THE ASYMPTOTIC BEHAVIOUR OF CERTAIN INTEGRAL FUNCTIONS

BY

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ABSTRACT. Let \( f(z) \) be an integral function satisfying

\[
\int_{0}^{\infty} \left\{ \log m(r,f) - \cos \pi \rho \log M(r,f) \right\} \frac{dr}{r^{\rho+1}} < \infty
\]

and

\[
0 < \lim_{r \to \infty} \frac{\log M(r,f)}{r^\rho} < \infty
\]

for some \( \rho : 0 < \rho < 1 \). It is shown that such functions have regular asymptotic behaviour outside a set of circles with centres \( \xi_j \) and radii \( t_j \) for which

\[
\sum_{j=1}^{\infty} \frac{t_j}{|\xi_j|} < \infty.
\]

1. Introduction. For an integral function \( f(z) \) let

\[
M(r,f) = \max_{|z|=r} |f(z)|, \quad m(r,f) = \min_{|z|=r} |f(z)|
\]

and let \( n(r,f) \) be the number of zeros of \( f \) in \( |z| < r \). The order \( \rho \) of \( f \) is

\[
\rho = \lim_{r \to \infty} \frac{\log \log M(r,f)}{\log r}.
\]

The following result appears in [6].

THEOREM A. Let \( \rho \) be a positive number less than one and let \( f(z) \) be an integral function of order \( \rho \) satisfying the following conditions:

(i) there is a finite constant \( K \) such that

\[
\lim_{r_i \to \infty} \int_{r_1}^{r_i} \left\{ \log m(r,f) - \cos \pi \rho \log M(r,f) \right\} \frac{dr}{r^{\rho+1}} < K;
\]

(ii) there are numbers \( \alpha \) and \( \beta \), with \( 0 < \alpha < \beta < \infty \), such that, for all large \( r \),

\[
ar^\alpha \leq n(r,f) \leq \beta r^\rho.
\]

Let

Received by the editors March 28, 1976 and, in revised form, April 1, 1977.

AMS (MOS) subject classifications (1970). Primary 30A64.
Then there is a curve $C$: $z = re^{i\phi(r)}$, where $\phi(r)$ is a continuous function satisfying

$$|\phi(R_1) - \phi(R_2)| = o\left(\log \frac{R_2}{R_1}\right)^{1/2} \text{ as min}(R_1, R_2) \to \infty,$$

a function $\epsilon(t)$ satisfying $\pi > \epsilon(t) > 0$ and $\epsilon(t) \to 0$ as $t \to \infty$, and a function $\nu(t)$ satisfying $1 > \nu(t) > 0$, $\nu(t) \to 0$ as $t \to \infty$ and

$$\int_{-\infty}^{\infty} \frac{\nu(t)}{t} \, dt < \infty,$$

for which the following is true. If $\xi$ is any point on $C$, then the set

$$\{ z : k^{-1} |\xi| < |z| < k |\xi| \text{ and } |\arg z \xi^{-1}| > \epsilon(|\xi|) \}$$

contains at most $\nu(|\xi|) N(|\xi|)$ zeros of $f$, where $N(|\xi|)$ is the number of zeros of $f$ in

$$\{ z : k^{-1} |\xi| < |z| < k |\xi| \}.$$

The equation (1.2) is a consequence of the following: there is a constant $A = A(k)$ and a function $\Delta(t)$ satisfying $\pi > \Delta(t) > 0$, $\Delta(t) \to 0$ and

$$\int_{-\infty}^{\infty} \frac{\Delta(t)^2}{t} \, dt < \infty$$

for which

$$|\psi'(t)| < A \frac{\Delta(t)}{t} \text{ for all large } t.$$
Theorem 1. Let $\rho$ be a positive number less than one and let $f(z)$ be an integral function satisfying conditions (i)' and (ii)' above. Then $f(z)$ satisfies the hypotheses of Theorem A and with $\phi(r)$ as in that theorem we have

$$
|\rho^r \log |f(r e^{(\phi(r) + \delta - \pi)})| - \beta \cos \rho \theta| = o(1)
$$

as $r$ tends to infinity outside a set of discs with centres $\zeta_i$, radii $t_i$, for which

$$
\sum_{i=1}^{\infty} \frac{t_i}{|\zeta_i|} < \infty.
$$

The exceptional set of Theorem 1 may be described briefly, following Hayman [7], as an $E$-set. Theorem 1 has much in common with results of Essén [4] and Essén and Lewis [5] on subharmonic functions. In [4] Essén is concerned with functions subharmonic in the plane slit along the negative real axis while [5] generalizes the considerations of [4] to functions subharmonic in $d$-dimensional cones and also establishes an improved estimate of the exceptional set. When restricted to integral functions the result of [4] combined with the estimate of the exceptional set of [5] may be viewed as a special case of Theorem 1, when $|f(r)| = M(r,f)$, $|f(-r)| = m(r,f)$ and $\log |f(-r)| < \cos \pi \log |f(r)|$.

The condition (i)' cannot be replaced with

$$
\lim_{r_1, r_2 \to \infty} \int_{r_1}^{r_2} \left\{ \log m(r,f) - \cos \pi \rho \log M(r,f) \right\} \frac{dr}{r^{\rho+1}} < 0,
$$

a condition arising in the work of Anderson [1]. For in [1], Anderson shows that

$$
\int_0^{\infty} \left\{ \log |f(-r)| - \cos \pi \rho \log |f(r)| \right\} \frac{dr}{r^{\rho+1}}
$$

exists (so that (1.8) certainly holds) for an integral function $f(z)$ with real negative zeros if

$$
\frac{\log f(r)}{r^\rho} \to A \quad (0 < A < \infty)
$$

for some $\rho$: $0 < \rho < 1$. It will be shown in §9, however, that there exists an integral function $f(z)$ with real negative zeros satisfying (1.9) and such that, for some $\epsilon > 0$,

$$
\log |f(-r)| < (A \cos \pi \rho - \epsilon) r^\rho
$$

for all $r$ in a set of infinite logarithmic measure. Since an $E$-set intersects every ray through the origin in a set of finite logarithmic measure (1.6) cannot hold outside an $E$-set.
2. Preliminaries. From (i)' it follows that

$$\lim_{r \to \infty} \int_{r_1}^{r_2} \{\log m(r, f) - \cos \pi \rho \log M(r, f)\} \frac{dr}{r^{\rho+1}} < 0$$

and from this together with (ii)' and the theorem of Anderson already mentioned [1, p. 154] we deduce that

(2.1) \( \log M(r, f) \sim \beta r^\rho \),
(2.2) \( \log f_1(r) \sim \beta r^\rho \),

where

(2.3) \( f_1(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{|a_n|}\right) \),

the numbers \( a_n \), \( n = 1, 2, 3, \ldots \), being the nonzero zeros of \( f \) arranged in order of increasing magnitude. A well-known consequence of (2.2) is that

(2.4) \( n(r, f) = n(r, f_1) \sim \pi^{-1} \beta \sin \pi \rho r^\rho \),

so that functions satisfying (i)' and (ii)' are of order \( \rho \) and satisfy (i) and (ii) of Theorem A.

In the course of the proof of Theorem 1 we shall find it convenient to refer to a result due to Kolomiiceva [9]. A complete discussion of Kolomiiceva’s theorem would involve us in needless complications but a simple consequence of it is

**Lemma 1.** Let \( g(z) \) be an integral function satisfying

$$\lim_{r \to \infty} \frac{\log M(r, g)}{r^\rho} = \beta,$$

where \( 0 < \rho < 1 \) and \( 0 < \beta < \infty \), which is such that, for each \( \eta > 0 \), the number of zeros of \( g \) in

$$\{ |z| < r \} \cap \{ |\arg z| < \pi - \eta \}$$

is \( o(r^\rho) \) as \( r \to \infty \). Then a necessary and sufficient condition that

$$\log |g(re^{i\theta})| = (\beta \cos \rho \theta + o(1))r^\rho$$

outside a set \( E \) is the following: given \( \varepsilon > 0 \), there exist \( \delta = \delta(\varepsilon) > 0 \) and \( r(\varepsilon) \) such that for all \( z \) outside \( E \) satisfying \( |z| > r(\varepsilon) \),

(2.5) \( \int_0^{\delta r} \frac{n_z(t, g)}{t} \, dt < \varepsilon r^\rho \),

where \( n_z(t, g) \) is the number of zeros of \( g \) contained in the open disc with centre \( z \) and radius \( t \).
3. An auxiliary function. We suppose without loss of generality that 
$f(0) = 1$ so that

\[ f(z) = \prod_{l=1}^{\infty} \left( 1 - \frac{z}{a_n} \right). \]

As was mentioned before the results of Theorem A hold for functions
satisfying the hypotheses of Theorem 1. Choose

\[ k = (1 2)^{1/p} \]

and let $C$ and $\phi(t)$ be as in Theorem A. We relabel $C$ as $C_{\theta}$
and for every $\theta$ satisfying $-\pi < \theta < \pi$ we define $C_{\theta}$ by

\[ C_{\theta}: z = re^{i(\phi(t) + \theta - \phi)} . \]

Let us rearrange the zeros of $f$ in the following way: if $a_n$ is a zero of $f$
lying on the curve $C_{\theta}$ say, we transfer it to the point $a''_n = |a_n|e^{i\theta}$
and define

\[ F(z) = \prod_{l=1}^{\infty} \left( 1 - \frac{z}{a''_n} \right). \]

Our first concern is to show that $\log|F(|z|e^{i\theta})|$ and $\log|f(z)|$
do not greatly differ. Later we shall show that $\log|F(z)|$ and $\log|f(z)|$
have similar asymptotic behavior and then, after estimating $\log|f(z)|$,
we shall appeal to the intermediate character of $F$ to estimate $\log|f(z)|$.

4. Comparison of $f$ and $F$. We shall prove

**Lemma 2.** Given any number $\epsilon > 0$, there exists a number $R(\epsilon)$
such that, if $f(z) \not= 0$,

\[ |\log|f(z)|| - |\log|F(|z|e^{i\theta})|| < \epsilon r^p \]

whenever $|z| > R(\epsilon)$, where $\theta$ satisfies $-\pi < \theta < \pi$ and is such
that $z$ lies on $C_{\theta}$.

Throughout the proof we suppose that $z = re^{i\theta}$ is not a zero of $f$. We have,
from (3.1) and (3.4),

\[ \log \left| \frac{f(z)}{F(re^{i\theta})} \right| = \sum_{l=1}^{\infty} \log \left| \left( 1 - \frac{z}{a_n} \right) \left( 1 - \frac{re^{i\theta}}{a_n} \right)^{-1} \right| \]

and we examine the sum of (4.2) in three parts. First, with $a_n = r_n e^{i\phi_n}$
consider, for $p > 1,$
Integrating by parts we obtain

\[(4.3)\quad S_1 = O\left(k^{-p(\rho - 1)} \rho^\rho\right).\]

Next consider

\[(4.4)\quad S_2 = \sum_{r_n < k^{-\rho r}} \log\left\{\frac{1 - \frac{z}{a_n}}{1 - \frac{r e^{i\theta}}{a_n}}\right\}^{-1} \]

\[\leq \sum_{r_n < k^{-\rho r}} \log\left\{\left(1 + \frac{r_n}{r}\right)\left(1 - \frac{r_n}{r}\right)^{-1}\right\} \]

\[\leq 2r^{-1}(1 - k^{-1})^{-1} \sum_{r_n < k^{-\rho r}} r_n \]

\[= 2r^{-1}(1 - k^{-1})^{-1} \int_{k^{-\rho r}}^\infty \frac{dn(t)}{t} \]

\[= O\left(k^{-p(\rho + 1)} \rho^\rho\right).\]

Finally we consider the remaining part of the sum, that for which \(k^{-\rho r} \leq r_n \leq k^{\rho r}\). Since \(\theta = \pi + \psi - \phi(r)\) and \(a'_n = r_n e^{i(\pi + \psi - \phi(r))}\),

\[J_n = \left|\left(1 - \frac{z}{a_n}\right)\left(1 - \frac{r e^{i\theta}}{a_n'}\right)\right|^2 \]

\[= \frac{(1 - r/r_n)^2 + 4r/r_n \sin^2\left(\frac{\psi - \phi_n}{2}\right)}{(1 - r/r_n)^2 + 4r/r_n \sin^2\left(\frac{\psi - \phi_n - \phi(r) + \phi(r_n)}{2}\right)} \]

Let us write \(t_n = r/r_n\), \(\psi - \phi_n = \psi_n\), \(\phi(r) - \phi(r_n) = \nu_n\). Then
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\[ J_n = \frac{(1 - t_n)^2 + 4t_n \sin^2(\psi_n/2)}{(1 - t_n)^2 + 4t_n \sin^2((\psi_n - \nu_n)/2)} \]

\[ = 1 + \frac{4t_n \sin(\psi_n - \frac{1}{2} \nu_n) \sin \frac{1}{2} \nu_n}{(1 - t_n)^2 + 4t_n \sin^2((\psi_n - \nu_n)/2)}. \]

Hence

\[ \log J_n \leq \frac{4t_n |\sin(\psi_n - \frac{1}{2} \nu_n) \sin \frac{1}{2} \nu_n|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)} \]

(4.5)

\[ \leq \frac{8t_n |\sin\left(\frac{\psi_n - \nu_n}{2}\right)| \sin \frac{1}{2} \nu_n + 8t_n |\sin \frac{1}{4} \nu_n \sin \frac{1}{2} \nu_n|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)} \]

since, for any real numbers \( a \) and \( b \),

\[ |\sin\left(a - \frac{1}{2} b\right)| \leq 2 |\sin\left(\frac{1}{2} (a - b) + \frac{1}{4} b\right)| \]

\[ \leq 2 |\sin \frac{1}{2} (a - b)| + 2 |\sin \frac{1}{4} b|. \]

Further, from (1.5),

\[ |\nu_n| = |\phi(r) - \phi(r_n)| \leq \left| \int_{r_n}^r |\phi'(t)| dt \right| \]

(4.6)

\[ \leq \left| \int_{r_n}^r \frac{\Delta(t)}{t} dt \right| \leq A |\log \frac{r}{r_n}| \left\{ \sup_{t \geq k - r_n} \Delta(t) \right\} \]

\[ \leq Ak^p \left| 1 - \frac{r}{r_n} \right| \left\{ \sup_{t \geq k - r_n} \Delta(t) \right\}. \]

Substituting (4.6) into (4.5) we obtain

\[ \log J_n \leq \frac{4Ak^p t_n |1 - t_n| \left| \sin\left(\frac{\psi_n - \nu_n}{2}\right)\right|}{(1 - t_n)^2 + 4t_n \sin^2\left(\frac{\psi_n - \nu_n}{2}\right)} \left\{ \sup_{t \geq k - r_n} \Delta(t) \right\} \]

\[ + A^2k^2p t_n \left\{ \sup_{t \geq k - r_n} \Delta(t)^2 \right\} \]

\[ \leq Ak^p t_n^{1/2} \left\{ \sup_{t \geq k - r_n} \Delta(t) \right\} + A^2k^2p t_n \left\{ \sup_{t \geq k - r_n} \Delta(t)^2 \right\} \]

\[ \leq A_1k^3p \left\{ \sup_{t \geq k - r_n} \Delta(t) \right\}. \]
where $A_1 = A + \pi A^2$. Hence, since from (2.4) the number of zeros of $f$ in $|z| \leq k^r r$ is at most $2ak^r p r^2$ for large $r$, where $a = \beta \pi^{-1} \sin \pi \rho$,

$$S_3 = \sum_{k^{-r} < r < k^r} \log \left( 1 - \frac{z}{a_n} \left( 1 - \frac{re^{i\theta}}{a_n^2} \right)^{-1} \right)$$

(4.7)

$$\leq 2aA_1 k^{p(3+3)} r^p \left\{ \sup_{t > k^{-r}} \Delta(t) \right\} .$$

Given $\epsilon > 0$ we may choose $p$ sufficiently large that $S_1 + S_2 < \epsilon r^p$ for all large $r$ and with this $p$ we may choose $r_0(\epsilon)$ so that $S_3 < \epsilon r^p$ for $r > r_0(\epsilon)$, since $\Delta(t) \to 0$ as $t \to \infty$, which proves one half of Lemma 1. The second half, that

$$\log \left| \frac{F(z)}{f(z)} \right| < \epsilon |z|^p ,$$

is proved similarly.

5. The zeros of $F(z)$. We shall prove

**Lemma 3.** Let $\delta$ be a fixed positive number less than $\pi$ and let $n_z(t, F, \delta)$ be the number of zeros of $F$ contained in

$$\left\{ \zeta : |\arg \zeta| < \pi - \frac{1}{2} \delta \right\} \cap \left\{ \zeta : |\zeta - z| < t \right\} .$$

(5.1)

Then given any positive number $\epsilon < \frac{1}{2}$ there exists a number $R(\epsilon, \delta)$ such that, with $|z| = r$,

$$\int_0^r n_z(t, F, \delta) \frac{dt}{t} < \epsilon r^p$$

(5.2)

for all $z$ outside a set $H_1$ (where $H_1$ depends only on $\delta$) and such that $|z| > R(\epsilon, \delta)$. Moreover $H_1$ is covered by a set of discs $C_i$, centres $\zeta_i$, radii $t_i$, $i = 1, 2, 3, \ldots$, such that $\sum_{i} t_i / |\zeta_i| < \infty$.

Throughout the proof of Lemma 3 we write $n_z(t), n_z(t, \delta)$ instead of $n_z(t, F), n_z(t, F, \delta)$.

We shall make use of an argument of Azarin [2] in which the following lemma is used.

**Lemma 4 ([10, Lemma 3.2]).** If a set $E$ in the complex plane is covered by discs of bounded radii such that each point of the set is the centre of a disc, then from this one may select a subsystem of discs which covers the set, each point of the plane being covered no more than $v$ times by the discs of this subsystem, where $v$ is an absolute constant.

Let $R_1 = R_1(\delta)$ be such that, for $r > R_1$ we have $\epsilon(r) < \frac{1}{2} \delta$, where $\epsilon(r)$ is
the function occurring in Theorem A. (We may note that, if \( z \in S(\delta, R_1) \), where

\[
S(\delta, R_1) = \{ z : |z| > R_1 \text{ and } |\arg z| < \pi - \delta \},
\]
then \( n_z(t, \delta) = n_z(t) \) certainly for \( 0 < t < \frac{1}{4} \delta |z| \). Let \( H_1 \) be the set of points \( z \) in \( |z| > R_1 \) at which, for some \( t = t(z) \) satisfying \( 0 < t < \frac{1}{2} |z| \), we have

\[
n_z(t, \delta) > t |z|^{\rho - 1}.
\]

Let \( E_n \) be the subset of \( H_1 \) contained in the annulus

\[
\{ z : 4^{n+1} > |z| > 4^n \}, \quad n = 0, 1, 2, \ldots
\]

We surround each point \( z \) of \( H_1 \) by a disc of radius \( t(z) \) and from the set of such discs surrounding points of \( E_n \) we select a subsystem \( K_n \) which covers \( E_n \), while covering each point of the plane at most \( v \) times. This can be done, by Lemma 4. We note that the members of \( K_n \) do not intersect the members of \( K_m \) if \( |n - m| > 2 \), and therefore \( K = \bigcup_{n=1}^{\infty} K_n \) is a set of discs the members of which cover each point of the plane at most \( 2v \) times.

Now, \( K \) is a countable set the members of which may be ordered: \( C_i, \ i = 1, 2, 3, \ldots \), where \( C_i \) is a disc with centre \( \xi_i \) and radius \( t_i \), where \( 0 < t_i < \frac{1}{2} |\xi_i|, i = 1, 2, 3, \ldots \); moreover, from (5.4) we have

\[
n_z(t_i, \delta) > t_i |\xi_i|^{\rho - 1}, \quad i = 1, 2, 3, \ldots
\]

Hence

\[
\sum_{i=1}^{\infty} \frac{t_i}{|\xi_i|} \leq \sum_{i=1}^{\infty} \frac{n_z(t_i, \delta)}{|\xi_i|^{\rho}}.
\]

Now, if \( z_n \) is one of the zeros of \( F \) contained in \( S(\frac{1}{2} \delta, R_1) \) and also in one of the discs, say \( C_i \), then \( |z_n| - |\xi_i| < |z_n - \xi_i| < t_i < \frac{1}{2} |\xi_i| \) so \( |z_n| < \frac{3}{2} |\xi_i| \). Hence, from (5.5) and the fact that \( K \) covers any point in the plane at most \( 2v \) times,

\[
\sum_{i=1}^{\infty} \frac{t_i}{|\xi_i|} \leq 2v \left( \frac{3}{2} \right)^\rho \sum \frac{1}{|z_n|^{\rho}},
\]

where the sum on the right-hand side is taken over those zeros of \( F \) which are contained in \( S(\frac{1}{2} \delta, R_1) \). We proceed to show that this sum is finite.

Let \( n \) be a nonnegative integer, and let \( b_n \) be a positive number satisfying \( k^n R_1 < b_n < k^{n+1} R_1 \) at which

\[
n(b_n) \log k < \int_{k^n R_1}^{k^{n+1} R_1} \nu(t) \, dt,
\]

where \( \nu(t) \) is the function occurring in Theorem A and \( k \) is given by (3.2). The number of zeros of \( F \) in

\[
\{ z : k^n R_1 < |z| < k^{n+1} R_1 \text{ and } |\arg z| < \pi - \frac{1}{2} \delta \}
\]
is no more than \( \nu(b_n)N(b_n) \), where \( N(b_n) \) is the number of zeros of \( F \) in \( \{z: k^{-1}b_n < |z| < kb_n\} \). Hence, making use of (5.7) we have, for some constant \( A \),

\[
\sum_{|z_n|} \frac{1}{|z_n|^\rho} \leq \sum_{0}^{\infty} \nu(b_n)N(b_n)(k^nR_1)^{-\rho} \leq \sum_{0}^{\infty} \nu(b_n)A b_n^\rho(k^nR_1)^{-\rho} \leq A k^\rho \sum_{0}^{\infty} \nu(b_n) \leq A k^\rho \int_{R_1}^{\infty} \nu(t) \frac{dt}{t} < \infty,
\]

from Theorem A. The sum on the left-hand side of (5.6) is thus finite.

Suppose that \( z \) is a point outside \( H_1 \) and satisfying \( |z| > R_1 \). Then, given any positive number \( \varepsilon < \frac{1}{2}, \int_{H_1}^{R_1} \frac{dt}{t} < \varepsilon |z|^\rho \). This proves Lemma 3.

6. The behaviour of \( f_1(z) \). Let \( f_1(z) \) be the function (2.3). Since

\[
\log m(r, f) - \cos \pi \rho \log M(r, f) > \log m(r, f_1) - \cos \pi \rho \log M(r, f_1)
\]

it follows from (i)' and Kjellberg’s Lemma [8, p. 193, formula (21)] that

\[
\int_{R_1}^{\infty} |\log m(r, f_1) - \cos \pi \rho \log M(r, f_1)| \frac{dr}{r^\rho+1} < \infty.
\]

Given a positive number \( \varepsilon > 0 \), it follows from (6.1) and (2.2) that

\[
\log m(r, f_1) > \left( \beta \cos \pi \rho - \frac{1}{2} \varepsilon \right) r^\rho
\]

for \( r \) outside a set \( E = E(\varepsilon) \) of finite logarithmic measure. Hence, for \( \delta = \varepsilon/2\beta \rho \),

\[
\log |f_1(re^{i\theta})| > \log m(r, f_1) > (\beta \cos \rho \theta - \varepsilon)r^\rho
\]

for \( \pi > |\theta| > \pi - \delta \) and for \( r \) outside \( E \).

It is well known (see e.g. [12, p. 272]) that

\[
|r^{-\rho} \log |f_1(re^{i\theta})| - \beta \cos \rho \theta| \to 0
\]

as \( r \to \infty \), uniformly for \( |\theta| < \pi - \delta \). In particular

\[
|r^{-\rho} \log |f_1(re^{i(\pi-\delta)})| - \beta \cos \rho (\pi - \delta)| \to 0
\]

as \( r \to \infty \). Hence, for \( \pi > |\theta| > \pi - \delta \) and for sufficiently large \( r \)

\[
r^{-\rho} \log |f_1(re^{i\theta})| < r^{-\rho} \log |f_1(re^{i(\pi-\delta)})|< \beta \cos \rho (\pi - \delta) + \frac{1}{2} \varepsilon
\]

\[
< \beta \cos \rho \theta + 2 \beta \sin \frac{1}{2} \rho (|\theta| - \pi + \delta) + \frac{1}{2} \varepsilon
\]

\[
< \beta \cos \rho \theta + \varepsilon.
\]

Taking (6.2) and (6.4) together, and taking account of (6.3) we obtain
Lemma 5. Let \( f_1(z) \) be the integral function (2.3). Given \( \epsilon > 0 \)
\[
|r^{-p} \log |f_1(re^{\theta})| - \beta \cos \rho \theta| < \epsilon \quad (-\pi < \theta < \pi)
\]
for all large \( r \) outside \( E \), a set of finite logarithmic measure.

From this together with Lemma 1 we deduce

Lemma 6. Given any \( \epsilon > 0 \) there exist positive numbers \( \delta = \delta(\epsilon) \) and \( r(\epsilon) \) such that

\[
(6.5) \quad \int_0^{\delta r} n_{-r}(t, f_1) \frac{dt}{t} < \epsilon r^p
\]

for all \( r > r(\epsilon) \) lying outside a set \( E_0 \) of finite logarithmic measure, where \( n_{-r}(t, f_1) \) is the number of zeros of \( f_1 \) in \((-r - t, -r + t)\). Further, \( E_0 \) is independent of \( \epsilon \) and is a union of disjoint intervals each of which contains more than one point.

We need verify only that \( E_0 \) may be taken to be a union of disjoint intervals each containing more than one point; Lemma 6 certainly holds for some set \( E_1 \) of finite logarithmic measure and some functions \( \delta(\epsilon), r(\epsilon) \), by Lemma 1.

To this end, given \( \epsilon > 0 \), let \( \eta = \delta(\epsilon) \), where \( \delta \) is the function known to exist and suppose that \( r_1 \) and \( r_2 \) are two points outside \( E_1 \) with \( 2r_1 > r_2 > r_1 > r(\epsilon) \) and such that \( f_1 \) has no zeros in \([r_1, r_2]\). Then, for \( r_1 < r < r_2 \),

\[
I = \int_0^{\eta r} n_{-r}(t, f_1) \frac{dt}{t}
\]

\[
(6.6) \quad = \sum_{1}^{n_1} \log \frac{\eta r}{r + x_n} + \sum_{1}^{n_2} \log \frac{-\eta r}{r + y_n},
\]

where \( x_1, \ldots, x_{n_1} \) are the zeros of \( f_1 \) in \((-r, -r + \eta r)\) and \( y_1, \ldots, y_{n_2} \) are the zeros of \( f_1 \) in \((-r - \eta r, -r)\). Now

\[
\frac{r}{r + x_n} = 1 - \frac{x_n}{r + x_n} < 1 - \frac{x_n}{r_1 + x_n} = \frac{r_1}{r_1 + x_n},
\]

so, in view of Lemma 1 and Lemma 5

\[
(6.7) \quad \sum_{1}^{n_1} \log \frac{\eta r}{r + x_n} < \sum_{1}^{n_1} \log \frac{\eta r_1}{r_1 + x_n} < \int_0^{\eta r_1} n_{-r_1}(t, f_1) \frac{dt}{t} < \frac{1}{3} \epsilon r_1^p.
\]

Similarly, \(-r/(r + y_n) < -r_2/(r_2 + y_n)\), so

\[
(6.8) \quad \sum_{1}^{n_2} \log \frac{-\eta r}{r + y_n} < \int_0^{\eta r_2} n_{-r_2}(t, f_1) \frac{dt}{t} < \frac{1}{3} \epsilon r_2^p < \frac{2}{3} \epsilon r_1^p.
\]
From (6.6), (6.7) and (6.8),

\[(6.9) \int_0^{\tau r} n_{-r}(t; f_1) \frac{dt}{t} < \varepsilon r^q < \varepsilon r^p\]

for \(r \in (r_1, r_2)\).

Let \(E_0\) be the set obtained by removing from \(E_1\) all points which are limit points both from the left and from the right of the complement of \(E_0\). Then \(E_0\) is a union of disjoint intervals each containing more than one point and is contained in \(E_1\). Moreover, if \(r > r(\frac{1}{e})\) and \(r\) lies outside \(E_0\), then (6.9) holds, with \(\eta = \eta(e) = \delta(\frac{1}{4}e)\). This completely proves Lemma 6.

7. Further consideration of the zeros of \(F\). We first observe that the set of intervals the union of which is \(E_0\) is countable since the logarithmic measure of \(E_0\) is finite and the logarithmic measure of each interval is positive. We may therefore regard each interval as closed without affecting the value of the logarithmic measure of \(E_0\). We write \(E_0 = \bigcup_{i=1}^{\infty} J_i\), where

\[J_i = [c_i, d_i], \quad c_{i+1} > d_i > c_i, \quad i = 1, 2, 3, \ldots\]

Let \(\epsilon\) be any positive number. With Lemmas 3 and 6 in view, let \(\tau = \tau(\epsilon) = \min(\frac{1}{2}, \epsilon, \delta(\epsilon))\) (where \(\delta(\epsilon)\) is the function of Lemma 6) and let \(r_0(\epsilon)\) be a number at least as large as \(\max(r(\epsilon), R(r, \frac{1}{8}\tau))\) (where \(r(\epsilon), R(x, y)\) are respectively the functions of Lemmas 6 and 3) such that \(r_0(\epsilon) \not\in E_0\) (\(E_0\) is the set of Lemma 6) and for which \(d_i < (1 + \frac{1}{4}\tau)c_i\) whenever \(c_i > r_0(\epsilon)\). Since \(E_0\) is of finite logarithmic measure this choice of \(r_0(\epsilon)\) is possible.

Define, for \(J_i = [c_i, d_i]\) where \(c_i > r_0(\epsilon)\),

\[B_i = \{z: |z| \in J_i\} \cap \{z: \pi > |\arg z| > \pi - \frac{1}{8}\tau\},\]
\[D_i = \{z: |z| \in J_i\} \cap \{z: \pi > |\arg z| > \pi - \frac{1}{8}\tau\} \setminus H_1\]

where \(H_1\) is the exceptional set of Lemma 3 corresponding to \(\delta = \frac{1}{8}\tau\).

Let \(z\) be any point in \(D_i\). Then, with \(r = |z|\),

\[I(z) = \int_0^{\tau r} n_z(t; F) \frac{dt}{t} = \log \frac{(\tau r)^{m+n+p+q}}{\prod |z - a_j| \prod |z - b_j| \prod |z - u_j|},\]

where the \(a_j, b_j, u_j, v_j\) are zeros of \(F\) in \(|z - z| < \tau r\) which are respectively in \(B_i\), in \(|z| \in J_i\) \(\setminus B_i\) in \(|z| < c_i\) and in \(|z| > d_i\). Then we have, recalling Lemma 3 and Lemma 6,

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\[ I(z) \leq \log \frac{(\tau)^m}{\prod_i |z - a_j|} + \log \frac{(\tau)^n}{\prod_{j=1}^n (r - |u_j|)} + \log \frac{(\tau)^p}{\prod_{j=1}^n (|v_j| - r)} + \int_0^r \frac{n_i(t, F, \frac{1}{8} \tau)}{t} \, dt \]
\[ < \log \frac{(\tau)^m}{\prod_i |z - a_j|} + \log \frac{(\tau c_j)^n}{\prod_{j=1}^n (c_j - |u_j|)} + \log \frac{(\tau d_i)^p}{\prod_{j=1}^n (|v_j| - d_i)} + \int_0^r \frac{n_i(t, F, \frac{1}{8} \tau)}{t} \, dt \]
\[ (7.1) \]
\[ < \log \frac{(\tau)^m}{\prod_i |z - a_j|} + \int_0^{\sigma} n_{-c_i}(t, f_1) \, dt \frac{t}{t} + \int_0^{\sigma} n_{-d_i}(t, f_1) \, dt \]
\[ + \int_0^r \frac{n_i(t, F, \frac{1}{8} \tau)}{t} \, dt \]
\[ < \log \frac{(\tau)^m}{\prod_i |z - a_j|} + \varepsilon [c_i^p + d_i^p] + \tau r^p \]
\[ < \log \frac{(\tau)^m}{\prod_i |z - a_j|} + 4\varepsilon r^p. \]

Now, \( B_i \) is contained in a rectangle the sides of which have length \( \frac{1}{4} \tau d_i \) \( \frac{3}{8} \tau c_i \) and \( d_i - c_i \cos \frac{1}{8} \tau \). Hence for any point \( z \) in \( D_i \) the circle \( |z - \xi| < \tau |z| \) contains all of \( B_i \) and so all the zeros of \( F \) in \( B_i \) (i.e. all the \( a_j \) appear in (7.1)). We can thus apply Cartan's Lemma [3, p. 75] to estimate (7.1) and obtain
\[ \prod_{j=1}^m |z - a_j| > \left\{ \tau d_i \exp \left(-\frac{e d_i^p}{m}\right) \right\} \]
outside a set of at most \( m \) discs \( C_j, j = 1, 2, \ldots, m \), the sum of the radii of which is at most \( A = 2\varepsilon d_i \exp(-e d_i^p / m) \). Hence, for all \( z \) in \( D_i \) outside these discs we have, from (7.1),
\[ \int_0^r \frac{n_i(t, F)}{t} \, dt < 4\varepsilon r^p + \log \left( \frac{r}{d_i} \right)^m + \varepsilon d_i^p < 6e r^p. \]

We must have \( A < 2e(d_i - c_i). \) For suppose that \( A > 2e(d_i - c_i). \) Then
\[ I(e, d_i) = \int_0^{\delta(e)} n_{-d_i}(t, f_1) \frac{dt}{t} > \int_0^{\kappa d} n_{-d_i}(t, f_1) \frac{dt}{t} \]

\[ > \log \frac{\tau d_i}{\Pi_{j=1}^m (d_i - |\sigma_j|)} \]

\[ > \log \frac{(\tau d_i)^m}{(d_i - c_i)^m} \]

\[ > \log \frac{(2\pi d_i)^m}{A^m} = ed^p, \]

a contradiction, since \( I(e, d_i) \leq ed^p, d_i \) being a boundary point of \( E_0 \). Hence

\[ A > 2e(d_i - c_i). \]

Suppose that \( \mathcal{C}_j \) has radius \( t_j \) and centre \( j, j = 1, 2, \ldots, m. \)

Also, since \( d_i < (1 + \frac{1}{4})c_i < 2c_i \) and since, for \( x > 1, \log x > (x - 1)/x, \)

\[ \frac{d_i - c_i}{c_i} \]

so

\[ \sum_{j=1}^m \frac{t_j}{|j'|} < 4e \log \frac{d_i}{c_i}. \]

We are thus able to prove

**Lemma 7.** Let \( \epsilon \) be any positive number, and let \( \tau = \min\left(\frac{1}{\delta}, \epsilon, \delta(e)\right) \), where \( \delta(e) \) is the function of Lemma 6. Let \( r_0(e) \) be a positive number greater than \( \max(r(e), R(\tau, \frac{1}{8} \tau)) \) such that \( r_0(e) \notin E_0 \) and for which \( d_i < (1 + \frac{1}{4})c_i \) whenever \( c_i > r_0(e) \), where \( r(e), R(\tau, \frac{1}{8} \tau) \) are respectively the functions of Lemmas 6, 3, and \( E_0 \) is the set of Lemma 6. Then for all \( z \) in

\[ T\left(\frac{1}{8} \tau, r_0(e)\right) = \{z: |z| > r_0(e) \text{ and } \pi > |\arg z| > \pi - \frac{1}{8} \tau\} \]

we have, with \( |z| = r, \)

\[ \int_0^{\pi} n_z(t, F) \frac{dt}{t} < 6er^p \]

except when \( z \) belongs to an \( E \)-set, \( H_2 = H_2(e) \).

Suppose first that \( z \), in \( T\left(\frac{1}{8} \tau, r_0(e)\right) \), lies in \( \bigcup \{z: |z| \in J_i\} \), where the union is over those \( J_i = [c_i, d_i] \) for which \( c_i > r_0(e) \). Then for all \( z \) outside \( H_1 \), the \( E \)-set of Lemma 3, and outside a set of discs centres \( \xi \), radii \( t \) for which
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\[ \sum \frac{t}{|z|} < 4e \sum \frac{\log \frac{d_i}{c_i}}{\log \meas E_0} < 4e \log \meas E_0 < \infty, \]

(7.5) holds. This follows from (7.2) and (7.4).

Suppose next that \( z \), in \( T\left(\frac{1}{8} \tau, r_0(\epsilon)\right) \), lies outside \( \bigcup \{ z : |z| \in J_i \} \). Then, with \( |z| = r \), we have from Lemma 6

\[ \int_0^r n_z(t, F) \frac{dt}{t} < \int_0^{\delta(z)r} \eta_{-r}(t, f_1) \frac{dt}{t} < e^\rho. \]

(7.5) thus holds for \( z \) in \( T\left(\frac{1}{8} \tau, r_0(\epsilon)\right) \) outside an E-set, and Lemma 7 is proved.

We prove

**Lemma 8.** Let \( \epsilon \) be any positive number and let \( \sigma = \sigma(\epsilon) = \frac{1}{32} \tau(\epsilon) \), where \( \tau(\epsilon) \) is the function of Lemma 7. There exists a number \( r_1(\epsilon) \) and an E-set, \( H_3 = H_3(\epsilon) \), such that

\[ \int_0^{r_1} n_z(t, F) \frac{dt}{t} < 6e^\rho, \]

whenever \( |z| = r > r_1(\epsilon) \) and \( z \) lies outside \( H_3 \).

For \( z \) in \( T\left(\frac{1}{8} \tau, r_0(\epsilon)\right) \) and outside \( H_2(\epsilon) \), (7.6) certainly holds, by Lemma 7.

Consider \( z \) outside \( T\left(\frac{1}{8} \tau, r_1(\epsilon)\right) \), where \( r_1(\epsilon) = \max(r_0(\epsilon), R\left(\frac{1}{32} \tau, \frac{1}{8} \tau\right)) \). Let \( H_4(\epsilon) \) be the E-set \( H_1\left(\frac{1}{8} \tau\right) \) of Lemma 3. Then, with \( \sigma = \frac{1}{32} \tau \) and \( r = |z| \), and \( z \) outside \( H_4(\epsilon) \), we have from Lemma 3

\[ \int_0^{r_1} n_z(t, F, \frac{1}{8} \tau) \frac{dt}{t} < \sigma e^\rho < e^\rho. \]

But for \( 0 < t < \frac{1}{32} \tau \), \( r > R\left(\frac{1}{32} \tau, \frac{1}{8} \tau\right) \), \( n_z(t, F) = n_z(t, F, \frac{1}{8} \tau) \) for \( z \) outside \( T\left(\frac{1}{8} \tau, r_1(\epsilon)\right) \). Hence \( \int_0^{r_1} n_z(t, F) dt/t < e^\rho \) for \( z \) outside \( T\left(\frac{1}{8} \tau, r_1(\epsilon)\right) \) and outside \( H_4(\epsilon) \), with \( |z| = r > R\left(\frac{1}{32} \tau, \frac{1}{8} \tau\right) \).

Lemma 8 then follows with \( H_3(\epsilon) = H_2(\epsilon) \cup H_4(\epsilon) \).

The following is an immediate consequence of Lemma 8.

**Lemma 9.** Let \( \epsilon \) be any positive number. There exist positive numbers \( a(\epsilon), r_2(\epsilon) \) and an E-set \( H_5 \), independent of \( \epsilon \), such that

\[ \int_0^{a(\epsilon)} n_z(t, F, \frac{1}{8} \tau) \frac{dt}{t} < e^\rho \]

when \( r = |z| > r_2(\epsilon) \) and \( z \) lies outside \( H_5 \).

For \( z \) such that \( r = |z| > r_1\left(\frac{1}{6} \tau \right) \) and \( z \) lies outside \( H_3\left(\frac{1}{8} \tau \right) \), where \( r_1 \) and \( H_3 \) are as in Lemma 8, we have, with \( \sigma = \sigma\left(\frac{1}{6} \tau\right) \), \( \int_0^{r_1} n_z(t, F) dt/t < r^\rho \). Given any integer \( n > 1 \), suppose that \( H_5(1/6n) \) is covered by the discs \( C_i(n) \), radii \( i(n) \) and centres \( \xi_i(n) \), \( i = 1, 2, 3, \ldots \). Let \( i_0 = i_0(\epsilon) \) be the smallest integer such that
Let $r_2(1) = r_1\left(\frac{1}{2}\right)$ and, supposing $r_2(m)$ defined, $m > 1$, let $r_2(m + 1)$ be the smallest number which is no less than $\max\{r_2(1/m) + 1, r_1(1/6(m + 1))\}$ and such that

\[ C_i(m + 1) \subset \{z : |z| < r_2(1/ (m + 1))\}, \quad i = 1, 2, \ldots, i_0(m + 1) - 1. \]

Let $H_4$ be given by

\[ H_4 = \left\{ \bigcup_{i=1}^{i_0(n)} C_i(1) \right\} \cup \left\{ \bigcup_{n=2}^{\infty} \bigcup_{i=i_0(n)} C_i(n) \right\}. \]

From (7.7), $H_4$ is an $E$-set.

Given any number $\epsilon$, $0 < \epsilon < 1$, let $m$ be the integer such that

\[ \frac{1}{m + 1} < \epsilon < \frac{1}{m}; \]

define $r_2(\epsilon) = r_2(1/(m + 1))$, $\alpha(\epsilon) = \sigma(1/6(m + 1))$. Let $\epsilon$ be any positive number, $0 < \epsilon < 1$, and let $z$ be outside $H_4$ and such that $r = |z| > r_2(\epsilon)$. Then, if $m$ is the integer satisfying (7.8), $z$ lies outside $H_3(1/6(m + 1))$ and $r = |z| > r_1(1/6(m + 1))$ so by Lemma 8,

\[ \int_{0}^{\alpha(\epsilon)r} n_z(t, F) \frac{dt}{t} = \int_{0}^{\sigma(1/6(m + 1))} n_z(t, F) \frac{dt}{t} < \frac{1}{m + 1} r^\rho < \epsilon r^\rho. \]

Lemma 9 is thus proved.

8. Completion of the proof of Theorem 1. By Lemma 1 and Lemma 9,

\[ |r^{-\rho} \log |F(re^{i\theta})| - \beta \cos \rho \theta| \to 0 \]

as $z = re^{i\theta}$ tends to infinity outside $H_4$. From (8.1) and Lemma 2, Theorem 1 follows.


\[ \lim_{r \to \infty} \frac{\log |f(r)|}{r^\rho} = A \quad (0 < A < \infty) \]

for some $\rho$ such that $0 < \rho < 1$ then

(i) $\lim_{r \to \infty} \log |f(-r)|/r^\rho = A \cos \pi \rho$; and

(ii) given $\epsilon > 0$,

\[ \log |f(-r)| > (A \cos \pi \rho - \epsilon)r^\rho \]

for all $r$ outside a set of linear density zero.
We shall show that the exceptional set of (ii) cannot be replaced with a set of finite logarithmic measure by constructing an integral function satisfying (9.1) for which (9.2) fails for some \( \varepsilon > 0 \) on a set of infinite logarithmic measure. The construction depends on Lemma 1.

Let \( A \) be any fixed positive number. Let \( (R_m) \) be an increasing sequence of positive numbers, let \( \eta_m = (\log m)^{-1} \) and let \( \delta_m = m^{-1/2}, \ m = 2, 3 \ldots \). Let \( f(z) \) be an integral function with real negative zeros for which the counting function \( n(r,f) \) satisfies

\[
n(r,f) \sim Ar^p.
\]

We introduce an integral function \( g(z) \) obtained from \( f(z) \) by placing \( 1 + [\eta_m R_m^p] \) additional zeros at \( -R_m \). It is clear that the sequence \( (R_m) \) may be chosen sparsely enough that \( n(r,g) \), the counting function of the zeros of \( g(z) \), satisfies \( n(r,g) \sim Ar^p \).

With Lemma 1 in view let us consider, for \( R_m < r < (1 - 1/m)^{-1}R_m \),

\[
\int_0^{\delta_m r} \frac{n_{-r}(t,g)}{t} dt.
\]

Since \( r - R_m < r - r(1 - m^{-1}) < \delta_m r \), each zero at \( -R_m \) contributes \( \log \{ \delta_m r / r - R_m \} \) to the integral (9.3). Hence we have, for \( R_m < r < (1 - 1/m)^{-1}R_m \),

\[
\int_0^{\delta_m r} \frac{n_{-r}(t,g)}{t} dt \geq \eta_m R_m^p \log \left( \frac{\delta_m r}{r - R_m} \right) \geq \eta_m R_m^p \log(m\delta_m) = \frac{1}{2} R_m^p > \frac{1}{4} r^p
\]

for all large \( m \), taking account of the definitions of \( \eta_m \) and \( \delta_m \). Further

\[
E = \bigcup_{m=3}^{\infty} \left\{ r: R_m < r < \left(1 - \frac{1}{m}\right)^{-1} R_m \right\}
\]

is a set of infinite logarithmic measure.

Now we appeal to Lemma 1 to conclude that there must be a number \( \varepsilon > 0 \) such that

\[
\log|f(-r)| < (A \cos \pi \rho - \varepsilon) r^p
\]

for all large \( r \) in \( E \). For suppose that there were a sequence \( (r_n) \) tending to infinity through \( E \) such that

\[
\log|f(-r_n)| > (A \cos \pi \rho - o(1)) r_n^p.
\]

From (i) of Titchmarsh's result, then,

\[
\log|f(-r_n)| = (A \cos \pi \rho + o(1)) r_n^p
\]
and so, from Lemma 1, there must exist $\delta > 0$ such that
\[ \int_0^{\delta r_n} \frac{n-r_n(t, \varphi)}{t} \, dt < \frac{1}{4} r_n^\varphi \]
for all large $n$, which contradicts (9.4). (9.5) thus holds for all large $r$ in $E$.

Theorem 1 is an improved version of a result which forms part of a thesis submitted for the degree of Ph.D at the University of London. It is a pleasure to express my gratitude to Professor W. K. Hayman of Imperial College, London, for his generous advice and encouragement.

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