} \to F^J/\mathcal{V}$ by $\tilde{\pi}((x_j)_{j\in J}) = (\pi(x_j))_{j\in J}$; clearly $\|\tilde{\pi}\| \leq 1$. Now let y be an element of $F^J/\mathcal{V}$ given by $(y_j)_{j\in J}$; if ε is a strictly positive real, then for each j we may pick an element x_j of E such that

$$\pi(x_j) = y_j \quad \text{and} \quad \|y_j\| \leq \|x_j\| < \|y_j\| + \varepsilon$$

so that if $x = (x_j)_{j\in J}$ we get

$$\tilde{\pi}(x) = y \quad \text{and} \quad \|y\| \leq \|x\| \leq \|y\| + \varepsilon.$$

This proves that $F^J/\mathcal{V}$ is the quotient space of $(E^I/\mathcal{U})^J/\mathcal{V}$ by the kernel of $\tilde{\pi}$. The following theorem establishes duality properties.

THEOREM 4.1. *Let E, F be Banach spaces*:

(i) *F is finitely representable in E if and only if F^* is a local quotient space of E^*.*

(ii) *If F is a local quotient space of E, then F^* is finitely representable in E^*.*

Theorem 4.1 is a consequence of Theorem 2.4 which shows that $(E^I/\mathfrak{U})^*$ is a complemented subspace of some ultrapower $(E^{*I}/\mathfrak{U})^J/\mathfrak{V}$.

We do not know if the converse of 4.1 (ii) is true and we ask the following question:

Problem 4.1. *Is E a local quotient space of E^{**}?*

4.2. *The classes* $\mathcal{C}$. For the rest of the section, we let C be a class of Banach spaces closed under ultrapowers. As in Definition 1.5, $\mathcal{C}$ is the class of Banach spaces E such that there exists a u-extension F of E, isometric to an element of C. We first state some basic facts about $\mathcal{C}$.

PROPOSITION 4.3. (i) *$\mathcal{C}$ is closed under ultrapowers.*

(ii) *If a Banach space E has a u-extension in $\mathcal{C}$ then E belongs to $\mathcal{C}$.*

(iii) *If C is closed under isomorphisms, then $\mathcal{C}$ is also closed under isomorphisms.*

As a corollary of 4.3(iii) we get

COROLLARY 4.1. *Assume C is closed under isomorphisms; then $\mathcal{C}$ is exactly the class of Banach spaces E such that there exists a weak-u-extension F of E, which belongs to C.*

We omit the proof of 4.3(ii) and (iii). To prove (i), we show that whenever F is a u-extension of E, $F^I/\mathfrak{U}$ is a u-extension of $E^I/\mathfrak{U}$. So we let A be a finite dimensional subspace of $F^I/\mathfrak{U}$ and $\varepsilon > 0$; we assume that $A \cap E^I/\mathfrak{U}$ is spanned by $a_1, \dots, a_k$ where $\|a_1\| = \cdots = \|a_k\| = 1$ and a_j, $1 \leq j \leq k$, is given by $(a_j^i)_{i\in I}$; $a_j^i \in E$; A is spanned by $a_1, \dots, a_k$; $b_1, \dots, b_p$ with $\|b_1\| = \cdots = \|b_p\|$ and with $b_1 = (b_1^i)_{i\in I}$ etc. We let A_i be the span of $a_1^i, \dots, a_k^i$; $b_1^i, \dots, b_p^i$. By Proposition 4.1(ii), for each i in I, we may pick T_i: $A_i \to E$ such that:

$T_i(a_j^i) = a_j^i$, $1 \leq j \leq k$.

T_i is a $1 + \varepsilon$-isomorphism from A_i onto $T_i(A_i)$.

If we let T: $\Pi_{i\in I}A_i/\mathfrak{U} \to E^I/\mathfrak{U}$ be defined by $T((x_i)_{i\in I}) = (T_i(x_i))_{i\in I}$ then, as $\Pi_{i\in I}A_i/\mathfrak{U}$ is A, T is a $1 + \varepsilon$-isomorphism from A onto $T(A)$ and T is the identity on $A \cap E^I/\mathfrak{U}$. By Proposition 4.1 this proves that $F^I/\mathfrak{U}$ is a u-extension of $E^I/\mathfrak{U}$.

We now give some examples; the following proposition has already been observed in **[15]**.

PROPOSITION 4.4. *Let E be a Banach space*; *the following conditions are equivalent*:

(i) *E has a u-extension isomorphic to an L_p-space, $1 < p < \infty$.*

(ii) *E is an $\mathcal{L}_p$-space, i.e. there exists λ such that for any finite dimensional subspace B of E there is a finite dimensional subspace C of E which includes B and is λ-isomorphic to an l_p^n.*

The above result shows that if C is the class of spaces isomorphic to an L_p-space, $1 < p < \infty$, $\mathcal{C}$ is the class of $\mathcal{L}_p$-spaces defined by Lindenstrauss and Pelczyński in [9]. Similarly if C is the class of spaces isomorphic to a $C(K)$-space, $\mathcal{C}$ is the class of $\mathcal{L}_\infty$-spaces (for a proof see [15]).

In order to give one more example, we need first a definition (due to Dubinsky, Pelczyński and Rosenthal [4]). Before we state this definition, we recall that if $x_1, \ldots, x_n$ is an algebraic basis of a finite dimensional space, the unconditional constant of the sequence $x_1, \ldots, x_n$ is the real number ρ

$$\rho = \sup\left\{ \left\| \sum_{i=1}^{n} \varepsilon_i \alpha_i x_i \right\| : \varepsilon_i = \pm 1, \alpha_i \in \mathbf{R}, \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| < 1 \right\}.$$

DEFINITION 4.2. A Banach space E has a *local unconditional structure* (l.u.s.) if there exists a real number λ such that for any finite dimensional subspace B of E there is a finite dimensional subspace C of E, which includes B and has unconditional constant at most λ.

It is proved in [5] that any Banach lattice has a l.u.s.

PROPOSITION 4.5. *A Banach space has a local unconditional structure if and only if it has a u-extension isomorphic to a Banach lattice.*

This result is implicit in [5], where the notion of u-extension is not used and we only give a sketch of the proof.

Assume E has a l.u.s. and for each finite dimensional subspace B of E, pick a finite dimensional subspace C_B of E such that $B \subseteq C_B$ and C_B has unconditional constant at most λ. Pick an ultrafilter $\mathfrak{U}$ on the set $\mathfrak{B}$ of finite dimensional subspaces of E such that for any B in $\mathfrak{B}$, $\hat{B} \in \mathfrak{U}$, where $\hat{B} = \{B' \colon B' \supseteq B\}$. Clearly, $\Pi_{B\in\mathfrak{B}} C_B/\mathfrak{U}$ contains an isometric copy of E and is a u-extension of it. Also, $\Pi_{B\in\mathfrak{B}} C_B/\mathfrak{U}$ is λ-isomorphic to a Banach lattice; this is because each C_B is λ-isomorphic to a Banach lattice and because the class of Banach lattices is closed under ultraproducts [2].

To prove the converse implication, it is enough to notice that if E has a u-extension with a l.u.s. then E itself has a l.u.s. (apply Proposition 3.1). Q.E.D.

From now on we assume C is closed under isomorphisms.

THEOREM 4.2. *Assume C is closed under isomorphisms, then a Banach space E belongs to $\mathcal{C}$ if and only if its second dual E^{**} belongs to $\mathcal{C}$.*

PROOF. Assume E belongs to $\mathcal{C}$; E^{**} is a complemented subspace of an ultrapower $E^I/\mathfrak{U}$ [15, Theorem 6.4]. By Theorem 3.1, $E^I/\mathfrak{U}$ is a weak-u-extension of E^{**}; but by Proposition 4.3(i), $\mathcal{C}$ is closed under ultrapowers so that $E^I/\mathfrak{U}$ belongs to $\mathcal{C}$. By Corollary 4.1, it follows that E^{**} belongs to $\mathcal{C}$. To prove the converse just notice that E^{**} is a u-extension of E and use 4.3(ii).

Using the same idea, we can obtain a more general result, which provides a reasonably simple characterization of $\mathcal{C}$.

THEOREM 4.3.[3] *Assume C is closed under isomorphisms; then a given Banach space E belongs to $\mathcal{C}$ if and only if there exists a Banach space L in C such that*

(1) *E^{**} is isometric to a complemented subspace of L.*

(2) *L is finitely representable in E.*

REMARK. If c_0 is finitely representable in E, then condition (2) is always satisfied.

PROOF. Assume first E belongs to $\mathcal{C}$; let F be a u-extension of E, $F \in C$; by Proposition 3.3 E^{**} is isometric to a complemented subspace of F^{**}; F^{**} in its turn is complemented in some ultrapower $F^I/\mathfrak{U}$. Let L be $F^I/\mathfrak{U}$; as C is closed under ultrapowers L belongs to C; clearly L is finitely representable in E and E^{**} is isometric to a complemented subspace of L.

Now, let E, L be given spaces, $L \in C$ and assume (1) and (2) hold. By Theorem 3.1 L is a weak-u-extension of E^{**} so that E^{**} is in $\mathcal{C}$, in view of Theorem 4.2; this implies $E \in \mathcal{C}$. Q.E.D.

4.3. *The classes $\tilde{\mathcal{C}}$, when C is the class of spaces isomorphic to an L_p-space.*

The class of $\mathfrak{L}_p$-spaces [9] is in some sense the class of Banach spaces which are locally isomorphic to an L_p-space. If we want to define a class of Banach spaces which share the "local properties" of the spaces isomorphic to an L_p-space and if we are ready to admit that a "local" property is a property preserved under ultrapowers (which is the underlying idea of the paper), then it is natural to substitute for $\mathfrak{L}_p$ the class $\tilde{\mathcal{C}}$ (where C is the class of Banach spaces isomorphic to an L_p-space). That these two classes coincide, at least if $1 < p < \infty$, is quite striking.

THEOREM 4.4.[4] *Assume $1 < p < \infty$; any $\mathfrak{L}_p$-space has an ultrapower isomorphic to an L_p-space.*

In the case $p = 1$ or $p = \infty$ we can only prove a weaker result:

[3] In the case of Banach spaces with a l.u.s. this is an improvement of results in [5].

[4] This result has an interesting meaning in the model-theory of Banach spaces. Let a $\forall\exists$-formula be a formula which consists of a string of universal quantifiers followed by a string of existential quantifiers and a quantifier free formula (we follow [15] for the terminology). Implicit in [8] is the fact that the class of $\mathfrak{L}_p$-spaces, $1 < p < \infty$, is the union of countably many classes of spaces defined by a set of $\forall\exists$-formulas. Theorem 4.4 suggests that considering more complicated formulas will not yield more information, contrary to what was expected.

THEOREM 4.5. *Let E be a Banach space*;

(i) *If E is isomorphic to a complemented subspace of an L_1-space, E has an ultrapower isomorphic to an L_1-space.*

(ii) *If E is isomorphic to a complemented subspace of a $C(K)$-space, E has an ultrapower isomorphic to a $C(K)$-space.*

REMARKS. (1) Theorem 4.5(ii) should be compared to the well-known question: is any complemented subspace of a $C(K)$-space isomorphic to a $C(K)$-space?

(2) We do not know if Theorem 4.4 is true for $p = 1$ or $p = \infty$.

Problem 4.2. *Find a Banach space E which is an $\mathfrak{L}_1$-space (resp. $\mathfrak{L}_\infty$-space) and such that no ultrapower of E is isomorphic to an L_1-space (resp. to a $C(K)$-space).*

(3) Theorem 4.5 shows that the second dual of an $\mathfrak{L}_1$-space (resp. $\mathfrak{L}_\infty$-space) has an ultrapower which is isomorphic to an L_1-space (resp. L_∞-space).

Before we prove Theorems 4.4 and 4.5 we state some simple lemmas.

LEMMA 4.1. (i) *Let $1 < p < \infty$; any $\mathfrak{L}_p$-space is isomorphic to a complemented subspace of an ultrapower $l_p^I/\mathfrak{U}$.*

(ii) *Any L_1-space is isomorphic to a complemented subspace of an ultrapower $l_1^I/\mathfrak{U}$.*

LEMMA 4.2. (i) *Let $1 \leqslant p < \infty$, if H is $(l_p)^K/\mathfrak{W}$, then $l^p(H)$ is isometric to a complemented subspace of H.*

(ii) *If H is $c_0^K/\mathfrak{W}$, then $c_0(H)$ is isometric to a complemented subspace of H.*

PROOF OF LEMMA 4.1. Let E be an $\mathfrak{L}_p$-space; from Proposition 4.4 we know that E has a u-extension isomorphic to an L_p-space, F. Actually the proof of this result (given in [15]) shows that we may assume $F = \prod_{i \in I} l_p^{n(i)}/\mathfrak{U}$. (To define F, pick for any finite dimensional subspace B of E a subspace C_B λ-isomorphic to $l_p^{\dim C_B}$ and such that $B \subseteq C_B$ and choose an ultrafilter $\mathfrak{U}$ such that $\hat{B} \in \mathfrak{U}$ where $\hat{B} = \{B' \colon B' \supseteq B\}$; F is $\prod_{B \in I} l_p^{\dim C_B}/\mathfrak{U}$.)

Now if $1 < p < \infty$, E is reflexive, so that, by Proposition 3.3, E is isomorphic to a complemented subspace of F; F is complemented in $l_p{}^I/\mathfrak{U}$; to prove this, let $\pi_i \colon l_p \to l_p^{n(i)}$ be a projection of norm 1 and let $\pi((x_i)_{i \in I}) = (\pi_i(x_i))_{i \in I}$. This finishes the proof of Lemma 4.1(i). The proof fails for $\mathfrak{L}_1$-spaces by lack of reflexivity; it can be carried through for L_1-spaces because any L_1-space is complemented in its second dual and therefore, by Proposition 3.3, in any u-extension.

PROOF OF LEMMA 4.2. We only prove the first statement; as $l_p(l_p)$ is exactly l_p, H is $l_p(l_p)^K/\mathfrak{U}$ so it is enough to prove that, for any E, $l_p(E^K/\mathfrak{W})$ is isometric to a complemented subspace of $(l_p(E))^K/\mathfrak{W}$. Any element of $l_p(E^K/\mathfrak{W})$ is a sequence $(x_n)_{n \in N}$, where $x_n \in E^K/\mathfrak{W}$ and $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$.

We let Z be the dense subspace of $l_p(E^K/\mathcal{U})$ which consists of those sequences which are eventually 0. (Z is not a closed subspace.) If x is an element of Z, $x = (x_1, \ldots, x_N, 0, \ldots, 0, \ldots)$ and if x_n, $1 \leq n \leq N$, is given by $(x_{nk})_{k\in K}$ we let $\phi_k(x)$ be $(x_{1k}, \ldots, x_{Nk}, 0, \ldots, 0, \ldots)$. $\phi_k(x)$ is an element of $l^p(E)$ and

$$\|\phi_k(x)\|^p = \|x_{1k}\|^p + \cdots + \|x_{Nk}\|^p.$$

We let $\phi(x) = (\phi_k(x))_{k\in K}$;

$$\|\phi(x)\| = \lim_{\mathcal{U}} \|\phi_k(x)\| = \left(\sum_{n=1}^{N} \lim_{\mathcal{U}} \|x_{nk}\|^p\right)^{1/p} = \|x\|$$

so that ϕ can be extended to an isometry $\bar{\phi}$: $l_p(E^K/\mathcal{U}) \to (l_p(E))^K/\mathcal{U}$. Now if y is an element of $(l_p(E))^K/\mathcal{U}$ given by $(y_k)_{k\in K}$ and if y_k is the sequence $(y_{kn})_{n\in N}$ we let $\pi_n(y) = (y_{kn})_{k\in K}$. π_n is a mapping from $(l_p(E))^K/\mathcal{U}$ into $E^K/\mathcal{U}$; furthermore

$$\sum_{n=1}^{N} \|\pi_n(y)\|^p = \sum_{n=1}^{N} \lim_{\mathcal{U}} \|y_{kn}\|^p = \lim_{\mathcal{U}} \sum_{n=1}^{N} \|y_{kn}\|^p \leq \|y\|^p$$

so that $(\sum_{n=1}^{\infty}\|\pi_n(y)\|^p)^{1/p} \leq \|y\|$. Finally if $\pi(y) = (\pi_n(y))_{n\in\mathbf{N}}$, π is an application from $(l_p(E))^K/\mathcal{U}$ into $l_p(E^K/\mathcal{U})$; it is easy to see that $\pi \circ \bar{\phi}$ is the identity on Z and therefore on $l_p(E)^K/\mathcal{U}$ so that $l_p(E)^K/\mathcal{U}$ is isometric to a complemented subspace of $(l_p(E))^K/\mathcal{U}$.

Before we give the proof of Theorem 4.4 we make one more observation which will be used repeatedly: if E is isomorphic to a complemented subspace of F, then $E^J/\mathcal{V}$ is isomorphic to a complemented subspace of $F^J/\mathcal{V}$. The proof is straightforward: if ϕ is an isomorphism from E onto a complemented subspace of F and π is a projection from F onto $\phi(E)$, then $\tilde{\phi}$ defined by $\tilde{\phi}((x_j)_{j\in J}) = (\phi(x_j))_{j\in J}$ is an isomorphism from $E^J/\mathcal{V}$ into $F^J/\mathcal{V}$ and $\tilde{\pi}$ defined by $\tilde{\pi}((y_j)_{j\in J}) = (\pi(y_j))_{j\in J}$ is a projection from $F^J/\mathcal{V}$ onto $\tilde{\phi}(E^J/\mathcal{V})$.

Proof of Theorem 4.4. It is known that any $\mathcal{L}_p$-space ($1 \leq p < \infty$) has a complemented subspace isomorphic to l_p (see [11, Proposition II, 5,5]). From Lemma 4.1, we know that any $\mathcal{L}_p$-space, E ($1 < p < \infty$), is isomorphic to a complemented subspace of an ultrapower $l_p^I/\mathcal{U}$.

By Theorem 2.1, there exists an ultrafilter $\mathcal{U}$ on a set K such that $l_p^K/\mathcal{U}$ and $(l_p^I/\mathcal{U})^K/\mathcal{U}$ are isometric. Let H be $l_p^K/\mathcal{U}$, H is an L_p-space and the following facts are true:

$E^K/\mathcal{U}$ is isomorphic to a complemented subspace of H.

H is isomorphic to a complemented subspace of $E^K/\mathcal{U}$.

$l_p(H)$ is isomorphic to a complemented subspace of H (this is proved in Lemma 4.2(i)).

The Pelczyński "decomposition method" [12] asserts precisely that the above facts imply that H and $E^K/\mathcal{U}$ are isomorphic. Q.E.D.

The proof of Theorem 4.5(i) is exactly similar. To prove 4.5(ii), we need one more result due to Pelczyński [13].

THEOREM 4.6 (PELCZYŃSKI). *Let K be a compact metric space and Y be a separable Banach space; if $C(K)$ is isomorphic to a subspace of Y, then $C(K)$ is isomorphic to a complemented subspace of Y.*

In the nonseparable case this theorem has the following consequence.

PROPOSITION 4.5. *Let Ω be a compact space and Z be a Banach space; if $C(\Omega)$ is isomorphic to a subspace of Z, then there exist an ultrapower of $C(\Omega)$, X_1, and an ultrapower Z_1, of Z, such that X_1 is isomorphic to a complemented subspace of Z_1.*

PROOF. Let X be a separable sublattice of $C(\Omega)$ which contains the unit as an element and is such that X and $C(\Omega)$ have isometric ultrapowers. That such an X exists is a consequence of Theorem 2.2. X is a closed sublattice of $C(\Omega)$ with a unit, therefore X is a $C(K)$-space for some compact K; as $C(K)$ is separable K may be chosen compact metric. $C(K)$ is isomorphic to a subspace of Z; we let ϕ stand for an isomorphism from $C(K)$ into Z. By Theorem 2.2, there is a separable Banach space Y such that

$\phi(C(K)) \subseteq Y \subseteq Z$ and

Y and Z have isometric ultrapowers.

By Theorem 4.6, $C(K)$ is isomorphic to a complemented subspace of Y. If $C(K)^I/\mathfrak{U}$ and $C(\Omega)^J/\mathfrak{V}$ are isometric it follows that $C(\Omega)^J/\mathfrak{V}$ is isomorphic to a complemented subspace of $Y^I/\mathfrak{U}$.

Now, it is an easy consequence of Theorem 2.1 that the relation $L \sim M$ defined by L and M have isometric ultrapowers is an equivalence relation. Therefore $Y^I/\mathfrak{U}$ and Z have isometric ultrapowers. Assume $(Y^I/\mathfrak{U})^{I'}/\mathfrak{U}'$ is isometric to an ultrapower of Z, it follows that $(C(\Omega)^J/\mathfrak{V})^{I'}/\mathfrak{U}'$ is isomorphic to a complemented subspace of an ultrapower of Z. Q.E.D.

We now prove Theorem 4.5(ii). Assume E is complemented in $C(\Omega)$. It is known that any infinite dimensional complemented subspace E of a $C(\Omega)$-space, where Ω is a compact set, has a subspace isomorphic to c_0 [11, Proposition II.4.33]. By replacing E by a suitable ultrapower of E, E_1, we may assume E_1 has a complemented subspace isomorphic to an ultrapower of c_0 and E_1 is complemented in a space $C(\Omega')$. Now $C(\Omega')$ is finitely representable in c_0 so it is a subspace of an ultrapower of c_0; by Proposition 4.5, there is an ultrapower of $C(\Omega')$ isomorphic to a complemented subspace of an ultrapower of c_0. Finally, by replacing E_1 by an ultrapower E_2, we get the following situation:

E_2 has a complemented subspace isomorphic to $c_0^I/\mathfrak{U}$.

E_2 is isomorphic to a complemented subspace of $c_0^J/\mathfrak{V}$. We let $\mathfrak{W}$ be an ultrafilter on a set K, such that $(c_0^I/\mathcal{U})^K/\mathcal{W}$ and $(c_0^J/\mathcal{V})^K/\mathcal{W}$ are isometric; if we let $H = (c_0^I/\mathcal{U})^K/\mathcal{W}$, $E_3 = E_2^K/\mathcal{W}$, we get

E_3 is isomorphic to a complemented subspace of H.

H is isomorphic to a complemented subspace of E_3.

$c_0(H)$ is isomorphic to a complemented subspace of H (by Lemma 4.2(ii)).

By the Pelczyński decomposition method it follows that E_3 and H are isomorphic. But E_3 is an ultrapower of E and H is isomorphic to a $C(K)$-space; this finishes the proof.

5. Appendix. Proof of Theorems 2.2 and 2.3. In this appendix we consider structures of the following type:

$$\mathfrak{A} = \left(|\mathfrak{A}|; +^{\mathfrak{A}}, (q^{\mathfrak{A}})_{q\in\mathbf{Q}}, B^{\mathfrak{A}}\right)$$

where

$|\mathfrak{A}|$ is a set (the domain of $\mathfrak{A}$)

$+^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|^2$ to $|\mathfrak{A}|$.

$q^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ to $|\mathfrak{A}|$ for each rational number q.

$B^{\mathfrak{A}}$ is a subset of $|\mathfrak{A}|$.

The appropriate language L to discuss such structures includes

variables $(v_i)_{i\in\mathbf{N}}$.

a binary function symbol $+$.

for each $q \in \mathbf{Q}$, a unary function symbol $q\,\cdot$.

a unary predicate symbol B.

We assume the reader is familiar with model theory.

By a *normed* **Q**-*space* we mean a **Q**-vector space E with a mapping $\|x\|$ from E into $\mathbf{R}^+$ such that

$\|x + y\| \leq \|x\| + \|y\|$

$\|qx\| = |q|\,\|x\|$, $q \in \mathbf{Q}$, and

$\|x\| = 0$ if and only if x is 0.

It is clear that to any normed **Q**-space E, one can associate a structure $\mathfrak{A}(E)$: $+$ is interpreted by the addition in E, $q\,\cdot$ by the multiplication by q and B by the set of elements of norm at most 1.

On the other hand if $\mathfrak{B}$ is elementarily equivalent to $\mathfrak{A}(E)$, it is possible to associate to $\mathfrak{B}$ a unique normed-**Q**-space $[\mathfrak{B}]$ as follows: we first let $\Pi_{\mathfrak{B}} = \{x$: $x \in |\mathfrak{B}|$ and for some $q > 0$ $(\mathfrak{B}, x) \models B(q\cdot v)\}$. $\Pi_{\mathfrak{B}}$ is closed under $+^{\mathfrak{B}}$ as the formula

$$\forall v_1\, \forall v_2\left(B(q_1\cdot v_1) \wedge B(q_2\cdot v_2) \rightarrow B(q_3\cdot(v_1 + v_2))\right)$$

is true in $\mathfrak{A}(E)$ with $q_1 > 0$, $q_2 > 0$ and $q_3 = q_1q_2/(q_1 + q_2)$. $\Pi_{\mathfrak{B}}$ is also closed under $q^{\mathfrak{B}}$, $q \in \mathbf{Q}$. We now let

$$N_{\mathfrak{B}} = \left\{x\colon x \in |\mathfrak{B}| \text{ and for all } q > 0 (\mathfrak{B}, x) \models B(q\cdot v)\right\}.$$

$N_{\mathfrak{B}}$ is closed under $+^{\mathfrak{B}}$ as the formula

$$\forall v_1\, \forall v_2\, (B(2q \cdot v_1) \wedge B(2q \cdot v_2) \to B(q \cdot (v_1 + v_2)))$$

holds in $\mathfrak{A}(E)$. Also $N_{\mathfrak{B}}$ is closed under $q^{\mathfrak{B}}$. $[\mathfrak{B}]$ is the quotient space $\Pi_{\mathfrak{B}}/N_{\mathfrak{B}}$.

For any element x in $\Pi_{\mathfrak{B}}$ we let

$$\|x\| = \left[\sup\{q\colon q > 0 \text{ and } (\mathfrak{B}, x) \models B(q \cdot v)\}\right]^{-1}.$$

To prove $\|x + y\| \leqslant \|x\| + \|y\|$, pick $\varepsilon > 0$ and let q_1, q_2 such that $1/q_1 \leqslant \|x\| + \varepsilon$, $1/q_2 \leqslant \|y\| + \varepsilon$, and

(1) $(\mathfrak{B}, x) \models B(q_1 \cdot v)$,

(2) $(\mathfrak{B}, y) \models B(q_2 \cdot v)$.

From (1) and (2) we get

(3) $(\mathfrak{B}, x + y) \models B(q_2 q_1/q_1 + q_2 \cdot v)$

because the formula

$$\forall v_1\, \forall v_2 \left(B(q_1 \cdot v_1) \wedge B(q_2 \cdot v_2) \to B\left(\frac{q_1 q_2}{q_1 + q_2} \cdot v_1 + v_2\right)\right)$$

holds in $\mathfrak{A}(E)$.

(3) implies

$$\|x + y\| \leqslant \frac{q_1 + q_2}{q_1 q_2} \leqslant \|x\| + \|y\| + 2\varepsilon,$$

as ε is arbitrary we get $\|x + y\| \leqslant \|x\| + \|y\|$. $\|qx\| = |q|\, \|x\|$ for any $q \in \mathbf{Q}$. $\|x\| = 0$ if and only if x is in $N_{\mathfrak{B}}$.

Finally the space $\Pi_{\mathfrak{B}}/N_{\mathfrak{B}}$ endowed with $\|x\|$ is a normed-$\mathbf{Q}$-space $[\mathfrak{B}]$.

Lemma 5.1. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are isomorphic and elementarily equivalent to $\mathfrak{A}(E)$, then $[\mathfrak{B}]$ and $[\mathfrak{B}']$ are isometric.*

Remark. An isometry from a normed $\mathbf{Q}$-space onto another is a $\mathbf{Q}$-linear mapping which is norm preserving.

Proof. We let ϕ be an isomorphism from $\mathfrak{B}$ onto $\mathfrak{B}'$. ϕ is $\mathbf{Q}$-linear and maps $\Pi_{\mathfrak{B}}$ one-one onto $\Pi_{\mathfrak{B}'}$, and $N_{\mathfrak{B}}$ one-one onto $N_{\mathfrak{B}'}$; therefore it induces a $\mathbf{Q}$-linear bijection $\tilde{\phi}$ from $[\mathfrak{B}]$ onto $[\mathfrak{B}']$. To see that $\tilde{\phi}$ is norm preserving consider a given x in $\Pi_{\mathfrak{B}}$

$$\|x\| = \left[\sup\{q\colon q > 0 \text{ and } (\mathfrak{B}, x) \models B(q \cdot v)\}\right]^{-1}$$
$$= \left[\sup\{q\colon q > 0 \text{ and } (\mathfrak{B}', \phi(x)) \models B(q \cdot v)\}\right]^{-1} = \|\phi(x)\|.$$

Lemma 5.2. *Let E be a Banach space; $\mathfrak{U}$ an ω-incomplete ultrafilter on a set I; if E_0 is a $\mathbf{Q}$-vector space which is a dense subset of E, $[\mathfrak{A}(E_0)^I/\mathfrak{U}]$ is isometric to $E^I/\mathfrak{U}$.*

Remark. An ultrafilter $\mathfrak{U}$ is ω-incomplete if there exists a countable set of pairwise disjoint sets $(X_n)_{n \in \mathbf{N}}$ such that $X_n \notin \mathfrak{U}$ and $\bigcup_1^\infty X_n = I$.

Proof of the lemma. We let $\mathfrak{B}$ stand for $\mathfrak{A}(E_0)^I/\mathfrak{U}$, and we define a mapping $\psi\colon \Pi_{\mathfrak{B}} \to E^I/\mathfrak{U}$; assume x is the element of $\Pi_{\mathfrak{B}}$, $(x_i)_{i\in I}$, then, because $x \in \Pi_{\mathfrak{B}}$ there is a $q > 0$ such that $(\mathfrak{B}, x) \models B(q \cdot v)$; this means that $\{i\colon \|qx_i\| \leqslant 1\} \in \mathfrak{U}$ so that $(x_i)_{i\in I}$ represents an element in $E^I/\mathfrak{U}$; we let $\psi(x)$ be this element. Clearly ψ is linear. ψ is norm preserving: to see that, we have to show

$$\lim_{\mathfrak{U}} \|x_i\| = \left[\sup\{q\colon q > 0 \text{ and } (\mathfrak{B}, x) \models B(q \cdot v)\}\right]^{-1}$$

but we have

$$\begin{aligned}\sup\{q\colon q > 0 \text{ and } (\mathfrak{B}, x) \models B(q \cdot v)\} &= \sup\{q\colon \{i\colon \|qx_i\| \leqslant 1\} \in \mathfrak{U}\}\\ &= \sup\Big\{q\colon q \lim_{\mathfrak{U}} \|x_i\| \leqslant 1\Big\} = \Big(\lim_{\mathfrak{U}} \|x_i\|\Big)^{-1}.\end{aligned}$$

ψ is onto: let y be in $E^I/\mathfrak{U}$, and assume y is given by $(y_i)_{i\in I}$; let $(X_n)_{n\in \mathbf{N}}$ be a sequence of pairwise disjoint elements, $X_n \notin \mathfrak{U}$ and $\bigcup_1^\infty X_n = I$; for a given $i \in I$, let $n(i)$ be the unique integer such that $i \in X_{n(i)}$; for each i pick x_i in E_0 such that $\|x_i - y_i\| \leqslant 1/n(i)$. Clearly $(x_i)_{i\in I}$ is in the range of ψ and represents y as well as $(y_i)_{i\in I}$.

To finish the proof of the lemma, it is enough to observe that ϕ induces an isometry from $\Pi_{\mathfrak{B}}/N_{\mathfrak{B}}$ onto $E^I/\mathfrak{U}$.

We now give the proofs of Theorems 2.1 and 2.2 which we restate.

THEOREM 2.1. *Let E be a Banach space, $\mathfrak{U}$ an ultrafilter on a set I; $\mathfrak{V}$ an ultrafilter on a set J. There exist a set K and an ultrafilter $\mathfrak{W}$ on K such that the spaces $(E^I/\mathfrak{U})^K/\mathfrak{W}$ and $(E^J/\mathfrak{V})^K/\mathfrak{W}$ are isometric.*

PROOF. The structures $\mathfrak{A}(E)^I/\mathfrak{U}$ and $\mathfrak{A}(E)^J/\mathfrak{V}$ are both elementarily equivalent to $\mathfrak{A}(E)$, therefore, by the Keisler-Shelah theorem, there exists an ultrafilter $\mathfrak{W}$ on a set K such that $\mathfrak{A}(E)^{I\times K}/\mathfrak{U} \times \mathfrak{W}$ and $\mathfrak{A}(E)^{J\times K}/\mathfrak{V} \times \mathfrak{W}$ are isomorphic. By Lemma 5.1 it follows that $[\mathfrak{A}(E)^{I\times K}/\mathfrak{U} \times \mathfrak{W}]$ and $[\mathfrak{A}(E)^{J\times K}/\mathfrak{V} \times \mathfrak{W}]$ are isometric (as normed $\mathbf{Q}$-spaces). By Lemma 5.2, this means that $E^{I\times K}/\mathfrak{U} \times \mathfrak{W}$ and $E^{J\times K}/\mathfrak{V} \times \mathfrak{W}$ are isometric as normed $\mathbf{Q}$-spaces and therefore as Banach spaces. But $E^{I\times K}/\mathfrak{U} \times \mathfrak{W}$ is $(E^I/\mathfrak{U})^K/\mathfrak{W}$ and $E^{J\times K}/\mathfrak{V} \times \mathfrak{W}$ is $(E^J/\mathfrak{V})^K/\mathfrak{W}$ (by Proposition 2.1).

THEOREM 2.2. *Let E be a Banach space, F a separable subspace of E; there exists a separable subspace L, $F \subseteq L \subseteq E$, such that L and E have isometric ultrapowers. Furthermore if E is a Banach lattice, we may assume L is a sublattice of E.*

PROOF. Let F_0 be a $\mathbf{Q}$-vector space which is a dense countable subset of F. By the Lowenheim-Skolem theorem there exists a countable structure, $\mathfrak{B}$ which is an elementary substructure of $\mathfrak{A}(E)$ and whose domain $|\mathfrak{B}|$ contains F_0; if E is a lattice we may assume $|\mathfrak{B}|$ is closed under the lattice operations.

We let L be the closure of $|\mathfrak{B}|$ in E; L is a separable Banach space and if E is a lattice L is a closed sublattice of E. By the Keisler-Shelah theorem, there exists an ω-incomplete ultrafilter $\mathcal{U}$ on a set K such that $\mathfrak{B}^K/\mathcal{U}$ and $\mathfrak{A}(E)^K/\mathcal{U}$ are isomorphic. By Lemma 5.1, it follows that $[\mathfrak{B}^K/\mathcal{U}]$ and $[\mathfrak{A}(E)^K/\mathcal{U}]$ are isometric. By Lemma 5.2 it means that $L^K/\mathcal{U}$ and $E^K/\mathcal{U}$ are isometric. Q.E.D.

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