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TOPOLOGIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS[1]

BY

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ABSTRACT. Topologies β_0, β_1, β, β_∞, $\beta_{\infty c}$ are defined on $C_b(X, E)$, the space of all bounded, continuous functions from a completely regular Hausdorff space X, into E, a normed space, and their duals are determined. Also many properties of these topologies are proved.

0. Introduction. Let X be a completely regular Hausdorff space, E a Hausdorff locally convex space, and $C_b(X, E)$ all continuous, bounded functions from X into E. Many authors have considered the so-called strict topologies on $C_b(X, E)$ and some subspaces of $C_b(X, E)$ **[4]**, **[12]**, **[13]**, **[14]**, **[15]**, **[23]**, **[28]**. In this paper we prove many properties of these and some other topologies on $C_b(X, E)$ and $C(X, E)$, the space of all continuous E-valued functions on X.

The paper is divided into 8 sections. §1 is concerned with notation and definitions, and in §2 we prove some results to be used throughout the paper. In §§3 and 4 we define topologies β_∞, $\beta_{\infty c}$ and generalize many results of **[18]**, **[24]**, **[28]**; in particular, we prove that $(C_b(X, E), \beta_\infty)$ and $(C(X, E), \beta_{\infty c})$ are Mackey. Denseness of $C_b(X) \otimes E$ in $(C_b(X, E), \beta_1)$ is proved, under fairly general conditions on X and E, in §5. In §6 it is proved that when X is paracompact (resp. when $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_1)$), $(C_b(X, E), \beta)$ (resp. $(C_b(X, E), \beta_1)$) is Mackey. A different definition of β_∞ is considered in §7 and it is proved that the two definitions are equivalent. In §8 we prove that all these topologies are 'locally solid'.

1. Notations and definitions. All linear spaces are taken over K, the scalar field of real or complex numbers. X always denotes a Hausdorff completely regular space and $C(X)$ $(C_b(X))$ all scalar-valued continuous (bounded continuous) functions on X. $\tilde{X}$ (νX) will denote Stone-Čech compactification (real compactification) of X **[6]**. A locally convex space, say G, is always

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assumed to be Hausdorff and over K, and G' will denote its topological dual. (The terminology of [22] will be used for locally convex paces.) N always denotes the set of natural numbers. A locally convex space F will be called strong Mackey if every $\sigma(F', F)$ relatively countably compact subset of F' is equicontinuous. For a locally convex space E, $C(X, E)$ will denote the set of all continuous functions from X into E and $C_b(X, E)$ ($C_{rc}(X, E)$) will denote the set of all continuous bounded (all continuous with relatively compact images into E) functions from X into E. For a function $f \in C(X)$, $\bar{f}$ and $\tilde{f}$ denote its unique extensions to νX and $\tilde{X}$ respectively (if it is meaningful). For a locally convex space E with dual E' and a subset $A \subset E$, we denote by $A^\circ = \{f \in E' \colon |\langle f, a\rangle| = |f(a)| \leqslant 1, \forall\, a \in A\}$. A° will be called the polar of A. For a $g\colon X \to E$, $\operatorname{supp}(g) = \{x \in X \colon g(x) \neq 0\}^-$, the closure being in X. For a family $\{A_\alpha\}$ of subsets of a locally convex space G, $\Gamma_\alpha A_\alpha$ will stand for the absolutely convex hull of $\bigcup A_\alpha$ [22].

Let $\mathfrak{A}$ be an algebra of subsets of a set Y, E, F locally convex spaces, $\mathfrak{L}(E, F)$ the set of all linear continuous mappings from E into F, $S = S(Y, \mathfrak{A}, E)$ the set of all E-valued $\mathfrak{A}$-simple functions on Y with the topology of uniform convergence on Y, and $\mu\colon \mathfrak{A} \to \mathfrak{L}(E, F)$, a finitely additive set function; then we shall call μ a measure if the corresponding mapping $\mu\colon S(Y, \mathfrak{A}, E) \to F$ is continuous [25, p. 375]. Denoting by $B(Y, \mathfrak{A}, E)$, the closure of $S(Y, \mathfrak{A}, E)$, in the space of all bounded functions from Y into E with the topology of uniform convergence, the measure μ can be uniquely extended to a linear continuous mapping $\mu\colon B(Y, \mathfrak{A}, E) \to \tilde{F}$, $\tilde{F}$ being the completion of F. Let $\mathcal{C}$ be the algebra generated by zero-subsets of X (a subset $Z = f^{-1}\{0\}$ of X, for some $f \in C_b(X)$, is called a zero-subset of X). The σ-algebra generated by zero-subsets of X will be denoted by Ba and its elements will be called Baire-subsets of X. Similarly the σ-algebra generated by open subsets of X will be denoted by Bo and its elements will be called Borel sets. It is a straightforward verification that $C_b(X) \otimes E \subset B(X, \mathcal{C}, E)$ and so $B(X, \mathcal{C}, E) \supset C_{rc}(X, E) = \{f \in C_b(X, E), f(X)$ is relatively compact in $E\}$, [14]. $M(X)$ will stand for K-valued, regular finitely additive measures on $\mathcal{C}$, and $M_\sigma(X)$ ($M_\tau(X)$, $M_t(X)$) σ-smooth (τ-smooth, tight), K-valued Baire (Borel) measures on X [23], [27]. Let $M(X, E') = \{\mu\colon \mathcal{C} \to \mathfrak{L}(E; K) = E'\colon \mu$ is a measure and $\forall x \in E$, μ_x, defined by $\mu_x(B) = \langle \mu(B), x\rangle$, is in $M(X)\}$. Similar meanings for $M_\sigma(X, E')$, $M_\tau(X, E')$, $M_t(X, E')$.

From this place onwards we assume $(E, \|\cdot\|)$ to be a normed space.

For a $\mu \in M_\sigma(X, E')$ ($\mu \in M_\tau(X, E')$, $\mu \in M_t(X, E')$) we define for any Baire (Borel) set A, $|\mu|(A) = \sup|\sum \mu(F_i)s_i|$, where sup is taken over all finite Baire (Borel) partitions $\{F_i\}$ of A and all finite collections $\{s_i\}$ in E with $\|s_i\| \leqslant 1$, $\forall\, i$. It is well known that $|\mu| \in M_\sigma(X)$ ($|\mu| \in M_\tau(X)$, $|\mu| \in M_t(X)$)

and $|\mu(f)| \leqslant |\mu|(\|f\|)$, $\forall f \in S(X, Ba, E)$ $(S(X, Bo, E))$ [**14**, pp. 315–316], [**4**, Proposition 3.9]. It is now easy to see that $M_\sigma(X, E) = \{\mu: Ba \to E': \mu$ finitely additive and $|\mu| \in M_\sigma(X)\}$. Similar results for $M_\tau(X, E')$ and $M_t(X, E')$. Also if $\mu \in M_\tau(X, E')$ then, denoting by μ_0 when μ is considered as an element of $M_\sigma(X, E)$ we have $|\mu_0| = |\mu|$ on Baire sets [**14**, Lemma 2.4, p. 317]. From this it is obvious that if $\mu \in M_t(X, E)$, then also $|\mu_0| = |\mu|$ on Baire sets.

Fix a $\mu \in M_\sigma(X, E')$ $(\mu \in M_\tau(X, E'), \mu \in M_t(X, E'))$ and let $\mathfrak{U}^\mu = \{g: X \to [0, \infty]: g = \sup g_n, \{g_n\}$ a sequence in $C_b(X)$, $g_n \geqslant 0, \forall n\}$ $(\{g: X \to [0, \infty], g = \sup g_\alpha, g_\alpha$ a net in $C_b(X)$, $g_\alpha \geqslant 0, \forall \alpha\})$ [**26**, pp. 57–58]. For $g \in \mathfrak{U}^\mu$, we have $|\mu|(g) = \sup\{|\mu|(h), 0 \leqslant h \leqslant g, h \in C_b(X)\}$; for any δ: $X \to [0, \infty]$, define $|\mu|^*(\delta) = \inf\{|\mu|(\xi): \xi \in \mathfrak{U}^\mu, \xi \geqslant \delta\}$. For g_j: $X \to [0, \infty]$, $j = 1, 2$,

(i) $|\mu|^*(g_1 + g_2) \leqslant |\mu|^*(g_1) + |\mu|^*(g_2)$;

(ii) $g_1 \leqslant g_2 \Rightarrow |\mu|^*(g_1) \leqslant |\mu|^*(g_2)$;

(iii) $g_1 \in \mathfrak{U}^\mu \Rightarrow |\mu|(g_1) = |\mu|^*(g_1)$;

(iv) $|\mu|^*(\alpha g_1) = \alpha|\mu|^*(g_1)$ $(\alpha > 0)$.

These are simple verifications (see [**26**]). For an f: $X \to E$, we denote by $\|f\| \in C_b(X)$, $\|f\|(x) = \|f(x)\|$.

Let $P = \{f: X \to E: |\mu|^*(\|f\|) < \infty\}$. Then P is a vector space and p, $p(f) = |\mu|^*(\|f\|)$, is a seminorm on P. Also $P \supset S$, where $S = S(X, Ba, E)$ $(S(X, Bo, E))$. Let $\mathfrak{L}_1 = \mathfrak{L}_1(\mu, X, E)$ be the closure of $S(X, Ba, E)$ in the seminormed space (P, p). Since μ: $S(X, Ba, E) \to K$ has the property that $|\mu(g)| \leqslant p(g)$, $\forall g \in S$, there exists a unique extension μ: $\mathfrak{L}_1 \to K$ such that $|\mu(f)| \leqslant p(f)$, $\forall f \in \mathfrak{L}_1$. If $f \in \mathfrak{L}_1$, then $\|f\| \in \mathfrak{L}_1(|\mu|)$. To prove this, let $\{f_n\}$ be a sequence in S such that $|\mu|^*(\|f_n - f\|) \to 0$. This implies $|\mu|^*|\,\|f_n\| - \|f\|\,| \to 0$ and so $\{\|f_n\|\}$ is Cauchy in $\mathfrak{L}_1(|\mu|)$. Thus there exists a $g \in \mathfrak{L}_1(|\mu|)$ such that $|\mu|(|\,\|f_n\| - g|) \to 0$. From this it follows that $|\mu|^*(|g - \|f\|\,|) = 0$ and so $\|f\| \in \mathfrak{L}_1(|\mu|)$ (note the space $\mathfrak{L}_1(|\mu|)$ is formed by first taking the completion of $|\mu|$ and then collecting integrable functions; cf. [**24**]), $|\mu(f)| \leqslant |\mu|(\|f\|)$, $\forall f \in \mathfrak{L}_1$. Evidently $\mathfrak{L}_1 \supset B(X, Ba, E)$ $(B(X, Bo, E))$; in particular, $\mathfrak{L}_1 \supset C_b(X) \otimes E$. With seminorm, $f \to |\mu|(\|f\|)$, $\mathfrak{L}_1$ is a seminormed space, containing $C_b(X) \otimes E$ and $S(X, Ba, E)$ $(S(X, Bo, E))$ as dense subspaces.

The topology β_0, on $C_b(X, E)$, is generated by the family of seminorms, $\|\ \|_h$, as h varies through scalar-valued functions on X, vanishing at infinity, $\|f\|_h = \sup_{x \in X}\|h(x)f(x)\|$, $f \in C_b(X, E)$. From [**4**], β_0 is the finest locally convex topology agreeing with itself on all $\|\cdot\|$ – bdd. subsets of $C_b(X, E)$.

It is clear from the definition that a net $f_\alpha \to 0$, in $(C_b(X, E), \beta_0)$ iff $\|f_\alpha\| \to 0$ in $(C_b(X), \beta_0)$. For an $f \in C_b(X, E)$ there exists a net $\{f_\alpha\} \subset C_b(X) \otimes E$ such that $f_\alpha \to f$ in β_0-topology ($C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$ [**4**]). This means $\|f_\alpha - f\| \to 0$ in $(C_b(X), \beta_0)$ and so, for a

$\mu \in M_t(X, E')$, $|\mu|(\|f_\alpha - f\|) \to 0$. From this it easily follows that $\mathfrak{L}_1(\mu, X, E) \supset C_b(X, E)$.

For a compact subset $Q \subset \tilde{X} \setminus X$, let $C_Q(X) = \{f|_X : f \in C(\tilde{X}), f \equiv 0 \text{ on } Q\}$. The topology β_Q, on $C_b(X, E)$ ($C_{rc}(X, E)$), is defined by the seminorms $\| \ \|_h$, h ranging through the elements of $C_Q(X)$, $\|f\|_h = \sup_{x \in X} \|h(x)f(x)\|$, $f \in C_b(X, E)$ ($f \in C_{rc}(X, E)$). The topology β (β_1) on $C_b(X, E)$ (on $C_{rc}(X, E)$) is defined to be the intersection of the topologies β_Q as Q ranges through compact (compact G_δ's in $\tilde{X}$ and) subsets of $\tilde{X} \setminus X$ (these definitions are given for the case $K = R$ in [4] or [23]). We list some properties of these topologies in the following theorem.

THEOREM 1.1. (i) *For any compact $Q \subset \tilde{X} \setminus X$, β_Q is the finest locally convex topology agreeing with itself on bounded sets; also β_0, β, β_1 are the finest locally convex topologies which agree with themselves on norm-bounded subsets of $C_b(X, E)$.*

(ii) 'p' $\leqslant$ 'k' $\leqslant \beta_0 \leqslant \beta \leqslant \beta_1 \leqslant \| \cdot \|$.

(iii) $(C_b(X, E), \beta_0)' = M_t(X, E')$.

(iv) $(C_{rc}(X, E), \beta)' = M_\tau(X, E')$.

(v) $(C_{rc}(X, E), \beta_1)' = M_\sigma(X, E')$.

(vi) *β_0-bounded sets are norm-bounded in $C_b(X, E)$.*

The proof given in [13] extends immediately to this case.

2. Preliminary results. In this section, we shall prove some results which we will be using later in the paper.

THEOREM 2.1. *For a $\mu \in M_\sigma(X, E')$ ($\mu \in M_\tau(X, E')$, $M_t(X, E')$) and $f \in C_b(X), f \geqslant 0$, $|\mu|(f) = \sup\{|\mu(g)| : g \in C_b(X) \otimes E, \|g\| \leqslant f\}$.*

PROOF. Fix $\varepsilon > 0$. Since $|\mu| \in M_\sigma(X)$, there exists a disjoint finite collection $\{Z_i\}$, $1 \leqslant i \leqslant n$, of zero sets in X and real numbers $\{\alpha_i\}$, $\alpha_i > 0$, $\forall i$, such that $\Sigma \alpha_i \chi_{Z_i} \leqslant f$, $| \, |\mu|(f) - \Sigma \alpha_i |\mu|(Z_i)| \leqslant \varepsilon/2$. This means, $\exists$, $\forall i$, a finite disjoint collection of Baire sets A_{ij}, $1 \leqslant j \leqslant p$, and elements x_{ij} in E, $\|x_{ij}\| \leqslant 1$, such that $\bigcup_j A_{ij} \subset Z_i$, and $| \, |\mu|(f) - \Sigma_{i=1}^n \alpha_i \Sigma_{j=1}^p \mu_{x_{ij}}(A_{ij})| < 3\varepsilon/4$ (same p, $\forall i$, becomes possible by allowing A_{ij} to be void for some i, j). By regularity of Baire measures $\mu_{x_{ij}}$, there exist $Z_{ij} \subset A_{ij}$, such that $| \, |\mu|(f) - \Sigma_{i=1}^n \alpha_i \Sigma \mu_{x_{ij}}(Z_{ij})| < 5\varepsilon/6$. Since the closures of $\{Z_{ij}\}$ in $\tilde{X}$ are mutually disjoint, we can define, $\forall i, j$, functions f_{ij} in $C_b(X), f_{ij} \geqslant 0$, $\operatorname{supp}(f_{ij})$ mutually disjoint, $f_{ij} \leqslant f/\alpha_i$ and $| \, |\mu|(f) - \Sigma_{1 \leqslant i \leqslant n; 1 \leqslant j \leqslant p} \mu_{x_{ij}}(f_{ij})| < \varepsilon$. Since $\|\Sigma x_{ij} \otimes f_{ij}\| \leqslant f$, we get $|\mu|(f) \leqslant \sup\{|\mu(g)| : g \in C_b(X) \otimes E, \|g\| \leqslant f\}$. Conversely, take a $g \in C_b(X) \otimes E$ with $\|g\| \leqslant f$. There exists $g_0 \in S(X, \mathfrak{B}_0, E)$ such that $\|g - g_0\| \leqslant \varepsilon$. This means $\|g_0\| \leqslant f + \varepsilon$ and so $|\mu|(f + \varepsilon) \geqslant |\mu(g_0)| \geqslant |\mu(g)| - |\mu(g - g_0)|$. Since μ is continuous on $B(X, Ba, E)$ with sup norm top it follows that $|\mu|(f) \geqslant |\mu(g)|$, which proves the theorem.

LEMMA 2.2. *Let λ_n: $2^N \to K$ be a sequence of continuous, with product topology on 2^N, the class of all subsets of N, finitely additive functions. (Note the continuity of λ_n is equivalent to its countable additivity.) If $\{\mu_n(A)\}$ is convergent in K, $\forall A \in 2^N$, then $\{\mu_n\}$ converges uniformly on 2^N. In particular, $\lambda_\infty = \lim \lambda_n$ is also countably additive and $\lambda_n(\{n\}) \to 0$.*

PROOF. This lemma is a particular case of [**16**, Lemma 1]. This lemma also follows from the Vitali-Hahn-Saks theorem, and the classical Phillips lemma [**29**].

LEMMA 2.3. *Let $\{f_n\}$ be a sequence in $C(X)$, converging point is 0. Then $\bar{f}_n$, the extension of f_n to νX, converges pointwise to zero.*

We omit the proof because the result is well known.

LEMMA 2.4. *If a net $\{f_\alpha\} \subset (C_b(X, E), \mathfrak{T})$, where $\mathfrak{T} = \beta_0, \beta$, or β_1, converges to 0, then $\|f_\alpha\| \to 0$ in $(C_b(X), \mathfrak{T})$.*

PROOF. It is easy to verify that the $C_b(X, R)$, with $K = R$ and topology induced by $(C_b(X), \mathfrak{T})$ is exactly $(C_b(X, R), \mathfrak{T})$ with $K = R$. So it will be enough to show that $\|f_\alpha\| \to 0$ in $(C_b(X, R), \mathfrak{T})$ with $K = R$.

For $\mathfrak{T} = \beta_0$ the result is trivial. Let W be an absolutely convex, solid [**23**] 0-nbd. in $(C_b(X, R), \beta)$. Thus for every compact set $Q \subset \tilde{X} \setminus X$, there exists an $h_Q \in C_Q(X)$ such that $W \supset \{g \in C_b(X), \sup_{x \in X} \|g(x)h_Q(x)\| \leqslant 1\}$. Define a β 0-nbd. W_0 in $C_b(X, E)$, $W_0 = \Gamma_Q\{f \in C_b(X, E)$: $\sup_{x \in X} \|f(x)h_Q(x)\| \leqslant 1\}$. Using the fact that $f \in W_0$ implies $\|f\| \in W$ and that W is solid we get $\|f_\alpha\| \in W$, for every $\alpha \geqslant$ some α_0. The case of β_1 is similar.

LEMMA 2.5. *Let T be a Hausdorff topological space having a σ-compact dense subset, $C(T)$ all scalar-valued continuous functions on T with topology induced by the product topology on $K^T \supset C(T)$, $A \subset C(T)$ such that every sequence in A has a cluster point in $C(T)$, and ψ an element in the closure $\bar{A}$ of A in K^T. Then there exists a sequence $\{\psi_n\} \subset A$ such that $\psi_n \to \psi$.*

This result is proved in [**20**].

3. The topology β_∞. In this section we define the topology β_∞ on $C_b(X, E)$ and prove some results about this topology. When $E = R$ this is discussed in [**21**], [**23**], [**24**], [**28**]. Let $\mathcal{H}^\infty = \mathcal{H}^\infty(X, E) = \{H \subset C_b(X, E)$, H pointwise equicontinuous and uniformly bounded$\}$. If $E = K$ we write $\mathcal{H}^\infty(X)$ in place of $\mathcal{H}^\infty(X, K)$. It is easily verified that $\mathcal{H}^\infty$ is closed under pointwise closure and absolutely convex hull in $C_b(X, E)$. β_∞ is defined to be the finest locally convex topology which agrees with pointwise topology on each $H \in \mathcal{H}^\infty$. It is easily verified that $\beta_\infty \leqslant$ sup norm topology on $C_b(X, E)$ and that β_∞ is

the finest locally convex topology agreeing with itself on the norm-bounded subsets of $C_b(X, E)$.

In [21], $(C_b(X), \beta_\infty)'$ is denoted by $M_\infty(X)$. The elements of $M_\infty(X)$ are called separable measures in [28]. We define $M_\infty(X, E') = \{\mu \in Ba \to E' = \mathfrak{L}(E, K)$: μ is a measure and $\forall x \in E'$, μ_x: $Ba \to K$, defined by $\mu_x(A) = \langle \mu(A), x\rangle$, is in $M_\infty(X)\}$.

We prove some properties of $M_\infty(X, E')$.

THEOREM 3.1. *If* $\mu \in M_\infty(X, E')$ *then* $|\mu| \in M_\infty(X)$.

PROOF. Since $M_\infty(X, E') \subset M_\sigma(X, E)$, $|\mu| \in M_\sigma(X)$. By [28, Proposition 4.1, p. 295] we need only prove that for any bounded continuous pseudometric d on X $\exists$ a d-separable, d-zero set Z_d such that $|\mu|(X \setminus Z_d) = 0$. Take a '$d$' and fix a positive integer n. There exist a Baire partition $\{A_{n,i}\}$ of X and elements $x_{n,i}$ $(1 \leqslant i \leqslant p(n))$ in E, with $\|x_{n,i}\| \leqslant 1$, having the property that $|\mu|(X) \leqslant |\sum_{i=1}^{p(n)} \mu(x_{n,i} \otimes \chi_{A_{n,i}})| + 1/n$. Since $\{\mu_{x_{n,i}}\}$ are in $M_\infty(X)$, $\exists$ d-separable, d-zero sets $Z_{n,i}$ such that $|\mu_{x_{n,i}}|(X \setminus Z_{n,i}) = 0$. Thus we get

$$\begin{aligned}
|\mu|(X) &\leqslant \sum_{i=1}^{p(n)} |\mu_{x_{n,i}}|(A_{n,i}) + \frac{1}{n} \\
&= \sum_{i=1}^{p(n)} |\mu_{x_{n,i}}|(A_{n,i} \cap Z_{n,i}) + \frac{1}{n} \\
&\leqslant \sum_{i=1}^{p(n)} |\mu|(A_{n,i} \cap Z_{n,i}) + \frac{1}{n}
\end{aligned}$$

(it is a simple verification that $|\mu_x| \leqslant |\mu|$, $\forall x \in E$ with $\|x\| \leqslant 1$). From this it follows that $|\mu|(X) \leqslant |\mu|(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{p(n)} Z_{n,i})$, and so taking $Z_d = d$-closure of $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{p(n)} Z_{n,i}$, we get the result.

THEOREM 3.2. $\beta_\infty \leqslant \beta_1$.

PROOF. Suppose W, an absolutely convex, absorbing subset of $C_b(X, E)$, is a β_∞ 0-nbd. but not β_1 0-nbd. This means for some compact-G_δ subset Z of $\tilde{X}$, $Z \subset \tilde{X} \setminus X$, W is not a β_Z 0-nbd. Let $\tilde{X} \setminus Z = \bigcup_{n=1}^{\infty} V_n$, where V_n's are open and increasing in $\tilde{X}$, and $\overline{V}_n \subset V_{n+1}$, $\forall n$ (closures in $\tilde{X}$). Take an increasing sequence $\{f_n\} \subset C(\tilde{X})$, $0 \leqslant f_n \leqslant 1$, $f_n(Z) = 0$ and $f_n(V_n) = 1$, $\forall n$. Since W is not β_Z 0-nbd. and β_Z is the finest locally convex topology which agrees with itself on norm-bounded subset of $C_b(X, E)$, there exists, relative to f_n, $\forall n$, a $g_n \in C_b(X, E)$, and $\alpha > 0$ such that $\|g_n\| \leqslant \alpha$, $\sup_{x \in X} \|f_n(x) g_n(x)\| < 1/n$, and $g_n \notin W$, $\forall n$. But this means that $\{g_n\} \in \mathcal{K}^\infty(X, E)$ and $g_n \to 0$ pointwise and so $g_n \in W$ from some n onwards, which is a contradiction.

THEOREM 3.3. *$C_B(X) \otimes E$ is dense in $(C_b(X, E), \beta_\infty)$ and, for $\mu \in M_\infty(X, E')$, $C_b(X, E) \subset \mathfrak{L}_1(\mu, X, E)$.*

PROOF. Take $h \in C_b(X, E)$ and define, on X, a semimetric $d(x, y) = \|h(x) - h(y)\|$. Defining an equivalence relation in X, $x \sim y \Leftrightarrow d(x, y) = 0$, we get a metric space $(X_d, \tilde{d})$ of equivalence classes, $\tilde{d}(\tilde{x}, \tilde{y}) = d(x, y)$ $(x \in \tilde{x}, y \in \tilde{y})$. Fix $\varepsilon > 0$ and let $\{\tilde{f}_\alpha\}_{\alpha \in I}$ be a partition of unity in X_d, subordinated to the open covering $\{B(\tilde{x}, \varepsilon): \tilde{x} \in X_d\}$, where $B(\tilde{x}, \varepsilon)$ is the open ball with center at $\tilde{x}$ and radius ε. For every α, choose $\tilde{x}_\alpha \in X_d$ such that $\tilde{f}_\alpha(\tilde{x}) = 0$ if $\tilde{x} \notin B(\tilde{x}_\alpha, \varepsilon)$. Denoting by π_d: $X \to X_d$, the canonical mapping, and taking $x_\alpha \in X$ such that $\pi_d(x_\alpha) = \tilde{x}_\alpha$, we get a partition of unity in X, $\{f_\alpha\}_{\alpha \in I}$, $f_\alpha = \tilde{f}_\alpha \circ \pi_d$ such that if $h^*(x) = \sum_{\alpha \in I} h(x_\alpha) f_\alpha(x)$ then $\|h - h^*\| < \varepsilon$ (to prove this, one has only to note that $h(x) = \sum_{\alpha \in I} h(x) f_\alpha(x)$). The collection $\{h_j = \sum_{\alpha \in J} h(x_\alpha) f_\alpha$: J a finite subset of $I\} \in \mathcal{H}^\infty$ and $h_J \to h^*$, pointwise and so h^* is in the β_∞-closure of $C_b(X) \otimes E$. Since $\varepsilon > 0$ is arbitrary and $\beta_\infty \leqslant$ sup norm topology, $h \in \beta_\infty$-closure of $C_b(X) \otimes E$, in $(C_b(X, E), \beta_\infty)$. Also since $\|h_J - h^*\| \to 0$ in $(C_b(X), \beta_\infty)$, for a $\mu \in M_\infty(X, E')$, $|\mu|(\|h_J - h^*\|) \to 0$. This gives $h^* \in \mathfrak{L}_1(\mu, X, E)$. Since $\|h - h^*\| < \varepsilon$, and ε is arbitrary, this gives $h \in \mathfrak{L}_1(\mu, X, E)$ and so $C_b(X, E) \subset \mathfrak{L}_1(\mu, X, E)$. This completes the theorem.

REMARK 3.4. What we have proved above is that given $f \in C_b(X, E)$ and $\varepsilon > 0$ there exist an $h \in C_b(X, E)$ and a net $\{h_\alpha\}_{\alpha \in I} \subset C_b(X) \otimes E$ such that $h_\alpha \to h$, pointwise, $\{h_\alpha: \alpha \in I\} \in \mathcal{H}^\infty(X)$, and $\|h - f\| < \varepsilon$.

THEOREM 3.5. *$(C_b(X, E), \beta_\infty)' = M_\infty(X, E')$, the element $L \in (C_b(X, E), \beta_\infty)'$ being related to the corresponding $\mu \in M_\infty(X, E')$, by $L(f) = \mu(f)$, $\forall f \in C_b(X, E)$.*

PROOF. Take $\mu \in M_\infty(X, E')$. Then by Theorem 3.3, $\mu(f)$ is defined $\forall f \in C_b(X, E)$. Suppose a net $\{f_\alpha\} \in \mathcal{H}^\infty$ and $f_\alpha \to 0$, pointwise, in $C_b(X, E)$. This means $\{\|f_\alpha\|\} \in \mathcal{H}^\infty(X)$ and $\|f_\alpha\| \to 0$, in $C_b(X)$, pointwise. Since $|\mu(f_\alpha)| \leqslant |\mu|(\|f_\alpha\|) \to 0$, we get $\mu(f_\alpha) \to 0$ and so $\mu \in (C_b(X, E), \beta_\infty)'$.

Conversely let $L \in (C_b(X, E), \beta_\infty)'$. This implies L is continuous with norm topology on $C_b(X, E)$. For $x \in E$, the measure μ_x, defined by $\mu_x(f) = L(f \otimes x)$, $\forall f \in C_b(X)$, is easily seen to be in $M_\infty(X)$ and so μ can be considered to be an element of $M_\infty(X, E')$ (first μ_x can be considered as μ_x: $Ba \to K$ and then $\langle \mu(A), x \rangle$ can be defined as $\mu_x(A)$; one needs the usual procedure to prove $\mu \in M_\infty(X, E')$ [27]). Extend μ to $\mathfrak{L}_1$. From the first part of the proof $\mu \in (C_b(X, E), \beta_\infty)'$. Since $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\infty)$, we get $\mu(f) = L(f)$, $\forall f \in C_b(X, E)$. This proves the theorem.

COROLLARY 3.6. *A bounded subset of $(C_b(X, E), \beta_\infty)$ is norm-bounded.*

PROOF. With norm topology, i.e., the topology induced by $(C_b(X, E), \|\cdot\|)'$, $M_\infty(X, E')$ is a Banach space. The result follows now easily by the uniform boundedness principle (see [**23**, Theorem 4.7]).

THEOREM 3.7. *Let A be $\sigma(F', F)$ relatively countably compact subset of F', where $F = C_b(X, E)$ and $F' = M_\infty(X, E')$. If A is norm-bounded in $F' \subset (C_b(X, E), \|\ \|)'$, then A is equicontinuous in $(C_b(X, E), \beta_\infty)$. In particular,* (i) *$(C_b(X, E), \beta_\infty)$ is always Mackey*; (ii) *if E is a Banach space $(C_b(X, E), \beta_\infty)$ is strongly Mackey.*

PROOF. We first prove $|A| = \{|\mu|: \mu \in A\}$ is equicontinuous in $(C_b(X), \beta_\infty)'$. Using the technique of [**9**, p. 4] (see also [**24**]), it is enough to prove that for any partition of unity $\{f_\alpha\}$ $(\alpha \in I)$ on X and $\varepsilon > 0$, $\exists$ a finite subset I_0 of I such that $\sum_{\alpha \in I_0} |\mu|(f_\alpha) > |\mu|(1) - \varepsilon$, $\forall \mu \in A$. If this is not true, there exists a sequence $\{\mu_n\} \subset A$, a strictly increasing sequence $\{p(n)\} \subset N$ with $p(0) = 1$, and a distinct countable set $\{\alpha_n\} \subset I$ such that $|\mu_n|(\sum_{i=0}^{p(n+1)-p(n)-1} f_{\alpha_{p(n)+i}}) > \varepsilon/2$, $\forall n$. Thus there exists a sequence $\{g_n\}$ in $C_b(X, E)$ (using Theorem 2.1), $\|g_n\| \leqslant \sum_{i=0}^{p(n+1)-p(n)-1} f_{\alpha_{p(n)+i}}$, and $|\mu_n(g_n)| > \varepsilon/2$, $\forall n$.

For any subset $M \subset N$, $g_M = \sum_{n \in M} g_n$ is in $C_b(X, E)$ (note $g_n(x) = 0$, except for a finite number of values of n, for every $x \in X$) and $\|\sum_{n \in M} g_n\| \leqslant 1$. The countable set $\{g_M: M \text{ finite}\}$ is dense in $P = \{g_M: M \subset N\} \subset (C_b(X, E), \beta_\infty)$ and so if μ, in $M_\infty(X, E')$, is an adherent point of $\{\mu_n\}$, there is a subsequence of $\{\mu_n\}$, which for notational convenience we again denote by $\{\mu_n\}$, such that $\mu_n \to \mu$, pointwise on P (Lemma 2.5). Define $\lambda_n: 2^N \to K$, $\lambda_n(M) = \mu_n(g_M)$. Using the dominated convergence and the relation $|\mu_n(g)| \leqslant |\mu_n|(\|g\|)$, $\forall g \in C_b(X, E)$, we prove that the λ_n's are countably additive. Also $\lambda_n(M)$ is convergent $\forall M$. Thus $\mu_n(g_n) \to 0$, by Lemma 2.2, which is a contradiction. So we have established that $|A|$ is equicontinuous in $(C_b(X), \beta_\infty)$. Suppose A is not equicontinuous. This means $A^\circ = \{f \in C_b(X, E): |\mu(f)| \leqslant 1, \forall \mu \in A\}$ is not a β_∞ 0-nbd. Thus there exists an absolutely convex, pointwise closed $H \in \mathcal{K}^\infty(X, E)$ such that for any finite subset $\alpha \subset X$ and any $n \in N$, $\exists f_{\alpha,n} \in H$ and $\mu_{\alpha,n} \in A$ with the property that $\sup_{y \in \alpha} |f_{\alpha,n}(y)| < 1/n$ and $|\mu_{\alpha,n}(f_{\alpha,n})| > 1$. It is easily verified that $\|H\| = \{\|h\|: h \in H\} \in \mathcal{K}^\infty(X)$. Since $f_{\alpha,n} \to 0$, we get $\|f_{\alpha,n}\| \to 0$, pointwise. Using the equicontinuity of $|A|$, we get $|\mu|(\|f_{\alpha,n}\|) \to 0$, uniformly for $\mu \in A$. This contradicts $|\mu_{\alpha,n}(f_{\alpha,n})| > 1$, $\forall \alpha$ and $\forall n$. If A is absolutely convex and $\sigma(F', F)$-compact then A will be strongly bounded in F' [**22**, Theorem 5.1, p. 141]. By Corollary 3.6, A is norm-bounded and so is equicontinuous. If E is a Banach space and A is $\sigma(F', F)$, relatively countably compact subset of F', then A is a relatively countably compact subset of $((C_b(X, E), \|\cdot\|)'$,

$\sigma((C_b(X, E), \|\cdot\|)', F))$ and so is norm-bounded. Thus A is equicontinuous. This completes the theorem.

THEOREM 3.8. $\beta \leqslant \beta_\infty$.

PROOF. Let $H \in \mathcal{H}^\infty(X, E)$, $\mu \in (C_b(X, E), \beta)'$ and $f_\alpha \to 0$, pointwise, in H.

Let $g_\alpha = \sup_{\gamma \geqslant \alpha} \|f_\gamma\| \in C_b(X)$. Fix $\eta > 0$. Since $\|f_\alpha/(g_\alpha + \eta)\| \leqslant 1$, $\forall \alpha$ and $g_\alpha \downarrow 0$, $\mu((f_\alpha/(g_\alpha + \eta))\, g_\alpha) \to 0$ [**4**, Theorem 2.3]. Since μ is continuous with norm topology on $C_b(X, E)$ it follows that $\mu(f_\alpha) \to 0$ (since $\eta > 0$ is arbitrary). This implies $f_\alpha \to 0$ weakly in $(C_b(X, E), \beta)$. Since β_∞ is Mackey (Theorem 3.7) we get $\beta_\infty \geqslant \beta$ [**22**, 7.4, (a) $\Rightarrow$ (b_1)].

COROLLARY 3.9. (i) *$C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta)$.*

(ii) *If $\mu \in M_\tau(X, E)$, then $\mathfrak{L}_1(X, E, \mu) \supset C_b(X, E)$.*

(iii) *$C_b(X, E)' = M_\tau(X, E')$, the element $L \in C_b(X, E)'$ being related to corresponding $\mu \in M_\tau(X, E')$, by $L(f) = \mu(f)$, $\forall f \in C_b(X, E)$.*

PROOF. (i) follows from Theorems 3.3 and 3.8.

(ii) Take an $f \in C_b(X, E)$. By (i) there exists a net $\{f_\alpha\} \subset C_b(X) \otimes E$ such that $f_\alpha \to f$ in β-topology. This means $\|f_\alpha - f\| \to 0$ in $(C_b(X), \beta)$ (Lemma 2.4). This implies $|\mu|(\|f_\alpha - f\|) \to 0$. From this the required result easily follows.

(iii) Take $\mu \in M_\tau(X, E')$ and define $L(f) = \mu(f)$, $\forall f \in C_b(X, E)$. If $f_\alpha \to 0$ in $(C_b(X, E), \beta)$ then, by Lemma 2.4, $\|f_\alpha\| \to 0$ in $(C_b(X), \beta)$ and so $|\mu|(\|f_\alpha\|) \to 0$. Since $|L(f_\alpha)| = |\mu(f_\alpha)| \leqslant |\mu|(\|f_\alpha\|)$, we get $L \in (C_b(X, E), \beta)'$. Since $\beta \leqslant \beta_\infty$, the converse is very similar to what is done in Theorem 3.5.

On $M_\infty(X, E')$, the topology $\mathcal{H}^\infty$ is defined to be the topology of uniform convergence on $\mathcal{H}^\infty$.

THEOREM 3.10. *$(M_\infty(X, E'), \mathcal{H}^\infty)$ is complete. If E is a reflexive Banach space each $H \in \mathcal{H}^\infty$ is relatively $\sigma(C_b(X, E), M_\infty(X, E'))$-compact; in this case $(M_\infty(X, E'), \mathcal{H}^\infty)' = C_b(X, E)$ and $L(X) \otimes E'$ is dense in $(M_\infty(X, E'), \mathcal{H}^\infty)$, $L(X)$ being the linear space of all discrete Baire measures on X* [**18**].

PROOF. The first statement follows from the Grothendieck completeness theorem [**22**, Theorem 6.2, p. 148]. Suppose E is a reflexive Banach space. Take an absolurely convex, pointwise closed H in $\mathcal{H}^\infty$. With the weak topology on E and with the topology of convergence pointwise on X, H is compact (Ascoli's theorem, [**1**, Chapter X]). Take a net $\{f_\alpha\} \subset H$ such that $f_\alpha(x) \to 0$, weakly in E, $\forall x \in X$, and let $\mu \in M_\infty(X, E')$. We claim $\mu(f_\alpha) \to 0$. Suppose $Rl(\mu(f_\alpha)) \geqslant \varepsilon$, $\forall \alpha \in I$ for some $\varepsilon > 0$. Let $H_0 = \{f \in H\colon Rl(\mu(f)) \geqslant \varepsilon\}$. With norm topology on E and product topology on E^X, H_0 is a closed convex

subset of E^X, $\{f_\alpha\} \subset H_0$, and $0 \notin H_0$. By the separation theorem, there exists $\{g_i\colon 1 \leqslant i \leqslant n\} \subset E'$, $\{x_i\colon 1 \leqslant i \leqslant n\} \subset X$, and $\eta > 0$ such that $0 > \sup\{Rl \sum_{1 \leqslant i \leqslant n} g_i \circ f(x_i)\colon f \in H_0\} + \eta$. This contradicts the weak convergence of $\{f_\alpha(x)\}$ in E, $\forall x \in X$. Taking $-f_\alpha$ and $\pm if_\alpha$, instead of f_α, and proceeding in a similar way, we justify the claim. This proves that H is $\sigma(C_b(X, E), M_\infty(X, E'))$-compact. By Mackey-Arens theorem [22, p. 131], $(M_\infty(X, E'), \mathcal{K}^\infty)' = C_b(X, E)$. If $L(X) \otimes E'$ is not dense in $(M_\infty(X, E'), \mathcal{K}^\infty)$ there exists a nonzero $f \in C_b(X, E)$ such that $f \equiv 0$ on $L(X) \otimes E'$. But $f \equiv 0$ on $L(X) \otimes E'$ means $f \equiv 0$ on X, a contradiction. This proves the theorem.

REMARK. This theorem generalizes [18, Theorem 8.6, p. 17].

4. The topology $\beta_{\infty c}$. For a $\mu \in M_\sigma(X)$, we get, by Riesz representation, $\tilde{\mu} \in M(\tilde{X})$, the set of all regular Borel measures on $\tilde{X}$, such that $\tilde{\mu}(g) = \mu(g|_X)$, $\forall g \in C(\tilde{X})$. It is a simple verification that for a Baire measurable f: $\tilde{X} \to [0, \infty]$, $\int f\, d\tilde{\mu} = \int f|_X\, d\mu$. Let $M_c(X) = \{\mu \in M_\sigma(X)\colon \operatorname{supp}|\mu|^\sim \subset \nu X\}$. Every $f \in C(X)$ is integrable with respect to every $\mu \in M_c(X)$ [10], [18]. We denote by $\mathcal{K}(X)$ the collection of all pointwise bounded and equicontinuous subsets of $C(X)$. We write $M_{\infty c}(X)$ for $M_\infty(X) \cap M_c(X)$.

LEMMA 4.1. *Every $\mu \in M_{\infty c}(X)$ is continuous on each $H \in \mathcal{K}(X)$ with pointwise topology on H.*

PROOF. $\mu \in M_{\infty c}(X)$ implies $|\mu| \in M_{\infty c}(X)$. Let G be the compact support of $|\mu|^\sim$, and suppose a net $\{f_\alpha\}_{\alpha \in I}$, $f_\alpha \geqslant 0$, converges to 0 pointwise, in some $H \in \mathcal{K}(X)$. Let $g_\alpha = \sup_{\gamma \geqslant \alpha} f_\gamma$. Then $\{g_\alpha\colon \alpha \in I\} \in \mathcal{K}(X)$ and $g_\alpha \downarrow 0$ pointwise. Let $\tilde{g}_\alpha\colon \tilde{X} \to [0, \infty)$ be the unique continuous extension of g_α. Fix an $\alpha_0 \in I$. Then $M = \sup g_{\alpha_0}(G) < \infty$, since $G \subset \nu X$. For every $\alpha \geqslant \alpha_0$, put $h_\alpha = \inf(g_\alpha, M)$, $\{h_\alpha\colon \alpha \geqslant \alpha_0\} \in \mathcal{K}^\infty(X)$ and $h_\alpha \to 0$ pointwise. Thus $|\mu|(h_\alpha) \to 0$. If $\tilde{h}_\alpha\colon \tilde{X} \to [0, \infty)$ is the unique continuous extension of h_α, then for every $\alpha \geqslant \alpha_0$, $|\mu|(g_\alpha) = \int g_\alpha^\sim\, d|\mu|^\sim = \int_G g_\alpha^\sim\, d|\mu|^\sim = \int \inf(g_\alpha^\sim, M)\, d|\mu|^\sim = \int h_\alpha^\sim\, d|\mu|^\sim = |\mu|(h_\alpha)$ and so $|\mu|(g_\alpha) \to 0$. This proves $|\mu|(f_\alpha) \to 0$. In the general case, if $f_\alpha \to 0$, pointwise, in some $H \in \mathcal{K}$, then $\{|f_\alpha|\} \in \mathcal{K}(X)$ and $|f_\alpha| \to 0$, pointwise. Thus $|\mu(f_\alpha)| \leqslant |\mu|(|f_\alpha|) \to 0$, from which the result follows.

We denote by $\mathcal{K} = \mathcal{K}(X, E)$ the collection of all pointwise bounded and equicontinuous subsets of $C(X, E)$. $\mathcal{K}(X, E)$ is closed under taking absolutely convex hulls and pointwise closure. We define $M_c(X, E') = \{\mu \in M_\sigma(X, E')\colon |\mu| \in M_c(X)\}$ and we put $M_{\infty c}(X, E') = M_c(X, E') \cap M_\infty(X, E')$ (if there are no measurable cardinals, $M_{\infty c}(X, E') = M_c(X, E')$ [18]). On $C(X, E)$, the topology $\beta_{\infty c}$ is defined to be the finest locally convex topology which agrees with pointwise topology on each $H \in \mathcal{K}$ (the existence of this topology is guaranteed by [5]).

THEOREM 4.2. (i) *The topology* β_∞ *on* $C_b(X, E)$ *is finer than the one induced by* $(C(X, E), \beta_{\infty c})$.

(ii) $C_b(X) \otimes E$ *is dense in* $(C(X, E), \beta_{\infty c})$.

(iii) $\mu \in M_{\infty c}(X, E')$ *implies* $C(X, E) \subset \mathfrak{L}_1(\mu, X, E)$.

(iv) *For an* $f \in C(X), f \geqslant 0$, *and* $\mu \in M_{\infty c}(X, E') = |\mu|(f) = \sup\{|\mu(g)|: \|g\| \leqslant f, g \in C(X, E)\}$.

PROOF. (i) This follows from the definition of $\beta_{\infty c}$.

(ii) Take $f \in C(X, E)$. For any $r > 0$, $P_r = \{x \in X: \|f(x)\| \leqslant r\}$ and $Q_r = \{x \in X: \|f(x)\| \geqslant r + 1\}$ are disjoint zero-sets and so there exists an $h_r \in C_b(X)$, $0 \leqslant h_r \leqslant 1$, $h_r(P_r) = 1$, and $h_r(Q_r) = 0$ [27]. Define, $\forall n \in N$, $g_n = \sup_{1 \leqslant i \leqslant n} h_i$. Then $\{g_n\} \subset C_b(X)$, $0 \leqslant g_n \leqslant 1$, $g_n \uparrow 1$, and $g_n f \in C_b(X, E)$, $\forall n$. Since $\{g_n f\} \in \mathcal{K}$ and $g_n f \to f$ pointwise, we get $C_b(X, E)$ is dense in $(C(X, E), \beta_{\infty c})$. Using (i) and Theorem 3.3, we get the result.

(iii) Since $\mu \in M_\infty(X, E')$, $\mathfrak{L}_1(X, E, \mu) \supset C_b(X, E)$. In the notations of (ii), $\|f - g_n f\| \to 0$, pointwise. Also $\|f - g_n f\| \leqslant 2\|f\|$ and so by dominated convergence $|\mu|(\|f - g_n f\|) \to 0$. From this we get $\mathfrak{L}_1(\mu, X, E) \supset C(X, E)$.

(iv) Using the technique of (ii), this follows from Theorem 2.1.

THEOREM 4.3. $(C(X, E), \beta_{\infty c})' = M_{\infty c}(X, E')$, *the element* $L \in (C(X, E), \beta_{\infty c})'$ *being related to the corresponding* $\mu \in M_{\infty c}(X, E')$, *by* $L(f) = \mu(f)$, $\forall f \in C(X, E)$.

PROOF. Take $\mu \in M_{\infty c}(X, E')$ and define $L(f) = \mu(f)$, $\forall f \in C(X, E)$. Proceeding as in Theorem 3.5 and using Lemma 4.1, we get $L \in (C(X, E), \beta_{\infty c})'$. Conversely take an $L \in (C(X, E), \beta_{\infty c})'$. Defining μ, as in Theorem 3.5, and proceeding in an exactly similar way, we get $\mu \in M_\infty(X, E')$ and $\mu(f) = L(f)$, $\forall f \in C_b(X, E)$ (we will need Theorem 4.2(i)). To establish $|\mu| \in M_c(X)$, it is enough to prove that every $f \in C(X), f \geqslant 0$, is $|\mu|$-integrable [**10**, Theorem 17, p. 172]. Take $f \in C(X)$, $f \geqslant 0$, and suppose $|\mu|(f) = +\infty$. There exists a sequence $\{f_n\} \subset C_b(X)$, $0 \leqslant f_n \uparrow f$, and $|\mu|(f_n) > 4^n$, $\forall n$. By Theorem 2.1, there exists a sequence $\{g_n\} \subset C_b(X) \otimes E$ such that $\|g_n\| \leqslant f_n$ and $|\mu(g_n)| > 4^n$, $\forall n$. Putting $h_n = g_n/2^n$, we get $\|h_n\| \leqslant f/2^n$ and $|\mu(h_n)| > 2^n$, $\forall n$. This means $\{h_n\} \in \mathcal{K}$ and $h_n \to 0$, pointwise, which implies that $\mu(h_n) \to 0$, a contradiction of $|\mu(h_n)| > 2^n$, $\forall n$. To prove $L(f) = \mu(f)$, $\forall f \in C(X, E)$, take a sequence $\{g_n\}$ as in Theorem 4.2(ii). It is easy to verify that $L(f) = \lim L(g_n f) = \lim \mu(g_n f) = \mu(f)$ (note $|\mu(g_n f - f)| \leqslant |\mu|(\|g_n f - f\|) \to 0$ by dominated convergence theorem, since $\|g_n f - f\| \leqslant 2\|f\|$). This completes the proof.

THEOREM 4.4. (i) *Every element of* $\mathcal{K}$ *is bounded in* $(C_b(X, E), \beta_{\infty c})$.

(ii) $(M_{\infty c}(X, E'), \mathcal{K})$ *is complete* ($\mathcal{K}$ *being the topology of uniform convergence on the elements of* $\mathcal{K}$).

(iii) *If E is a reflexive Banach space, each $H \in \mathcal{H}$ is relatively $\sigma(C(X, E'), M_\infty(X, E'))$-compact, $(M_{\infty c}(X, E'), \mathcal{H})' = C(X, E)$, and $(M_{\infty c}(X, E'), \mathcal{H})$ is the completion of $(L(X) \otimes E', \mathcal{H})$.*

PROOF. (i) follows from the definition of $\beta_{\infty c}$. (ii) is immediate from [**22**, Theorem 6.2, p. 148], once we use the fact that $\mathcal{H}$ is saturated. Proof of (iii) is very similar to the corresponding result of Theorem 3.10.

REMARK. This theorem generalizes [**18**, Theorem 9.5, p. 23].

THEOREM 4.5. *If $A \subset M_{\infty c}(X, E')$ is norm-bounded in $(C_b(X, E), \|\cdot\|)'$ and is $\sigma(M_{\infty c}(X, E'), C(X, E))$ relatively countably compact, then A is equicontinuous in $(C(X, E), \beta_{\infty c})$. In particular, $(C(X, E), \beta_{\infty c})$ is Mackey and in case E is a Banach space $(C(X, E), \beta_{\infty c})$ is strongly Mackey.*

PROOF. We first prove, as in Theorem 3.7, that $|A|$ is equicontinuous on $(C(X), \beta_{\infty c})$. For this we take $H \in \mathcal{H}(X)$. Using the technique of [**9**, p. 4] it is enough to take H of the form $H = \{\sum_{\alpha \in I} c_\alpha f_\alpha : |c_\alpha| \leqslant p_\alpha\}$, where $\{f_\alpha\}_{\alpha \in I}$ is a partition of unity in X and $\{p_\alpha : \alpha \in I\}$ a set of positive real numbers. As in Theorem 3.7 it is enough to show that given $\varepsilon > 0$ there exists a finite subset $I_0 \subset I$ such that $|\mu|(\sum_{\alpha \in I \setminus I_0} p_\alpha f_\alpha) < \varepsilon$, $\forall\, \mu \in A$. Hereafter the procedure is very similar to Theorem 3.7. Details are omitted.

5. Denseness of $C_b(X) \otimes E$ in $(C_b(X, E), \beta_1)$. It was proved in Corollary 3.9 that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta)$. In this section we consider the same problem for β_1.

THEOREM 5.1. *If X has a σ-compact dense subset (e.g., X separable) then $\beta_1 = \beta_\infty$ on $C_b(X, E)$, in particular, $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_1)$.*

PROOF. Take $H \in \mathcal{H}^\infty(X, E)$ and suppose a net $\{f_\alpha\} \subset H$ converges to 0 pointwise. Take $\mu \in (C_b(X, E), \beta_1)'$. If $\mu(f_\alpha) \not\to 0$, by taking subnets if necessary we assume $|\mu(f_\alpha)| \geqslant \varepsilon$, $\forall \alpha$, for some $\varepsilon > 0$.

Taking $g_\alpha = \sup\{\|f_\gamma\| : \gamma \geqslant \alpha\}$, we get $\{g_\alpha\} \in \mathcal{H}^\infty(X)$ and $g_\alpha \downarrow 0$. By the Ascoli theorem [**1**, Chapter X] and [**20**], $\exists$ a subsequence $\{g_{\alpha(n)}\}$ of $\{g_\alpha\}$ such that $g_{\alpha(n)} \downarrow 0$. By [**4**, Theorem 2.3], for any $\eta > 0$, $\mu((f_{\alpha(n)}/(g_{\alpha(n)} + \eta))\, g_{\alpha(n)}) \to 0$, from which it easily follows that $\mu(f_{\alpha(n)}) \to 0$, a contradiction. Thus $(C_b(X, E), \beta_1)' \subset M_\infty(X, E')$. Since β_∞ is Mackey, $\beta_\infty \geqslant \beta_1$. Using Theorem 3.2 we get $\beta_1 = \beta_\infty$.

REMARK. The assumptions of this theorem are needed so that results of Lemma 2.5 may be applicable.

THEOREM 5.2. *If X or E is a D-space* [7], *then $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_1)$.*

PROOF. Take an $h \in C_b(X, E)$. Fixing $\varepsilon > 0$ and using the notation of

Theorem 3.3 we get an $h^* \in C_b(X, E)$, $\|h - h^*\| \leqslant \varepsilon$ and $h^*(x) = \sum_{\alpha \in I} h(x_\alpha) f_\alpha(x)$, for a partition of unity $\{f_\alpha\}_{\alpha \in I}$ in X. If E (resp. X) is a D-space, there exists, in E (in X), a dense (a d-dense, d being as in Theorem 3.3) $\{p_\gamma : \gamma \in P\}$, P being of nonmeasurable cardinal [7, p. 11, footnote (6)]. This means $\exists h^{**} \in C_b(X, E)$ and a partition $\{I_\gamma : \gamma \in P_0\}$ of I, cardinality of P_0 not exceeding that of P, such that $h^{**}(x) = \sum_{\gamma \in P_0} z_\gamma g_\gamma(x)$, $z_\gamma \in E$ with $\|z_\gamma\| \leqslant \sup_{x \in X} \|h(x)\| + \varepsilon$, $g_\gamma = \sum_{\alpha \in I_\gamma} f_\alpha$, and $\|h^* - h^{**}\| \leqslant \varepsilon$. Fix a $\mu \in (C_b(X, E), \beta_1)'$, $\mu \equiv 0$ on $C_b(X) \otimes E$, and define $\nu(A) = \mu(\sum_{\gamma \in A} z_\gamma \otimes g_\gamma)$ for any $A \subset P_0$. Proceeding as in Theorem 5.1 we prove that ν is countably additive; also ν is bounded and $\nu(\{p\}) = 0$, $\forall p \in P_0$. Since P_0 is of nonmeasurable cardinal, $\nu \equiv 0$ and so $\mu(h^{**}) = 0$. Since $\beta_1 \leqslant \|\cdot\|$ on $C_b(X, E)$, we have $\mu \in (C_b(X, E), \|\cdot\|)'$. Thus $|\mu(h)| \leqslant |\mu(h - h^{**})| \leqslant \|\mu\| \cdot 2\varepsilon$. Since ε is arbitrary we get $\mu(h) = 0$. This proves the theorem.

THEOREM 5.3. *If $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_1)$, then*
(i) *for any $\mu \in M_\sigma(X, E')$, $\mathfrak{L}_1(\mu, X, E) \supset C_b(X, E)$;*
(ii) *$(C_b(X, E), \beta_1)' = M_\sigma(X, E')$, $L \in (C_b(X, E), \beta_1)'$ being related to corresponding $\mu \in M_\sigma(X, E')$ by $L(f) = \mu(f)$, $\forall f \in C_b(X, E)$.*

Proof is similar to Theorem 3.9 and is omitted.

6. Some sufficient conditions for $(C_b(X, E), \beta)$ and $(C_b(X, E), \beta_1)$ to be Mackey. In Theorem 3.7 we proved that $(C_b(X, E), \beta_\infty)$ is strongly Mackey in case E is a Banach space. In this section we prove similar results for other topologies on $C_b(X, E)$.

THEOREM 6.1. *If X is a paracompact Hausdorff space and A is a norm-bounded, $\sigma(M_\tau(X, E'), C_b(X, E))$ relatively countably compact subset of $M_\tau(X, E')$, then A is equicontinuous on $(C_b(X, E), \beta)$. Consequently $(C_b(X, E), \beta)$ is Mackey, and in case E is a Banach space it is strongly Mackey.*

PROOF. Since $A \subset M_\infty(X, E')$, A is $\sigma(M_\infty(X, E'), C_b(X, E))$ relatively countably compact. This implies (proof of Theorem 3.7) that $|A|$ is β_∞-equicontinuous in $M_\infty(X)$. Since X is paracompact, $\beta = \beta_\infty$ and $M_\infty(X) = M_\tau(X)$. Thus $|A|$ is β-equicontinuous in $M_\tau(X)$. By the Lemma 2.4, A is β-equicontinuous.

THEOREM 6.2. *Suppose $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_1)$, and let A be a norm-bounded and $\sigma(M_\sigma(X, E'), C_b(X, E))$ a relatively countably compact subset of $M_\sigma(X, E')$. Then A is equicontinuous on $(C_b(X, E), \beta_1)$. Thus $(C_b(X, E), \beta_1)$ is Mackey and in case E is a Banach space it is strongly Mackey.*

PROOF. Suppose $A^\circ = \{f \in C_b(X, E) : |\mu(f)| \leqslant 1, \forall \mu \in A\}$ is not β_1

$0 =$ nbd. There exist a $k > 0$ and a zero-set Z in $\tilde{X}$, $Z \subset \tilde{X} \setminus X$ such that $A^\circ \not\supset S_k \cap V$, for any β_Z 0-nbd. V in $C_b(X, E)$, where $S_k = \{f \in C_b(X, E)$: $\|f\| \leqslant k\}$. This means there exists an increasing sequence $\{V_n\}$ of open sets in $\tilde{X}$, $\bar{V}_n \subset V_{n+1}$ (closure in $\tilde{X}$), and $\tilde{X} \setminus Z = \bigcup_{n=1}^{\infty} V_n$. Thus $\tilde{X} \setminus Z$ is paracompact and contains X. Take a partition of unity $\{f_\alpha\}_{\alpha \in I}$ in X, subordinate to the covering $\{V_n \cap X\}_{n \in N}$. Denoting by $\tilde{f}_\alpha$, the extension of f_α to $\tilde{X}$, we get $\tilde{f}_\alpha \equiv 0$, on Z, $\forall \alpha$. Thus for any finite subset $\gamma \subset I$ and $n \in N$, $\exists g_{\gamma,n} \in S_k$ and $\mu_{\gamma,n} \in A$ such that $\|g_{\gamma,n}(\Sigma_{\alpha \in \gamma} f_\alpha)\| \leqslant 1/2^n$ and $|\mu_{\gamma,n}(g_{\gamma,n})| > 1$. We claim, given $\varepsilon > 0$, there exists a finite subset $I_{(\varepsilon)} \subset I$ such that $|\mu|(\Sigma_{\alpha \in I_{(\varepsilon)}} f_\alpha) > |\mu|(1) - \varepsilon/(1 + k)$, $\forall \mu \in A$. If this is not true, there exist a sequence $\{h_n\} \subset C_b(X) \otimes E$, a sequence $\{\mu_n\} \subset A$, a strictly increasing sequence $\{p_n\} \subset N$, and distinct countable set $\{\alpha_n\} \subset I$ satisfying the conditions, $|\mu_n(h_n)| > \varepsilon/(1 + k)$ and $\|h_n\| \leqslant \Sigma_{i=0}^{p(n+1)-p(n)-1} f_{\alpha_{p(n)+i}}$. Proceeding as in Theorem 3.7 we get a contradiction and so the claim is established.

Let $\theta_0 = \sup\{\|\mu\|: \mu \in A\}$. Take $n_0 \in N$ such that $2^{n_0} > 4\theta_0$ and fix $\gamma_0 = I_{(1/4)}$. Putting $q_1 = \Sigma_{\alpha \in I_{(1/4)}} f_\alpha$ and $q_2 = \Sigma_{\alpha \in I \setminus I_{(1/4)}} f_\alpha$, we get $g_{\gamma_0,n_0} = q_1 g_{\gamma_0,n_0} + q_2 g_{\gamma_0,n_0}$. For a $\mu \in A$,

$$|\mu(g_{\gamma_0,n_0})| \leqslant |\mu|(q_1 \| g_{\gamma_0,n_0}\|) + |\mu|(q_2 \| g_{\gamma_0,n_0}\|)$$
$$\leqslant \theta_0 \frac{1}{2^{n_0}} + \frac{1}{4(1+k)} k \leqslant \frac{1}{2},$$

a contradiction to $|\mu_{\gamma,n}(g_{\gamma,n})| > 1$. This proves the result. The rest is similar to Theorem 3.7.

7. A different definition for β_∞. In case E is reals a different definition for β_∞ is given in [**28**, Definition 3.2, p. 292] and then it is proved that our definition is equivalent with that. In this section we prove that a similar result holds when E is a normed space. For a continuous semimetric d on X we define an equivalence relation, in X, $x \sim y \Leftrightarrow d(x, y) = 0$. This gives us a metric space $(X_d, \tilde{d})$ of equivalence classes, $d(\tilde{x}, \tilde{y}) = d(x, y)$ $(x \in \tilde{x}, y \in \tilde{y})$.

THEOREM 7.1. *β_∞ is the finest locally convex topology on $C_b(X, E)$ for which the canonical mappings $(C_b(X_d, E), \beta) \to C_b(X, E)$ are continuous for all continuous semimetrics d on X.*

PROOF. Let τ be the finest locally convex topology on $C_b(X, E)$ satisfying the above condition. If $\{f_\alpha\} \in \mathcal{K}^\infty(X, E)$ and $f_\alpha \to 0$ pointwise, by defining $d(x, y) = \sup_\alpha \|f_\alpha(x) - f_\alpha(y)\|$ we see $\{\tilde{f}_\alpha\} \in \mathcal{K}^\infty(X_d, E)$ and $f_\alpha^\sim \to 0$ pointwise, $f_\alpha^\sim$ being defined by $\tilde{f}_\alpha(\tilde{x}) = f_\alpha(x)$ $(x \in \tilde{x})$. This means $\tilde{f}_\alpha \to 0$ in $(C_b(X_d, E), \beta_\infty)$ and so $\tilde{f}_\alpha \to 0$ in $(C_b(X_d, E), \beta)$, since $\beta \leqslant \beta_\infty$. Thus $f_\alpha \to 0$ in $(C_b(X, E), \tau)$ and so $\tau \leqslant \beta_\infty$. Now for any continuous semimetric d, $(C_b(X_d, E), \beta)' = M_\tau(X_d, E') = M_\infty(X_d, E') = (C_b(X_d, E), \beta_\infty)'$ (we are using the fact that for metric space Y, $M_\tau(Y) = M_\infty(Y)$ and therefore

$M_\tau(Y, E') = M_\infty(Y, E')$ [28]). Since $(C_b(X_d, E), \beta)$ and $(C_b(X_d, E), \beta_\infty)$ are both Mackey (Theorems 6.1 and 3.7) we get $\beta = \beta_\infty$ on $C_b(X_d, E)$. We want to prove $\tau \geqslant \beta_\infty$, i.e., the mapping $(C_b(X, E), \tau) \to (C_b(X, E), \beta_\infty)$ is continuous. For this it is enough to prove that the canonical mapping $(C_b(X_d, E), \beta) \to (C_b(X, E), \beta_\infty)$ is continuous for any continuous semimetric d on X. Since $\beta = \beta_\infty$ on $C_b(X_d, E)$, this is equivalent to proving that the mapping $(C_b(X_d, E), \beta_\infty) \to (C_b(X, E), \beta_\infty)$ is continuous.

Take a net $\{f_\alpha\} \in \mathcal{K}^\infty(X_d, E)$ such that $f_\alpha \to 0$ pointwise. This means $\{f_\alpha \circ \varphi\} \in \mathcal{K}^\infty(X, d)$ and $f_\alpha \circ \varphi \to 0$ pointwise, $\varphi\colon X \to X_d$ being the canonical mapping. Thus $f_\alpha \circ \varphi \to 0$ in $(C_b(X, E), \beta_\infty)$. This proves the result.

8. 'Locally solid' property. It is easily seen that the topology β_0 has a 0-nbd. base consisting of sets V which are absolutely convex and has the additional property that $f \in V$, $g \in C_b(X, E)$ with $\|g\| \leqslant \|f\|$ implies $g \in V$; we shall call this property locally solid and V will be called a solid set (this terminology is used in ordered locally convex spaces [19], [22]). With $E = K = R$ it is known [19], [22] that $\beta_0, \beta, \beta_1, \beta_\infty$ have 0-nbd. bases consisting of sets which are absolutely convex and locally solid. In this section we generalize those results to the case when E is a normed space.

THEOREM 8.1. *The topologies $\beta_0, \beta, \beta_1, \beta_\infty$ on $C_b(X, E)$ are locally solid and the topology $\beta_{\infty c}$ on $C(X, E)$ is locally solid.*

PROOF. The result is obvious for β_0. We consider the topology β. Let $\mathcal{K}$ be all compact subsets of $\tilde{X} \setminus X$. For a $Q \in \mathcal{K}$ and an $h_Q \in C_Q(X)$, let $V_Q = \{f \in C_b(X, E), \|fh_Q\| \leqslant 1\}$. Then V_Q is solid and the polar V° of $V = \bigcup \{V_Q\colon Q \in \mathcal{K}\}$, in $M_\tau(X, E')$ satisfies the result, $V^0 = \bigcap \{V_Q^0\colon Q \in \mathcal{K}\}$. For a $\mu \in V^\circ$ and $Q \in \mathcal{K}$, $\mu \in V_Q^\circ$, and since V_Q is solid, $|\mu|(\|f\|) = \sup\{|\mu(g)|\colon g \in C_b(X, E), \|g\| \leqslant \|f\|\} \leqslant 1$, $\forall f \in V_Q$. From this it follows that the absolutely convex solid set $W = \{f \in C_b(X, E)\colon |\mu|(\|f\|) \leqslant 1, \forall \mu \in V^\circ\}$ contains $\bigcup \{V_Q\colon Q \in \mathcal{K}\}$ and is contained in $V^{\circ\circ}$. This proves that the topology β is locally solid. The case of β_1 is similar and is omitted. Let $\mathcal{T}$ be the locally convex, locally solid topology on $C_b(X, E)$ with nbd. base consisting of all sets $A_1 = \{f \in C_b(X, E)\colon |\mu|(\|f\|) \leqslant 1, \forall \mu \in A\}$, where A is β_∞-equicontinuous in $M_\infty(X, E')$. Evidently $\beta_\infty \leqslant \mathcal{T}$. Take a $\mu \in (C_b(X, E), \mathcal{T})'$. If $\{f_\alpha\} \subset \mathcal{K}^\infty(X, E)$ and $f_\alpha \to 0$, pointwise, then $\{\|f_\alpha\|\} \subset \mathcal{K}^\infty(X)$ and $\|f_\alpha\| \to 0$, in $(C_b(X), \beta_\infty)$. Fix $\varepsilon > 0$. There exists a β_∞-equicontinuous set $A \subset M_\infty(X, E')$, such that $|\mu(A_1)| \leqslant \varepsilon$. Since the equicontinuity of A implies norm-boundedness of A, $|A|$ is β_∞-equicontinuous in $M_\infty(X)$. Thus $|\nu|(\|f_\alpha\|) \to 0$ uniformly for $\nu \in A$. This means $f_\alpha \in A_1$ for $\alpha \geqslant$ some α_0 and so $|\mu(f_\alpha)| \leqslant \varepsilon$, $\forall\, \alpha \geqslant \alpha_0$. This means $\mu \in M_\infty(X, E')$. By Theorem 3.7, $\beta_\infty \geqslant \mathcal{T}$ and so $\beta_\infty = \mathcal{T}$. The case of $\beta_{\infty c}$ is similar to β_∞, the only difference being that we have to use the results of §4.

COROLLARY 8.2. *If* $f_\alpha \to 0$ *in* $(C_b(X, E), \beta_\infty)$ (*resp.* $(C(X, E), \beta_{\infty c})$), $\|f_\alpha\| \to 0$ *in* $(C_b(X, \beta_{\infty c}))$, (*resp.* $(C(X), \beta_{\infty c})$).

PROOF. Let $\mathfrak{A}_1 = \{H \in \mathcal{H}^\infty(X, E)$: H absolutely convex, equicontinuous, and pointwise closed$\}$ and $\mathfrak{A}_2 = \{H \in \mathcal{H}^\infty(X)$, H absolutely convex, equicontinuous, and pointwise closed$\}$. For every $H \in \mathfrak{A}_1$, take $\rho(H) \in \mathfrak{A}_2$ such that $\rho(H) \supset \|H\|$. Let V be an absolutely convex, solid 0-nbd. in $(C_b(X), \beta_\infty)$. Then for every $H \in \mathfrak{A}_2$, there exists a finite subset $\eta(H) \subset X$ and $\gamma(H) > 0$ such that $V \supset \bigcup_{H \in \mathfrak{A}_0}\{f \in H$: $\sup|f(\eta(H))| < \gamma(H)\}$. Then $W =$ the absolutely convex hull of $\bigcup_{H \in \mathfrak{A}_1}\{f \in H$: $\sup\|f(\eta(\rho(H)))\| < \gamma(\rho(H))\}$ is a 0-nbd. in $(C_b(X, E), \beta_\infty)$. Thus $f_\alpha \in W$, $\forall \alpha \geqslant \alpha_0$. Fix $\alpha \geqslant \alpha_0$. There exists a finite collection $\{\lambda_i\} \subset K$ and $h_i \in H_i \in \mathfrak{A}_1$, with $\Sigma|\lambda_i| \leqslant 1$, $\sup\|h_i(\eta(\rho(H_i)))\| < \gamma(\rho(H_i))$, and $f_\alpha = \Sigma\lambda_i h_i$. This means $\|h_i\| \in \rho(H_i)$ and $\sup\|h_i\|(\eta(\rho(H_i))) < \gamma(\rho(H_i))$. Thus $\Sigma|\lambda_i|\ \|h_i\| \in V$ and since V is solid $\|f_\alpha\| \in V$. This proves the result for β_∞. The case of $\beta_{\infty c}$ is similar.

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