Curves with Large Tangent Space

By

Joseph Becker¹ and Rajendra Gurjar²

Abstract. Theorem. Let \( V \) be a complex analytic variety irreducible at a point \( p \in V \). Given any integer \( I \), there exists an analytic curve \( C_i \) on \( V \) passing through \( p \) and irreducible at \( p \) such that the germs of \( C_i \) and \( V \) at \( p \) are isomorphic up to order \( I \).

In [6] Hironaka and Rossi have proven the following result: Let \( V \) be a complex analytic variety of pure dimension \( r \) and \( p \) an isolated singular point of \( V \). Then there exists an integer \( v_0 \) so that if \( p' \) is an isolated singular point of a complex analytic variety \( V' \) of pure dimension \( r \) and if \( \mathcal{O}_p(V)/m_p^r \cong \mathcal{O}_{p'}(V')/m_{p'}^r \), for some \( r > v_0 \), where the isomorphism is as \( \mathbb{C} \) algebras and \( m_p \) is the maximal ideal of the local ring \( \mathcal{O}_p(V) \) of \( V \) at \( p \), then \( \mathcal{O}_p(V) \cong \mathcal{O}_{p'}(V') \).

We show here that it is necessary for \( V \) and \( V' \) to be of the same dimension.

Theorem. Let \( V \) be a complex analytic variety irreducible at a point \( p \in V \). Given any integer \( I \), there exists an analytic curve \( C_i \) on \( V \) passing through \( p \) and irreducible at \( p \) such that \( \mathcal{O}_p(V)/m_p(V)^I \cong \mathcal{O}_{p'}(C_i)/m_{p'}(C_i)^I \) as \( \mathbb{C} \) algebras. In particular, there exists an irreducible analytic curve \( C_2 \) on \( V \) having the same tangent space at \( p \) as \( V \) does, where the tangent space to \( V \) at \( p = \mathcal{O}_p(V)/m_p(V)^2 \).

In §§2 and 3, we generalize this result to complete domains and finally to arbitrary analytically irreducible local Noetherian rings to read as follows: Let \( R \) be a local Noetherian ring, whose completion \( \hat{R} \) with respect to the maximal ideal \( M \) is a domain. Then for every integer \( I > 0 \), there exists a prime ideal \( P_i \) in \( R \), with \( P_i \subset M^I \) and \( \text{dim } R/P_i = 1 \). The results of §§1 and 2 can be deduced from those of §3; we include both proofs to illustrate the various techniques. The argument presented in Theorem 3 is due to M. Hochster, and the authors are grateful for his permission to reproduce it here.

In §4, we give an application of these results to local differentiable
embeddings of a real or complex analytic variety. We show how to recover several other known results, due to Risler [12] and Merrien [8]. Also these curves have application to the solution of power series equations [5].

1.

**Proof of Theorem.** We may assume that a neighborhood of \( p \) in \( V \) is embedded in \( \mathbb{C}^n \), \( n = \text{embedding dim of } V \) at \( p \), and \( p \) is the origin in \( \mathbb{C}^n \). We consider the blowing up of the maximal ideal, that is the quadratic transform \( B \) of \( V \) with center at \( p \); \( B \) is the closure in \( V \times \mathbb{P}^{n-1} \) of the set \( B' \) in \( V \times \mathbb{P}^{n-1} \) where \( B' = \{(y_1, \ldots, y_n, z_1, \ldots, z_n) \in V - \{0\} \times \mathbb{P}^{n-1} : z_jy_i = z_iy_j \text{ for all } 1 \leq i, j \leq n \} \). It is known that the natural projection \( \phi : B \rightarrow V \) has the following properties: \( F = \phi^{-1}(p) \subset \mathbb{P}^{n-1} \) is the tangent cone to \( V \) at \( p \) so \( \dim F = r - 1 \). \( \phi : B - \phi^{-1}(p) \rightarrow V - p \) is a biholomorphism, so \( B - \phi^{-1}(p) \) is of pure dim \( r \).

We also consider the normalization \( X \) of \( B \). There exists a holomorphic map \( \pi : X \rightarrow B \) such that: \( \pi \) is a proper map with finite fibers. If \( S \) is the singular locus of \( B \), then \( \pi^{-1}(B - S) \) is dense in \( X \), \( \pi : \pi^{-1}(B - S) \rightarrow B - S \) is biholomorphic, and \( X \) is locally irreducible. The local ring of \( X \) at any point \( x \), \( \Theta_x(X) \) is the integral closure of \( \Theta_{\pi(x)}(B) \) in its full quotient ring. If \( S(X) \) denotes the singular locus of \( X \), then \( \dim_x \Theta_x(X) \leq \dim_x X - 2 \).

Hence there exist \( x \in \pi^{-1}(F) \) such that \( x \) is a simple point of \( X \). Now \( \Theta_x(X) \approx \mathbb{C}(x_1, \ldots, x_r) \), the convergent power series in \( r \) variables so there exist convergent power series \( f_1(x_1, \ldots, x_r), \ldots, f_n(x_1, \ldots, x_r) \) all vanishing at the origin such that \( \Theta_p(V) = \mathbb{C}(y_1, \ldots, y_n) = \mathbb{C}(f_1, \ldots, f_n) \). We now show that \( \Theta_p(V) \) is a subring of \( \Theta_x(X) \), that is the canonical homomorphism \( \Theta_p(V) \rightarrow \Theta_x(X) \) is an injection. Let \( B_\pi(x) = B_1 \cup B_2 \cup \cdots \cup B_n \) be a decomposition into germs of irreducible components (which all have the same dimension since the connected manifold \( \phi^{-1}(\text{Reg } V) \) is dense in \( B \)) such that the germ of \( X \) at \( x \) is the normalization of \( B_1 \). \( \phi|X - \phi^{-1}(F) \) is an open map and \( \pi|B_1 - F \) is an open map so \( \pi \circ \phi(X) \) contains an open set of \( V \); hence any analytic function vanishing on \( \pi(B_1) \) must vanish identically on the irreducible variety \( V \), so it is an injection.

It is clear that \( m_p(V) = m_x(X) \cap \Theta_p(V) \). The Krull topology of \( \Theta_x(X) \) defined by powers of the maximal ideal induces a topology \( T_2 \) on \( \Theta_p(V) \) which is clearly stronger than the natural topology \( T_1 \) on \( \Theta_p(V) \).

**Lemma 1.** \( T_1 = T_2 \), that is there is an increasing function \( h : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) such that \( m_x(X)^{h(n)} \cap \Theta_p(V) \subset m_p(V)^n \).

**Lemma 2.** Given integers \( r \) and \( N > 0 \), there exist integers \( n_1, n_2, \ldots, n_r \), all \( > 0 \) so that for any formal power series \( h(x_1, \ldots, x_r), h(t^{n_1}, \ldots, t^{n_r}) \equiv 0 \) implies \( \text{ord } h(x_1, \ldots, x_r) > N \).
We will finish deducing the theorem before giving the proofs of these lemmas. We will pick suitable convergent power series $x(t) = (x_1(t), \ldots, x_r(t))$ in one variable $t$ without constant terms and let $C_i$ be the image of $f(x(i))$ in $V$; then the domain

$$T = C\{ f_1(x_1(t), \ldots, x_r(t)), \ldots, f_n(x_1(t), \ldots, x_r(t)) \}$$

will be the local ring $\Theta_p(C_i)$ in the theorem. Let $R = \Theta_p(V)$, $\psi: R \to T$ be the canonical surjection given by substitution ($\phi \to \phi(f(x(i)))$, and $I_i$ denote the kernel of $\psi$. For $g \in R$, by ord $g$ we mean the order of $g$ considered as an element of $\Theta_p(X)$.

Let $N = h(I) + 1$. Now choose the monomials $t^{n_1}, \ldots, t^{n_r}$ which satisfy the property in Lemma 2. If $g \in I_I$, then $g(f(t^{n_1}, \ldots, t^{n_r})) = 0$ so by Lemma 2, ord $g > N > h(I)$, so $g \in m_p(V)^I$. Hence $I_I \subset m_p(V)^I$. Now $T = R/I_I \cong \Theta(C_i)$ and the maximal ideal $m(T)$ of $T$ is $m(R)/I_I$. Hence

$$\Theta_p(C_i)/m_p(C_i)^I = T/m(T)^I \cong (R/I_I)/(m(R)/I_I)^I = R/(m(R)^I + I_I) = R/m(R)^I$$

as $I_I \subset m(R)^I$. Q.E.D.

Proof of Lemma 2. Suppose $n_1, \ldots, n_r$ are any positive integers and $h(x_1, \ldots, x_r)$ is a nonzero formal power series such that $h(t^{n_1}, \ldots, t^{n_r}) = 0$. Then there exist two monomials $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $x_1^{\beta_1} \cdots x_r^{\beta_r}$ appearing in $h$ such that $\alpha_1 n_1 + \cdots + \alpha_r n_r = \beta_1 n_1 + \cdots + \beta_r n_r$ and each $\alpha_i < N$. Letting $\gamma_i = \alpha_i - \beta_i$, we have each $\gamma_i < N$ too. So it is enough to choose $n_i$ so that $\Sigma \gamma_i n_i = 0$ and $\gamma_i < N$ implies each $\gamma_i = 0$.

Let $p$ be a prime $> N$, $l_i, 1 < i < r$, be primes chosen inductively, $l_1 > p$, so that $p^{2l_i} < l_{i+1}$, and $n_i = p^{r-i}l_i$. Suppose $\Sigma_{i=1}^r \gamma_i n_i = 0$ and each $\gamma_i < p$, let $k$ be the largest integer for which $\gamma_k \neq 0$. By our choice $p^{r-k+1}$ divides $n_1, \ldots, n_{k-1}$ so $p^{r-k+1}$ divides $n_k = p^{r-k}l_k \gamma_k$ so $p$ divides $l_k \gamma_k$; since $p$ and $l_k$ are relatively prime, $p$ divides $\gamma_k$. We know $\gamma_k < p$ so a contradiction will follow when we show $\gamma_k > 0$.

Now $n_i = p^{r-i}l_i$, $n_{i+1} = p^{r-i-1}l_{i+1}$, and $l_{i+1} > p^{2l_i} \Rightarrow rp^n < n_{i+1} \Rightarrow n_i < n_k$ for $i < k - 1$. Since $\gamma_i < p$ for all $i$, we have

$$\left| \sum_{i=1}^{k-1} \gamma_i n_i \right| < \sum_{i=1}^{k-1} |\gamma_i n_i| < p(k - 1)n_{k-1}$$

$$< p(k - 1)n_k/rp < n_k.$$
the Krull topology of $S'_1$. We now recall:

(A) **Zariski Subspace Theorem** [2]. Let $R$ be an analytically irreducible domain (completion is a domain) and $S$ a local ring birational with $R$ (means same quotient field) and dominating $R$, and $S$ a spot over $R$ (means $S$ is the localization of a finite algebra over $R$), then $R$ is a subspace of $S$.

(B) The ring of convergent power series at an irreducible point of a complex analytic variety is analytically irreducible. [7, p. 89].

(C) [3, Lemma 1.11]. Let $R$ be an analytic local ring and $S$ a spot over $R$. Then there is a smallest analytic ring $S'$ containing $S$ and $\hat{S} = \hat{S}'$, where hat denotes completion in the maximal ideal topology.

Now let $(p; a_1, a_2, \ldots, a_n)$ denote the coordinates of the point $\pi(x)$ on $B$. By linear change of coordinates, we may assume $a_1 = 1$. Let

$$S = \mathcal{O}_p(V)[y_2/y_1, \ldots, y_n/y_1](z_1 - a_1, \ldots, z_n - a_n),$$

where $z_i = y_i/y_1$. We first show that $\mathcal{O}_{\pi(x)}(B)$, the local analytic ring of $B$ at the point $\pi(x)$, is the smallest analytic ring $S'$ containing $S$. We have a commutative diagram:

$$\begin{array}{ccc}
S & \xrightarrow{\phi} & \mathcal{O}_{\pi(x)}(B) \\
\cap & & \uparrow \\
S' & \xrightarrow{\phi} & \hat{S}'
\end{array}$$

Since $y_1, \ldots, y_n, z_1 - a_1, \ldots, z_n - a_n$ generate the maximal ideal of both $S$ and $\mathcal{O}_{\pi(x)}(B)$ we have that $\phi(M_S) = M_{\mathcal{O}_{\pi(x)}(B)}$. Since $S'$ and $\mathcal{O}_{\pi(x)}(B)$ are both analytic rings, it follows [9] that $\phi(S') = \mathcal{O}_{\pi(x)}(B)$.

Now $S$ is a spot over $\mathcal{O}_p(V)$ so by (B) and (A) above, $\mathcal{O}_p(V)$ is a subspace of $S$. Since $S$ is obviously a subspace of $\hat{S}$, we have $\mathcal{O}_p(V)$ is a subspace of $\hat{S}$. Hence $\mathcal{O}_p(V)$ is a subspace of $\hat{S}'$. But $\mathcal{O}_p(V) \subset S'$ so it follows that $\mathcal{O}_p(V)$ is a subspace of $S'$. By similar techniques we could show that $\mathcal{O}_p(V)$ is a subspace of $S'_1$.

**Remark.** Alternately we could go directly to the normalization and show $\mathcal{O}_p(V)$ is a subspace of $\mathcal{O}_x(X)$ by applying (A), (B), and (C). This proof would proceed by using the fact that $\mathcal{O}_x(X)$ is a finite $\mathcal{O}_{\pi(x)}(B)$ module and constructing a spot over $\mathcal{O}_p(V)$ for which $\mathcal{O}_x(X)$ is the smallest analytic ring containing the spot. Q.E.D.

2. By techniques similar to those employed in §1, one can prove a formal version of the theorem.

**Theorem 2.** If $k$ is a field of characteristic zero and $R = k[[y_1, \ldots, y_n]]/I$ is a complete domain, and $m$ is the maximal ideal of $R$, then for every integer
There is a prime ideal $P_l$ of depth one with $P_l \subseteq m^l$.

Before starting on the proof of this we need to establish some preliminaries. For an arbitrary Noetherian ring $S$ and field $k \subset S$, let $D^1(S/k)$ be the module of Kahler differentials of $S$ over $k$. If $S$ is local with max ideal $m$, let

$$\hat{D}(S/k) = D^1(S/k)/\bigcap_{i=1}^{\infty} m^i D^1(S/k).$$

It is well known that for a local ring $S$ containing its residue field of characteristic zero, with $\hat{D}(S/k)$ a finite $S$ module, we have $\hat{D}(S/k)$ is a free $S$ module if and only if $S$ is regular. If $S$ is finitely generated over $k$ (respectively complete) then $D(S/k)$ (respectively $\hat{D}(S(k))$) is a finite module over $S$. If $M$ is a finitely generated $S$ module, generated by say $e_1, \ldots, e_m$, let $(a_{ij})$ be a matrix with entries in $S$ such that for each $i$, $\sum_{j=1}^{m} a_{ij} e_i = 0$, and such that any row vector $(a_1, a_2, \ldots, a_n)$ for each $\sum a_j e_j = 0$ is a linear combination with coefficients in $S$ of the rows of $(a_{ij})$; in other words the rows of $(a_{ij})$ generate the module of relations of $e_1, \ldots, e_m$. For an integer $p \geq 0$, the $(n-p) \times (n-p)$ minors of $(a_{ij})$ generate an ideal $I_p(M)$; by convention $I_p(M) = S$ for $p > n$. The ideal does not depend on either the choice of basis $\{e_1, \ldots, e_n\}$ or the relation matrix $(a_{ij})$; $I_p(M)$ is the $p$th fitting of $M$. We have $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = S$. By the jacobian ideal $J$ of $S$ we will mean the first nonzero fitting ideal of $D(S/k)$. One knows that $S$ is regular if and only if $J$ is the unit ideal. We will denote $\hat{D}(S/k)$ by $\Omega(S/k)$.

It follows immediately from the Serre criteria for normality that if $S$ is a normal Noetherian ring and $k \subset S$, then every minimal prime of $\text{Jac}(\Omega(S/k))$ has height $\geq 2$. (We will denote this by saying $\text{ht} J > 2$.)

**Lemma 3.** Let $S$ be a Noetherian ring, $k \subset S$, $J = \text{Jac}(\Omega(S/k))$, $M$ a maximal ideal of $S$ with $J \not\subset M$, $S/M = F$, $F \subset S$, and $F$ a finite field extension of $k$. Then $S_M$ is regular.

**Proof.** $\text{Jac}(\Omega(S_M/k)) = \text{Jac}(\Omega(S/k)) \cdot S_M = J S_M = S_M$ as $J \not\subset M$. In char 0 all extensions are separable and for a finite separable field extension, $\Omega(F/k) = 0$. So by the exact sequence $0 \rightarrow S \otimes_F \Omega(F/k) \rightarrow \Omega(S/k) \rightarrow \Omega(S/F) \rightarrow 0$ we see that $\Omega(S/k) = \Omega(S/F)$ so $\text{Jac}(\Omega(S_M/F)) = S_M$. Hence $S_M$ is regular.

**Lemma 4.** Let $k \subset S \subset F[[t]]$, where $k \subset F$ are fields, $[F:k] < \infty$, $S \subset F$, and $S$ is a Noetherian complete ring. Then $\text{Krull dim } S < 1$.

**Proof.** We will show that $F[[t]]$ is a finite extension over $S$ and hence integral over $S$ so by the going up and down theorem for integral extensions, $\text{Krull dim } S = \text{Krull dim } F[[t]]$.

Because $[F:k] < \infty$, it is easy to see that the map
is not injective: Let \( g \in S \), \( g \notin F \), and \( \nu(g) \neq 0 \), then \( \nu(g) \) satisfies some minimal polynomial over \( k \), \( \nu(g)^r + k_1\nu(g)^{r-1} + \cdots + k_r = 0 \) with \( k_i \in k \) and \( k_r \neq 0 \). So \( f = g^r + k_1g^{r-1} + \cdots + k_{r-1}g + k_r \in S \), \( \nu(f) = 0 \), and \( f \neq 0 \). To see the last statement, let \( g = \sum a_i t^i \), \( a_i \in F \), and \( a_n \) be the first nonzero term with \( m > 0 \); then the \( m \)th coefficient of \( f \) is \( a_m(k_r + 2k_{r-1}a_0 + 3k_{r-2}a_0^2 + \cdots + rk_0a_0^{r-1}) \) which is nonzero since the polynomial \( \sum_{i=0}^{r} ik_{r-i}a_0^{i-1} \) has degree less than the minimal polynomial for \( a_0 \).

Since \( 1 \in S \), \( m(F[[t]]) \cap S \subseteq m(S) \). Let \( g \) be an element of the kernel of \( \nu \), \( g = t^n u(t) \), \( n > 0 \), \( u \) a unit in \( F[[t]] \). We can subject \( F[[t]] \) to an \( F \) isomorphism \( x = \phi(t) \) so that \( \phi(g) = x^n \); hence we may assume \( k \subseteq S \subseteq F[[x]] \) and \( x^n \in S \). Now \( x^n \in m(F[[x]]) \) and \( x^n \in S \) so \( x^n \in m(S) \); since \( S \) is complete, \( k[[x^n]] \subseteq S \). Let \( f_1, \ldots, f_r \) be a basis of \( F \) over \( k \). Then \( \{f_j x^{i-j} \mid 1 < j < r, 1 < i < n\} \) is a basis of \( F[[x]] \) over \( k[[x^n]] \). Since \( k[[x^n]] \subseteq S \), \( F[[x]] \) is also a finite \( S \) module.

**Remark.** The hypothesis of complete is critical here as the following example shows. There exist in \( k[[t]] \) an infinite set \( \{f_i\}_{i=1}^\infty \) of algebraically independent elements \([1]\); like \( e^x - 1, \ e^{x^2} - 1, \ldots \) for example. Then \( k[f_1, f_2, f_3, \ldots] \) is not Noetherian and has infinite Krull dim. And \( S' = k[f_1, \ldots, f_m] \) is local, Noetherian, not complete, and has Krull dim \( n \).

Note that \( k[[f_1, \ldots, f_m]] \) has Krull dim one. (Any two power series in one variable are analytically dependent.) The problem here is that the completion of \( k[f_1, \ldots, f_n] \) does not inject into \( k[[f_1, \ldots, f_n]] \).

**Proof of Theorem 2.** Let \( B \) be the blowing up of the maximal ideal in \( R \), and \( N \) be the integral closure of \( B \) in its quotient field. Clearly \( B \) and \( N \) are contained in the quotient field of \( R \), and \( B \) is a finite algebra over \( N \). That \( N \) is a finite \( B \) module follows from the below fact.

**Fact D** [9, 36.1, 36.5]. A ring \( A \) is pseudogeometric if \( A \) is Noetherian and if for every prime ideal \( p \) of \( A \) and ring \( E \), \( A/p \subseteq E \), with the quotient field of \( E \) a finite extension of the quotient field of \( A/p \), and \( E \) integral over \( A/p \), we have that \( E \) is a finite \( A/p \) module. In characteristic zero, complete rings are pseudogeometric. If \( A \) is pseudogeometric, then every localization of a finite algebra over \( A \) is pseudogeometric.

Let \( M(R) = (y_1, \ldots, y_m)R, \ B = R[y_2/y_1, \ldots, y_m/y_1], \) and \( I = (y_1, \ldots, y_m)B = y_1B \) and \( IN \) be the extension of \( I \) to \( N \). \( N \) is a finite \( B \) module so \( N/IN \) is a finite \( B/I \) module. However \( B/I \) does not inject into \( N/IN \).

**Example.** \( R = k[[x, y, z]]/(x^4 + y^4 - z^2), \) let \( z = xv, \ y = xu, \) \( B = k[[x]][u, v]/(x^2(1 + u^4) - v^2). \) Then \( v/x \) is in the quotient field of \( B, (v/x)^2 - (1 + u^4) = 0 \) so \( v/x \in N. \) However \( v/x = \sqrt[4]{1 + u^4} \) is in \( \hat{B} \) but not in \( B. \)
Hence $x$ divides $v$ in $N$, but $x$ does not divide $v$ in $B$. So $v$ is an element of the kernel $B/I \rightarrow N/IN$.

Now $I$ is principal and a proper ideal. Furthermore it can be shown that $\dim N/IN = \dim N - 1$. (It should be pointed out that $B$ and $N$ have problems with their Krull dimension. In opposition to the case of local rings or affine rings, $B$ does not have the property that every maximal chain of primes has the same length.)

**Example.** Let $R = k[[x, y]]$ and $B = k[[x, y]][y/x]$. Then $(0) \subset (x) \subset (x, y/x)$ is a chain of primes of length 2 in $B$. But $(0) \subset I = (1 - y(y/x))B$ is a maximal chain of primes of length one in $B$: $I$ is principal so of height one; it suffices to show $I$ is a maximal ideal. Now $x - y^2 \in I$ so $B/I = k[[y^2, y]][1/y] = k[[y]][1/y] = k((y))$ which is a field.

Let $J$ be the jacobian ideal of $N$. Now $\text{ht} J > 2$, and $\text{ht} IN = 1$ so $J \not\subset IN$; hence $J' = J/IN$ is not zero in $N' = N/IN$. Clearly $N'$ is an affine ring, so there exists a maximal ideal $M'$ of $N/IN$ with $J' \not\subset M'$. (This is easily seen by tensoring $N'$ with the algebraic closure $\bar{k}$ of $k$, finding a maximal ideal $M_0$ of $N' \otimes_k \bar{k}$ with $J' \otimes_k \bar{k} \not\subset M_0$ via the Hilbert Nullstellensatz, and letting $M' = M_0 \cap N'$.) Now let $M$ be the contraction of $M'$ to $N$ and $S = N/M$. Clearly the contraction of $M$ to $R$ is the maximal ideal of $R$. Since $N_M$ is a spot over $R$, the residue field $F$ of $S$ is finite over the residue field $k$ of $R$. By Lemma 3, $S$ is regular. By Cohen’s structure theorem for complete local rings, $S = F[[t_1, \ldots, t_r]]$, where $r = \dim R$. Let $\eta: S \rightarrow F[[t]]$ be a homomorphism as in Lemma 2 with $\ker \eta \subset M(S)^\prime$. Let $i$ be the inclusion $R \rightarrow B \rightarrow N \rightarrow S$, and $\psi = \eta(i)$. By Lemma 4, $\dim \psi(R) = 1$; hence $\dim R/\ker \psi = 1$. The theorem now follows from the Chevalley subspace theorem:

**Fact E** [13, p. 255]. Let $R \subset S$ be complete local rings with $m(S) \cap R = m(R)$. Then $R$ is a subspace of $S$. Q.E.D.

3. We would now like to improve the results of §§1 and 2 to yield the following:

**Proposition 1.** If $R$ is the field of real numbers and $A = R\{x_1, \ldots, x_n\}/I$ is the local ring of germs of convergent power series over $R$ at a point $p$ on a real analytic variety $V$, and $l > 0$, then there is a curve $C_l$ lying on $V$ and passing through $p$ such that $V$ and $C_l$ are isomorphic up to order $l$ at $p$. This curve is given by a $R$ algebra homomorphism $A \rightarrow R\{t\}$.

Of course the ideal $I$ must be the ideal of a real analytic variety, that is the real Hilbert Nullstellensatz must hold for $I$. That is every $f \in A$ which vanishes on the locus of $I$ belongs to $I$. It is known that this is equivalent to the following condition on $I$: for every $f_1, \ldots, f_p \in A$ with $f_1^2 + \cdots + f_p^2 \in I$, we have each $f_i \in A$.

Unfortunately this does not seem to follow immediately because when
blowing up and normalizing the residue field is likely to change from \( \mathbb{R} \) to \( \mathbb{C} \), yielding a complex analytic curve on the complexification, \( V \otimes_{\mathbb{R}} \mathbb{C} \) of \( V \), rather than a real analytic curve on \( V \). For this reason, we give a different proof of Theorem 2 which works for a more general class of rings.

**Theorem 3.** Let \( S \) be an analytically irreducible domain with residue field of characteristic zero. Then for every \( l > 0 \), there is a prime \( P_l \) with \( P_l \subset M^l \) and \( \dim S/P_l = 1 \).

**Remark.** Actually this theorem holds in mixed characteristic as well, but we are not interested in that here.

We first recall [13, pp. 86–92, Vol. I] the properties of the norm map. Let \( K \) be a finite algebraic extension of a field \( k \) of degree \( n \) and \( y \in K \). Fixing a basis \( w_1, w_2, \ldots, w_n \) of \( K \) over \( k \), we write \( \sum_{j=1}^{n} a_j w_j \), \( a_j \in k \), \( 1 \leq i < n \), or in matrix notation \( y \Omega = A \Omega \), where \( A = (a_{ij}) \) is an \( n \times n \) matrix and \( \Omega \) is the column matrix with entries \( w_i \). The characteristic polynomial \( \det(A - XI) = X^n + a_1 X^{n-1} + \cdots + a_n \) has \( y \) as root. It is not necessarily the minimal polynomial of \( y \) over \( k \). It is not hard to see that the polynomial does not depend on the choice of basis of \( K \) over \( k \). Note that \( a_n = (-1)^n \det A \). We set the norm of \( y \), \( N_{K/k}(y) = (-1)^n a_n \). The following properties are well known:

(a) \( N_{K/k}(xy) = N_{K/k}(x) N_{K/k}(y) \).

(b) If \( y \in k \), then \( N_{K/k}(y) = y^n \).

(c) If \( R \subset S \) are domains with quotient fields \( k \subset K \) respectively, with \( S \) finite over \( R \), \( R \) normal, and \( y \in S \), then \( N_{K/k}(y) \in R \) (because the norm is just the product of all the conjugates of \( x \) in some normal extension of \( K \)). Also if \( p \) is an ideal of \( S \), then \( N_{K/k}(p) \subset R \cap p \).

Also recall [11] Pfister and Poperin have generalized Artin approximation to show the following: Let \( (R, m) \) be a complete local ring. Let \( f_1, \ldots, f_m \in R[x] = R[x_1, \ldots, x_n] \) be \( m \) polynomials in \( n \) variables over \( R \). Then for all \( c \in N \), \( \exists N_c \in \mathbb{N} \) (depending on \( c \) and \( f_1, \ldots, f_m \)) such that if \( y = (y_1, \ldots, y_n) \in R^n \) and each \( f_i(y) = 0 \mod m^{N_c} \), then \( \exists (\lambda_1, \ldots, \lambda_n) \in R^n \) such that \( \lambda_i = y_i \mod m^c \) for each \( i \), and each \( f_i(\lambda) = 0 \).

**Lemma 5.** Let \( (R, m) \) be analytically irreducible local domain (i.e. \( \hat{R} \) is a domain) and let \( a_1, \ldots, a_n \in \mathbb{N} \) be given. Then there exist \( N \in \mathbb{N} \) such that

\[
(R - m^{a_1}) \cdots (R - m^{a_n}) \subset R - m^N.
\]

**Proof.** Pick \( c = \max\{a_1, \ldots, a_n\} \) and apply the above to the equation \( X_1 \cdot X_2 \cdots X_n = 0 \) over \( \hat{R} \), i.e. let \( h = 1 \) and \( f_1(X) = X_1 \cdot \cdots \cdot X_n \). Choose \( N = N_c \) as guaranteed by the theorem. Now suppose \( y_i \in R - m^{a_i} \). It then follows that \( y_1 \cdot y_2 \cdots y_m \in m^N \). For if not \( f_1(y) = 0 \mod m^n \hat{R} \) and we could choose \( \lambda_1, \ldots, \lambda_n \in \hat{R} \) such that \( \lambda_i = y_i \mod m^c \hat{R} \) and \( \lambda_1 \cdot \cdots \cdot \lambda_n = 0 \).
Now \( \lambda_i = y_i \mod m^a \hat{R}, y_i \notin m^a \hat{R} \), and \( c > a \) implies \( \lambda_i \notin m^a \hat{R} \) which implies \( \lambda_i \neq 0 \). But \( \hat{R} \) is a domain. This is a contradiction. Q.E.D.

**Lemma 6.** Let \((S, n)\) be a complete local domain, \((R, m)\) be a normal local domain, with \( R \subset S \), and \( S \) a finite \( R \)-module. Let the quotient fields of \( R \) and \( S \) be \( F \) and \( G \) respectively. Let \( \mathcal{U} = \text{Norm}_{G/F} \). Then \( \mathcal{U}(S) \subset R \) and for all \( c \in \mathbb{N} \), there exist \( N_c \in \mathbb{N} \) such that \( \mathcal{U}^{-1}(m^N) \subset n^c \).

**Proof.** We first assume that \( G \) is a normal field extension of \( F \), that is the fixed field of the Galois group of \( G \) over \( F \) is precisely \( F \). Let \( \{\phi_1, \ldots, \phi_n\} = \text{Gal}(G/F) \), say \( \phi_1 = \text{identity} \). Each \( \phi_i \) induces an \( R \)-automorphism of \( S \). For all \( s \in S \), \( \mathcal{U}(s) = \prod_{i=1}^{n} \phi_i(s) \). Hence if \( s = \phi_1(s) \notin n^c \), we have \( \phi_i(s) \notin n^c \) for each \( i \) and by Lemma 5, there exist \( N_c \) such that \( (S - n^c)^n \subset S - n^{N_c} \). But then \( \mathcal{U}(s) \notin n^{N_c} \). Hence \( \mathcal{U}^{-1}(m^N) \subset n^c \).

In the general case, let \( H \) be a finite normal field extension of \( F \) containing \( G \) and let \((T, q)\) = integral closure of \( R \) in \( H \). Since \( T \) is finite over \( S \), by the Artin-Rees lemma, the natural Krull topology on \( S \) is the same as the topology induced from \( T \). Hence for every \( c \) there exists \( d \) such that \( q^d \cap S \subset n^c \). By the special case, there exists \( N \) so \( \mathcal{U}^{-1}(m^N) \subset q^d \). This \( N \) also works for \( S \): If \( s \in S \) and \( \mathcal{U}_{S/R}(s) \subset m^N \), then

\[
\mathcal{U}_{T/R}(s) = \mathcal{U}_{S/R}(\mathcal{U}_{T/S}(s)) = \mathcal{U}_{S/R}(s^{[H:G]}) = (\mathcal{U}_{S/R}(s))^{[H:G]} \subset m^{N[H:G]} \subset m^N \Rightarrow s \in q^d \cap S \subset n^c.
\]

**Proof of Theorem 3.** Let the Krull dimension of \( S = d \). We will prove that for all \( i, 0 < i < d \), and all \( c \), there exists a prime \( p \) in \( S \) with \( \text{ht} p = i \) and \( p \subset m^c \). By a trivial induction on \( i \), we may assume \( i = 1 \). Let \( X_1, \ldots, X_d \) be a system of parameters for \( S \), then \( \hat{S} \) is a finite extension of \( R = k[[X_1, \ldots, X_d]] \). Let \( \mathcal{U} = \text{Norm}_{\hat{S}/R} \). Pick \( N \) so large that \( \mathcal{U}^{-1}(m^N) \subset \hat{n}^c \). Let \( P_0 \) be the prime in \( \hat{R} \) generated by \( X^p_1 - X^p_2 \), where \( p_1, p_2 \) are large prime integers; \( P_0 \subset m^N \). Let \( \hat{p} \) be a prime of \( \hat{S} \) lying over \( P_0 \). Then \( \hat{p} \subset \hat{n}^c \), for \( s \in \hat{p} \Rightarrow \mathcal{U}(s) \subset \hat{p} \cap R \subset P_0 \subset m^N \Rightarrow s = \mathcal{U}^{-1}(\mathcal{U}(s)) \subset n^c \). Now let \( p = \hat{p} \cap S \). Clearly \( p \subset n^c \), and \( p \cap S \neq (0) \) because \( X^p_1 - X^p_2 \in p \cap S \).

4. It is clear that by the techniques of the last section, we can prove Proposition 1. (The projection of the local parametrization of a real analytic variety is semianalytic so one just picks a high order curve in this semianalytic set not lying in the discriminant locii and lifts it to a real analytic curve on \( V \). Details are omitted and left to the reader. See for instance [8].)

We now show how Proposition 1 can be used to study the ring of \( C^\infty \) and \( C^k \) functions on an irreducible real analytic set. Let \( X \) be a real analytic set in \( \mathbb{R}^n \), irreducible at \( x \), and \( C^\infty_x(X) \) be the ring of germs at \( x \) of infinitely differentiable functions on \( X \). Let \( T : C^\infty_x(X) \to \mathbb{R}[[x_1, \ldots, x_n]]/I(X) \) be the Taylor map, taking a \( C^\infty \) function to its Taylor series. Then \( T \) is clearly an \( \mathbb{R} \).
algebra homomorphism. It is well known that $T$ is surjective. Let $m_1$ be the maximal ideal of $\mathcal{F} = R[[x_1, \ldots, x_n]]/I(X)$ and $m_2$ be the maximal ideal of $C^\infty_x(X)$. Let $p_i \subset m_1^i$ be as in Proposition 1. Since there is a one to one correspondence between primes of $C^\infty_x(X)$ containing $\ker T$ and primes of $\mathcal{F}$, $q_i = T^{-1}(p_i)$ is a prime of $C^\infty_x(X)$ of depth one with $q_i \subset m_2^i$.

**Corollary 1.** Let $X$ be germ of a real analytic subvariety in $\mathbb{R}^n$, irreducible at $x$, and $C_x(X)$ the ring of germs at $x$ of real valued infinitely differentiable functions on $X$. Then there is an irreducible real analytic curve $C$ in $X$ passing through $x$ so that their $C^\infty$ tangent spaces are the same at $x$, that is $T(V, C^\infty_x) = T(C, C^\infty)$, where $T(V, C^\infty_x) = \{ r \in \mathbb{R}^n | \sum_{i=1}^n r_i \partial f(x) = 0 \forall f \in I(V, C^\infty_x) \}$.

**Remark.** We have not made use of the fact that $T(V, \mathcal{O}_x) = T(V, C^\infty_x)$ in the above corollary, and in fact our theorem can be used to prove results of this nature.

**Corollary 2.** Let $X$ be a germ of a real analytic subvariety in $\mathbb{R}^n$, irreducible at $x$, $C^k_x(X)$ the ring of germs at $x$ of real valued $k$ times continuously differentiable functions on $X$, and $\mathcal{O}_x(X)$ the ring of germs of real analytic functions on $X$. Then there exist $k > 0$ so that $T(X, C^k_x) = T(X, \mathcal{O}_x)$, where

$$T(X, C^k_x) = \left\{ r \in \mathbb{R}^n | \sum_{i=1}^n r_i \partial f(x) = 0 \forall f \in I(X, C^k_x) \right\}$$

and

$$T(X, \mathcal{O}_x) = \left\{ r \in \mathbb{R}^n | \sum_{i=1}^n r_i \partial f(x) = 0 \forall f \in I(X, \mathcal{O}_x) \right\}.$$

**Proof.** Pick a curve $C$ in $X$ as in Corollary 1. Then clearly:

$$T(C, C^k_x) \cap T(C, \mathcal{O}_x) = T(C, \mathcal{O}_x)$$

so it suffices to see that these exist $k > 0$ so that $T(C, C^k_x) = T(C, \mathcal{O}_x)$. We sketch the proof of this fact below. Assume the point $x$ is the origin. Let $\phi: \mathbb{R} \to C$ be the desingularization of the irreducible curve $C$ (obtained by complexifying $C$, normalizing the new complex curve and then restricting to the real part). Without loss of generality, one may assume that $C$ is embedded real analytically in minimal dimension. One may make a real analytic change of coordinates so that $\phi$ has the form $\phi(t) = (t^{q_1}u_1(t), \ldots, t^{q_n}u_n(t))$, where the $u_i$ are units $q_1 < q_2 < \cdots < q_n$, and there is no polynomial in $\phi_1(t), \ldots, \phi_{k-1}(t)$ whose order is precisely $q_k$. (If this condition is not
satisfied, make inductively a sequence of transformations until it is.) Pick
$k > q_n/q_1 + 1$. If $f \in I(X, C^*_x)$ and $T_k$ is the $k$th order Taylor series about
0, then $f - T_k = o(|x|^k)$ on $C$. Comparing with $\phi(t)$, we get $\sum_{|a| \leq k} a_\ell^a \phi(t)^a$
$= o(t^{a_k})$ where $T_k = \sum_{|a| \leq k} a_\ell^a x^a$. Then $T_k$ can have no linear terms because
each $q_i < kq_1$ and no polynomial in the $\phi_j(t), j \neq i$, can have order $q_i$.

For additional details of the argument above, see [4].

Straightforward generalization of these arguments yields the following
result claimed by the author in [4] and proven by Risler in [12]: Let $X$ be the
ger of a real analytic set in $R^n$ and $I(X)$ the ideal of $X$ in $R\{x_1, \ldots, x_n\}$. Then
there is a function $\lambda: N \rightarrow N$, with $\lim_{r \rightarrow \infty} \lambda(r) = r$, so that if $F$ is a
function of class $C^r$ on $R^n$ vanishing on $X$, then $T_{\lambda(r)} F \in I(X) + m^\lambda(r)$.

**Corollary 3.** In [8] Merrien has proven that if $I$ is an ideal of $\Theta = \Theta_n = \R\{x_1, \ldots, x_n\}$ with $I = \text{ideal}(\text{locus}(I))$, then $I = \bigcap_{\gamma \in \Gamma} \ker \gamma$, where $\Gamma$ is the
set of all $\R$ algebra homomorphisms $\Theta \rightarrow \R\{t\}$ with $I \subseteq \ker \gamma$. This follows
immediately from Proposition 1 because the condition on the ideal is precisely
that $I$ is the ideal of a real analytic variety; hence for every $i > 0$, there exists
$\gamma_i: \Theta \rightarrow \R[[t]]$ with $I \subseteq \ker \gamma_i \subseteq I + m^i$. Hence

$$I \subseteq \bigcap_{i=1}^{\infty} \ker \gamma_i \subseteq \bigcap_{i=1}^{\infty} (I + m^i) = I$$
as $\Theta_n$ is a local Noetherian ring.

Now suppose $\phi: X \rightarrow Y$ is a map of irreducible analytic varieties and $\phi: \Theta(Y, q) \rightarrow \Theta(X, p)$ the induced map on the associated local analytic rings. It
is not necessarily true that the image of $\phi$ is closed in $\Theta(X, p)$ in the Krull
topology. (The map $(x, y) \rightarrow (x, xy, x e^y)$ provides an easy counterexample.)

**Definition.** Let $\phi: R \rightarrow S$ be a local algebra homomorphism of analytic
domains. By a curve in $S$, we mean a local algebra homom $\gamma: S \rightarrow \Theta_1$. Clearly a
curve in $S$ induces a curve in $R$. We say that $\phi$ is nice if for every
$f \in S - \phi R$, there is a nonzero curve $\gamma$ in $S$ such that $\gamma(f) \in \Theta_1 - \gamma \phi R$.
That is a function factors through $\phi$ if and only if it factors through every
curve.

**Corollary 4.** $\phi$ is nice if and only if $\phi(R)$ is closed in $S$ in the Krull
 topology.

**Proof.** Suppose $\phi(R)$ is closed in $S$. Since the closure of $\phi(R)$ in $S$ is
$\bigcap_{k=1}^{\infty} (\phi(R) + M(S)^k)$, for every $f \in S - \phi(R)$, there exists a $k > 0$ so
$f \notin \phi(R) + M(S)^k$. Now let $\gamma$ be a curve in $S$ so that $\ker \gamma \subseteq M(S)^k$. We have a commutative diagram:
296

It is clear that $\pi_2(f) \in \phi(R/\ker \gamma \phi) = \pi_1 \phi(R)$. Conversely assume $\phi$ is nice. Let $f \in S - R$ and $\gamma$ curve in $S$ with $\gamma(f) \notin \gamma \phi R$. Now mappings of one dim analytic rings are always finite of zero, so by the Artin Reese lemma $\gamma R$ is closed in $S$. Since $\gamma$ is continuous, $\gamma^{-1} \gamma R$ is closed in $S$. Also $f / \gamma^{-1} \gamma R$ and $\phi R \subset \gamma^{-1} \gamma R$. Hence $\phi R = \bigcap \gamma \gamma^{-1} \gamma R$ is closed in $S$.

For additional applications of these ideas, see [5].

ADDED IN PROOF. After this paper was accepted for publication, the authors discovered that Theorem 3 had been proven by R. Berger (Zur Ideal theorie analytisch normaler Stellenringe, J. Reine Angew. Math. 201 (1958), 172–177) by different techniques.

REFERENCES


DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637