THE PRODUCT OF NONPLANAR COMPLEXES 
DOES NOT IMBED IN 4-SPACE

BY 
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Abstract. We prove that if $K_1$ and $K_2$ are nonplanar simplicial complexes, 
then $K_1 \times K_2$ does not imbed in $\mathbb{R}^4$.

In this paper a proof is given of the following theorem:

**Theorem P.** If $K_1$ and $K_2$ are finite simplicial complexes neither of which is 
homeomorphic to a subset of the euclidean plane $\mathbb{R}^2$, then their cartesian product 
$K_1 \times K_2$ is not homeomorphic to any subset of euclidean 4-space $\mathbb{R}^4$.

This result answers a question originally posed by Professor Karl Menger in [5]. I wish to thank Professor Joseph Zaks for showing me this problem.

1. Preliminaries. We say a space $X$ imbeds in euclidean $n$-space $\mathbb{R}^n$ if there 
is an imbedding (i.e., homeomorphism into) $f: X \to \mathbb{R}^n$. If $X$ imbeds in $\mathbb{R}^2$ we 
say that $X$ is planar. For a proof of the following see [4]:

**Proposition 1.1.** If $K$ is a finite nonplanar simplicial complex then $K$ 
contains a subspace homeomorphic to one of the following spaces:

a. $K_5$, the complete graph on 5 vertices (or, if you prefer, the 1-skeleton of a 
4-simplex);
b. $K_{3,3}$, the join of 2 3-point sets;
c. $S^2$, the 2-sphere; or
d. $Q^2 = \{ x \in \mathbb{R}^3 : x_3 = 0 \text{ and } x_1^2 + x_2^2 < 1 \text{ or } x_1 = x_2 = 0 \text{ and } 0 < x_3 < 1 \}$.

We will henceforth assume that the complexes $K_1$ and $K_2$ of Theorem P are 
chosen from the list of Proposition 1.1, and that we have chosen imbeddings 
$f_1: K_1 \to \mathbb{R}^3$ and $f_2: K_2 \to \mathbb{R}^3$.

To clarify notations we recall some standard definitions. Let $\pi = \{1, \tau\}$ be 
the multiplicative group of order 2. A $\pi$-space $X$ is a Hausdorff space together 
with a fixed point free involution $\tau: X \to X$; this involution defines a free
\(\pi\)-action on \(X\) and we denote the orbit space of this action by \(X/\pi\). The natural projection \(p: X \to X/\pi\) is then a 2-fold covering. If \(X\) and \(Y\) are \(\pi\)-spaces then a map \(f: X \to Y\) is \(\pi\)-equivariant if \(f \cdot \tau = \tau \cdot f\). Homotopies are equivariant if they are \(\pi\)-equivariant at each stage. If \(K\) is a Hausdorff space, the deleted product of \(K\) is

\[D_2K = \{(x,y) \in K \times K : x \neq y\} .\]

Using the action \(\tau(x,y) = (y,x)\), \(D_2K\) is a \(\pi\)-space and we denote the orbit space by \(\Sigma_2K\). Let \(S^\infty = \text{proj lim } S^n\) under the natural inclusions and \(\tau: S^\infty \to S^\infty\) be the limit of the antipodal maps; set \(P^\infty = S^\infty/\tau\). If \(K\) is paracompact there is a \(\pi\)-equivariant map \(\hat{k}: D_2K \to S^\infty\) and any two such maps are \(\pi\)-equivariantly homotopic (cf. [2]). Using the induced map \(k: \Sigma_2K \to P^\infty\) and singular cohomology with \(Z_2\) coefficients we define the \(n\)th mod-2 imbedding class of \(K\) by

\[\Phi^\infty_2(K) = k^*(w^n) \in H^n(\Sigma_2K; Z_2)\]

where \(w^n\) is the nonzero element of \(H^n(P^\infty; Z_2)\), \(n > 0\). The following is an immediate consequence of the definition.

**Proposition 1.2.**

a. If \(K\) and \(L\) are paracompact, \(f: K \to L\) is an imbedding, and \(F: \Sigma_2K \to \Sigma_2L\) is the induced map, then \(F^*(\Phi^\infty_2(L)) = \Phi^\infty_2(K)\).

b. For \(n > 0\), \(D_2\mathbb{R}^n\) is \(\pi\)-equivariantly homotopy equivalent to \(S^{n-1}\) (with antipodal action); thus \(\Phi^\infty_2(\mathbb{R}^n) \neq 0\) iff \(0 < i < n - 1\).

Thus \(\Phi^\infty_2(K) = 0\) is a necessary condition for a paracompact space to imbed in \(\mathbb{R}^n\). In §3 we prove Theorem P by showing that \(\Phi^\infty_2(K_1 \times K_2) \neq 0\).

The information we need about the deleted products of \(K_1\) and \(K_2\) is summarized in the following:

**Proposition 1.3.** If \(K\) is one of the four complexes \(K_1^1, K_3^1, S^2\) or \(Q^2\) of Proposition 1.1, then

a. \(\Phi^\infty_2(K) \neq 0\);

b. \(D_2K\) is \(\pi\)-equivariantly homotopy equivalent to a closed 2-manifold of genus \(g\), where \(g = 6\) if \(K = K_1^1\), \(g = 4\) if \(K = K_3^1\) and \(g = 0\) if \(K = S^2\) or \(Q^2\);

c. if \(f: K \to \mathbb{R}^3\) is an imbedding and \(\hat{F}: D_2K \to D_2\mathbb{R}^3\) is the induced map, then \(\hat{F}^*: H^2(D_2\mathbb{R}^3) \to H^2(D_2K)\) is an isomorphism.

**Proof.** For a and b see [7] and [8]. For c we have the following commutative diagram whose rows are exact Gysin sequences (where we interpret a 2-fold covering as a 0-sphere bundle cf. [6]):

\[
\begin{align*}
\text{\(D_2K\)} & \quad \xrightarrow{p} \quad \text{\(H^2(\Sigma_2K)\)} \\
\downarrow \hat{F}^* & \quad \downarrow \hat{F}^* \\
\text{\(H^2(D_2K)\)} & \quad \xrightarrow{p} \quad \text{\(H^2(\Sigma_2K)\)} \\
\end{align*}
\]

\[
\begin{align*}
\text{\(H^2(D_2\mathbb{R}^3)\)} & \quad \xrightarrow{p} \quad \text{\(H^2(D_2K)\)} \\
\downarrow F^* & \quad \downarrow F^* \\
\text{\(H^2(\Sigma_2\mathbb{R}^3)\)} & \quad \xrightarrow{p} \quad \text{\(H^2(\Sigma_2K)\)} \\
\end{align*}
\]

\[\to 0\]
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where $F: \Sigma_2 K \to \Sigma_2 \mathbb{R}^3$ is the map induced by $\hat{F}$. All six groups in the diagram are isomorphic to $\mathbb{Z}_2$ and so $\rho$ and $\rho'$ are isomorphisms. Thus $\hat{F}^*$ is an isomorphism.

Using $[g_1, \ldots, g_n]$ to denote the $\mathbb{Z}_2$-module with basis $\{g_1, \ldots, g_n\}$ or the zero module if $n = 0$, we can write

$$H^0(K_1) = [\omega^0], \quad H^0(K_2) = [\mu^0],$$

$$H^1(K_1) = [\omega_1, \ldots, \omega_n], \quad H^1(K_2) = [\mu_1, \ldots, \mu_n],$$

where $\eta$ or $\sigma$ is 0, 0, 4, or 6 depending upon whether $K_1$ or $K_2$ is $S^2$, $Q^2$, $K_{1,3}$, or $K_2$. Here the superscripts denote dimension rather than exponents. If $K_1$ or $K_2$ is $S^2$ or $Q^2$ we denote $H^2(K_1) = [\omega^2] \{H^2(K_2) = [\mu^2]\}$; otherwise $H^2(K_1) = 0 \{H^2(K_2) = 0\}$. We also need to assume that if $\eta \neq 0 \{\sigma \neq 0\}$ then the above basis for $H^1(K_1)$ \{H^1(K_2)\} is dual to a basis which satisfies the following:

**Lemma 1.4.** If $K$ is a finite 1-dimensional simplicial complex and $i: D_2K \to K \times K$ is the inclusion map, then there is a basis $\{\beta_1, \ldots, \beta_m\}$ for $H_1(K)$ such that if $\beta \in H_2(D_2K)$ then $i_*(\beta) = \sum c_{ij}(\beta_i \times \beta_j)$ where $c_{ij} \in \mathbb{Z}_2$, $c_{ii} = 0$ for $i = 1, \ldots, m$ and "\times" denotes cross product.

**Proof.** Let $D_2^\theta(K) = \{(x_1, x_2) \in K \times K: c(x_1) \cap c(x_2) = \emptyset\}$ where $c(x_j)$ is the smallest closed simplex of $K$ containing $x_j$. Then, by [9], $D_2^\theta(K)$ is a strong $\pi$-equivariant deformation retract of $D_2K$, and so we can use the inclusion $j: D_2^\theta K \to K \times K$ instead of $i$. Let $\{\sigma_1, \ldots, \sigma_{m+n}\}$ be the 1-simplices of $K$ numbered so that $\{\sigma_{m+1}, \ldots, \sigma_{m+n}\}$ form a maximal tree $T$ of $K$. We also use $\sigma_i$ to denote the linear singular 1-simplex whose image is $\sigma_i$; there is no orientation problem since we are using $\mathbb{Z}_2$ coefficients. For $i = 1, \ldots, m$ set $\beta_i = [\sigma_i + \lambda_i] \in H_1(K_1)$ where $\lambda_i$ is a sum of simplices of $T$. For $i > m$ we set $\lambda_i = \sigma_i$. Suppose $\beta \in H_2(D_2^\theta K)$. Then $\beta = \sum k_{ij}(\sigma_i \times \sigma_j)$ where $k_{ij} \in \mathbb{Z}_2$ and $k_{ii} = 0$ for all $i$. We have

$$\sigma_i \times \sigma_j = (\sigma_i + \lambda_i - \lambda_j) \times (\sigma_j + \lambda_j - \lambda_i)$$

$$= (\sigma_i \times \lambda_i) \times (\sigma_j \times \lambda_j) - \lambda_i \times \sigma_j - \sigma_i \times \lambda + \lambda_i + \lambda_j.$$ 

So if $i \neq j$, $\sigma_i \times \sigma_j = (\sigma_i + \lambda_i) + \gamma_{ij}$ where $\gamma_{ij}$ is a 2-chain of $X \times \Gamma$ \{H_2(K) \times \Gamma \times \Gamma\}. So $\beta = \sum k_{ij}(\sigma_i + \lambda_i) \times (\sigma_j + \lambda_j) \pm \gamma$ where $\gamma$ is a 2-chain of $\Gamma \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \time
by [8] $D_2(CK)$ is $\pi$-equivariantly homotopy equivalent to $S^2$. So in the exact sequence (cf. [1])

$$
\rightarrow H^n(D_2CK) \rightarrow H^n(K) \oplus H^n(K) \xrightarrow{\alpha} H^n(D_2K) \rightarrow H^{n+1}(D_2CK) \rightarrow
$$

where $\alpha(u, v) = q_1^*(u) + q_2^*(v)$, $q_i: D_2K \rightarrow K$ given by $q_i(x_1, x_2) = x_i$, we have $\alpha$ is an isomorphism if $n = 1$. Using this and the Künneth Theorem proves Lemma 1.5.

Using Lemma 1.5, Proposition 1.5 and the above bases we have:

$$
H^0(D_2K_1) = \{\omega^0 \times \omega^0\}, \quad H^1(D_2K_1) = \{\omega^0 \times \omega^1, \omega^1 \times \omega^0; i = 1, \ldots, \eta\},
$$

and

$$
H^2(D_2K_1) = \{\Omega^2\},
$$

and

$$
H^0(D_2K_2) = \{\mu^0 \times \mu^0\}, \quad H^1(D_2K_2) = \{\mu^0 \times \mu^1, \mu^1 \times \mu^0; i = 1, \ldots, \sigma\},
$$

$$
H^2(D_2K_2) = \{\Lambda^2\}
$$

where "\times" denotes cross product followed by restriction.

For Hausdorff spaces $K$ and $L$ we define

$$
\hat{J}_0(K, L) = D_2K \times D_2L, \quad \hat{J}_1(K, L) = D_2K \times (L \times L),
$$

$$
\hat{J}_2(K, L) = (K \times K) \times D_2L.
$$

Using $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$, $\hat{J}_k(K, L)$ becomes a $\pi$-space and we denote the quotient spaces by $J_k(K, L)$ for $k = 0, 1, 2$.

**Lemma 1.6.** If $K$ and $L$ are Hausdorff spaces then $D_2(K \times L)$ is $\pi$-equivariantly homeomorphic to $\hat{J}_1(K, L) \cup \hat{J}_2(K, L)$. Moreover, $\{J_1(K, L), J_2(K, L)\}$ is an excisive couple and $J_1(K, L) \cap J_2(K, L) = J_0(K, L)$.

**Proof.** Clearly $\phi: D_2(K \times L) \rightarrow \hat{J}_1(K, L) \cup \hat{J}_2(K, L)$ defined by

$$
\phi(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2)
$$

is a $\pi$-equivariant homeomorphism. Since $J_1(K, L)$ and $J_2(K, L)$ are open in their union, the couple is excisive.

For $k = 0, 1, 2$ let

$$
\hat{J}_k = \hat{J}_k(K_1, K_2), \quad \hat{J}'_k = \hat{J}_k(\mathbb{R}^3; \mathbb{R}^3),
$$

$$
J_k = J_k(K_1, K_2), \quad J'_k = J_k(\mathbb{R}^3; \mathbb{R}^3),
$$

and $\hat{F}: \hat{J}_k \rightarrow J_k$, $F_k: J_k \rightarrow J'_k$ denote the maps induced by the imbeddings $f_j$: $K_j \rightarrow \mathbb{R}^3, j = 1, 2$. We also have maps

$$
\hat{F}: D_2(K_1 \times K_2) \rightarrow D_2(\mathbb{R}^3 \times \mathbb{R}^3)
$$

and

$$
F: \Sigma_2(K_1 \times K_2) \rightarrow \Sigma_2(\mathbb{R}^3 \times \mathbb{R}^3).$$
Finally let \( \tilde{i}_k: J_0 \rightarrow \tilde{J}_k \) and \( i_k: J_0 \rightarrow J_k \) be the inclusions for \( k = 1, 2 \) and \( p_k: \tilde{J}_k \rightarrow J_k, p_k': \tilde{J}_k' \rightarrow J_k' \) be the natural projections for \( j = 0, 1, 2 \).

**Lemma 1.7.** \( F_0^*: H^4(J_0) \rightarrow H^4(J'_0) \) is an isomorphism.

**Proof.** \( \tilde{J}'_0 \) and \( \tilde{J}_0 \) are \( \pi \)-equivariantly homotopy equivalent to closed 4-manifolds; hence \( J_0 \) and \( J_0' \) are homotopy equivalent to closed 4-manifolds. Thus in the commutative diagram

\[
\begin{array}{ccc}
H^4(\tilde{J}'_0) & \xrightarrow{\rho'} & H^4(J_0) \\
\downarrow F_0 & & \downarrow F_0 \\
H^4(\tilde{J}_0) & \xrightarrow{\rho} & H^4(J_0)
\end{array}
\]

where \( \rho' \) and \( \rho \) are from the appropriate Gysin sequences, and Proposition 1.3.c, \( F_0^* \) is an isomorphism. This proves Lemma 1.7.

**2. The spectral sequence of a double covering.** The proof of Theorem P requires the following in which we use the notation of §1.

**Lemma 2.1.** \( \text{Ker}(p_1^*: H^4(J_1) \rightarrow H^4(J'_1)) \subseteq \text{Im}(i_1^*: H^4(J_1) \rightarrow H^4(J_0)) \).

The proof of Lemma 2.1 requires using the cohomology spectral sequence of a covering (cf. [3]) specialized to the case of a double covering which allows explicit identification of the \( E_1 \)-term and the \( E_1 \) differential operators. The properties of this spectral sequence are summarized in the following:

**Proposition 2.2.** If \( X \) is a \( \pi \)-space, there is a natural first quadrant \( E_1 \)-spectral sequence \( \{ E^{p,q}_r(X), d^{p,q}_r \}_{r=1} \) convergent to \( H^*(X/\pi; Z_2) \) with the following properties:

a. \( E^{p,q}_1(X) = H^q(X; Z_2) \),

b. \( d^{p,q}_r: E^{p,q}_r(X) \rightarrow E^{p+1,q}_r(X) \) is given by \( d^{p,q}_1(\alpha) = \alpha + \tau \alpha \) where \( \tau: H^1(H; Z_2) = H^1(X, Z_2) \) is the homomorphism induced by the involution \( \tau: X \rightarrow X \).

c. For each \( n \) there is a natural decreasing filtration \( \{ F_p H^n(X/n) \}_{p=0}^{\infty} \) of \( H^n(X/\pi; Z_2) \) such that \( F_0 H^n(X/\pi; Z_2) \rightarrow F_n H^n(X/\pi; Z_2) = 0 \) and for each \( p > 0 \) there is a natural short exact sequence

\[
0 \rightarrow F_{p+1} H^n(X/\pi) \rightarrow F_p H^n(X/\pi) \xrightarrow{q} E^n_{\infty,p} (X) \rightarrow 0.
\]

d. The projection induced map \( p^*: H^n(X/\pi; Z_2) \rightarrow H^n(X; Z_2) \) is the composition:

\[
H^n(X/\pi) = F_0 H^n(X/\pi) \xrightarrow{q} E^n_{\infty,0} (X) \subseteq E_1^{0,n} (X) = H^*(X).
\]

For the remainder of this section and the next we assume all coefficients to be \( Z_2 \) and suppress this in the notation.
Lemma 2.3. If \( p > 1 \) then \( d_{31}^{\mathfrak{2}}: E_3^{p,2}(D_2K_1) \to E_3^{p+3,0}(D_2K_1) \) is an isomorphism with \( E_3^{p,2}(D_2K_1) = [\Omega^2] \).

Proof. Using the calculations of §1 and Proposition 2.2.b,
\[
E^p_0(D_2K_1) = [\omega^0 \times \omega^0], \quad p > 0, \quad E^p_1(D_2K_1) = 0, \quad p > 1,
\]
\[
E_3^{p,2}(D_2K_1) = [\Omega^2], \quad p > 0.
\]
Thus \( E_3^{p,0}(D_2K_1) = E_3^{p,0}(D_2K_1) = [\omega^0 \times \omega^0] \) if \( p > 3 \) and \( E_3^{p,2}(D_2K_1) = E_3^{p,2}(D_2K_1) = [\Omega^2] \) if \( p > 0 \). Since \( H^n(\Sigma_2K_1) = 0 \) if \( n > 2 \),
\[
d_{31}^{\mathfrak{2}}: E_3^{p,2}(D_2K_1) \to E_3^{p+3,0}(D_2K_1)
\]
must be an isomorphism. This proves Lemma 2.3.

Lemma 2.4. \( E_{\infty,4}^{p-q}(\hat{J}_0) = 0 \) if \( p > 3 \).

Proof. Using the calculations of §1 and the Künneth formula,
\[
H^0(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0],
\]
\[
H^1(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu^1 \times \mu^0, \mu^0 \times \mu^1; i = 1, \ldots, \sigma] \oplus [\omega^1 \times \omega^0, \omega^0 \times \omega^1; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],
\]
where \( \eta \) and \( \sigma \) are the ranks of \( H^1(K_1) \) and \( H^1(K_2) \) respectively. Using the diagonal \( \tau \)-action and Proposition 2.2.b
\[
E_2^{p,0}(\hat{J}_0) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0], \quad E_2^{p,1}(\hat{J}_0) = 0 \quad \text{if} \quad p > 0.
\]
Since \( E_2^{p,1}(\hat{J}_0) = 0, E_3^{4,0}(\hat{J}_0) = E_2^{4,0}(\hat{J}_0) \) and
\[
\pi_1^\tau: E_3^{4,0}(D_2K_1) \to E_3^{4,0}(\hat{J}_0)
\]
is an isomorphism where \( \pi_1^\tau: \hat{J}_0 \to D_2K_1 \) is the projection. From the commutative diagram
\[
\begin{array}{ccc}
E_3^{1,2}(D_2K_1) & \xrightarrow{\pi_1^\tau} & E_3^{1,2}(\hat{J}_0) \\
d_{31}^{2} \downarrow & & \downarrow d_{31}^{2} \\
E_3^{4,0}(D_2K_1) & \xrightarrow{\pi_1^\tau} & E_3^{4,0}(\hat{J}_0)
\end{array}
\]
and Lemma 2.3, \( d_{31}^{2}: E_3^{1,2}(\hat{J}_0) \to E_3^{4,0}(\hat{J}_0) \) is surjective. Thus \( E_\infty^{4,0}(\hat{J}_0) = E_4^{4,0}(\hat{J}_0) = E_3^{4,0}(\hat{J}_0) = 0 \). From above, \( E_\infty^{3,1}(\hat{J}_0) = 0 \). This proves Lemma 2.4.

Lemma 2.5. \( E_\infty^{3,1}(\hat{J}_1) = 0 \) and \( i^*: E_\infty^{2,2}(\hat{J}_0) \) is the zero homomorphism.

Proof. We first compute \( E_\infty^{p,q}(\hat{J}_1) \) in 3-cases. If \( K_2 = K_3^1 \) or \( K_3^1 \), then
$H^1(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^1, \mu^1 \times \mu^0; i = 1, \ldots, \sigma]$
\[\oplus [\omega^0 \times \omega^0, \omega^1 \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],\]

$H^2(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0] \otimes [\mu^1 \times \mu^1; i, j = 1, \ldots, \sigma]$
\[\oplus [\omega^0 \times \omega^1, \omega^1 \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0; i = 1, \ldots, \sigma] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0].\]

Using the diagonal $\pi$-action on $\hat{\mathcal{J}}_1$ and Proposition 2.2

$E^2_{p,1}(\hat{\mathcal{J}}_1) = 0$ if $p > 0,$

$E^2_{2,2}(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0] \otimes [\mu^1 \times \mu^1; i = 1, \ldots, \sigma] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0],$

$E^2_{1,3}(\hat{\mathcal{J}}_1) = 0.$

If $K_2 = S^2$ and we set $H_2(K_2) = [\mu^2]$ then

$H^1(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0, \omega^1 \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],$

$H^2(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^2, \mu^2 \times \mu^0] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0].$

Therefore

$E^2_{p,1}(\hat{\mathcal{J}}_1) = 0$ if $p > 0,$ $E^2_{2,2}(\hat{\mathcal{J}}_1) = [\Omega^2 \times \mu^0 \times \mu^0],$

$E^2_{1,3}(\hat{\mathcal{J}}_1) = 0.$

Finally if $K_2 = Q_2,$ then

$E^2_{p,1}(\hat{\mathcal{J}}_1) = 0$ if $p > 0,$ $E^2_{2,2}(\hat{\mathcal{J}}_1) = H^2(\hat{\mathcal{J}}_1) = [\Omega^2 \times \mu^0 \times \mu^0],$

$E^2_{1,3}(\hat{\mathcal{J}}_1) = H^3(\hat{\mathcal{J}}_1) = 0.$

In all cases $E^2_{p,0}(\hat{\mathcal{J}}_1) = [\omega^0 \times \omega^0 \times \mu^0 \times \mu^0].$ Since $\pi_i^*: H^1(D_2K_1) \rightarrow H^2(\hat{\mathcal{J}}_1)$ is an isomorphism, where $\pi_i: \hat{\mathcal{J}}_1 \rightarrow D_2K_1$ is the projection, and $E^2_{p,1}(\hat{\mathcal{J}}_1) = 0$ if $p > 0,$ we have

$\pi_i^*: E^p_{3,0}(D_2K_1) \rightarrow E^p_{3,0}(\hat{\mathcal{J}}_1)$

is an isomorphism for $p > 2.$ Consider the commutative diagram

$E^2_{2,2}(D_2K_1) \xrightarrow{\pi^*} E^2_{3,2}(\hat{\mathcal{J}}_1)$

$\downarrow d^2_{2,2} \quad \downarrow d^2_{3,2}$

$E^3_{p,0}(D_2K_1) \xrightarrow{\pi^*} E^3_{3,0}(\hat{\mathcal{J}}_1)$

Using Lemma 2.3, and $\pi_i^*([\Omega^2]) = [\Omega^2 \times \omega^0 \times \omega^0]$ we have

$d^2_{3,2}([\Omega^2 \times \omega^0 \times \omega^0]) \neq 0.$
Therefore

\[ E^{2,2}_4(\hat{J}_1) = \begin{cases} 
[\omega^0 \times \omega^0 \times \mu^1_i \times \mu^1_i; i = 1, \ldots, \sigma] & \text{if } K_2 = K_3 \text{ or } K_{2,3}, \\
0 & \text{if } K_2 = S^2 \text{ or } Q^2.
\end{cases} \]

To show that \( i^*[\omega^0 \times \omega^0 \times \mu^1_i \times \mu^1_i] = 0 \), let \( \{\alpha_1, \ldots, \alpha_\sigma\} \) be the basis of \( H_1(K_2) \) dual to \( \{\mu^1_i, \ldots, \mu^1_i\} \) as in Lemma 1.4 and let \( \alpha^2 \) denote the nonzero element of \( H^2(D_2K_2) \). If \( j: D_2K_2 \to K_2 \times K_2 \) is the inclusion, then by Lemma 1.4

\[ \langle j^*(\mu^1_i \times \mu^1_i), \alpha^2 \rangle = \langle \mu^1_i \times \mu^1_i, \sum_{j \neq k} c_{jk}(\alpha^1_j \times \alpha^1_k) \rangle = \sum_{j \neq k} c_{jk} \langle \mu^1_i, \alpha^1_j \rangle \langle \mu^1_i, \alpha^1_k \rangle = \sum_{j \neq k} c_{jk} \delta_{jk} = 0. \]

Thus \( j^*(\mu^1_i \times \mu^1_i) = 0 \). So \( i^*[\omega^0 \times \omega^0 \times \mu^1_i \times \mu^1_i] = 0 \). This proves Lemma 2.5.

**Proof of Lemma 2.1.** By Proposition 2.2.d, if \( p_1: \hat{J}_1 \to J_1 \) is the natural projection then \( p_1^*: H^4(J_1) \to H^4(\hat{J}_1) \) is the composition

\[ H^4(J_1) = F^0H^4(J) \to E_0^{0,4}(\hat{J}_1) \subseteq E_1^{0,4}(\hat{J}_1) = H^4(J_1). \]

So using Proposition 2.2.c we have

\[ \ker[p_1^*: H^4(J_1) \to H^4(\hat{J}_1)] = \ker[q: F^0H^4(J_1) \to E_0^{0,4}(\hat{J}_1)] = F_1H^4(J_1). \]

By Lemma 2.5, \( E_\infty^{1,3}(\hat{J}_1) = 0 \); thus \( F_2H^4(J_1) = F_1H^4(J_1) \). Thus \( i_1^*[\ker p_1^*] = 0 \) if and only if \( i_1^*[F_2H^4(J_1)] = 0 \). Now consider the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 \to F_3H^4(J_1) \to F_2H^4(J_1) \to E_{\infty}^{2,2}(\hat{J}_1) \to 0 \\
\downarrow i_1^* \quad \downarrow i_1^* \quad \downarrow i_1^* \\
0 \to F_3H^4(J_0) \to F_2H^4(J_0) \to E_{\infty}^{2,2}(\hat{J}_0) \to 0
\end{array}
\]

By Lemma 2.4, \( E_\infty^{4,1}(\hat{J}_0) = 0 \); thus \( F_3H^4(J_0) = 0 \) and so the projection

\[ F_2H^4(J_0) \to E_{\infty}^{2,2}(\hat{J}_0) \]

is an isomorphism. But by Lemma 2.5, \( i_1^*: E_{\infty}^{2,2}(\hat{J}_1) \to E_{\infty}^{2,2}(\hat{J}_0) \) is zero. Thus \( i_1^*: F_2H^4(J_1) \to F_2H^4(J_0) \) is zero. This proves Lemma 2.1.

**3. Proof of Theorem P.** We use the notation of §1 including the assumptions that \( K_1 \) and \( K_2 \) are complexes selected from the list in Proposition 1.1 and that all coefficients are \( \mathbb{Z}_2 \). We will prove that \( \Phi_2(K_1 \times K_2) \neq 0 \).

Consider first the following commutative diagram in which the rows are the
exact Mayer Vietoris sequences given by Lemma 1.6:

\[ H^3(J'_1) \oplus H^3(J'_2) \to H^4(J'_0) \xrightarrow{\delta'^*} H^4(\Sigma_2(\mathbb{R}^3 \times \mathbb{R}^3)) \to H^4(J'_1) \oplus H^4(J'_2) \]

\[ F'_0 \downarrow \quad F'^* \]

\[ H^3(J'_1) \oplus H^3(J'_2) \xrightarrow{\delta'^*} H^4(J'_0) \xrightarrow{\delta'^*} H^4(\Sigma_2(\mathbb{K}_1 \times \mathbb{K}_2)) \]

Since \( H^j(J'_1) \approx H^j(J'_2) \approx H^j(\mathbb{R}P^2) = 0 \) for \( j > 2 \), \( \delta'^* \) is an isomorphism.

Since \( \Phi'_2(\mathbb{K}_1 \times \mathbb{K}_2) \neq 0 \) if \( F^* \neq 0 \), we have \( \Phi'_2(\mathbb{K}_1 \times \mathbb{K}_2) \neq 0 \) if \( \text{Im}(F'_0) \subsetneq \text{Im}(i'_1 + i'_2) \).

Now consider the following commutative diagram in which the rows are exact Gysin sequences or sums of Gysin sequences:

\[
\begin{array}{cccccc}
H^3(J'_0) & \xrightarrow{\Delta'^*} & H^4(J'_0) & \xrightarrow{F'_0*} & H^4(J'_0) & \xrightarrow{p'} \to H^4(J'_0) \\
F'_0 & \downarrow & F'_0 & \downarrow & F'_0 & \downarrow \\
H^3(J'_0) & \xrightarrow{\Delta^*} & H^4(J'_0) & \xrightarrow{P'_0*} & H^4(J'_0) & \xrightarrow{p} \to H^4(J'_0) \\
& & i'_1 + i'_2 & \xrightarrow{i'_1 + i'_2} & & \hat{i}'_1 + \hat{i}'_2 \\
& & & \xrightarrow{\Delta'_1 + \Delta'_2} & & \\
H^3(J'_1) \oplus H^3(J'_2) & \xrightarrow{\Delta^*} & H^4(J'_1) \oplus H^4(J'_2) & \xrightarrow{P'_1 \oplus P'_2} & H^4(J'_1) \oplus H^4(J'_2) \\
& & & & \end{array}
\]

Since \( \hat{J}'_0 \) is \( \pi \)-equivariantly homotopy equivalent to \( S^2 \times S^2 \), \( H^4(\hat{J}'_0) \approx H^4(J'_0) \approx H^3(J'_0) = \mathbb{Z}_2 \) and \( \Delta'^* \) and \( \rho' \) are isomorphisms. By Proposition 1.3, \( H^4(\hat{J}'_0) \approx H^4(J'_0) \approx \mathbb{Z}_2 \) and by exactness \( \rho \) is an isomorphism. By Lemma 1.7, \( F'_0* : H^4(\hat{J}'_0) \to H^4(J'_0) \) is an isomorphism. Let \( \alpha \) be the nonzero element of \( H^3(J'_0) \). We need to show there do not exist elements \( \alpha_1 \in H^3(J'_1) \) and \( \alpha_2 \in H^3(J'_2) \) such that \( i'_1(\alpha_1) + i'_2(\alpha_2) = F'_0* (\alpha) \). Since \( F'_0* (\Delta'^* (\alpha)) \neq 0 \), it suffices to show that

\[ i'_1 (\Delta'_1 (\alpha_1)) + i'_2 (\Delta'_2 (\alpha_2)) = 0. \]

By exactness and symmetry this follows if we prove

\[ \ker [p'_1 : H^4(J'_1) \to H^4(\hat{J}'_1)] \subseteq \ker [i'_1 : H^4(J'_1) \to H^4(J'_0)]. \]

This is exactly what was proven in Lemma 2.1. Thus the proof of Theorem P is complete.

**References**


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