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THE TRACES OF HOLOMORPHIC FUNCTIONS ON REAL SUBMANIFOLDS

BY

GARY ALVIN HARRIS[1]

ABSTRACT. Suppose M is a real-analytic submanifold of complex Euclidean n = space and consider the following question: Given a real-analytic function f defined on M, is f the restriction to M of an ambient holomorphic function? If M is a C.R. submanifold the question has been answered completely. Namely, f is the trace of a holomorphic function if and only if f is a C.R. function. The more general situation in which M need not be a C.R. submanifold is discussed in this paper.

A complete answer is obtained in case the dimension of M is larger than or equal to n and M is generic in some neighborhood of each point off its C.R. singularities. The solution is of infinite order and follows from a consideration of the following problem: Given a holomorphic function f and a holomorphic mapping Φ, when does there exist a holomorphic mapping F such that $f = F \circ \Phi$?

1. Introduction. Given a submanifold M of complex Euclidean n-space $\mathbf{C}^n$, it is desired to characterize all complex valued functions on M which are the restriction to M of ambient holomorphic functions. Such a function is called the *trace* of a holomorphic function, and we will refer to the characterization of these functions as the *holomorphic trace problem*.

In general, the holomorphic trace problem appears to be quite difficult; in fact, even the case of a C^∞ curve in $\mathbf{C}$ seems untractable. Here we will consider only real-analytic submanifolds of $\mathbf{C}^n$ and address the local problem:

Which complex valued functions in a neighborhood of some point p belonging to a real-analytic submanifold $M \subset \mathbf{C}^n$ are the restriction to M of a holomorphic function defined in some ambient neighborhood of p?

Here we should note that if M is a uniqueness set for holomorphic functions on $\mathbf{C}^n$, then a solution to the local holomorphic trace problem implies a solution to the global problem.

Because holomorphic functions are real-analytic on their domains of definition in $\mathbf{C}^n$, their traces must be real-analytic functions on M. Thus we

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need only consider germs of real-analytic functions on M.

If M is a totally real real-analytic submanifold of $\mathbf{C}^n$, it is known that every complex valued real-analytic function on M is locally the trace of an ambient holomorphic function. For example, suppose $M = \mathbf{R} \subset \mathbf{C}$ and f is real-analytic near $\mathbf{O}$. Then f has a power series representation $f = \sum_{j=0}^{\infty} a_j x^j$, and clearly the ambient holomorphic function $\tilde{f} \equiv \sum_{j=0}^{\infty} a_j z^j$ agrees with f on M. But any totally real real-analytic submanifold $M \subset \mathbf{C}^n$ is locally biholomorphically equivalent to a real linear subspace of $\mathbf{C}^n$ and a similar argument will work.

In 1966, Tomassini showed that if M is a C.R. real-analytic submanifold of $\mathbf{C}^n$, then a real-analytic function f: $M \to \mathbf{C}$ is locally the trace on M of an ambient holomorphic function exactly when f satisfies the Tangential Cauchy Riemann equations (i.e. f is a C.R. function) [T]. Here we are interested in the more general situation in which the Tangential Cauchy Riemann equations for M may have singularities. The following example serves to illustrate the phenomena which can occur.

EXAMPLE 1.1 ("cup" in $\mathbf{C}^2$). $M \equiv \{(z_1, z_1\bar{z}_1) | z_1 \in \mathbf{C}\} \subset \mathbf{C}^2$.

M is totally real at every point except $(0, 0)$, where M is tangent to $\{(z_1, 0) | z_1 \in \mathbf{C}\}$. Suppose f: $\mathbf{C}^2 \to \mathbf{C}$ is real-analytic in a neighborhood of $(0, 0)$. Then f has a power series representation

$$f = \sum_{|\alpha|=0}^{\infty} a_\alpha z_1^{\alpha_1} \bar{z}_1^{\alpha_2} z_2^{\alpha_3} \bar{z}_2^{\alpha_4}$$

in some neighborhood of $(0, 0)$ and

$$f|_M = \sum_{|\beta|=0}^{\infty} b_\beta z_1^{\beta_1} \bar{z}_1^{\beta_2}.$$

Here, as elsewhere in the paper, we use the usual multi-index notation for power series in several variables.

We observe that in some neighborhood of each point of $M \setminus \{(0, 0)\}$, $\bar{z}_1$ agrees on M with the holomorphic function z_2/z_1.

Because M is a uniqueness set for holomorphic functions on $\mathbf{C}^2$, f is the trace of an ambient holomorphic function in a neighborhood of $(0, 0)$ if and only if $b_\beta = 0$ for all $\beta = (\beta_1, \beta_2)$ such that $\beta_1 < \beta_2$.

This solves the holomorphic trace problem for the cup in $\mathbf{C}^2$ and also suggests the possible nature of the general solution; namely, an infinite collection of conditions on the coefficients of a power series representation for the given function. Examples 5.2 and 5.3 will serve to further illustrate this phenomenon. Theorem 4.6 is the solution to the holomorphic trace problem for a general class of submanifolds.

To demonstrate the flavor of Theorem 4.6 we provide an alternative approach to the trace problem for the "cup", $M \subset \mathbf{C}^2$. Consider the vector fields

$$Y_1 \equiv \frac{\partial}{\partial z_1} - \frac{\bar{z}_1}{z_1}\frac{\partial}{\partial \bar{z}_1} \quad \text{and} \quad Y_2 \equiv \frac{\partial}{\partial z_2} + \frac{1}{z_1}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}. \tag{1.1}$$

By direct calculations one can show for each $\mathbf{q} \in M \setminus \{(0, 0)\}$, $\{Y_{1\mathbf{q}}, Y_{2\mathbf{q}}\}$ is a basis for the tangent space to M at $\mathbf{q}$ and $Y_1Y_2 = Y_2Y_1$. A quick inspection of the form of Y_1 and Y_2 in (1.1) yields necessary conditions for a given function f to be the trace of a holomorphic function on M; namely, for each pair of nonnegative integers α_1 and α_2, $Y_1^{\alpha_1}Y_2^{\alpha_2}f|M \setminus \{(0, 0)\}$ extends real-analytically to all of M. The sufficiency of these conditions is guaranteed by Theorem 4.6, and the calculations in Example 5.1 show that this solution agrees with our previous solution for the cup.

§2 reformulates the holomorphic trace problem in terms of a local parametric representation for the submanifold M. This reformulation is Problem 3, which leads to the algebraic problem of factoring a given holomorphic function through a given holomorphic mapping. §3 is concerned with this problem and contains the key ideas of this paper, with Theorem 3.5 being the main result of the section.

§4 contains the technical machinery necessary to interpret analytically the algebraic results of §3. The above-mentioned Theorem 4.6 is the solution to the holomorphic trace problem for about half of all possible real-analytic submanifolds, and is the major result of this paper. In §5 the theory is explicitly applied to the case of a general 2-dimensional real-analytic submanifold of $\mathbf{C}^2$. Here three examples are presented, including Example 1.1 above. §6 concludes the paper with a brief discussion of the complex envelope and comments on the cases left open by Theorem 4.6.

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2. The holomorphic trace problem. We must establish some notation which will be used throughout the remainder of the paper. For any positive integer m, $\mathbf{R}^m$, respectively $\mathbf{C}^m$, will denote the germ of real, respectively complex, Euclidean m-space at the origin $\mathbf{O}$. For notational convenience we will also let $\mathbf{R}^m$, respectively $\mathbf{C}^m$, denote a particular representative for the germ $\mathbf{R}^m$, respectively $\mathbf{C}^m$, whose choice is determined by the context. We will let $\dim_r$, respectively $\dim_c$, denote real, respectively complex, dimension. In addition, $\mathbf{x} \equiv (x_1, \ldots, x_m)$ or $\mathbf{y} \equiv (y_1, \ldots, y_m)$ will consistently denote vectors with real coordinates, while $\mathbf{z} \equiv (z_1, \ldots, z_m)$ or $\mathbf{w} \equiv (w_1, \ldots, w_m)$ will denote vectors with complex coordinates. $\mathbf{C}[\langle x_1, \ldots, x_m\rangle]$ will denote the ring of germs at $\mathbf{O}$ of complex valued real-analytic functions on $\mathbf{R}^m$ and $\mathbf{C}[\langle z_1, \ldots, z_m\rangle]$ the ring of germs at $\mathbf{O}$ of holomorphic functions on $\mathbf{C}^m$. When convenient, $\mathbf{C}[\langle x_1, \ldots, x_m\rangle]$, respectively $\mathbf{C}[\langle z_1, \ldots, z_m\rangle]$, will be identified with the ring of convergent power series in the variables $x_1, \ldots, x_m$, respectively $z_1, \ldots, z_m$, centered at $\mathbf{O}$ with complex coefficients. It should be understood that the particular interpretation of $\mathbf{C}[\langle x_1, \ldots, x_m\rangle]$ or $\mathbf{C}[\langle z_1, \ldots, z_m\rangle]$ will be determined by the context. Finally, for any positive integer k, M^k will denote the germ at $\mathbf{O}$ of a k-dimensional real-analytic submanifold. As above, M^k may also denote a particular representative for the germ M^k, hence we can always assume $\mathbf{O} \in M^k$ and $\dim_r M^k = k$.

We can now formulate the holomorphic trace problem.

Problem 1. *Given* $M^k \subset \mathbf{C}^n$ *and* $f \in \mathbf{C}[\langle x_1, y_1, \ldots, x_n, y_n\rangle]$, *when does there exist* $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$ *such that* $f - F \in \mathbf{O}(M^k)$ (*i.e.*, $f - F$ *vanishes on* M^k)?

Let $\Phi \equiv (\varphi_1, \ldots, \varphi_n)$: $\mathbf{R}^k \to \mathbf{C}^n$ be any real-analytic parametric representation for M^k in some neighborhood of $\mathbf{O}$. We assume without loss of generality that $\Phi(\mathbf{O}) = \mathbf{O}$.

Clearly we can reformulate the holomorphic trace problem as follows.

Problem 2. *Given* $f \in \mathbf{C}[\langle x_1, \ldots, x_k\rangle]$, *when does there exist* $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$ *such that* $f = F \circ \Phi$?

Let ν: $\mathbf{R}^k \to \mathbf{C}^k$ be the embedding given by

$$\nu: (x_1, \ldots, x_k) \in \mathbf{R}^k \to (x_1, \ldots, x_k) \in \mathbf{C}^k. \tag{2.1}$$

If $f \in \mathbf{C}[\langle x_1, \ldots, x_k\rangle]$ then f has a power series representation

$$f = \sum_{|\alpha|=0}^{\infty} a_\alpha \mathbf{x}^\alpha.$$

We define $\tilde{f} \in \mathbf{C}[\langle z_1, \ldots, z_k\rangle]$ by

$$\tilde{f} \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \mathbf{z}^\alpha. \tag{2.2}$$

Here, as elsewhere, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $x = (x_1, \ldots, x_k)$ then $\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$. It follows that $\tilde{f}$ is the unique holomorphic function on $\mathbf{C}^k$ such that

$$\tilde{f} \circ \nu = f \tag{2.3}$$

and, furthermore, the mapping $f \to \tilde{f}$ is an isomorphism from $\mathbf{C}[\langle x_1, \ldots, x_k\rangle]$ onto $\mathbf{C}[\langle z_1, \ldots, z_k\rangle]$ with inverse given by $g \to g \circ \nu$.

For the parameterization Φ: $\mathbf{R}^k \to \mathbf{C}^k$ given above define $\tilde{\Phi}$: $\mathbf{C}^k \to \mathbf{C}^n$ by

$$\tilde{\Phi} \equiv (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n). \tag{2.4}$$

Thus, for each $j = 1, 2, \ldots, n$, $\tilde{\varphi}_j \in \mathbf{C}[\langle z_1, \ldots, z_k\rangle]$ is a nonunit and

$$\tilde{\Phi} \circ \nu = \Phi. \tag{2.5}$$

For a given $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$ it is an easy exercise to show $\widetilde{F \circ \Phi} = F \circ \tilde{\Phi}$. (For a general real-analytic F: $\mathbf{C}^n \to \mathbf{C}$ this need not be true.) Thus, given $f \in \mathbf{C}[\langle x_1, \ldots, x_k\rangle]$ and $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$,

$$f = F \circ \Phi \quad \textit{if and only if} \quad \tilde{f} = F \circ \tilde{\Phi}. \tag{2.6}$$

We now state our final equivalent reformulation of the holomorphic trace problem.

Problem 3. *Given* $f \in \mathbf{C}[\langle x_1, \ldots, x_k\rangle]$, *hence* $\tilde{f} \in \mathbf{C}[\langle z_1, \ldots, z_k\rangle]$, *when does there exist* $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$ *such that* $\tilde{f} = F \circ \tilde{\Phi}$?

We have reduced the original Problem 1 to the "new" Problem 3, in which we need consider only germs of holomorphic functions. We are thus motivated to suppress the submanifold altogether and consider a more general algebraic question concerning holomorphic functions. We will return to the consideration of a submanifold in §4.

3. Necessary and sufficient conditions that a holomorphic function factor through a holomorphic mapping. The question posed for the given $\tilde{\Phi}$ in Problem 3 can be posed for any holomorphic map. In this section we drop the "~" and relax any rank conditions Φ must satisfy in order to come from a parametrization. For convenience it is still assumed that $\Phi(\mathbf{O}) = \mathbf{O}$.

Let k and n be positive integers and for each $j = 1, 2, \ldots, n$ suppose $\varphi_j \in \mathbf{C}[\langle z_1, \ldots, z_k\rangle]$ is a nonunit. Let Φ: $\mathbf{C}^k \to \mathbf{C}^n$ be defined by

$$\Phi = (\varphi_1, \ldots, \varphi_n). \tag{3.1}$$

Question. If $f \in \mathbf{C}[\langle z_1, \ldots, z_k\rangle]$ *does there exist* $F \in \mathbf{C}[\langle w_1, \ldots, w_n\rangle]$ *such that* $f = F \circ \Phi \equiv F(\varphi_1, \ldots, \varphi_n)$?

Let $\mathcal{J}[\Phi]$ denote the complex Jacobian matrix $[\partial\varphi_i/\partial z_j]$ and for each $\mathbf{z} \in \mathbf{C}^k$, $\mathcal{J}[\Phi](\mathbf{z}) \equiv [\partial\varphi_i/\partial z_j(z)]$. We denote the usual *pointwise* rank of $\mathcal{J}[\Phi]$ at $\mathbf{p}$ by rk $\mathcal{J}[\Phi](\mathbf{p})$. It must be carefully distinguished from the following notion.

DEFINITION 3.1. The *generic rank of* Φ, denoted rk Φ, is the order of the largest minor subdeterminant of $\mathcal{J}[\Phi]$ whose germ at $\mathbf{O}$ does not vanish.

Thus rk $\Phi = s$ means there is an $s \times s$ minor A of $\mathcal{J}[\Phi]$ such that det A does not vanish identically on any neighborhood of $\mathbf{O}$, and rk $\mathcal{J}[\Phi](\mathbf{p}) \leqslant s$ at all points $\mathbf{p}$ in some neighborhood of $\mathbf{O}$.

REMARK 3.2. If rk $\Phi = s$ and rk $\mathcal{J}[\Phi](\mathbf{O}) = r$, then $r \leqslant s \leqslant \min\{k, n\}$ and $r = s$ if and only if rk $\mathcal{J}[\Phi](z)$ is constant in some neighborhood of $\mathbf{O}$.

An immediate consequence of Remark 3.2 is

$$\text{rk } \Phi = s \textit{ and } \text{rk } \mathcal{J}[\Phi](\mathbf{z}) \textit{ not constant} \Rightarrow \text{rk } \mathcal{J}[\Phi](\mathbf{O}) < s. \tag{3.2}$$

For example, Φ: $\mathbf{C}^2 \to \mathbf{C}^2$ defined by $\Phi(z_1, z_2) \equiv (z_1, z_1 z_2)$ has rk $\Phi = 2$;

however, $\operatorname{rk} \mathcal{J}[\Phi](\mathbf{O}) = 1$. Notice that $\operatorname{rk} \mathcal{J}[\Phi](\mathbf{z}) = 2$ for each $\mathbf{z} \in \mathbf{C}^2 \setminus \{(0, z_2)\}$.

DEFINITION 3.3. The *exceptional set of* Φ, denoted $E(\Phi)$, is defined to be $\{z \in \mathbf{C}^k | \operatorname{rk} \mathcal{J}[\Phi](z) < \operatorname{rk} \Phi\}$. If $\operatorname{rk} \Phi = s$ and A is any $s \times s$ minor of $\mathcal{J}[\Phi]$, we let $E(\Phi; A)$ denote $\{\mathbf{z} \in \mathbf{C}^k | \det A(\mathbf{z}) = 0\}$.

REMARK 3.4. For any such A we clearly have $E(\Phi) \subset E(\Phi; A)$ and it follows from the holomorphy of $\det A(z)$ that $E(\Phi; A)$ is closed and nowhere dense in $\mathbf{C}^k$. Moreover, it is clear that $E(\Phi) = \bigcap E(\Phi; B)$ where B ranges over all $s \times s$ minors of $\mathcal{J}[\Phi]$.

Suppose Φ is as above, (3.1), and $\operatorname{rk} \Phi = s$. For notational convenience we will assume

$$\det\left[\frac{\partial \varphi_i}{\partial z_j}(\mathbf{z})\right]_{i,j=1}^{s} \not\equiv 0. \tag{3.3}$$

Define a holomorphic mapping $H\colon \mathbf{C}^k \to \mathbf{C}^k$ by

$$H \equiv (\varphi_1, \ldots, \varphi_s, z_{s+1}, \ldots, z_k). \tag{3.4}$$

We see immediately that $\operatorname{rk} H = k$; in fact, $\operatorname{rk} \mathcal{J}[H](\mathbf{z}) = k$ for every $\mathbf{z} \in \mathbf{C}^k \setminus E(\Phi; A)$ where

$$A \equiv \left[\frac{\partial \varphi_i}{\partial z_j}\right]_{i,j=1}^{s}$$

and $\operatorname{rk} \mathcal{J}[H](\mathbf{z}) < k$ for every $z \in E(\Phi; A)$. In particular, H is a local biholomorphism defined on $\mathbf{C}^k \setminus E(\Phi; A)$. Thus for each $j = 1, 2, \ldots, k$ there exists a unique holomorphic vector field Z_j defined on $\mathbf{C}^k \setminus E(\Phi; A)$ such that

$$H_*(Z_j) = \frac{\partial}{\partial z_j}\Big|_{H(\mathbf{C}^k \setminus E(\Phi;A))}. \tag{3.5}$$

For any $\mathbf{q} \in \mathbf{C}^k \setminus E(\Phi; A)$ and $j = 1, 2, \ldots, k$ we let $Z_{j\mathbf{q}}$ denote the vector associated to $\mathbf{q}$ by the field Z_j.

We now list several important properties of $\{Z_1, \ldots, Z_n\}$ which follow directly from (3.5) and the fact that H is locally biholomorphic.

(3.6) *For any* $\mathbf{q} \in \mathbf{C}^k \setminus E(\Phi; A)$, $Z_{1\mathbf{q}}, \ldots, Z_{k\mathbf{q}}$ *are linearly independent.*

(3.7) $Z_i Z_j = Z_j Z_i$ *for every* $i, j = 1, 2, \ldots, k$.

(3.8) $Z_j(H_i) = \delta_i^j$ *for every* $i, j = 1, 2, \ldots, k$. (δ_i^j *is the Kronecker Delta.*)

Given $f \in \mathbf{C}[\langle z_1, \ldots, z_k \rangle]$ we will now seek necessary conditions on f in order that there may exist a solution to the question posed at the beginning of this section. Thus suppose $F \in \mathbf{C}[\langle w_1, \ldots, w_n \rangle]$ and $f = F \circ \Phi$.

We assume the following restrictive hypothesis on Φ.

HYPOTHESIS. $\operatorname{rk} \Phi = n$.

By Remark 3.2 we must then have $k \geqslant n$; however, we may still have rk $\mathcal{J}[\Phi](\mathbf{O}) < n$. We will see in §4 that this hypothesis is quiite reasonable for our application to submanifolds.

Under the hypothesis rk $\Phi = n$ it follows from (3.4) and (3.8) that

$$Z_j(\varphi_i) = \delta_i^j \textit{ for each } i, j = 1, 2, \ldots, n, \quad \textit{and}$$
$$Z_j(\varphi_i) = 0 \textit{ for each } i = 1, 2, \ldots, n \textit{ and } j = n+1, \ldots, k. \tag{3.9}$$

Suppose E is a closed nowhere dense subset of $\mathbf{C}^k$ and $\{Z_1, \ldots, Z_k\}$ is a collection of holomorphic vector fields defined on $\mathbf{C}^k \setminus E$ which satisfies (3.6), (3.7), and (3.9). $f = F \circ \Phi = F(\varphi_1, \ldots, \varphi_n)$ is a power series in $\varphi_1, \ldots, \varphi_n$ and it follows from (3.7) and (3.9) that

$$Z_j(f) = \begin{cases} \partial F/\partial w_j \circ \Phi & \text{if } j = 1, 2, \ldots, n, \\ 0 & \text{if } j = n+1, \ldots, k. \end{cases} \tag{3.10}$$

The left side of (3.10) is initially defined only on $\mathbf{C}^k \setminus E$, whereas the right side is defined holomorphically on all of $\mathbf{C}^k$. Thus we have a nontrivial condition on f, namely for each $j = 1, 2, \ldots, n$, $Z_j(f)$ extends holomorphically to all of $\mathbf{C}^k$ (i.e. across E) and for each $j = n+1, \ldots, k$, $Z_j(f) \equiv 0$. Notice, for each $j = 1, 2, \ldots, n$ not only does $Z_j(f)$ extend holomorphically to all of $\mathbf{C}^k$, but it is also a holomorphic function, $\partial F/\partial w_j$, composed with Φ.

Suppose for some positive integer ν and every n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leqslant \nu$ we have

$$Z_1^{\alpha_1} Z_2^{\alpha_2} \cdots Z_n^{\alpha_n}(f) = \frac{\partial^{|\alpha|} F}{\partial w_1^{\alpha_1} \ldots \partial w_n^{\alpha_n}} \circ \Phi.$$

For any $j = 1, 2, \ldots, k$, the arguments applied to f above can now be applied to $Z_1^{\alpha_1} \ldots Z_n^{\alpha_n}(f)$ to yield

$$Z_j Z_1^{\alpha_1} \ldots Z_n^{\alpha_n}(f) = \begin{cases} \dfrac{\partial}{\partial w_j}\left[\dfrac{\partial^{|\alpha|} F}{\partial w_1^{\alpha_1} \ldots \partial w_n^{\alpha_n}}\right] \circ \Phi & \text{if } j = 1, 2, \ldots, n, \\ 0 & \text{if } j = n+1, \ldots, k. \end{cases} \tag{3 11}$$

We conclude the following set of necessary conditions on f.

Conditions C. *For any n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, $Z_1^{\alpha_1} Z_2^{\alpha_2} \ldots Z_n^{\alpha_n}(f)$ is holomorphically extendible across E to all of $\mathbf{C}^k$, and for each $j = n+1, \ldots, k$, $Z_j(f) \equiv 0$.*

We now provide an answer to the question that began this section.

Theorem 3.5. *Let Φ: $\mathbf{C}^k \to \mathbf{C}^n$ be a holomorphic mapping with $\Phi(\mathbf{O}) = \mathbf{O}$ and* rk $\Phi = n$. *Then there exists a closed nowhere dense subset $E \subset \mathbf{C}^k$ and a collection of holomorphic vector fields $\{Z_1, \ldots, Z_k\}$ defined on $\mathbf{C}^k \setminus E$ which satisfies* (3.6), (3.7), *and* (3.9). *Moreover, if E is any closed nowhere dense subset of $\mathbf{C}^k$, $\{Z_1, \ldots, Z_k\}$ is any such collection of vector fields, and*

$f \in \mathbf{C}[\langle z_1, \ldots, z_k \rangle]$, *then*

(i) *there exists* $F \in \mathbf{C}[\langle w_1, \ldots, w_n \rangle]$ *such that* $f = F \circ \Phi$ *if and only if* f *satisfies Condition* C.

In addition,

(ii) *if such* F *exists, then for each n-tuple of nonnegative integers* $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$\frac{\partial^{|\alpha|} F}{\partial w_1^{\alpha_1} \ldots \partial w_n^{\alpha_n}}(\mathbf{O}) = Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}(f)(\mathbf{O}). \tag{3.12}$$

We have just shown the existence of the set E and the vector fields $\{Z_1, \ldots, Z_k\}$, as well as the necessity of Condition C in part (i). Moreover, (ii) follows immediately from (3.11). Thus the theorem will be proved when we show the sufficiency of Condition C in (i). To do this we will need some algebraic machinery.

Let $\mathfrak{F}$ denote $\{f \in \mathbf{C}[\langle z_1, \ldots, z_k \rangle] | f$ satisfies Condition C$\}$ and let $\mathbf{C}[[w_1, \ldots, w_n]]$ denote the ring of formal power series in the variables $w_1, \ldots, w_n$ centered at $\mathbf{O}$ with complex coefficients. For $f \in \mathfrak{F}$ define $H(f, \mathbf{w}) \in \mathbf{C}[[w_1, \ldots, w_n]]$ by

$$H(f; \mathbf{w}) \equiv \sum_{|\alpha|=0}^{\infty} \frac{Z_1^{\alpha_1} \cdots Z_n^{\alpha_n}(f)(\mathbf{O})}{\alpha_1! \cdots \alpha_n!} w_1^{\alpha_1} \cdots w_n^{\alpha_n}. \tag{3.13}$$

We adopt the usual convention $Z_1^0 \cdots Z_k^0(f) = f$.

The composition $H(f; \mathbf{w}) \circ \Phi \equiv H(f; \varphi_1, \ldots, \varphi_n)$ is clearly well defined. We will first verify the identity $H(f; \mathbf{w}) \circ \Phi = f$ and then use this result to conclude $H(f; \mathbf{w}) \in \mathbf{C}[\langle w_1, \ldots, w_n \rangle]$.

LEMMA 3.6. *For each* $f \in \mathfrak{F}$ *and* $j = 1, 2, \ldots, n$,

$$\partial H(f; \mathbf{w}) / \partial w_j = H(Z_j f; \mathbf{w}). \tag{3.14}$$

PROOF. For each $f \in \mathfrak{F}$ and $j = 1, 2, \ldots, n$, $Z_j f \in \mathfrak{F}$ and, hence, $H(Z_j f; \mathbf{w})$ is well defined. (3.14) follows from a straightforward calculation. □

LEMMA 3.7. *For each* $f \in \mathfrak{F}$, $H(f; \mathbf{w}) \circ \Phi = f$.

PROOF. As mentioned above $H(f; \mathbf{w}) \circ \Phi$ is well defined. Let $\mathfrak{M}$ be the maximal ideal of $\mathbf{C}[[z_1, \ldots, z_k]]$ and let $\mathcal{S} \subset \mathbf{R}$ be defined as follows:

$$\mathcal{S} \equiv \{\text{nonnegative integers } m | \text{for each } f \in \mathfrak{F}, H(f; \mathbf{w}) \circ \Phi = f + \mathfrak{M}^{m+1}\}.$$

Lemma 3.7 is proved by showing that $\mathcal{S}$ contains all nonnegative integers. Clearly $0 \in \mathcal{S}$. Suppose $m \in \mathcal{S}$ and f is any member of $\mathfrak{F}$. For each $j = 1, 2, \ldots, k$, we get

$$\frac{\partial[H(f;\mathbf{w})\circ\Phi]}{\partial z_j}=\sum_{\nu=1}^{n}\frac{\partial H(f;\mathbf{w})}{\partial w_\nu}\circ\Phi\frac{\partial\varphi_\nu}{\partial z_j}.$$

By (3.14) and the fact that $m\in\mathcal{S}$ we have

$$\frac{\partial[H(f;\mathbf{w})\circ\Phi]}{\partial z_j}=\sum_{\nu=1}^{n}Z_\nu(f)\frac{\partial\varphi_\nu}{\partial z_j}+\mathfrak{M}^{m+1}.$$

For each $j=n+1,\ldots,k$, $Z_j(f)=0$. Thus for each $j=1,2,\ldots,k$,

$$\sum_{\nu=1}^{n}Z_\nu(f)\frac{\partial\varphi_\nu}{\partial z_j}=\sum_{\nu=1}^{k}Z_\nu(f)\frac{\partial H_\nu}{\partial z_j}.$$

Let $\mathbf{p}\in\mathbf{C}^k\setminus E$. By (3.8) and the chain rule it follows that

$$\sum_{\nu=1}^{k}Z_\nu(f)\frac{\partial H_\nu}{\partial z_j}=\frac{\partial f}{\partial z_j}\tag{3.15}$$

in some neighborhood of $\mathbf{p}$. Thus (3.15) must hold in some neighborhood of $\mathbf{O}$, $\mathbf{C}^k$. Hence

$$\partial[H(f;\mathbf{w})\circ\Phi]/\partial z_j=\partial f/\partial z_j+\mathfrak{M}^{m+1}.\tag{3.16}$$

But $j=1,2,\ldots,k$ is arbitrary in (3.16) and $H(f;\mathbf{w})\circ\Phi=f+\mathfrak{M}^{m+1}$; thus $H(f;\mathbf{w})\circ\Phi=f+\mathfrak{M}^{m+2}$. Since $f\in\mathfrak{F}$ was arbitrary we have $m+1\in\mathcal{S}$. □

Proof of Theorem 3.5. By Lemma 3.7 we have the formal identity $H(f;\mathbf{w})\circ\Phi=f$ and it remains to show that $H(f;\mathbf{w})$ has positive radius of convergence. The convergence follows from $\operatorname{rk}\Phi=n$; as shown by the following result of Paul Eakin and this author [E–H]. □

Theorem (P. Eakin and G. Harris). *Let Φ be as in* (3.1). $\operatorname{rk}\Phi=n$ *if and only if for each* $F\in\mathbf{C}[[w_1,\ldots,w_n]]$, $F\circ\Phi\in\mathbf{C}[\langle z_1,\ldots,z_k\rangle]$ *implies* $F\in\mathbf{C}[\langle w_1,\ldots,w_n\rangle]$ (*i.e. the homomorphism $F\to F\circ\Phi$ is strongly injective*).

Corollary 3.8. *If* $\operatorname{rk}\Phi=n$ *and* $f=F\circ\Phi$ *then F is unique.*

Proof. Corollary 3.8 follows immediately from Theorem 3.5(ii). □

Remark 3.9. Suppose $\operatorname{rk}\Phi=s<n$, $Z_1,\ldots,Z_k$ are defined by (3.5), and $f\in\mathbf{C}[\langle z_1,\ldots,z_k\rangle]$. If for any s-tuple of nonnegative integers $\alpha=(\alpha_1,\ldots,\alpha_s)$, $Z_1^{\alpha_1}\cdots Z_s^{\alpha_s}(f)$ is holomorphically extendible to $\mathbf{C}^k$ and for each $j=s+1,\ldots,k$, $Z_j(f)=0$, then there exists $F\in\mathbf{C}[\langle w_1,\ldots,w_s\rangle]$ such that $F\circ\Phi=f$. In fact, by arguments similar to those used above we can show

$$F\equiv\sum_{|\alpha|=0}^{\infty}\frac{Z_1^{\alpha_1}\cdots Z_s^{\alpha_s}(f)(\mathbf{O})}{\alpha_1!\cdots\alpha_s!}w_1^{\alpha_1}\cdots w_s^{\alpha_s}$$

works.

For a general Φ with rk $\Phi < n$, the sufficient conditions of Remark 3.9 are not necessary.

EXAMPLE 3.10. $\Phi \equiv (z_1, z_1 z_2, z_1 z_2^2)$.

Let f: $\mathbf{C}^2 \to \mathbf{C}$ be given by $f(z_1, z_2) \equiv z_1 + z_1 z_2 + z_1 z_2^2$. Clearly there is an $F \in \mathbf{C}[\langle w_1, w_2, w_3\rangle]$ such that $F \circ \Phi = f$; namely, $F(w_1, w_2, w_3) \equiv w_1 + w_2 + w_3$. However, there does not exist G in either $\mathbf{C}[\langle w_1, w_2\rangle]$, $\mathbf{C}[\langle w_1, w_3\rangle]$, or $\mathbf{C}[\langle w_2, w_3\rangle]$ such that $f = G \circ \Phi$. Thus f does not satisfy the conditions of Remark 3.9.

Nevertheless, if rk $\mathfrak{J}[\Phi](z)$ is constantly s then we may assume $E(\Phi; A)$ is empty and the first set of conditions in Remark 3.9 is trivially satisfied. Moreover, the conditions $Z_{s+1}(f) = \cdots = Z_k(f) = 0$ are "equivalent" to the Rank Theorem; that is, Φ is biholomorphically equivalent to projection onto the first s-coordinates and $f = F \circ \Phi$ for some F if and only if f is independent of the last $k - s$ coordinates.

Theorem 3.5 answers the algebraic question which began this section in the situation rk $\Phi = n$. We saw in §2 that this algebraic question is related to the holomorphic trace problem as stated in Problem 3. Thus we will use the results of this section, especially Theorem 3.5, to derive a solution to the trace problem.

4. A solution to the holomorphic trace problem. We now return to the setting of §2. Thus M^k is a k-dimensional real-analytic submanifold of $\mathbf{C}^n$ with $\mathbf{O} \in M^k$ and Φ: $\mathbf{R}^k \to \mathbf{C}^n$ a real analytic parametrization of M^k with $\Phi(\mathbf{O}) = \mathbf{O}$. Suppose $\tilde{\Phi}$: $\mathbf{C}^k \to \mathbf{R}^k$ is defined by (2.4) and rk $\tilde{\Phi} = n$. Let f: $\mathbf{C}^n \to \mathbf{C}$ be real-analytic. Then Theorem 3.5, replacing f by $\widetilde{f \circ \Phi}$ and Φ by $\tilde{\Phi}$, gives necessary and sufficient conditions on $\widetilde{f \circ \Phi}$ such that there exists a holomorphic function F: $\mathbf{C}^n \to \mathbf{C}$ with $\widetilde{f \circ \Phi} = F \circ \tilde{\Phi}$. However, if such F exists, then by (2.6) $f \circ \Phi = F \circ \Phi$; that is, f is the trace of F on M^k.

Our task is two-fold. We must first interpret the condition "rk $\tilde{\Phi} = n$". Secondly, we must translate Condition C applied to $\widetilde{f \circ \Phi}$ into conditions on the original function f. The idea is simple. Suppose $Z_1, \ldots, Z_k$ are the holomorphic vector fields defined on an open dense subset of $\mathbf{C}^k$ and used in Condition C. Their "real parts" $X_1, \ldots, X_k$ will be defined on an open dense subset of $\mathbf{R}^k$. We use Φ to push $X_1, \ldots, X_k$ forward to M^k. The result will be real-analytic vector fields $Y_1, \ldots, Y_k$, defined on an open dense subset of M^k, which yield conditions on f analogous to Condition C applied to $\widetilde{f \circ \Phi}$.

For any point $\mathbf{q}$ belonging to $\mathbf{C}^k$ (respectively, $\mathbf{R}^k$ or M^k), $T_{\mathbf{q}}\mathbf{C}^k$ (respectively, $T_{\mathbf{q}}\mathbf{R}^k$ or $T_{\mathbf{q}}M^k$) will denote the complexified tangent space to $\mathbf{C}^k$ (respectively, $\mathbf{R}^k$ or M^k) at $\mathbf{q}$. We always let $\mathbf{i}$ denote the complex structure arising from complexification while $\mathbf{J}$ denotes the complex structure induced on the tangent space by the complex structure of the underlying space. For example, suppose $\{z_1, \ldots, z_k\}$ are the coordinates of $\mathbf{C}^k$ and $z_j = x_j + \mathbf{i}y_j$ for each $j = 1, 2, \ldots, k$. Then we define $\mathbf{J}$ by the relation

$$\mathbf{J}\,\partial/\partial x_j = \partial/\partial y_j \quad \text{for each } j = 1, 2, \ldots, k. \tag{4.1}$$

We will employ the notation $T'_{\mathbf{q}}\mathbf{C}^k \equiv \{Z_{\mathbf{q}} \in T_{\mathbf{q}}\mathbf{C}^k | \mathbf{i}Z_{\mathbf{q}} = \mathbf{J}Z_{\mathbf{q}}\}$ and $T''_{\mathbf{q}}\mathbf{C}^k \equiv \{Z_{\mathbf{q}} \in T_{\mathbf{q}}\mathbf{C}^k | \mathbf{i}Z_{\mathbf{q}} = -\mathbf{J}Z_{\mathbf{q}}\}$. It is well known that $T_{\mathbf{q}}\mathbf{C}^k = T'_{\mathbf{q}}\mathbf{C}^k \oplus T''_{\mathbf{q}}\mathbf{C}^k$ and $\{\partial/\partial z_1, \ldots, \partial/\partial z_k\}$ is a basis for $T'_{\mathbf{q}}\mathbf{C}^k$, while $\{\partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_k\}$ is a basis for $T''_{\mathbf{q}}\mathbf{C}^k$. Thus for $Z_{\mathbf{q}} \in T_{\mathbf{q}}\mathbf{C}^k$ there exist $\{a_j\}_{j=1}^k$ and $\{b_j\}_{j=1}^k$ contained in $\mathbf{C}$ such that

$$Z_{\mathbf{q}} = \sum_{j=1}^{k} a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j}. \tag{4.2}$$

We let $\bar{Z}_{\mathbf{q}}^{\mathbf{i}}$ denote conjugation with respect to the $\mathbf{i}$-structure and $\bar{Z}_{\mathbf{q}}^{\mathbf{J}}$ denote conjugation with respect to the $\mathbf{J}$-structure; that is,

$$\bar{Z}_{\mathbf{q}}^{\mathbf{i}} \equiv \sum_{j=1}^{k} \bar{a}_j \frac{\partial}{\partial \bar{z}_j} + \bar{b}_j \frac{\partial}{\partial z_j} \quad \text{and} \quad \bar{Z}_{\mathbf{q}}^{\mathbf{J}} \equiv \sum_{j=1}^{k} a_j \frac{\partial}{\partial \bar{z}_j} + b_j \frac{\partial}{\partial z_j}. \tag{4.3}$$

Thus we have

$$\overline{T'_{\mathbf{q}}\mathbf{C}^k}^{\mathbf{i}} = T''_{\mathbf{q}}\mathbf{C}^k = \overline{T'_{\mathbf{q}}\mathbf{C}^k}^{\mathbf{J}}. \tag{4.4}$$

In addition, we let $H_{\mathbf{q}}M^k$ denote the largest $\mathbf{J}$-linear subspace of $T_{\mathbf{q}}M^k$; that is, $H_{\mathbf{q}}M^k = T_{\mathbf{q}}M^k \cap \mathbf{J}T_{\mathbf{q}}M^k$, the *complex tangent space* to M^k at $\mathbf{q}$. It is well known that $H_{\mathbf{q}}M^k = H'_{\mathbf{q}}M^k \oplus H''_{\mathbf{q}}M^k$, where

$$H'_{\mathbf{q}}M^k \equiv T'_{\mathbf{q}}\mathbf{C}^n \cap T_{\mathbf{q}}M^k \quad \text{and} \quad H''_{\mathbf{q}}M^k \equiv T''_{\mathbf{q}}\mathbf{C}^n \cap T_{\mathbf{q}}M^k.$$

$H''_{\mathbf{q}}M^k$ defines the system of *Tangential Cauchy Riemann equations* for M^k at $\mathbf{q}$.

For notational convenience we will consistently let $\mathbf{p}$ denote the unique point in $\mathbf{R}^k$ such that $\Phi(\mathbf{p}) = \mathbf{q}$. We then define an integer valued function on $\mathbf{R}^k$ by

$$l(\mathbf{p}) \equiv \tfrac{1}{2}\dim_c H_{\mathbf{q}}M^k. \tag{4.5}$$

Thus $l(\mathbf{p}) = \dim_c H'_{\mathbf{q}}M^k = \dim_c H''_{\mathbf{q}}M^k$.

REMARK 4.1. M^k is a C. R. submanifold if and only if the function $\mathbf{p} \to l(\mathbf{p})$ is identically equal to $l(\mathbf{O})$.

PROPOSITION 4.2. $\operatorname{rk} \mathcal{J}[\tilde{\Phi}](\nu(\mathbf{p})) = k - l(\mathbf{p})$.

PROOF. Let $\nu_*\colon T_{\mathbf{p}}\mathbf{R}^k \to T_{\nu(\mathbf{p})}\mathbf{R}^k \subset T_{\nu(\mathbf{p})}\mathbf{C}^k$, $\Phi_*\colon T_{\mathbf{p}}\mathbf{R}^k \to T_{\mathbf{q}}M^k$, and $\tilde{\Phi}_*\colon T_{\nu(\mathbf{p})}\mathbf{R}^k \to T_{\mathbf{q}}\mathbf{C}^k$ be the usual push forwards. It follows that

$$T_{\nu(\mathbf{p})}\mathbf{C}^k = \nu_* T_{\mathbf{p}}\mathbf{R}^k \oplus \mathbf{J}\nu_* T_{\mathbf{p}}\mathbf{R}^k. \tag{4.6}$$

Moreover, it follows from (2.5) that

$$\tilde{\Phi}_*\nu_* = \Phi_*, \tag{4.7}$$

and hence,

$$T_{\mathbf{q}}M^k + \mathbf{J}T_{\mathbf{q}}M^k = \tilde{\Phi}_*\left[T_{\nu(\mathbf{p})}\mathbf{C}^k\right].$$

Thus the rank of the map $\tilde{\Phi}_*\colon T_{\nu(\mathbf{p})}\mathbf{C}^k \to T_{\mathbf{q}}\mathbf{C}^n$ is $2(k - l(\mathbf{p}))$. Therefore, $\operatorname{rk} \mathfrak{J}[\tilde{\Phi}](\nu(\mathbf{p})) = k - l(\mathbf{p})$. $\square$

Proposition 4.2 shows that the pointwise rank of $\tilde{\Phi}$ is k minus the dimension of the complex tangent space to M^k, and hence that $\operatorname{rk}\tilde{\Phi}$ is a biholomorphic invariant.

It follows from (4.5) and Proposition 4.2 that

$$k - [k/2] \leqslant \operatorname{rk} \mathfrak{J}[\tilde{\Phi}](\nu(\mathbf{p})) \leqslant k - \max\{k - n, 0\}. \tag{4.8}$$

($[\cdot]$ is the greatest integer function.)

Definition 4.3. A submanifold M^k of dimension k is *generic at* $\mathbf{q} \in M^k$ provided $\dim_c H_{\mathbf{q}}M^k = 2\max\{k - n, 0\}$. M^k is said to be *generic* if it is generic at each of its points.

It follows immediately from (4.8) and the continuity of $\mathfrak{J}[\tilde{\Phi}]$ that

$$\begin{gathered}\operatorname{rk} \mathfrak{J}[\tilde{\Phi}](\nu(\mathbf{p})) = n \text{ if and only if } k \geqslant n \text{ and}\\ M^k \text{ is generic in some neighborhood of } \Phi(\mathbf{p}).\end{gathered} \tag{4.9}$$

Definition 4.4. A point $\mathbf{q} \in M^k$ is called a *C.R. point of* M^k provided there is an open neighborhood U of $\mathbf{q}$ such that $M^k \cap U$ is a C.R. submanifold of $\mathbf{C}^n$ ($\dim_c H_{\mathbf{z}}M^k$ is constant for each $\mathbf{z} \in M^k \cap U$). Let $\mathrm{CR}(M^k)$ denote $\{\mathbf{q} \in M^k | \mathbf{q} \text{ is a CR point of } M^k\}$ and $E(M^k)$ denote $M^k \setminus \mathrm{CR}(M^k)$. We call $E(M^k)$ the set of CR *singular points of* M^k.

Remark 4.5. By the holomorphy of the components of $\mathfrak{J}[\tilde{\Phi}]$ and Proposition 4.2 it follows that $\mathrm{CR}(M^k)$ is an open dense subset of M^k. Moreover, $\dim_c H_{\mathbf{z}}M^k = 2[k - \operatorname{rk}\tilde{\Phi}]$ for each $\mathbf{z} \in \mathrm{CR}(M^k)$. Thus we have the appropriate interpretation of the hypothesis "$\operatorname{rk}\tilde{\Phi} = n$"; namely,

$$k \geqslant n \quad \textit{and} \quad \mathrm{CR}(M^k) \textit{ is generic.} \tag{4.10}$$

We now begin the task of translating Condition C applied to $\widetilde{f \circ \Phi}$ into conditions on f. Let $E(\tilde{\Phi})$ be the nowhere dense subset of $\mathbf{C}^k$ defined in Definition 3.3 by replacing Φ with $\tilde{\Phi}$. Proposition 4.2 implies

$$\nu^{-1}\left[\mathbf{C}^k \setminus E(\tilde{\Phi})\right] = \Phi^{-1}\left(\mathrm{CR}(M^k)\right). \tag{4.11}$$

Let $\mathbf{q}$ belong to $\mathrm{CR}(M^k)$. By (4.11) $\nu(\mathbf{p})$ belongs to $\mathbf{C}^k \setminus E(\tilde{\Phi})$. (Recall $\Phi(\mathbf{p}) = \mathbf{q}$ by convention.) By Remark 3.4 there exists an $s \times s$ minor A of $\mathfrak{J}[\tilde{\Phi}]$ such that $\nu(\mathbf{p}) \in \mathbf{C}^k \setminus E(\tilde{\Phi}; A)$, in which $E(\tilde{\Phi}; A)$ is defined by replacing Φ by $\tilde{\Phi}$ in Definition 3.3. For each $j = 1, 2, \ldots, k$ let $Z_{j\nu(\mathbf{p})}$ be the vector defined by (3.5). Then $Z_1, \ldots, Z_k$ are the holomorphic vector fields used in Condition C and it is the "real part" of these fields that we wish to push forward to M^k.

For each $j = 1, 2, \ldots, k$, (4.3) implies

$$Z_{j\nu(\mathbf{p})} + \bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})} \in \nu_* T_{\mathbf{p}}\mathbf{R}^k. \tag{4.12}$$

Thus for each $j = 1, 2, \ldots, k$ there exists a unique vector $X_{j\mathbf{p}} \in T_{\mathbf{p}}\mathbf{R}^k$ such that

$$\nu_* X_{j\mathbf{p}} = Z_{j\nu(\mathbf{p})} + \bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})}. \tag{4.13}$$

The real-analytic vector fields $X_1, \ldots, X_k$ are defined on $\mathbf{R}^k \setminus \nu^{-1}(E(\tilde{\Phi}; A))$, an open dense subset of $\mathbf{R}^k$ which is contained in $\Phi^{-1}(\mathrm{CR}(M^k))$.

We adopt the following notation:

$$\mathfrak{N} \equiv \Phi\left[\nu^{-1}\left(\mathbf{C}^k \setminus E(\tilde{\Phi}; A)\right)\right].$$

Thus $\mathfrak{N}$ is open and dense in M^k and contained in $\mathrm{CR}(M^k)$. Let $Y_1, \ldots, Y_k$ be the push forwards of $X_1, \ldots, X_k$. Hence $Y_1, \ldots, Y_k$ are real-analytic vector fields defined on $\mathfrak{N}$. Since $Z_1, \ldots, Z_k$ are holomorphic and pointwise linearly independent (recall (3.6)) on $\mathbf{C}^k \setminus E(\tilde{\Phi}; A)$, it follows that $\{X_{1\mathbf{p}}, \ldots, X_{k\mathbf{p}}\}$ is a basis for $T_{\mathbf{p}}\mathbf{R}^k$ for each $\mathbf{p} \in \Phi^{-1}(\mathfrak{N})$. Thus for each $\mathbf{q} \in \mathfrak{N}$, $\{Y_{1\mathbf{q}}, \ldots, Y_{k\mathbf{q}}\}$ is a basis for $T_{\mathbf{q}}M^k$. Moreover, by (4.7) and (4.13) we have

$$Y_{1\mathbf{q}} = \tilde{\Phi}_*(Z_{j\nu(\mathbf{p})}) + \tilde{\Phi}_*\left(\bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})}\right) \quad \text{for each } j = 1, 2, \ldots, k. \tag{4.14}$$

We now assume hypothesis (4.10), namely, $k \geqslant n$ and $\mathrm{CR}(M^k)$ is generic. Thus $\mathrm{rk}\, \tilde{\Phi} = n$. If $w_1, \ldots, w_n$ are the coordinates for $\mathbf{C}^n$ then (4.14) and (3.9) imply

$$Y_{j\mathbf{q}} = \begin{cases} \partial/\partial w_j + \tilde{\Phi}_*\left(\bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})}\right) & \text{if } j = 1, 2, \ldots, n, \\ \tilde{\Phi}_*\left(\bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})}\right) & \text{if } j = n+1, \ldots, k. \end{cases} \tag{4.15}$$

Thus $Y_{j\mathbf{q}} \in T''_{\mathbf{q}}\mathbf{C}^n \cap T_{\mathbf{q}}M^k = H''_{\mathbf{q}}M^k$ for each $j = n+1, \ldots, k$. Hence $\{Y_{n+1\mathbf{q}}, \ldots, Y_{k\mathbf{q}}\}$ is a basis for $H''_{\mathbf{q}}M^k$.

Suppose $f \in \mathbf{C}[\langle x_1, \ldots, x_k \rangle]$ and $\tilde{f} \in \mathbf{C}[\langle z_1, \ldots, z_k \rangle]$ is the extension of f; that is, $f = \tilde{f} \circ \nu$. Because $Z_j\tilde{f}$ is holomorphic on $\mathbf{C}^k \setminus E(\tilde{\Phi}; A)$ we have

$$\widetilde{X_j f} = Z_j \tilde{f} \quad \text{for each } j = 1, 2, \ldots, k. \tag{4.16}$$

Commutativity of $X_1, \ldots, X_k$ follows from commutativity of $Z_1, \ldots, Z_k$ (recall (3.7)) and (4.16). Hence $Y_1, \ldots, Y_k$ commute.

We now present the main result of the paper.

THEOREM 4.6. *Suppose M^k is a k-dimensional real-analytic submanifold of $\mathbf{C}^n$ with $k \geqslant n$ and $\mathrm{CR}(M^k)$ generic. There exist holomorphic coordinates $w_1, \ldots, w_n$ for $\mathbf{C}^n$, an open dense subset $\mathfrak{N} \subset \mathrm{CR}(M^k)$, and real-analytic vector fields $Y_1, \ldots, Y_k$ defined on $\mathfrak{N}$ which satisfy the following properties*:

(1) $Y_iY_j = Y_jY_i$ *for each* $i, j = 1, 2, \ldots, k$.

(2) *For each* $\mathbf{q} \in \mathfrak{N}$, $\{Y_{1\mathbf{q}}, \ldots, Y_{k\mathbf{q}}\}$ *is a basis for* $T_{\mathbf{q}}M^k$ *and* $\{Y_{n+1\mathbf{q}}, \ldots, Y_{k\mathbf{q}}\}$ *is a basis for* $H''_{\mathbf{q}}M^k$. *In addition, for each* $j = 1, 2, \ldots, n$,

$$Y_{j\mathbf{q}} = \partial/\partial w_j|_{\mathbf{q}} \bmod T''_{\mathbf{q}}\mathbf{C}^n.$$

Moreover, suppose $\mathfrak{N}$ *is any open dense subset of* $CR(M^k)$ *and* $\{Y_1, \ldots, Y_k\}$ *any collection of real-analytic vector fields defined on* $\mathfrak{N}$ *satisfying* (1) *and* (2). *Then a given real-analytic function* f: $M^k \to \mathbf{C}$ *is the trace of a holomorphic function if and only if*

(a) $Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}(f)|_{\mathfrak{N}}$ *is real-analytically extendible to all of* M^k *for each* n*-tuple of positive integers* $\alpha = (\alpha_1, \ldots, \alpha_n)$; *and*

(b) $Y_j(f)|_{\mathfrak{N}} \equiv 0$ *for each* $j = n + 1, \ldots, k$.

Remark 4.7. If $M^k = \mathrm{CR}(M^k)$ then we may assume $\mathfrak{N} = M^k$. Thus (a) is vacuous and (b) recovers the Tangential Cauchy Riemann equations. That is, f is a CR function.

Proof of Theorem 4.6. The existence part of Theorem 4.6 follows immediately from the preceding discussion. To prove the second part of Theorem 4.6, suppose $\mathfrak{N}$ is an open dense subset of $\mathrm{CR}(M^k)$ and $Y_1, \ldots, Y_k$ are real-analytic vector fields on $\mathfrak{N}$ which satisfy (1) and (2). The necessity of conditions (a) and (b) follows immediately from (1) and (2). To prove sufficiency we simply reverse the discussion leading to Theorem 4.6.

Let $\Phi \equiv (\varphi_1, \ldots, \varphi_n)$: $\mathbf{R}^k \to \mathbf{C}^n$ be any real-analytic parametrization of M^k and assume $\Phi(\mathbf{O}) = \mathbf{O}$. Let $X_1, \ldots, X_k$ be the real-analytic vectors on $\Phi^{-1}(\mathfrak{N})$ such that $\Phi_* X_{j\mathbf{p}} = Y_{j\mathbf{q}}$ for each $j = 1, 2, \ldots; k$ and $\mathbf{p} \in \Phi^{-1}(\mathfrak{N})$. We wish to define holomorphic vector fields $Z_1, \ldots, Z_k$ on an open subset U of $\mathbf{C}^k$ such that $\nu^{-1}(U) = \Phi^{-1}(\mathfrak{N})$ and

$$\nu_* X_{j\mathbf{p}} = Z_{j\nu(\mathbf{p})} + \bar{Z}^{\mathbf{J}}_{j\nu(\mathbf{p})}. \tag{4.17}$$

We can do this as follows: if $x_1, \ldots, x_k$ are the coordinates of $\mathbf{R}^k$ and $\{a_{ij}\}_{i,j=1}^k$ is a set of real-analytic functions on $\Phi^{-1}(\mathfrak{N})$ such that

$$X_j = \sum_{i=1}^k a_{ij} \frac{\partial}{\partial x_i}$$

for each $j = 1, 2, \ldots, k$, then define for each $j = 1, 2, \ldots, k$,

$$Z_j \equiv \sum_{i=1}^k \tilde{a}_{ij} \frac{\partial}{\partial z_i}, \qquad z_i \equiv x_i + \mathrm{i}y_i.$$

It is easy to see that $Z_1, \ldots, Z_k$ defined above satisfies (4.17), and hence for any real-analytic g: $\mathbf{R}^k \to \mathbf{C}$,

$$X_j g = Z_j \tilde{g} \circ \nu \quad \text{for each } j = 1, 2, \ldots, k.$$

Thus the fields $Z_1, \ldots, Z_k$ commute. Moreover,

$$\tilde{\Phi}_*(Z_{j\nu(\mathbf{p})}) = \begin{cases} \partial/\partial w_j|_{\mathbf{q}} & \text{if } j = 1, 2, \ldots, n, \\ 0 & \text{if } j = n+1, \ldots, k. \end{cases}$$

Hence $Z_1, \ldots, Z_k$ satisfy the conditions of Theorem 3.5. Suppose f: $M^k \to \mathbf{C}$ is real-analytic and satisfies conditions (a) and (b). Thus $\widetilde{f \circ \Phi}$ satisfies Condition C with respect to $Z_1, \ldots, Z_k$. Hence, by Theorem 3.5, there exists $F \in \mathbf{C}[\langle w_1, \ldots, w_n \rangle]$ such that $\widetilde{f \circ \Phi} = F \circ \tilde{\Phi}$. Thus $f \circ \Phi = F \circ \Phi$. $\square$

Theorem 4.6 is the solution to the holomorphic trace problem in case $\mathrm{CR}(M^k)$ is generic and $k > n$. Two questions are immediately raised. First, what happens if we relax the hypothesis on M^k, and, secondly, how applicable is the solution in particular examples? The latter question is considered for a general 2-dimensional submanifold of $\mathbf{C}^2$ in §5. The former question is considered briefly in §6.

5. Application of results to examples of 2-dimensional submanifolds of $\mathbf{C}^2$. In order to apply the results of §4 to a given submanifold we must construct vector fields $Y_1, \ldots, Y_k$ satisfying properties (1) and (2) of Theorem 4.6. The proofs of Theorems 3.5 and 4.6 provide a method whereby such fields can be constructed. We will now explicitly construct such vector fields for a general 2-dimensional real-analytic submanifold of $\mathbf{C}^2$.

Let M^2 be a 2-dimensional real-analytic submanifold of $\mathbf{C}^2$. Further suppose $\Phi \equiv (\varphi_1, \varphi_2)$: $\mathbf{R}^2 \to \mathbf{C}^2$ is a real-analytic parametrization for M^2 with $\Phi(\mathbf{O}) = \mathbf{O}$. Let $\tilde{\Phi} \equiv (\tilde{\varphi}_1, \tilde{\varphi}_2)$: $\mathbf{C}^2 \to \mathbf{C}^2$ be the holomorphic extension of Φ. Let

$$A \equiv \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial x_1} & \dfrac{\partial \varphi_1}{\partial x_2} \\[2ex] \dfrac{\partial \varphi_2}{\partial x_1} & \dfrac{\partial \varphi_2}{\partial x_2} \end{bmatrix};$$

then

$$\tilde{A} \equiv \begin{bmatrix} \widetilde{\dfrac{\partial \varphi_1}{\partial x_1}} & \widetilde{\dfrac{\partial \varphi_1}{\partial x_2}} \\[2ex] \widetilde{\dfrac{\partial \varphi_2}{\partial x_1}} & \widetilde{\dfrac{\partial \varphi_2}{\partial x_2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \tilde{\varphi}_1}{\partial z_1} & \dfrac{\partial \tilde{\varphi}_1}{\partial z_2} \\[2ex] \dfrac{\partial \tilde{\varphi}_2}{\partial z_1} & \dfrac{\partial \tilde{\varphi}_2}{\partial z_2} \end{bmatrix}.$$

That is, $\tilde{A} = \mathcal{J}[\tilde{\Phi}]$. We now assume the hypothesis "$\operatorname{rk} \tilde{\Phi} = 2$." Note $\widetilde{\det A} = \det \tilde{A}$, $E(\tilde{\Phi}) = \{\mathbf{z} \in \mathbf{C}^2 | \det \tilde{A}(\mathbf{z}) = 0\}$, and $\tilde{A}^{-1} = \widetilde{A^{-1}}$ off $E(\tilde{\Phi})$. Let Z_1, Z_2 be the vector fields defined by (3.5); recall for this case $H \equiv (\tilde{\varphi}_1, \tilde{\varphi}_2)$. We compute

$$(Z_1, Z_2) = (\partial/\partial z_1, \partial/\partial z_2)\widetilde{A^{-1}}$$

on $\mathbf{C}^2 \setminus E(\tilde{\Phi})$. It follows from (4.3) and (4.13) that

$$(X_1, X_2) = (\partial/\partial x_1, \partial/\partial x_2)A^{-1}.$$

Thus

$$(Y_1, Y_2) = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}\right)\begin{bmatrix} I \\ \bar{A}A^{-1} \end{bmatrix} \tag{5.1}$$

on $M^2 \setminus E(M^2)$.

We will now provide several examples, beginning with the example which started this paper. In particular, we will derive the vector fields (1.1).

EXAMPLE 5.1. "Cup": $M^2 \equiv \{(x_1 + \mathbf{i}x_2, x_1^2 + x_2^2)|x_1, x_2 \in \mathbf{R}\}$.

M^2 in Example 5.1 is parametrized by Φ: $\mathbf{R}^2 \to \mathbf{C}^2$ where $\Phi(x_1, x_2) \equiv (x_1 + \mathbf{i}x_2, x_1^2 + x_2^2)$. Letting $z_1 \equiv x_1 + \mathbf{i}x_2$, (5.1) yields

$$Y_1 \equiv \frac{\partial}{\partial z_1} - \frac{\bar{z}_1}{z_1}\frac{\partial}{\partial \bar{z}_1} \quad \text{and} \quad Y_2 \equiv \frac{\partial}{\partial z_2} + \frac{1}{z_1}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2} \quad \text{on } M^2 \setminus \{(0, 0)\},$$

which are the vectors in (1.1). Given a real-analytic function f: $\mathbf{C}^2 \to \mathbf{C}$, $f \circ \Phi$ is a function of z_1 and $\bar{z}_1$; thus we assume f is independent of z_2. It is easy to show that for any (α_1, α_2) with $\alpha_1 + \alpha_2 = n$, $Y_1^{\alpha_1}Y_2^{\alpha_2}f \in \mathbf{C}[\langle x_1, x_2\rangle]$ if and only if $Y_2^n f \in \mathbf{C}[\langle x_1, x_2\rangle]$. Thus we conclude by Theorem 4.6 that f has a holomorphic extension to $\mathbf{C}^2$ if and only if for each $n = 0, 1, 2, \ldots,$

$$\frac{1}{z_1^n}\frac{\partial}{\partial \bar{z}_1^n}(f) \in \mathbf{C}[\langle x_1, x_2\rangle].$$

But this is equivalent to

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial z_1^{\alpha_1}\partial \bar{z}_1^{\alpha_2}}(f)(\mathbf{O}) = 0 \quad \text{for each } \alpha_1 < \alpha_2. \quad \square$$

EXAMPLE 5.2. "Saddle": $M^2 \equiv \{(x_1 + \mathbf{i}x_2, x_1^2 - x_2^2)|x_1, x_2 \in \mathbf{R}\}$.

M^2 in Example 5.2 is parametrized by Φ: $\mathbf{R}^2 \to \mathbf{C}^2$ where $\Phi(x_1, x_2) \equiv (x_1 + \mathbf{i}x_2, x_1^2 - x_2^2)$. Proceeding as for the "cup" we obtain

$$Y_1 = \frac{\partial}{\partial z_1} - \frac{z_1}{\bar{z}_1}\frac{\partial}{\partial \bar{z}_1} \quad \text{and} \quad Y_2 = -\frac{\partial}{\partial z_2} + \frac{1}{\bar{z}_1}\frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}.$$

Again assume f is independent of z_2. Because of the special nature of Y_1 and Y_2 we conclude that f has a holomorphic extension if and only if $Y_2^n(f) \in \mathbf{C}[\langle x_1, x_2\rangle]$ for each $n = 0, 1, 2, \ldots$. A simple calculation shows this is equivalent to

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial z_1^{\alpha_1}\partial \bar{z}_1^{\alpha_2}}(f)(\mathbf{O}) = 0 \quad \text{for each } (\alpha_1, \alpha_2) \textit{ with } \alpha_1 \textit{ odd.} \quad \square$$

EXAMPLE 5.3. "Parabola": $M^2 \equiv \{(x_1 + \mathbf{i}x_2, x_1^2)|x_1, x_2 \in \mathbf{R}^2\}$.

For Example 5.3 we consider Φ: $\mathbf{R}^2 \to \mathbf{C}^2$ given by $\Phi(x_1, x_2) = (x_1 + \mathbf{i}x_2, x_1^2)$. Again assuming f is independent of z_2 we compute

$$Y_1 \equiv -i\frac{\partial}{\partial x_2} \quad \text{and} \quad Y_2 \equiv \frac{1}{x_1}\left[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right].$$

Notice in this case $E(M^2) = \{(0 + ix_2, 0) | x_2 \in \mathbf{R}\}$. Suppose f: $\mathbf{C}^2 \to \mathbf{C}$ is real-analytic and represented by a power series $\Sigma a_{mn} x_1^m x_2^n$, $z_1 \equiv x_1 + \mathrm{i}x_2$. By direct computation we conclude $Y_2(f) \in \mathbf{C}[\langle x_1, x_2 \rangle]$ if and only if

$$a_{1,n} + \mathrm{i}(n+1)a_{0,n} = 0 \quad \text{for each } n = 0, 1, 2, \ldots .$$

Thus we have necessary conditions on f. To obtain more conditions we would have to compute $Y_1 Y_2(f)$, $Y_2^2(f)$, etc. □

It is interesting to notice that in each of the above examples the conditions turned out to be an infinite set of linear relations among the coefficients of the power series expansion for f. One might hope that some finite number of the conditions imply the rest; however, in general this is false. For example, suppose N is any positive integer and let $f(z_1, z_2) \equiv z_1^{N-1}\bar{z}_1^N$. By the arguments in Example 5.1, f is not the trace of a holomorphic function on the "cup"; nevertheless, $Y_2^j(f) \in \mathbf{C}[\langle x_1, x_2 \rangle]$ for each $j = 1, 2, \ldots, N-1$.

6. The complex envelope. We conclude this paper with some observations concerning the cases not covered by Theorem 4.6. Thus suppose M^k, Φ, and $\tilde{\Phi}$ are as in §4, except rk $\tilde{\Phi} = s < n$.

DEFINITION 6.1. $\tilde{M}$ is the *complex envelope* for M^k provided $\tilde{M}$ is a complex submanifold of $\mathbf{C}^n$, $M^k \subset \tilde{M}$, and for each $\mathbf{q} \in \mathrm{CR}(M^k)$, $T_\mathbf{q}M^k + \mathbf{J}T_\mathbf{q}M^k = T_\mathbf{q}\tilde{M}$.

If there exists a biholomorphic coordinate change G; $\mathbf{C}^n \to \mathbf{C}^n$ such that $G(M^k) \subset \mathbf{C}^s \to \mathbf{C}^s \times \mathbf{C}^{n-s}$, then the above arguments solve the holomorphic trace problem relative to $\mathbf{C}^s$. However, we may then extend to a solution relative to $\mathbf{C}^n$ by including the Tangential Cauchy Riemann equations for $\mathbf{C}^s$. It can be shown that the existence of such a coordinate change is equivalent to the existence of a complex envelope, $\tilde{M}$, for M^k. Thus we are naturally interested in the following question.

Question. Given $M^k \subset \mathbf{C}^n$ does there exist a complex envelope for M^k?

The answer to this question is known to be "yes" if M^k is a C.R. submanifold. Indeed, if rk $\mathfrak{J}[\tilde{\Phi}](\mathbf{z})$ is constant then we let $\tilde{M} \equiv \tilde{\Phi}(\mathbf{C}^k)$. In general, the answer is "no," and we will provide an example; however, to understand the example we need the following proposition, the proof of which is trivial.

PROPOSITION 6.2. *Let M^k be a k-dimensional real-analytic submanifold of $\mathbf{C}^n$ and let Φ: $\mathbf{R}^k \to \mathbf{C}^n$ be a parametrization of M^k with $\Phi(\mathbf{O}) = \mathbf{O}$. M^k is a uniqueness set for holomorphic functions on $\mathbf{C}^n$ if and only if the homomorphism from $\mathbf{C}[\langle w_1, \ldots, w_n \rangle]$ to $\mathbf{C}[\langle z_1, \ldots, z_k \rangle]$ defined by $F \to F \circ \tilde{\Phi}$ is injective.*

EXAMPLE 6.3. $M^2 \equiv \{(x_1 + \mathbf{i}x_2, (x_1 + \mathbf{i}x_2)x_2, (x_1 + \mathbf{i}x_2)x_2e^{x_2}) | x_1, x_2 \in \mathbf{R}\} \subset \mathbf{C}^3$.

M^2 is parametrized by $\Phi: \mathbf{R}^2 \to \mathbf{C}^3$ where

$$\Phi(x_1, x_2) \equiv (x_1 + \mathbf{i}x_2, (x_1 + \mathbf{i}x_2)x_2, (x_1 + \mathbf{i}x_2)x_2e^{x_2}).$$

Thus we have $\tilde{\Phi}: \mathbf{C}^2 \to \mathbf{C}^3$ given by

$$\tilde{\Phi}(z_1, z_2) \equiv (z_1 + \mathbf{i}z_2, (z_1 + \mathbf{i}z_2)z_2, (z_1 + \mathbf{i}z_2)z_2e^{z_2}),$$

which can be viewed as a composition with the holomorphic coordinate change $(z_1, z_2) \to ((z_1 + \mathbf{i}z_2), z_2)$. Thus we may assume

$$\tilde{\Phi}(z_1, z_2) \equiv (z_1, z_1z_2, z_1z_2e^{z_2}).$$

Notice that $\operatorname{rk} \tilde{\Phi} = 2 < 3$. The homomorphism $F \to F \circ \tilde{\Phi}$ from $\mathbf{C}[\langle z_1, z_2, z_3 \rangle]$ is known to be injective **[G-R**, p. 121]. Thus M^2 is a uniqueness set for holomorphic functions of $\mathbf{C}^3$. Hence $\mathbf{C}^3$ is the smallest complex subvariety of $\mathbf{C}^3$ which contains M^2. □

REMARK 6.4. Using the above ideas we can construct a 2-dimensional real-analytic submanifold M^2 of $\mathbf{C}^n$ which is a uniqueness set for holomorphic functions on $\mathbf{C}^n$. Let

$$\Phi(x_1, x_2) \equiv \Big((x_1 + \mathbf{i}x_2), (x_1 + \mathbf{i}x_2)x_2, (x_1 + \mathbf{i}x_2)x_2e^{x_2}, \\ (x_1 + \mathbf{i}x_2)x_2e^{\sqrt{2}\,x_2}, \dots, (x_1 + \mathbf{i}x_2)x_2e^{\sqrt{p_{n-2}}\,x_2}\Big)$$

where p_n is the nth positive prime integer. Consider

$$\tilde{\Phi}(z_1, z_2) \equiv \Big(z_1, z_1z_2, z_1z_2e^{z_2}, z_1z_2e^{\sqrt{2}\,z_2}, \dots, z_1z_2e^{\sqrt{p_{n-2}}\,z_2}\Big).$$

The homomorphism $F \to F \circ \tilde{\Phi}$ is shown to be injective by the same procedure as in **[G-R**, p. 121] and the fact that $1, e^{z_2}, e^{\sqrt{2}z_2}, \dots, e^{\sqrt{p_{n-2}}z_2}$ are algebraically independent.

There appear to be two avenues of approach to the further study of the holomorphic trace problem for submanifolds M^k without restriction on their dimension k or genericity. One approach is to seek conditions which guarantee the existence of a complex envelope, $\tilde{M}$, for M^k. For example, as mentioned above there exists such $\tilde{M}$ if and only if there exists a holomorphic coordinate change $G: \mathbf{C}^n \to \mathbf{C}^n$ such that $G \circ \tilde{\Phi}$ is of the form $(\gamma_1, \dots, \gamma_s, 0, \dots, 0)$ for some $\gamma_1, \dots, \gamma_s \in \mathbf{C}[\langle z_1, \dots, z_k \rangle]$. Thus the question becomes a question of functional dependence.

The second approach is to seek a direct extension of Theroem 4.6. This could be achieved by extending Theorem 3.5 to the case "$\operatorname{rk} \Phi < n$". By Remark 3.9 and appropriate alterations in the arguments of §4, we can derive sufficient conditions that a given function $f: M \to \mathbf{C}$ be the trace of a holomorphic function. However, in general these conditions need not be necessary as is shown by Example 6.3. For suppose f: $\mathbf{C}^3 \to \mathbf{C}$ is the function defined by $f(w_1, w_2, w_3) = w_1 + w_2 + w_3$. Then

$$\widetilde{f \circ \Phi}(z_1, z_2) = (z_1 + \mathbf{i}z_2) + (z_1 + \mathbf{i}z_2)z_2 + (z_1 + \mathbf{i}z_2)z_2 e^{z_2}.$$

If $\widetilde{f \circ \Phi}$ satisfies the condition of Remark 3.9, then there is a holomorphic function F: $\mathbf{C}^3 \to \mathbf{C}$ which is independent of one of w_1, w_2 or w_3 such that $\widetilde{f \circ \Phi} = F \circ \tilde{\Phi}$. But this contradicts the algebraic independence of $z_1 + \mathbf{i}z_2$, $(z_1 + \mathbf{i}z_2)z_2$, and $(z_1 + \mathbf{i}z_2)z_2 e^{z_2}$. Thus $\widetilde{f \circ \Phi}$ does not satisfy the conditions of Remark 3.9 even though f is clearly the trace of a holomorphic function on M^2.

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Department of Mathematics, Texas Tech University, Lubbock, Texas 79409