STRONG DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

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Abstract. Let $F$ be a differentiation basis in $\mathbb{R}^n$, i.e., a family of measurable sets $S$ contracting to 0 such that $\|M_{\mathcal{F}f}\|_p < A_p\|f\|_p$, where $M_{\mathcal{F}}$ is the Hardy-Littlewood maximal operator. For $f \in L^p_\mathcal{F}$, we let $E_F(f)$ be the complement of the Lebesgue set of $f$ relative to $F$, and we show that $E_F$ has $L^q_\mathcal{F}$-capacity 0, where $L^q_\mathcal{F}$ is a capacity associated with $L^p_\mathcal{F}$ in much the same way as the Bessel capacity $B^r_\lambda$ is associated with $L^r_\lambda$.

1. With the Bessel potential space $L^p_\alpha(R^n)$ there is associated the Bessel capacity $B^r_\lambda$ which is defined for $F \subset \mathbb{R}^n$ by

$$B^r_\lambda(E) = \inf \{ \|g\|_{L^p_\alpha}; g \in L^p_\alpha, g > 0, g \ast g > 1 \text{ on } E \},$$

where $G_\alpha$ is the Bessel kernel given by $G_\alpha(x) = (1 + 4\pi^2|x|^2)^{-\alpha/2}$ [5, p. 132]. The capacity $B_{\mathcal{F}}$ is an outer measure on $\mathbb{R}^n$ and its relation to $H^r$, Hausdorff measure of dimension $r$, is given by the following [4]. If $p > 1$, $\alpha p < n$, then $H^{n-\alpha p}(F) = 0$ implies $B_{\mathcal{F}}(E) = 0$, and $B_{\mathcal{F}}(E) = 0$ implies $H^{n-\alpha p+\epsilon}(E) = 0$ for every $\epsilon > 0$.

Let $F$ be a family of measurable sets $S \subset \mathbb{R}^n$ with $0 < |S| < \infty$, and let $S \to 0$ stand for the generic notation for the limit as $j \to \infty$ of any sequence $\{S_j\} \subset F$ such that, for any $\epsilon > 0$, $S_j \subset \{|x| < \epsilon\}$, $j \geq j(\epsilon)$. With such a family we associate the Hardy-Littlewood maximal operator

$$M_{\mathcal{F}f}(x) = \sup \left\{ \frac{1}{|S|} \int_{S+x} |f(y)| \, dy : S \in F \right\}.$$

The behavior of $M_{\mathcal{F}f}$ is decisive in the study of the differentiability of the integral. If, for example, $\|M_{\mathcal{F}f}\|_p < A_p\|f\|_p$ (there are many interesting families $F$ with this property [3]), then the set

$$E_F(f) = \left\{ x : \frac{1}{|S|} \int_{S+x} |f(y) - f(x)| \, dy \not\to 0 \text{ as } S \to 0 \right\}$$

has measure 0 for $f \in L^p$. If $f \in L^p_\alpha$, then as shown in [2] the exceptional set $E_F(f)$ has even $B_{\mathcal{F}}$-capacity 0.

The purpose of this paper is to prove an analogous result for the Lipschitz spaces $\Lambda^p_\alpha(R^n)$ (see [5], [6]), using instead of $B_{\mathcal{F}}$ a "Lipschitz" capacity $L^p_\alpha$. 

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In view of the inclusion relations

\[ L_a^p \subset \Lambda_{a}^{pq} \subset L_b^p, \quad p > 1, \]

\[ r = \max(p, 2), \quad q > r, \quad \beta < \alpha \]

our differentiability result lies then intermediate to those of [2].

We recall that, if \( F \) is the family of all oriented rectangles, a nonregular family, then \( \|M_Ff\|_p < A_p\|f\|_p, \quad 1 < p < \infty \) [5]. For regular families \( F \), i.e., those for which there is \( c > 0 \) such that every \( E \in F \) lies in a sphere \( S \) with \( |S| < c|E| \), the size of \( E_F(f) \) for \( f \in \Lambda_{a}^{pq} \) can be studied with the techniques developed in [8], especially the mean value property \( \int_{|y| < r} G_a(x - y) \, dy < cG_a(x)r^\alpha \). One sees easily that this tool is no longer available for nonregular families, and its substitute \( \|M_Ff\|_p < A_p\|f\|_p \), which is not a condition on the kernel \( G_a \), requires one to study the interplay between \( \Lambda_{a}^{pq} \) and \( M_F \).

2. Let \( F \) be a family as in \$1\$, and let \( tF = \{tS: S \in F\}, \) \( 0 < t < 1 \), where \( tS = \{ts: s \in S\} \). Let

\[ M_f(x) = \sup \left\{ \frac{1}{|S|} \int_{S+x} |f(y)| \, dy: S \in F \right\}. \]

**Lemma 1.** If \( \|M_Ff\|_p < A_p\|f\|_p, \quad f \in L^p \), then \( \|M_f\|_p < A_p\|f\|_p, \quad 0 < t < 1 \).

**Proof.**

\[ |ts|^{-1} \int_{S+x} f(u) \, du = |s|^{-1} \int_{S+x/t} f(tv) \, dv < M_F(\delta f)(x/t), \]

where \( (\delta f)(x) = f(tx) \). Hence

\[ \|M_f\|_p < \left\{ \int_{R^n} M_F(\delta f)(\frac{x}{t}) \, dx \right\}^{1/p} \]

\[ = \left\{ t^n \int_{R^n} M_F(\delta f)(u)^p \, du \right\}^{1/p} < t^n/p A_p \left\{ \int_{R^n} (\delta f)(x)^p \, dx \right\}^{1/p} \]

\[ = A_p\|f\|_p. \]

We note that \( A_p \) is independent of \( t \).

For \( 0 < \alpha \leq 1 \), the Lipschitz space \( \Lambda_{a}^{pq}(R^n) \) consists of all functions \( f \in L^p(R^n) \) for which the norm

\[ \|f\|_{\Lambda_{a}^{pq}} \]

is finite [5, p. 151]; here \( 1 < p, \quad q < \infty \). If \( 0 < \alpha < 1 \), the above norm is equivalent to

\[ \|f\|_{\Lambda_{a}^{pq}} \]

\[ \|f\|_{p} + \left\{ \int_{R^n} \frac{\|f(x + t) + f(x - t) - 2f(x)\|_p^q}{|t|^{n+aq}} \, dt \right\}^{1/q} \]

is finite [5, p. 151]; here \( 1 < p, \quad q < \infty \). If \( 0 < \alpha < 1 \), the above norm is equivalent to

\[ \|f\|_{p} + \left\{ \int_{R^n} \frac{\|f(x + t) - f(x)\|_p^q}{|t|^{n+aq}} \, dt \right\}^{1/q} \]
If $\alpha > 1$, $\Lambda^q_\alpha$ is the collection of $f \in L^p(R^n)$ for which the norm $\|f\|_p + \sum_{j=1}^n |\partial f/\partial x_j|_{L^q_{\alpha^{-1}}} < \infty$ [5, p. 153], where $\partial f/\partial x_j$ is taken in the sense of distribution.

**Lemma 2.** Let $0 < \alpha < 1$ and $\|M_f\|_p < A_p \|f\|_p$, $f \in L^p$. Then $\|M_f\|_{L^q_\alpha} < A_p \|f\|_{L^q_\alpha}$.

**Proof.** By Lemma 1 we only need to show that $\|M_f\|_{L^q_\alpha} < A_p \|f\|_{L^q_\alpha}$. If $f(x) = f(x + t)$, one easily verifies that $\|M_f(x) - M_f(x)\|_{L^q_\alpha} < M \|f(x) - f(x)\|_{L^q_\alpha}$, and the result follows from (ii).

3. If $G_\alpha$ is the Bessel potential of order $\alpha > 0$, and if $J_\alpha f = G_\alpha \ast f$, then $J_\alpha : \Lambda^q_\beta \to \Lambda^q_{\alpha+\beta}$ is an isomorphism and the norms $\|f\|_{L^q_\alpha}$, $\|J_\alpha f\|_{L^q_{\alpha+\beta}}$ are equivalent [5, Chapter 5]. This result will be used frequently in the sequel.

For $0 < \alpha < \infty$, $1 < p, q < \infty$, we define, for $E \subset R^n$,

$$L^p_\alpha(E) = \inf \{\|g\|_{L^q_\alpha} : g > 0 \text{ and } g > 1 \text{ on } E\}.$$

For $0 < \gamma < \alpha$ we define, for $E \subset R^n$,

$$B^q_\gamma(E) = \inf \{\|g\|_{L^q_\gamma} : g > 0 \text{ and } G_\alpha^{-\gamma} \ast g > 1 \text{ on } E\}.$$

It is easily verified that $L^p_\alpha$, $B^q_\gamma$ are capacities, i.e., they are monotone, countably subadditive, and assign 0 to $E = \emptyset$ (see [4, p. 251]).

The usefulness of $B^q_\gamma$ is exhibited in the following lemma.

**Lemma 3.** $L^p_\alpha(E) = 0$ if and only if there exists $0 < \gamma < \alpha$ such that $B^q_\gamma(E) = 0$.

**Proof.** ($\Leftarrow$) Let $0 < \gamma < \alpha$ and let $\varepsilon > 0$. Choose $g \in \Lambda^q_\alpha$ so that $g > 0$, $g > 1$ on $E$, and $\|g\|_{L^q_\alpha} < \varepsilon$. If $g = G_\alpha^{-\gamma} \ast \psi$, $\psi \in \Lambda^q_\gamma$, then $\|\psi\|_{L^q_\gamma} < K\varepsilon$, and $B^q_\gamma(E) = 0$.

($\Rightarrow$) If $B^q_\gamma(E) = 0$, then there is $g > 0$ in $\Lambda^q_\gamma$ with $G_\alpha^{-\gamma} \ast g > 1$ on $E$ and $\|g\|_{L^q_\gamma} < \varepsilon$. As before $\|G_\alpha^{-\gamma} \ast g\|_{L^q_\gamma} < K\varepsilon$, and $L^p_\alpha(E) = 0$.

**Lemma 4.** Let $1 < p < \infty$. The relation between the Bessel capacity $B^q_\gamma$ and the Lipschitz capacity $L^p_\alpha$ is given by

(i) $B^q_\gamma(E) = 0$ implies $L^p_\alpha(E) = 0$ if $q > \max(p, 2)$.

(ii) $L^p_\alpha(E) = 0$ implies $B^q_\gamma(E) = 0$, $0 < \gamma < \alpha$.

**Proof.** (i) We use Lemma 3 and verify that $B^q_\gamma(E) = 0$ for $0 < \gamma < \alpha$. Since $B^q_\gamma(E) = 0$, we have $g > 0$ in $L^p$ such that $\|g\|_p < \varepsilon$ and $G_\alpha \ast g(x) > 1$, $x \in E$. If $\psi = G_\gamma \ast g$, then $\|\psi\|_{L^q_\gamma} < M \|g\|_p < M \cdot \varepsilon$ [6, p. 452].

(ii) If $0 < \gamma < \alpha$, we have $B^q_\gamma(E) = 0$, and hence there is $g > 0$ with $\|g\|_{L^q_\gamma} < \varepsilon$ and $G_\gamma \ast g(x) > 1$, $x \in E$. Since $\|g\|_p < \varepsilon$, we get $B^q_\gamma(E) = 0$.

**Corollary.** If $L^p_\alpha(E) = 0$ and $\alpha p > 1$, then $H^{n-1}(E) = 0$.

**Proof.** Let $0 < \gamma < \alpha$ with $\gamma p > 1$. Since $B^q_\gamma(E) = 0$, we see from [4, Theorem 22] that $H^{n-1}(E) = 0$. 


The next lemma shows that $f \in \Lambda^p_\alpha$ can be defined absolutely modulo sets of $L^p_\alpha$-capacity 0.

**Lemma 5.** Let $f \in \Lambda^p_\alpha$, $0 < \gamma < \alpha$, and $f = G_{\alpha - \gamma} \ast \psi$, $\psi \in \Lambda^p_\alpha$. Then $G_{\alpha - \gamma} \ast |\psi|(x) < \infty$ for $L^p_\alpha$-a.e. $x$.

**Proof.** Let $E = \{x: G_{\alpha - \gamma} \ast |\psi|(x) = \infty\}$. Then

\[ B^p_\gamma(E) < B^p_\gamma \left\{ x: G_{\alpha - \gamma} \ast |\psi|(x) > k \right\} \]

\[ = B^p_\gamma \left\{ x: G_{\alpha - \gamma} \ast \frac{|\psi|}{k}(x) > 1 \right\} < \frac{1}{k} \|\psi\|_{\Lambda^p_\alpha} \to 0 \quad \text{as } k \to \infty. \]

The result now follows from Lemma 3.

**Lemma 6.** Let $\psi_j \to f(\Lambda^p_\alpha)$. Then there exists a subsequence $\{\psi_{j_i}\}$ such that $\psi_{j_i} \to f$ for $L^p_\alpha$-a.e. $x$.

**Proof.** The proof is standard and we give it for the sake of completeness. Let $0 < \gamma < \alpha, f = G_{\alpha - \gamma} \ast g, \psi_j = G_{\alpha - \gamma} \ast \phi_j, g, \phi_j \in \Lambda^p_\alpha$. Then

\[ B^p_\gamma \left\{ x: |\psi_j - f|(x) > \epsilon \right\} < B^p_\gamma \left\{ x: G_{\alpha - \gamma} \ast (|\phi_j - g|/\epsilon)(x) > 1 \right\} \]

\[ < \|\phi_j - g\|_{\Lambda^p_\alpha}/\epsilon \to 0 \quad \text{as } j \to \infty. \]

If we select now $\{j_i\}$ such that for $A_i = \{x: |\psi_{j_i} - f|(x) > 1/2^i\}$ we have $B^p_\gamma(A_i) < 1/2^i$, then on $A = \cap_{k=1}^\infty \bigcup_{i \geq k} A_i, \psi_{j_i} \to f$ and $B^p_\gamma(A) = 0$.

**Lemma 7.** If $\|M_Ff\|_p < A_p\|f\|_p$ and $0 < \alpha < 1$, then $\|\int_0^1 M_Fg(x) \, dt\|_{\Lambda^p_\alpha} < A_p\|g\|_{\Lambda^p_\alpha}$ where $M_F$ and $M_t$ are the maximal operators associated with $F$ and $tF$.

**Proof.** Let $\psi(x) = \int_0^1 M_Fg(x) \, dt$. Then $\|\psi\|_p < A_p\|g\|_p$ (Lemma 1), and

\[ \|\psi(x + \tau) - \psi(x)\|_p \leq \int_0^1 \|M_t(g_\tau - g)\|_p \, dt \]

\[ < A_p\|g(x + \tau) - g(x)\|_p, \]

where $g_\tau(x) = g(x + \tau)$.

From Lemma 5 we deduce that for $g \in \Lambda^p_\alpha, M_Fg(x) \in L^1(dt, [0, 1])$ for $L^p_\alpha = \text{a.e. } x$.

4. We are now ready to state and prove our differentiability results. We let $F$ be a family of sets as in §1, and we let $tF = \{tS: S \in F\}, 0 < t < 1$.

**Theorem 1.** Assume that for $f \in L^p(R^n), \|M_Ff\|_p < A_p\|f\|_p$. For $f \in \Lambda^p_\alpha(R^n)$, let
E_\eta(f) = \left\{ x: \limsup_{S \to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| \, dy > \eta \right\}.

Then L^{pq}_\alpha(E_\eta) = 0.

Proof. We have to verify that, for \eta > 0, L^{pq}_\alpha(E_\eta) = 0, where

E_\eta = \left\{ x \in E: \lim sup_{S \to 0} \frac{1}{|tS|} \int_{tS+x} |f(y) - f(x)| \, dy > \eta \right\}.

Let 0 < \beta < \min(1, \alpha). By Lemma 3 we need to show that \( B^{pq}_\beta(E) = 0. \) Let \eta > 0 be given.

Let \( C^\infty_0(R^n) \) be the space of infinitely differentiable functions with compact support. By [6, p. 444] there is a sequence \( \{\psi_j\} \subset C^\infty_0 \) such that \( \psi_j \to f(A^{pq}_\alpha) \).

Choose now \( \psi^\beta, f^\beta \in \Lambda^{pq}_\beta \) with \( G_{\alpha-\beta} \ast \psi^\beta = \psi_j \) and \( G_{\alpha-\beta} \ast f^\beta = f \). Then \( \psi^\beta \to f^\beta(\Lambda^{pq}_\beta) \), and from this we get that

\[ B^{pq}_\beta \left\{ x: |\psi - f(x)| > \delta \right\} < \|\psi^\beta - f^\beta\|_{\Lambda^{pq}_\beta} / \delta = o(1) \quad \text{as} \quad j \to \infty. \]

If \( \Sigma = tS \), we have

\[ \frac{1}{|\Sigma|} \int_{\Sigma + x} |f(y) - f(x)| \, dy < \frac{1}{|\Sigma|} \int_{\Sigma + x} |f(y) - \psi_j(y)| \, dy + \frac{1}{|\Sigma|} \int_{\Sigma + x} |\psi_j(y) - \psi_j(x)| \, dy + |\psi_j(x) - f(x)| = A_j(x) + A_2(x) + A_3(x). \]

(i) Since \( \{ x: M_i(f - \psi_j)(x) > \sigma/3 \} \subset \{ x: G_{\alpha-\beta} \ast M_i(f^\beta - \psi^\beta_j)(x) > \sigma/3 \} \), we see that

\[ B^{pq}_\beta \left\{ x: M_i(f - \psi_j)(x) > \sigma/3 \right\} < (3/\sigma)\|M_i(f^\beta - \psi^\beta_j)\|_{\Lambda^{pq}_\beta} < (3/\sigma)A_j \|f^\beta - \psi^\beta_j\|_{\Lambda^{pq}_\beta} < \eta/2, \quad \text{if} \quad j > j_1(\eta). \]

The next to the last inequality follows from Lemma 2 (applicable since \( \beta < 1 \)). Since \( A_j(\Sigma) < M_i(f - \psi_j)(x) \), there is a set \( E_{A_j} \) such that \( B^{pq}_\beta(E_{A_j}) < \eta/3, j > j_1(\eta), \) and \( x \notin E_{A_j} \) implies \( A_j(x) < \sigma/3. \)

(ii) Since \( B^{pq}_\beta \left\{ x: A_3(\Sigma) > \sigma/3 \right\} < (3/\sigma)\|\psi^\beta_j - f^\beta\|_{\Lambda^{pq}_\beta} \), there is \( j_2(\eta) \) such that for \( j > j_2(\eta), A_3(\Sigma) < \sigma/3 \) except for \( x \in E_{A_3} \) with \( B^{pq}_\beta(E_{A_3}) < \eta/2. \)

(iii) Let now \( j_0 > \max(j_1(\eta), j_2(\eta)) \). Then there is \( \tau(\sigma) \) such that \( S \subset B(0, \tau) \) implies \( |\psi_{\alpha_j}(y) - \psi_{\alpha_j}(x)| < \sigma/3, y \in tS + x \). If then \( x \notin E_{A_j} \cup E_{A_3} \) and \( S \subset B(0, \tau) \), we get \( |\Sigma|^{-1} \int_{\Sigma + x} |f(y) - f(x)| \, dy < \sigma, \Sigma = tS, \) and hence \( E_\eta \subset E_{A_j} \cup E_{A_3}. \) Then \( B^{pq}_\beta(E_\eta) < \eta, \) and the proof is complete.

**Theorem 2.** Under the same hypothesis as for Theorem 1, we have for \( L^{pq}_a \)-a.e. \( x, \)**
\[ \lim_{S \to 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)| \, dy = 0, \quad a.e. \, t, \quad 0 < t < 1. \]

(ii) \[ \lim_{S \to 0} \int_0^1 \frac{1}{|S|} \int_{S+x} |f(y) - f(x)| \, dy \, dt = 0. \]

PROOF. (i) Let
\[ E = \left\{ (t, x): \limsup_{S \to 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)| \, dy > 0 \right\}, \]
and let \( E_x = \{ t: (t, x) \in E \} \). We must show that \( |E_x| = 0 \) for \( L_\alpha^{pq} \)-a.e. \( x \).

Let \( A_\sigma = \{ x: |E_x| > \sigma \} \). Let \( 0 < \beta < \min(1, \alpha) \) and choose \( \{ \psi_j \} \subset C_0^\infty \) so that \( \psi_j \to f(A_\alpha^{pq}) \). By Lemma 6 we may assume that \( \psi_j(x) \to f(x) \) for \( L_\alpha^{pq} \)-a.e. \( x \). Let \( E_1 \) be the exceptional set, i.e., \( L_\alpha^{pq}(E_1) = 0 \) and \( x \notin E_1 \) implies \( \psi_j(x) \to f(x) \).

We need another exceptional set \( E_2 \). Since \( M_t(f - \psi_j)(x) \leq G_{\alpha - \beta} * M_t(f^\beta - \psi^\beta)(x) \) (notation as in the proof of Theorem 1), and \( \psi_j \to f^\beta \) (notation as in the proof of Theorem 1), we see that
\[ \int_0^1 M_t(f - \psi_j)(x) \, dt \leq G_{\alpha - \beta} * \int_0^1 M_t(f^\beta - \psi^\beta)(x) \, dt = \Psi_j(x). \]

By Lemma 7, \( f_0 M_t(f^\beta - \psi^\beta)(x) \, dt \in A_\alpha^{pq} \), and hence \( \Psi_j \in A_\alpha^{pq} \).

Next
\[ \left\{ x: \int_0^1 M_t(f - \psi_j)(x) \, dt > \epsilon \right\} \subset \left\{ x: G_{\alpha - \beta} * \frac{1}{\epsilon} \int_0^1 M_t(f^\beta - \psi^\beta)(x) \, dt > 1 \right\}, \]
and hence
\[ B_\epsilon^{pq} \left\{ x: \int_0^1 M_t(f - \psi_j)(x) \, dt > \epsilon \right\} < \frac{1}{\epsilon} \left\| \int_0^1 M_t(f^\beta - \psi^\beta)(x) \, dt \right\|_{A_\epsilon^{pq}} \]
\[ < \frac{A_\epsilon}{\epsilon} \left\| f^\beta - \psi^\beta \right\|_{A_\epsilon^{pq}} = o(1) \quad \text{as } j \to \infty. \]

We can now choose a subsequence \( \{ j_i \} \) so that \( \int_0^1 M_t(f - \psi_j)(x) \, dt \to 0 \) for \( L_\alpha^{pq} \)-a.e. \( x \) (see e.g. the proof of Lemma 6). We may assume that \( \{ j_i \} = \{ j \} \).

Let
\[ E_2 = \left\{ x: \int_0^1 M_t(f - \psi_j)(x) \, dt \to 0 \right\}. \]

Then \( L_\alpha^{pq}(E_2) = 0 \).

We return now to the set \( A_\sigma \) introduced at the beginning of the proof, and we claim that, for every \( \sigma > 0 \), \( A_\sigma \subset E_1 \cup E_2 \).

If we deny this, then there is \( x_0 \in A_\sigma \) and \( x_0 \notin E_1 \cup E_2 \). Then \( |E_{x_0\lambda}| > \sigma \). If
\[ E_{x_0\lambda} = \left\{ t: \limsup_{S \to 0} \frac{1}{|S|} \int_{S+x_0} |f(y) - f(x_0)| \, dy > \lambda \right\}, \]
then there is \( \lambda > 0 \) with \( |E_{x_0}| > \sigma \). We choose now \( j_0 \) so large that

(i) \( \int_0^1 M_t(f - \psi_{j_0})(x_0) \, dt < \sigma \lambda / 3 \),
(ii) \( |\psi_{j_0}(x_0) - f(x_0)| < \lambda / 3 \),

and we choose \( \tau_0 > 0 \) such that \( S \subset B(0, \tau_0) \) implies

(iii) \( |\psi_{j_0}(y) - \psi_{j_0}(x_0)| < \lambda / 3 \), \( y \in tS + x_0 \), \( 0 < t < 1 \).

Since for \( t \in E_{x_0} \),

\[
\lambda < M_t(f - \psi_{j_0})(x_0) + \frac{2\lambda}{3}, \quad S \subset B(0, \tau_0),
\]

we see that \( E_{x_0} \subset \{ t : M_t(f - \psi_{j_0})(x_0) > \lambda / 3 \} \). From this \( |E_{x_0}| < (3/\lambda) \int_0^1 M_t(f - \psi_{j_0})(x_0) \, dt < \sigma \), a contradiction.

To prove (ii), we first observe that \( \int_0^1 M_t^\beta f(x) \, dt \in \Lambda_\beta^p \) (Lemma 7), and hence \( G_{a-\beta} \int_0^1 M_t^\beta f(x) \, dt < \infty \) for \( L_{pq}^\alpha \)-a.e. \( x \) (Lemma 5). Since \( M_t f(x) < G_{a-\beta} M_t^\beta f(x) \), we see that \( M_t f(x) \in L^1(dt, [0, 1]) \) for \( L_{pq}^\alpha \)-a.e. \( x \). Finally,

\[
\left| f(x) \right| + M_t f(x) > \frac{1}{|S|} \int_{tS + x} |f(y) - f(x)| \, dy \to 0 \quad \text{as} \quad S \to 0,
\]

and we only need to apply the Lebesgue dominated convergence theorem to obtain (ii).

**Corollary.** Under the same hypothesis,

\[
\lim_{S \to 0} \int_0^1 \cdots \int_0^1 \frac{1}{|t_1 t_2 \cdots t_k S|} \int_{t_1 t_2 \cdots kS + x} |f(y) - f(x)| \, dy \, dt_1 \cdots dt_k = 0
\]

for \( L_{pq}^\alpha \)-a.e. \( x \).

5. In this section we will study higher order differentiability. Let \( F \) be a differentiation basis as in §1, and let \( \delta(S) \) denote the diameter of \( S \). We say that a function \( f \in L^p(R^n) \) is in \( t_k^\alpha(x_0) \) with respect to \( F \) if there exists a polynomial \( \Pi_{x_0}(y) \) of degree \( < k \) such that

\[
\left\{ \frac{1}{|S|} \int_{S + x_0} |f(y) - \Pi_{x_0}(y)|^p \, dy \right\}^{1/p} = o(\delta(S)^k), \quad \text{as} \quad S \to 0.
\]

If \( F \) is the family of all balls centered at the origin, this notion is due to Calderón and Zygmund [1], and for a general \( F \) was introduced in [2].

**Theorem 3.** Let \( \|M_t f\|_p \leq A_p \|f\|_p \). If \( f \in \Lambda_{\alpha}^p \), and \( k \) is a nonnegative integer \( < \alpha \), then \( f \in t_k^\alpha(x_0) \) with respect to \( F \) for \( L_{p}^{\alpha} \)-a.e. \( x \).

Of course, the special case \( k = 0 \) is Theorem 1. The proof of Theorem 3 proceeds along the lines of the corresponding theorem for \( L^p \) [2]. We shall see that with the help of Theorem 2 it is possible to omit the hypothesis \( tF \subset F \), \( 0 < t < 1 \), made in [2]. We need

**Lemma 8.** Let \( f \in \Lambda^p_\alpha \), \( 1 < \alpha < \infty \). Then for \( L_{p\alpha}^\alpha \)-a.e. \( x \), \( f \) is absolutely
continuous on $H^{n-1}$-a.e. ray from $x$, and on such a ray fix $f(x + z) = f(x) = \int f(x + tz) \cdot z \, dt$.

PROOF. The proof is precisely the same as the proof of the corresponding lemma for $L^p_\alpha$ [2] as soon as we establish the existence of a sequence \( \{f_j\} \subset C_0^\infty \) such that for $L^p_\alpha$-a.e. $x$,

\[ G_1 * |\nabla f_j - \nabla f_j(x)| \to 0 \quad \text{as} \quad j \to \infty. \]

Let $0 < \gamma < \min(1, \alpha - 1)$, and choose $g \in \Lambda_\alpha^{p_\alpha}$ such that $f = G_{\alpha - \gamma} * g$. Let \( \{f_j\} \subset C_0^\infty \) such that $f_j \to f(A_{p_\alpha}^\alpha)$, and let $g_i \in \Lambda_\alpha^{p_\alpha}$ with $f_i = G_{\alpha - \gamma} * g_i$. Then $g_i \to g(\Lambda_\alpha^{p_\alpha})$. By Lemma 6, $f_k(x) \to f(x)$ for $L^p_\alpha$-a.e. $x$, for some subsequence \( \{f_k\} \). Since $\alpha > 1$, $\nabla f_k \to \nabla f(\Lambda_\alpha^{p_\alpha})$, and hence $\nabla f_k - \nabla f = G_{\alpha - \gamma - 1} * h_k$, $h_k \to 0$ (\( \Lambda_\alpha^{p_\alpha} \)). Since $\gamma < 1$, we easily see that $h_k \to 0$ (\( \Lambda_\alpha^{p_\alpha} \)), and hence for some subsequence \( \{h_j\} \), $G_{\alpha - \gamma} * h_j(x) \to 0$ for $L^p_\alpha$-a.e. $x$. Since $G_1 * G_{\alpha - \gamma - 1} * h_j(x) = G_{\alpha - \gamma} * h_j(x)$, the sequence \( \{f_j\} \) has the desired property.

Proof of Theorem 3. Since $\alpha < 1$ is Theorem 1, we may assume that $1 < \alpha$. Let $f \in \Lambda_\alpha^{p_\alpha}$, and let $\Pi_\alpha(y) = \sum_{0<|\beta|<\alpha} (D^\beta f(x)/\beta!)(y - x)^\beta$; we observe that $D^\beta f \in \Lambda_\alpha^{p_\alpha}$. By Lemma 6, $f_k(x) \to f(x)$ for $L^p_\alpha$-a.e. $x$, for some subsequence \( \{f_k\} \). Since $\alpha > 1$, $\nabla f_k \to \nabla f(\Lambda_\alpha^{p_\alpha})$, and hence $\nabla f_k - \nabla f = G_{\alpha - \gamma - 1} * h_k$, $h_k \to 0$ (\( \Lambda_\alpha^{p_\alpha} \)). Since $\gamma < 1$, we easily see that $h_k \to 0$ (\( \Lambda_\alpha^{p_\alpha} \)), and hence for some subsequence \( \{h_j\} \), $G_{\alpha - \gamma} * h_j(x) \to 0$ for $L^p_\alpha$-a.e. $x$. Since $G_1 * G_{\alpha - \gamma - 1} * h_j(x) = G_{\alpha - \gamma} * h_j(x)$, the sequence \( \{f_j\} \) has the desired property.

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and the corollary to Theorem 2 proves (ii).

If now \( r \) is a multi-index, \(|r| = k - 2\), we get as before

\[
\int_0^1 \frac{1}{\delta(\tau S)} \frac{1}{|rS|} \int_{rS+x} |D^\eta R_x(z)| \; dz \; d\tau
\]

and hence for any \( S \),

\[
\frac{1}{\delta(S)} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)| \; dz < \int_0^1 \frac{1}{|rS|} \int_{rS+x} |\nabla (D^\eta R_x)(z)| \; dz \; dt.
\]

If we divide this by \( \delta(S) > \delta(tS) \), we finally obtain

\[
\frac{1}{\delta(S)^2} \frac{1}{|S|} \int_{S+x} |D^\eta R_x(z)| \; dz < \int_0^1 \frac{1}{|rS|} \int_{rS+x} |\nabla (D^\eta R_x)(z)| \; dz \; dt.
\]

We apply now (ii).

For the step treating a multi-index of length \( k - 3 \), we need to know that for \( L_{a-k}^\infty \text{-a.e. } x \)

\[
\int_0^1 \frac{1}{\delta(\alphaS)}^2 \frac{1}{|\alphaS|} \int_{\alphaS+x} |D^\eta R_x(z)| \; dz \; d\alpha \to 0
\]

as \( S \to 0 \), \( S \in F \). This integral is majorized by

\[
\int_0^1 \int_0^1 \frac{1}{\delta(t\alphaS)} \frac{1}{|t\alphaS|} \int_{t\alphaS+x} |\nabla (D^\eta R_x)(z)| \; dz \; dt \; d\alpha
\]

\[
< \sum \int_0^1 \int_0^1 \frac{1}{\delta(t\alphaS)} \frac{1}{|t\alphaS|} \int_{t\alphaS+x} |\nabla (D^\eta R_x)(z)| \; dz \; dt \; d\alpha \; d\tau,
\]

where \( \Sigma \) extends over all \( \rho \) with \(|\rho| = k - 1\). By the corollary to Theorem 2 this is \( o(1) \) as \( S \to 0 \), \( S \in F \), for \( L_{a-k}^\infty \text{-a.e. } x \). The proof of Theorem 3 is now complete.

6. We let \( F \) be a family as in §1 and we shall assume now that

(i) \( \|M_Ff\|_p < A_p \|f\|_p \), \( 1 < p < \infty \).

The family of all oriented rectangles is an example satisfying (i). If (i) holds, then for a.e. \( x \),

\[
\lim_{S \to 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r \; dy = 0,
\]
$f \in L^p$ and $1 < r < p$. This is no longer true if $r = p$ as the example due to Saks [7] shows.

If $f \in \mathcal{L}^p$, then $f \in L^p$, $\alpha > \beta$, and hence by Sobolov's theorem $f \in L^r$, where $1/r = 1/p - \beta/n > 1/p - \alpha/n$. Here, of course, we assume that $\alpha p < n$. As in Theorem 1, we wish to study the size of the exceptional set (for which (ii) does not hold) under the hypothesis that $f \in \mathcal{L}_\alpha^p$. For this purpose we need a lemma.

**Lemma 9.** Let $f \in \mathcal{L}_\alpha^p$, $0 < \alpha < 1$, and let $1 < r < p$. Then

$$\|f\|_{\mathcal{L}_\alpha^r} \leq A_r \|f\|_{L^p}^{r-1} \|f\|_{\mathcal{L}_\alpha^p}^r.$$

**Proof.** Recall that

$$\|f\|_{\mathcal{L}_\alpha^r} = \|f\|_p + \left\{ \int_{R^n} \frac{|f(x + t) - f(x)|^r}{|t|^{n+\alpha r}} \, dt \right\}^{1/r}.$$ 

Now $|f(x + t)|^r - |f(x)|^r = (|f(x + t)| - |f(x)|) r \cdot \chi^{r-1}$, where $\chi$ is between $|f(x + t)|$, $|f(x)|$. Hence, if $\psi(x, t) = \max(|f(x + t)|, |f(x)|)$, we get

$$\|f(x + t)|^r - |f(x)|| < r |f(x + t) - f(x)|\psi(x, t)^{r-1}.$$ 

For $\beta < p$ we have

$$\|f(x + t)|^r - |f(x)|| < r \left\{ \int |f(x + t) - f(x)|^r \psi(x, t)^{\beta(r-1)} \, dx \right\}^{1/\beta}.$$ 

If we let $s = p/\beta$, $1/t = 1 - 1/s = (p - \beta)/p$, and apply Hölder's inequality, we get

$$\|f(x + t)|^r - |f(x)|| < r \left[ \left\{ \int |f(x + t) - f(x)|^s \, dx \right\}^{\beta/p} \left\{ \int \psi(x, t)^{\beta(r-1)p/(p-\beta)} \, dx \right\}^{(p-\beta)/p} \right]^{1/\beta}.$$ 

We let now $\beta = p/r$. Then $\beta(r - 1)p/(p - \beta) = p$, and $p \cdot (p - \beta)/p\beta = r - 1$, and hence

$$\|f(x + t)|^r - |f(x)||_{p/r} < r \|f(x + t) - f(x)\|_p \|\psi(x, t)\|_{\mathcal{L}_\alpha^p}.$$ 

Since $\psi(x, t) < |f(x)| + |f(x + t)|$, we see that $\|\psi(x, t)\|_p < 2\|f\|_p$. Since, clearly, $\|f\|_{p/r} = \|f\|_p \|f\|_{\mathcal{L}_\alpha^p}^{-1}$, the proof of the lemma is complete.

Let $C_1, C_2$ be two capacities on $R^n$, and let $[C_1, C_2](E) = \inf\{C_1(E') + C_2(E'')\}$, where the inf is extended over all $E'$, $E''$ with $E = E' \cup E''$. It is easy to check that this is a capacity, and that $[C_1, C_2](E) < \min(C_1(E), C_2(E))$. If $\{C_\alpha\}$ is a collection of capacities on $R^n$, then $C(E) = \sup_\alpha C_\alpha(E)$ is also a capacity on $R^n$.

Let $f \in \mathcal{L}_\alpha^p$, $\alpha p < n$. As we have seen, $f \in L^r$ for $1/r > 1/p - \alpha/n$. If $p < r < s$, $1/s = 1/p - \alpha/n$, let for $r < t < s$, $\alpha_t = \alpha - n/p + n/t$. We
define now the capacity $C_{ra}$ by

$$C_{ra}(E) = \begin{cases} \left[ L^{p, q}_a, L^{p, r, q}_a \right](E), & 1 < r < p, \\ \sup_{t > r} \left[ L^{p, q}_a, L^{t, r, q}_a \right](E), & p < r < s. \end{cases}$$

**Theorem 4.** Assume that $\| M_F \|_p < A_p \| f \|_p$, $1 < p < \infty$. Let $f \in L^p_\alpha$, $0 < \alpha < 1$, $\alpha p < n$, and let $1/r > 1/p - \alpha/n$. Then for $C_{ra}$-a.e. $x$,

$$\frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r \, dy \to 0 \quad \text{as } S \to 0.$$

**Proof.** We assume first that $1 < r < p$. Let

$$E = \left\{ x : \limsup_{S \to 0} \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r \, dy > 0 \right\},$$

and let $0 < \beta < \alpha$. By Lemma 3 it suffices to show that there exists a decomposition $E = E' \cup E''$ such that $B^{p, q}_\beta(E') = 0$ and $B^{p, r, q}_\beta(E'') = 0$.

Let $\{ \psi_j \} \subset C^\infty_0$ such that $\psi_j \to f(L^p_\alpha)$. Then for each $j$

$$\left\{ \frac{1}{|S|} \int_{S+x} |f(y) - f(x)|^r \, dy \right\}^{1/r} \leq \left\{ M_F(|f - \psi_j|)(x) \right\}^{1/r}

+ \left\{ \frac{1}{|S|} \int_{S+x} |\psi_j(y) - \psi_j(x)|^r \, dy \right\}^{1/r} + |\psi_j(x) - f(x)| = A_{1j}(x) + A_{2j}(x) + A_{3j}(x).$$

We claim that there is $j_1 < j_2 < \cdots$ such that $M_F(|f - \psi_j|)(x) \to 0$ as $i \to \infty$ for $B^{p, r, q}_\beta$-a.e. $x$.

By Lemma 9 $|f - \psi_j| \in L^{p, q}_\alpha$, and hence $|f - \psi_j| = G_{a-\beta} * g_j$, $g_j \in L^{p, q}_\beta$. Since

$$\{ x : A_{1j}(x) > \varepsilon' \} \subset \{ x : G_{a-\beta} * M_F g_j(x) > \varepsilon' \},$$

we obtain

$$B^{p, r, q}_\beta \{ x : A_{1j}(x) > \varepsilon' \} \leq \frac{1}{\varepsilon'} \| M_F g_j \|_{M^{p, r, q}_\beta} \leq \frac{A}{\varepsilon'} \| g_j \|_{M^{p, r, q}_\beta} \quad \text{(Lemma 2)}.$$

By Lemma 9, $|f - \psi_j| \to 0 (L^{p, q}_\alpha)$, and hence $g_j \to 0 (L^{p, r, q}_\beta)$. The claim now follows (see Lemma 6).

Let $E'' = \{ x : \limsup_{i \to \infty} M_F(|f - \psi_i|)(x) > 0 \}$. Then $B^{p, r, q}_\beta(E'') = 0$. We may assume that $\psi_j(x) \to f(x)$ for $L^{p, q}_\alpha$-a.e. $x$, and if $E'$ is the exceptional set, we have $B^{p, q}_\beta(E') = 0$. We show that $E \subset E' \cup E''$.

Let $x \notin E' \cup E''$, and let $\varepsilon > 0$. Choose $j_0$ such that $A_{1j_0}(x) < \varepsilon/3$, $A_{3j_0}(x) < \varepsilon/3$, and then choose $\tau_0$ such that $S \subset B(0, \tau_0)$ implies $A_{2j_0}(x) < \varepsilon/3$. It follows then that $x \notin E$.

If $p < r < s$, $1/s = 1/p - \alpha/n$, let $r < t < s$, and let $\alpha = \alpha - n/p +$
\( n/t \). By [6, p. 441], \( f \in \Lambda_t^q \). Consequently by the first part of the proof, 
\( [L_{\alpha}^q, L_{\alpha}^{1/q}](E) = 0 \), and the proof of the theorem is complete.

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