GROUPS OF PL $\Lambda$-HOMOLOGY SPHERES

BY

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Abstract. Let $\Lambda = \mathbb{Z}_K$ be a subring of $\mathbb{Q}$. The group of $H_\Lambda$-cobordism classes of closed PL $n$-manifolds with the $\Lambda$-homology of $S^n$ is computed for $n > 4$ (modulo $K$-torsion). The simply connected version is also computed.

1. Introduction. Let $K$ be a set of primes and $\Lambda = \mathbb{Z}[1/p: p \in K]$ the localization of the integers away from $K$. This paper is devoted to the computation of the groups $\psi_n^\Lambda$ of piecewise-linear $\Lambda$-homology $n$-spheres, modulo $H_\Lambda$-cobordism.

In their classic paper [12], Kervaire and Milnor computed the group of $h$-cobordism classes of smooth homotopy spheres. This was generalized to homology spheres by Kervaire [11]. In two independent contributions, the group of $H_\Lambda$-cobordism classes of smooth $\Lambda$-homology spheres was computed by Alexander, Hamrick and Vick [1], in the case $K = \{\text{odd primes}\}$ and by Barge, Lannes, Latour and Vogel [5] in the arbitrary case.

In §2, we define $\psi_n^\Lambda$ and the corresponding group $\theta_n^\Lambda$ of $\Lambda$-homotopy spheres, requiring $\Lambda$-homology spheres and $H_\Lambda$-cobordisms to be simply connected. We show that if $2 \notin K$, then $\theta_4^\Lambda = \psi_4^\Lambda = 0$ and $\theta_n^\Lambda \cong \psi_n^\Lambda$ if $n > 5$.

In §3, we construct an embedding $L_{n+1}(1; \Lambda)/L_{n+1}(1) \to \theta_n^\Lambda$ (or $\psi_n^\Lambda$) and prove

$$\psi_n^\Lambda \otimes \Lambda \cong (L_{n+1}(1; \Lambda)/L_{n+1}(1)) \otimes \Lambda \text{ for } n > 5.$$  

We then show that every $\Lambda$-homology sphere is stably $K$-parallelizable. This is done by computing the mod($K$) homotopy groups of the fiber of $B\widetilde{PL} \to BH(K)$, where $H(K)$ is the structure monoid of $\Lambda$-homology cobordism bundles.

In §4, we prove our main result: If $n > 4$, $2 \notin K$, then

$$\theta_n^\Lambda = \begin{cases} 0, & n \equiv 3 \bmod(4), \\ W(\Lambda, \mathbb{Z}) \oplus \bigoplus_{\pi(n)-1} \Lambda/\mathbb{Z}, & n = 4k - 1, \end{cases}$$

modulo the Serre class of finite $K$-torsion groups.

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Three dimensional \( \Lambda \)-homology spheres are considered in \( \S 5 \). We construct elements of \( \psi^K_3 \) corresponding to the \( \text{mod}(K) \) surgery obstructions as above, and exhibit an epimorphism \( \alpha_K: \psi^K_3 \to \mathbb{Z}/(16/a_K)\mathbb{Z} \). Finally, we compute the fundamental groups of these manifolds.

2. Groups of homotopy and homology spheres. Let \( H = \text{DIFF}, \text{PL} \) or \( \text{TOP} \). A closed \( H \)-n-manifold \( \Sigma \) is an \( H \)-\( \Lambda \)-homology sphere if \( H_\ast(\Sigma; \Lambda) \cong H_\ast(S^n; \Lambda) \). Note that \( \Sigma \) is orientable since \( H_\ast(\Sigma) \otimes \Lambda = \Lambda \). An \( H \)-\( \Lambda \)-homology sphere \( \Sigma \) is an \( H \)-\( \Lambda \)-homotopy sphere if \( \pi_\ast(\Sigma) = 0 \).

A cobordism \((W; M, M')\) is an \( H_K \)-cobordism if \( H_\ast(W, M; \Lambda) = 0 \); an \( H_K \)-cobordism is an \( h_K \)-cobordism if, in addition, \( \pi_1(M) \cong \pi_1(W) \cong \pi_1(M') \).

Let \( \psi^K_n(H) \) be the set of \( H_K \)-cobordism classes of oriented \( H \)-\( \Lambda \)-homology \( n \)-spheres, and \( \theta^K_n(H) \) the set of \( h_K \)-cobordism classes of oriented \( H \)-\( \Lambda \)-homotopy \( n \)-spheres. Both \( \psi^K_n(H) \) and \( \theta^K_n(H) \) are abelian groups under the operation of connected sum. There is clearly a homomorphism \( \theta^K_n(H) \rightarrow \psi^K_n(H) \). We let \( \theta^K_n = \theta^K_n(\text{PL}), \psi^K_n = \psi^K_n(\text{PL}) \).

An oriented \( H \)-manifold \( M \) is stably \( K \)-parallelizable if the map

\[
M^\varphi \rightarrow \mathbb{B}H \rightarrow (\mathbb{B}H)_K
\]

is null-homotopic. We have the following mild generalization of Lemma 1.1 of [1]:

**Lemma 2.1.** Let \( \Sigma \) be a smooth \( \Lambda \)-homology sphere. Then \( \Sigma \) is stably \( K \)-parallelizable.

The proof is immediate from [1]. Compare [12, Theorem 3.1], and [11, Theorem 3].

**Proposition 2.2.** Suppose \( \pi \) is a finitely presented group so that \( H_1(\pi; \Lambda) = H_2(\pi; \Lambda) = 0 \). Then for \( n > 5 \), there is a smooth \( \Lambda \)-homology sphere \( \Sigma^n \) with \( \pi_1(\Sigma) = \pi \).

**Proof.** Suppose \( M \) is a smooth manifold. Then

\[
H_2(\pi_1(M); \Lambda) = H_2(\pi_1(M)) \otimes \Lambda \quad \text{by the universal coefficient theorem, since } \Lambda \text{ is torsion free}
\]

\[
\cong (H_2(M)/\text{Im}(h)) \otimes \Lambda \quad \text{where } h \text{ is the Hurewicz homomorphism, by [8]}
\]

\[
\cong H_2(M; \Lambda)/\text{Im}(h_K),
\]

\[
\pi_2(M) \xrightarrow{h} H_2(M) \rightarrow H_2(M; \Lambda).
\]

The result now follows exactly as in [11, Theorem 1].

Our main result of this section is
Theorem 2.3. Suppose $2 \notin K$. Then:

(i) $\theta_n^K = \psi_n^K$ for $n > 5$;

(ii) $\theta_4^K = \psi_4^K = 0$.

Proof. We prove (ii) first. Let $\Sigma$ be a PL $\Lambda$-homology sphere. By [9], $\Sigma$ is smoothable. Furthermore, $\Sigma$ has trivial normal bundle, since $H_2(\Sigma; \mathbb{Z}/2\mathbb{Z}) = H_3(\Sigma; \mathbb{Z}/2\mathbb{Z}) = 0$, and the obstruction in $H^4(\Sigma; \mathbb{Z})$ is given by a multiple of $p_1(\Sigma) = 3 \text{ Sign}(\Sigma) = 0$. Thus $\Sigma$ bounds a parallelizable manifold, since $\Omega_4^K = 0$. Since $L_5(1; \Lambda) = 0$, by [3] $\theta_4^K = 0$. To see that $\psi_4^K = 0$, let $\mathcal{F}: \Lambda[\pi_1(\Sigma)] \to \Lambda$ be the coefficient homomorphism and $\Gamma_5(\mathcal{F})$ the surgery obstruction group of [7]. By [7], $\Gamma_5(\mathcal{F}) = 0$, and $\Sigma$ bounds a $\Lambda$-acyclic manifold.

To prove (i) assume $n > 5$ and $\Sigma'$ is a smooth $\Lambda$-homology sphere. By Lemma 2.1, $\Sigma$ is stably $K$-parallelizable, and so $T_\Sigma \in KO(\Sigma)$ has order in $\Lambda$. Let $\alpha_1, \ldots, \alpha_m$ be generators for $\pi_1(\Sigma)$, represented by disjoint embeddings $\phi_i: S^1 \to \Sigma$. Then $\phi_1^*(T_\Sigma) \in KO(S^1) = \mathbb{Z}/2\mathbb{Z}$ has odd order, and so the normal bundle of $\phi_i$ is trivial. Attach handles along $\text{Im}(\phi_i)$; let $N$ be the trace of these surgeries.

Both $N$ and $\Sigma' = \partial_+ N$ and stably $K$-parallelizable, and

$$\pi_1(\Sigma') = 0, \quad H_i(\Sigma'; \Lambda) = 0, \quad 3 < i < n - 3,$$

and $H_2(\Sigma'; \Lambda), H_{n-2}(\Sigma'; \Lambda)$ are free $\Lambda$-modules of rank $m$. Do surgery on a basis for $H_2(\Sigma'; \Lambda)$; the resultant is a $\Lambda$-homotopy sphere which is $H_K$-cobordant to $\Sigma$.

Now, if $\Sigma$ is not smooth, then $\Sigma$ admits a smoothing in a neighborhood of its 2-skeleton, and the argument goes through as above.

Suppose finally that $\Sigma$ is a $\Lambda$-homotopy sphere and $W$ is an $H_K$-cobordism to $S^n$. Then $W$ is stably $K$-parallelizable and we may do surgery on a set of generators of $\pi_1(W) \cong \pi_1(W, \Sigma)$. Let $W'$ be the resultant. Then the result of doing surgery on a basis for $H_2(W', \Sigma; \Lambda)$ gives an $H_K$-cobordism between $\Sigma$ and $S^n$.

3. Stable $K$-parallelizability and $\psi_n^K \otimes \Lambda$. Recall that the simply connected surgery obstruction groups with coefficients in $\Lambda$, as computed in [3], [4], are given by

$$L_n(1; \Lambda) = \begin{cases} 0, & n \text{ odd}, \\ (\mathbb{Z}/2\mathbb{Z}) \otimes \Lambda, & n \equiv 2 \mod(4), \\ \overline{W}(\Lambda), & n \equiv 0 \mod(4), \end{cases}$$

where $\overline{W}(\Lambda) \subseteq \mathbb{Z} \oplus \bigoplus_{p \in K} W(\mathbb{F}_p)$ is the Witt-Wall group of even quadratic forms over $\Lambda$, modulo kernels, and
The embedding above is given by \((1/a_k)\text{Sign}\) and the Hasse-Minkowski invariants \(\beta_p: W(\mathbb{Q}) \to W(F_p)\), where

$$a_k = \begin{cases} 1, & K \equiv 0 \mod(2), \\ 2, & K \equiv 3 \mod(4), K \not\equiv 0 \mod(2), \\ 4, & K \not\equiv \emptyset, K \equiv 3 \mod(4), K \not\equiv 0 \mod(2), \\ 8, & K = \emptyset. \end{cases}$$

(We write \(K \equiv a \mod(b)\) if \(p \equiv a \mod(b)\) for some \(p \in K\).)

We have the following result from [3].

**Proposition 3.1.** Let \(x \in \overline{W}(\Lambda)\) and \(k > 1\). Then there is a smooth manifold triad \((M; \partial_+M, \partial_-M)\) and a normal map \(\phi: (M; \partial_+M, \partial_-M) \to (S^{4k-1} \times I; S^{4k-1} \times 0, S^{4k-1} \times 1)\) so that

(i) \(\phi|_{\partial_+M}\) is the identity,
(ii) \(\pi_1(\partial_-M) = 0\) and \(\phi|_{\partial_-M}\) induces isomorphisms on \(\Lambda\)-homology, and
(iii) the cup product pairing of \(M\) is equivalent to \(x\) in \(\overline{W}(\Lambda)\).

We let \(\Sigma_x = \partial_-M\) and \(M_x = M \cup c(\partial_+M) \cup c(\partial_-M); \Sigma_x\) is a smooth \(\Lambda\)-homotopy \((4k - 1)\)-sphere and \(M_x\) is a closed \(\Lambda\)-homology manifold.

Let \(\Omega^{G,K}_*\) be the cobordism group of oriented \(\Lambda\)-Poincaré complexes, and \(MSG(K)\) the Thom spectrum associated to \(\Lambda\)-spherical fibrations [19]. Since closed \(\Lambda\)-homology manifolds satisfy Poincaré duality with \(\Lambda\)-coefficients, the spaces \(M_x\) generate a subgroup \(A_{4k}\) of \(\Omega^{G,K}_*\). Let \(A_{2n+1} = 0\) and \(A_{4k+2} = (\mathbb{Z}/2\mathbb{Z}) \otimes \Lambda\).

**Proposition 3.2.** \(A_{4k} \cong \overline{W}(\Lambda)\) and for \(n > 5\), there is an exact sequence

\[0 \to A_n \to \Omega^{G,K}_n \to \pi_n(\text{MSG}(K)) \to 0.\]

**Proof.** Define \(f: \overline{W}(\Lambda) \to A_{4k}\) by \(f(x) = [M_x]\). By an elementary surgery argument, \(f\) is a homomorphism, which is clearly surjective. Furthermore, if \(M_x\) is a boundary, then the standard argument (e.g. [6, Theorem III.2.4]) shows that \(x\) is a kernel, so that \(f\) is injective. The second statement is immediate from [10, §7.11].

Let \(\overline{W}(\Lambda, \mathbb{Z}) = \text{coker}(\overline{W}(\mathbb{Z}) \to \overline{W}(\Lambda))\).

**Proposition 3.3.** There is an embedding \(\overline{W}(\Lambda, \mathbb{Z}) \to \theta^K_{4k-1}, k > 1\).

**Proof.** Define \(r: \overline{W}(\Lambda) \to \theta^K_{4k-1}\) by \(r(x) = [\Sigma_x]\). Surgery arguments show that \(r\) is a well-defined homomorphism. Suppose \(r(x) = 0\). Then there is an
h\_K-cobordism W from \( \Sigma_x \) to \( S^{4k-1} \). Furthermore, the quadratic form on \( H^{2k}(N_x; Q) \) represents x, where \( N_x = D^{4k} \cup M \cup W \cup D^{4k} \) and M is the manifold of Proposition 3.1 associated to x. Since \( N_x \) is a closed manifold, \( x \in \overline{W}(\mathbb{Z}) \).

**Remark.** The same proof shows that \( \overline{W}(\Lambda, \mathbb{Z}) \) embeds in \( \psi_{4k-1}^K \), \( k > 1 \).

**Proposition 3.4.** Let \( \Sigma^n \) be a stably parallelizable \( \Lambda \)-homotopy sphere, \( n > 4 \). Then \( [\Sigma] = 0 \) in \( \theta_n^K \) if \( n \equiv 3 \mod(4) \), and is \( h_K \)-cobordant to some \( \Sigma_x \) otherwise.

**Proof.** Let \( f: \Sigma \to S^n \) collapse the exterior of a disc to a point. Then \( f \) and \( l_{S^n} \) determine the same PL-normal invariant, and so are normally cobordant, by, say, \( F: W \to S^n \times I \). By adding a certain manifold \( M \), as constructed above, to \( W \), we may assume the surgery obstruction of \( F \) vanishes, so that \( \Sigma \) is \( h_K \)-cobordant to \( \partial M \). Clearly \( \partial M = S^{n-1} \) if \( n \equiv 3 \mod(4) \) and some \( \Sigma_x \) otherwise.

The following is implicit in Quinn [16].

**Theorem 3.5.** For \( n > 5 \), \( \psi_n^K \otimes \Lambda \cong (L_{n+1}(1; \Lambda)/L_{n+1}(1)) \otimes \Lambda \).

**Proof.** We sketch the proof. Let \( S_{PL}^K(S^n) \) be the set of \( \Lambda \)-homology triangulations of \( S^n \), i.e., degree 1 \( \Lambda \)-homology equivalences \( f: M^n \to S^n \) modulo \( H_K \)-cobordisms. Define \( \phi: \psi_n^K \to S_{PL}^K(S^n) \) by sending \( \Sigma \) to the map \( \Sigma \to S^n \), collapsing the exterior of an embedded \( n \)-disc to a point. (If \( W \) is an \( H_K \)-cobordism between \( \Sigma \) and \( \Sigma' \), then collapsing the exterior of a regular neighborhood of an embedded path between \( \Sigma \) and \( \Sigma' \) shows that \( \phi \) is well defined.) \( \phi \) is a clearly a bijection.

Let \( \mathcal{N}_{PL}^K(S^n) \) be the set of \( K \)-normal invariants of \( S^n \), cobordism classes of degree 1 mappings \( f: M^n \to S^n \) together with a factorization

\[ M \xrightarrow{\nu_M} (BSPL)_K \]

\[ f \]

\[ S^n \]

By [16], there are exact sequences for \( n > 5 \),

\[ \cdots \to L_{n+1}(1; \Lambda) \to S_{PL}^K(S^n) \to \mathcal{N}_{PL}^K(S^n) \to L_n(1; \Lambda), \]

\[ \cdots \to \Omega_n^K \to \mathcal{N}_{PL}^K(S^n) \to \pi_n(G/PL) \otimes \Lambda \to \Omega_{n-1}^K \to \cdots, \]

where \( \Omega_n^K = \pi_n^S(MSPL^K) \), and \( MSPL^K \) is the fiber of \( MSPL \to (MSPL)_K \).

Since \( \Omega_n^K \otimes \Lambda = 0 \), and \( \pi_n(G/PL) = L_n(1) \) for \( n > 5 \), the result follows.

To prove that PL \( \Lambda \)-homology spheres are stably \( K \)-parallelizable, we introduce \( \Lambda \)-homology cobordism bundles.

A polyhedron \( M \) is called a \( \Lambda \)-homology manifold of dimension \( n \) if \( M \) has a
subdivision $M'$ so that $\tilde{H}_*(Lk(x, M'); \Lambda) \cong \tilde{H}_*(S^{n-1}; \Lambda)$ or 0. The boundary of $M$,
$$\partial M = \{ x \in M': \tilde{H}_*(Lk(x, M'); \Lambda) = 0 \}$$
is a $\Lambda$-homology manifold of dimension $n - 1$.

A $\Lambda$-homology $n$-sphere is a $\Lambda$-homology $n$-manifold $\Sigma$ so that $H_*(\Sigma; \Lambda) \cong H_*(S^n; \Lambda)$; a $\Lambda$-homology $n$-disc is a compact $\Lambda$-acyclic $\Lambda$-homology $n$-manifold $\Delta$. The prefix "PL" indicates that $\Sigma$ or $\Delta$ is a PL manifold. A $\Lambda$ $n$-cell is the cone $cM$ over a $\Lambda$-homology $(n-1)$-sphere or $(n-1)$-disc $M$; such a $\Lambda$ $n$-cell is a $\Lambda$-homology $n$-manifold with boundary $M$ or $M \cup c(\partial M)$. An $H_K$-cobordism is a $\Lambda$-homology manifold triad $(W; M_+, M_-)$ with $H_*(W; M_\pm; \Lambda) = 0$. (Again the prefix "PL" means that $W$ is a PL-manifold.)

A $\Lambda$-cell decomposition of a simplicial complex $X$ is a collection $\emptyset$ of subpolyhedra of $X$ so that
(i) each $\Delta \in \emptyset$ is a $\Lambda$-cell,
(ii) $X$ has a subdivision $X'$ so that every simplex of $X'$ lies in the interior of a unique element of $\emptyset$, and
(iii) if $\Delta \in \emptyset$, $\partial \Delta$ is a union of elements of $\emptyset$.

Let $X$ be a simplicial complex with a $\Lambda$-cell decomposition $\emptyset$. A $\Lambda$-homology cobordism (n-sphere) bundle $\xi$ over $X$ is a complex $E = E(\xi)$ over $\emptyset$ (see [15, p. 96]) so that for each $\Delta^m \in \emptyset$:
(i) $E(\Delta)$ is a $\Lambda$-homology $(n + m)$-manifold with
$$\partial E(\Delta) = E(\partial \Delta) = \bigcup_{\Delta_0 \in \emptyset | \Delta \atop \Delta_0 \neq \Delta} E(\Delta_0),$$
and
(ii) there is a complex $W$ over $\emptyset | \Delta$ so that $W(\Delta_0)$ is an $H_K$-cobordism between $E(\Delta_0)$ and $\Delta_0 \times S^n$ for each $\Delta_0 \in \emptyset | \Delta$.

Here $\emptyset | \Delta$ denotes the $\Lambda$-cell decomposition of $\Delta$ consisting of those $\Delta_0 \in \emptyset$ with $\Delta_0 \subset \Delta$.

Two $\Lambda$-homology cobordism bundles $\xi^n$, $\eta^n$ over $X$ are isomorphic, written, $\xi \cong \eta$, if there is a complex $G$ over $\emptyset$ so that for each $\Delta \in \emptyset$, $G(\Delta)$ is an $H_K$-cobordism between $E(\xi)(\Delta)$ and $E(\eta)(\Delta)$.

These bundles satisfy the analogous properties that the homology cobordism bundles of [15] enjoys. In particular, isomorphism classes on $\Lambda$-homology cobordism $n$-sphere bundles are classified by $BH(K)_{n+1}$, where $H(K)_{n+1}$ is the $\Delta$-monoid with $i$-simplexes given by isomorphisms of the trivial bundle $\Delta^i \times S^n$ over $\Delta^i$. There are stabilization maps $BH(K)_n \to BH(K)_{n+1}$ and we let $BH(K) = \lim BH(K)_n$.

Let $\mathcal{PL}_n$ be the $\Delta$-set with $i$-simplexes PL $n$-block bundles over $\Delta^i \times I$ which are trivial over $\Delta^i \times \dot{I}$. Then $\mathcal{PL}_n$ and $\tilde{\mathcal{PL}}_n$ are homotopy equivalent.
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(see [14]), and $\overline{\text{PL}}_n \subset H(K)_n$. Therefore there is a natural map $B\overline{\text{PL}} \to BH(K)$, with fiber denoted by $H(K)/\overline{\text{PL}}$.

**Theorem 3.6.** $\pi_n(H(K)/\overline{\text{PL}}) \otimes \Lambda \cong \psi^n_+ \otimes \Lambda$.

The proof is immediate from the following two lemmas. Recall from [18] that a $\Lambda$-acyclic resolution $f: N^n \to M^n$ between $\Lambda$-homology manifolds is a proper, PL surjection so that $\tilde{H}_*(f^{-1}(x); \Lambda) = 0$ for every $x \in M$, and $f|\partial N: \partial N \to \partial M$ is also a $\Lambda$-acyclic resolution, and that if $M$ is orientable and $\partial M$ is a PL-manifold, then there is a $\Lambda$-acyclic resolution, $\text{rel}(\partial M)$, to a PL-manifold $N$ provided obstructions $\mu_j \in H_j(M; \psi^K_{n-j-1})$, $j = n - 1, \ldots, 0$, vanish.

Let $\text{PL}H(K)_n$ be the $\Delta$-set of which a typical $i$-simplex is a block-preserving PL $H_K$-cobordism between $\Delta^i \times S^{n-1}$ and itself. We have $\overline{\text{PL}}_n \subset \text{PL}H(K)_n \subset H(K)_n$.

**Lemma 3.7.** $\pi_n(H(K)/\text{PL}H(K)) \otimes \Lambda \cong \psi^n_+ \otimes \Lambda$.

**Proof.** Let $x \in \pi_n(H(K)/\text{PL}H(K))$. Then $x$ is represented by a $\Lambda$-homology cobordism $i$-sphere bundle $W$ over $\Delta^n \times I$, trivial over $\Delta^n \times \bar{I}$, and a PL $H_K$-cobordism over $\Delta^{n-1} \times I$; $W$ is a $\Lambda$-homology $(n + i + 1)$-manifold with PL boundary, and $H_*(W; \Lambda) \cong H_*(S^i; \Lambda)$.

Let $\mu_i(x) \in H_i(W; \psi^K_{n+i-i})$ be the first nonzero obstruction to resolving $W$ rel$(\partial W)$. If $j > i$, then $k\mu_i(x) = 0$ for some $k \in \Lambda$, and by naturality of the obstructions, $\mu_j(kx) = 0$. Continuing in this way, there are two obstructions $\mu_i \in \psi^n_+ \otimes \Lambda$, $\mu_0 \in \psi^K_{n+i} \otimes \Lambda$ to resolving an isomorphism representing $kx$ for some $k \in \Lambda$.

Define $\phi_i: \pi_n(H(K)_i/\text{PL}H(K)_i) \otimes \Lambda \to \psi^K_+ \otimes \Lambda$ by $\phi_i(x) = k^{-1}\mu_i(kx)$, where $k$ is chosen as above. It follows easily that $\phi_i$ is a well-defined homomorphism and that

$$
\pi_n(H(K)_i/\text{PL}H(K)_i) \otimes \Lambda \xrightarrow{\phi_i} \psi^K_+ \otimes \Lambda \\
\downarrow \quad \downarrow \\
\pi_n(H(K)_{i+1}/\text{PL}H(K)_{i+1}) \otimes \Lambda \xrightarrow{\phi_{i+1}} \psi^K_+ \otimes \Lambda
$$

commutes. Define

$$
\phi = \lim \phi_i: \pi_n(H(K)/\text{PL}H(K)) \otimes \Lambda \to \psi^K_+ \otimes \Lambda.
$$

To see that $\phi$ is injective, let $x \in \ker(\phi)$, and represent $kx$ by a $\Lambda$-homology cobordism $i$-sphere bundle with $n + i \equiv 3 \mod(4)$ and $\mu_i(kx) = 0$ for $j > i$. Then $\phi(x) = k^{-1}\mu_i(kx) = 0$, so that $\mu_i(kx)$ is $K$-torsion. Since $n + i \equiv 3 \mod(4)$, $\psi^K_{n+i} \otimes \Lambda = 0$ by Theorem 3.5, and so all obstructions $\mu_i, \mu_{i-1}, \ldots, \mu_0$ are $K$-torsion. As above, there exists $k' \in \Lambda$ so that $k'kx$ is represented by a bundle $W$ with all obstructions to resolving $W$ rel$(\partial W)$ vanishing. Thus there is a PL-manifold $V$ and a $\Lambda$-acyclic resolution $f$:
$V \to W$, identity on the boundary. But $V$ represents 0 in $\pi_n(H(K)_i/\text{PLH}(K)_i)$, and the mapping cylinder of $f$ represents a homotopy from $k'kx$ to $V$. Therefore $x = 0$.

Finally, $\phi$ is surjective: Let $y \in \psi^K_n \otimes \Lambda$ and $\Sigma$ a PL $\Lambda$-homology sphere representing $ky$ for some $k \in \Lambda$. By choosing an embedded $n$-disc in $\Sigma$, we can regard $c\Sigma$ as a space over $\Delta^n \times I$, as in [14], and $c\Sigma \times S^i$ represents an element $x \in \pi_n(H(K)_i/\text{PLH}(K)_i)$ with a single resolution obstruction $\mu_j(x) = [\Sigma]$. Then $\phi(x/k) = y$.

**Lemma 3.8.** $\pi_n(\text{PLH}(K)/\text{PL}) \otimes \Lambda = 0$.

**Proof.** Let $x \in \pi_n(\text{PLH}(K)/\text{PL})$ be represented by a PL $H_K$-cobordism $W$ of $\Delta^n \times S^i$ that is a PL block bundle over $\Delta^n \times I$ and the product bundle over $\Delta^{n-1} \times I$, with $n + i \equiv 2 \mod(4)$. Extend $W\mid \partial(\Delta^n \times I)$ to a disc bundle $V$, and let $\Sigma = W \cup V$. Then $\Sigma$ is a PL $\Lambda$-homology $(n + i + 1)$-sphere, and by Theorem 3.5, $\#_k \Sigma$ bounds a $\Lambda$-acyclic manifold $H$ for some $k \in \Lambda$. By replacing $x$ by $kx$, we may assume $W \cup V$ bounds a $\Lambda$-acyclic manifold $H$; by Corollary 3.3 of [5] we may further assume that $\pi_i(H) = 0$.

Let $i: \partial(\Delta^n \times I) \to V$ be the zero section. By the Hurewicz theorem, $\pi_n(H) \otimes \Lambda = 0$, and so $k'[i] = 0$ in $\pi_n(H)$ for some $k' \in \Lambda$. Again, replacing $x$ with $k'x$, we may assume $i$ is null-homotopic. The remainder of the proof now follows exactly as the proof of Lemma 2.1 of [14].

**Theorem 3.9.** Let $\Sigma^n$ be a PL $\Lambda$-homology sphere. Then $\Sigma$ is stably $K$-parallelizable.

**Proof.** Case 1. $n \not\equiv 3 \mod(4)$.

Let $\Sigma \to BSH(K)$ classify the stable normal $H_K$-homology cobordism bundle of $\Sigma$. Since $\Sigma$ bounds the contractible $\Lambda$-homology manifold $c\Sigma$, $v$ is null-homotopic. The obstructions to lifting this null-homotopy to $(BS\text{PL})_K$ lie in

$$H^i(\Sigma; \pi_i(SH(K)/\text{SPL}) \otimes \Lambda) = \begin{cases} 0, & i \neq n, \\ \psi^K_n \otimes \Lambda, & i = n, \end{cases}$$

by Theorem 3.6. Since $n \not\equiv 3 \mod(4)$, $\psi^K_n \otimes \Lambda = 0$.

Case 2. $n \equiv 3 \mod(4)$.

The argument of Case 1 shows that $\Sigma \to (BS\Sigma)_K$ is null-homotopic, and this null-homotopy lifts to $(BS\text{PL})_K$ since $\pi_n(G/\text{PL})_K = 0$.

4. The calculation of $\theta^K_n$, $2 \not\in K$. In this section, we sharpen Theorem 3.4 in the odd case.

**Theorem 4.1.** Let $2 \not\in K$. Then for $n \geq 4$,
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Here the \( \pi(k) \) denotes the number of partitions of \( k \).

**Proof.** Let \( T^H \) denote the spectrum for the cobordism group \( \Omega_{fr,K}^*(H) \) of stably \( K \)-parallelizable \( H \)-manifolds, \( H = \text{DIFF} \) or \( \text{PL} \). Let \( MSH \) be the Thom spectrum associated to \( BSH_n \) and \( S \) the sphere spectrum.

There is a homotopy equivalence \( T^H/S \to (MSH/S)^{(K)} \) (compare [2, p. 85]), and we have an exact sequence

\[
0 \to \Omega_n^{SO} \to \Omega_n^{SO,fr} \to \Omega_{n-1}^{fr}(\text{DIFF}) \to 0
\]

\[\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}\]

\[
0 \to \Omega_n^{SPL} \to \Omega_n^{SPL,fr} \to \Omega_{n-1}^{fr}(\text{PL}) \to 0
\]

where \( \Omega_n^{SPL,fr} = \pi_n(MSH/S) \) is the cobordism group of oriented \( H \)-manifolds with framed boundaries.

By [9], \( \gamma \) is an isomorphism. Also, \( \alpha \) is injective with cokernel a finite group.

Let \( F \) be the fiber of \( MSO/S \to MSPL/S \); then \( F^{(K)} \) is the fiber of \( T^0/S \to T^{PL}/S \). Furthermore, the map \( \pi_n(F^{(K)}) \to \pi_n(T^0/S) \) is 0 for all \( n \). This is clear if \( n \equiv 3 \ mod(4) \), since \( \pi_n(T^0/S) \to \pi_n(MSO/S) \) is then injective and \( \pi_n(F) \to \pi_n(MSO/S) \) is 0. If \( n \equiv 3 \ mod(4) \), then we have

\[
K\text{-torsion in } \Omega_{n+1}^{SPL,fr}/\Omega_n^{SO,fr} = \pi_n(F^{(K)})
\]

\[
\to \pi_n(MSO/S)
\]

\[
= \text{coker} \left[ \Omega_{n+1}^{SO,fr} \to \left( \Omega_n^{SO,fr} \right)_K \right]
\]

which is clearly 0.

Thus we have an exact sequence

\[
0 \to \pi_n(T^0/S) \to \pi_n(T^{PL}/S) \to K\text{-torsion in } \Omega_n^{SPL,fr}/\Omega_n^{SO,fr} \to 0.
\]

The homotopy sequences of the pairs \( (T^0, S) \), \( (T^{PL}, S) \) now imply that there is an exact sequence

\[
0 \to \Omega_n^{fr,K}(\text{DIFF}) \to \Omega_n^{fr,K}(\text{PL}) \to K\text{-torsion in } \Omega_n^{SPL,fr}/\Omega_n^{SO,fr} \to 0.
\]

Consider the commutative diagram

\[
0 \to \theta_n^K(\text{DIFF}) \to \theta_n^K(\text{PL}) \to 0
\]

\[\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}\]

\[
0 \to bP_n^K \to \Omega_n^{fr,K}(\text{DIFF}) \to \Omega_n^{fr,K}(\text{PL})
\]

where \( bP_n^K \) is the subgroup of smooth \( \Lambda \)-homology spheres that bound \( K \)-parallelizable manifolds. The top row is clearly exact, and the map \( bP_n^K \)}
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\[ \rightarrow L_{n+1}(1; \Lambda)/L_{n+1}(1), \] defined by the surgery obstruction, is surjective, since the manifolds \( \Sigma_x \) are smooth, with kernel a finite group \([1]\). Again by \([16]\),

\[ L_{n+1}(1; \Lambda) \rightarrow \psi_n^K \rightarrow \Omega^K_{PL}(S^n) \]
is exact, and so

\[ \ker(\theta_n^K \rightarrow \Omega^K_{PL}(PL)) = \ker(\psi_n^K \rightarrow \Omega^K_{PL}(S^n)) = L_{n+1}(1; \Lambda)/L_{n+1}(1). \]

Thus \( \theta_n^K(DIFF) \rightarrow \theta_n^K \) has cokernel a finite \( K \)-torsion group. Furthermore, by \([1]\), there is an exact sequence

\[ \theta_n(DIFF) \rightarrow \theta_n^K(DIFF) \rightarrow L_{n+1}(1; \Lambda)/L_{n+1}(1) \oplus G_n \rightarrow 0 \]

where

\[ G_n = \begin{cases} 0, & n \equiv 3 \mod(4), \\ \bigoplus \Lambda/\mathbb{Z}, & n = 4k - 1. \end{cases} \]

Since \( \theta_n(DIFF) \rightarrow \theta_n^K \) is 0, \( \theta_n^K \) is given as stated.

5. The 3-dimensional case. In general, the methods of the preceding sections do not apply in dimension 3, due to the lack of a well-behaved surgery theory. In this section we investigate the \( \Lambda \)-homology 3-spheres obtained from the plumbing theorem and compute their fundamental groups.

We define the \( \alpha_K \)-invariant, \( 2 \not\in K \), as follows. Let \( \Sigma^3 \) be a PL \( \Lambda \)-homology sphere. By Lemma 2.1, there is a homomorphism \( \psi_3^K \rightarrow \Omega^K_3 \cong \Lambda/\mathbb{Z} \) \([2]\). Thus for some integer \( k \in \Lambda \), \( \#_K \Sigma \) bounds a \( K \)-parallelizable manifold \( W \). Define \( k\alpha_K(\Sigma) = (1/\alpha_K)\text{Sign}(W) \mod(16/\alpha_K) \). We show \( \alpha_K \) is well defined. Suppose \( \#_K \Sigma = \partial W_i \), \( i = 1, 2 \). Let \( W = (\#_{k_1}W_2) \cup (\#_{k_2} - W_1) \) identified along the common boundary \( \#_{k_1 + k_2} \Sigma \). Then

\[ k_1 \text{Sign}(W_2) - k_2 \text{Sign}(W_1) = \text{Sign}(W) = 0 \mod(16) \] by \([17]\).

Let \( A \) be a symmetric integral matrix with even diagonal entries and \( \det(A) \in \Lambda^* \). We may apply the plumbing theorem to construct a smooth manifold \( M_A^3 \) with intersection pairing \( A \) and \( \Sigma^3_A = \partial M_A^3 \) a \( \Lambda \)-homology sphere (this is done explicitly in \([11]\) for \( K = \emptyset \)). By definition of \( \alpha_K \), there is a matrix \( A \) as above with \( \text{Sign}(A) = \alpha_K \). Thus we have

**PROPOSITION 5.1.** There is a surjection \( \alpha_K: \psi_3^K \rightarrow \mathbb{Z}/(16/\alpha_K)\mathbb{Z} \).

We now compute \( \pi_1(\Sigma_A^3) \) for the set of generators \( A \) of \( \mathcal{W}(\Lambda) \) described in \([4]\).

1. \( A = \) the Milnor matrix: By \([11]\),

\[ \pi_1(\Sigma_A^3) = \langle x, y : x^3 = y^2 = (xy)^5 \rangle. \]

2. \( A = \begin{pmatrix} 1 & 1 \\ 2k \end{pmatrix} \): Let \( S_1, S_2 \) be the attaching spheres for the 2-handles of \( M_A^3 \), and \( x, y \) the loops indicated below:
We clearly have that \( \pi_1(\Sigma_A) \) is generated by \( x \) and \( y \), and \( x^{2k} = y = xy^{-1} \). Thus \( \pi_1(\Sigma_A) \cong \mathbb{Z}/(4k - 1)\mathbb{Z} \).

(3) \( A = \left( \begin{smallmatrix} 1 & 2 & 1 \\ -2 & 1 & 2k \\ 1 \\ 2k \\ 1 \\ 0 \end{smallmatrix} \right) \): As in (2), \( \pi_1(\Sigma_A) \cong \mathbb{Z}/(4k + 1)\mathbb{Z} \).

(4) \( A = \left( \begin{smallmatrix} 2a & a \\ -2a & -2ak \end{smallmatrix} \right) \): Letting \( x \) and \( y \) be as in (2), we have
\[
\pi_1(\Sigma_A) \cong \langle x, y \colon x^{2ak} = y^a, y^{2a} = x^a \rangle \cong \mathbb{Z}/(4k - 1)a\mathbb{Z}.
\]

(5) \[
A = \begin{bmatrix}
2 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2k 
\end{bmatrix}:
\]

Let \( S_1, S_2, S_3, S_4 \) be the attaching spheres and \( x, y \) as indicated:

Then \( \pi_1(\Sigma_A) \cong \langle x, y \colon x^{4k} = y, y^2 = x^3 \rangle \cong \mathbb{Z}/(8k - 3)\mathbb{Z} \).
Example. Let \( p = 4k - 1 \) be a prime and \( M^3 \) the rational homology sphere obtained by intersecting the unit 5-sphere in \( \mathbb{C}^3 \) with algebraic variety defined by \( z_1^2 + z_2^2 + z_3^2 = 0 \). The origin is an isolated singularity of this variety and its minimal resolution has the \((p - 1) \times (p - 1)\) intersection matrix

\[
A = \begin{pmatrix}
-2 & 1 & & \\
1 & -2 & 1 & \\
 & & \ddots & \\
& & & -2 & 1 \\
& & & 1 & -2
\end{pmatrix}
\]

which can be diagonalized as \((-2, -3/2, -4/3, \ldots, -p/(p - 1))\). Thus \( \text{Sign}(A) = -p + 1 \), \( \beta_p(A) = (- (p - 1)) = 1 \), \( \beta_p'(A) = 0 \), \( p' \neq p \). It follows that \( M \cong (\#_n \Sigma_1) \neq \Sigma_2 \), where \( \Sigma_1 \) is the manifold of (1) above, \( \Sigma_2 \) the manifold of (2) (with orientations reversed) and \( n = [k/2] \).

References

4. _____, *Calculation of the surgery obstruction groups \(L_{4k}(1; \mathbb{Z}_p)\)* (to appear).


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