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THE MINIMUM NORM PROJECTION ON C^2-MANIFOLDS IN R^n

BY

THEAGENIS J. ABATZOGLOU

ABSTRACT. We study the notion of best approximation from a point $x \in R^n$ to a C^2-manifold. Using the concept of radius of curvature, introduced by J. R. Rice, we obtain a formula for the Fréchet derivative of the minimum norm projection (best approximation) of $x \in R^n$ into the manifold. We also compute the norm of this derivative in terms of the radius of curvature.

1. Introduction. Let $x \in R^n$ and M be a C^2 manifold of dimension k, $k < n$. The minimum norm projection, whenever it is defined, is the map P_M which takes x into the element of M closest to x, i.e. $\min_{m \in M}\|x - m\| = \|x - P_M(x)\|$, where the norm is the Euclidean one. Define $A = \{y | y \in R^n, P_M(y) \text{ is multivalued}\}$. Let $U = (\bar{A})^C$.

J. R. Rice has studied existence of the map P_M in terms of the radius of curvature and established continuity of P_M on general grounds. In [1], [2], [4] and [5], we have an investigation of the existence of P_M in a Banach space setting. The results use the notion of curvature.

In this paper we will examine the existence and differentiability properties of P_M around point x, in relation to the radius of curvature of M at m, where $m = P_M(x)$.

2. Definitions. Let f be a local representation of the manifold M around m. We assume the following:

(1) f is an open map in its domain of definition, i.e. some open set in R^k.

(2) f is C^2.

(3) $f'(a) \cdot R^k = R^k$.

Furthermore, assuming $f(a) = m$, we define the tangent plane of M at m to be $T_m \equiv m + f'(a) \cdot R^k$.

A vector $v = y - m$ is orthogonal to M at m if v is orthogonal to T_m.

3. Radius of curvature. Consider a vector $v = (y - m)/\|y - m\|$ orthogonal to M at m.

We then consider the ray $m + tv$, $t > 0$, and points $\mu \in M$ close to m such that $\|(m + tv) - m\| = \|(m + tv) - \mu\|$ holds for some $0 < t < \infty$. We solve for t and obtain

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$$t^2 = \|(m - \mu) + tv\|^2 = \|m - \mu\|^2 + 2t\langle v, m - \mu\rangle + t^2,$$

$$t = \|m - \mu\|^2 / 2\langle v, \mu - m\rangle. \tag{3.1}$$

We now define the radius of curvature of M at m in the direction v to be

$$\rho(m, v) = \liminf_{\mu \to m} \left\{ \|m - \mu\|^2 / 2\langle v, \mu - m\rangle \,|\, \langle v, \mu - m\rangle > 0 \right\}.$$

REMARK. If $\langle v, \mu - m\rangle < 0$ for all μ near m in M then we define $\rho(m, v) = \infty$.

For Hilbert spaces, this definition is equivalent to that given in [1], [2], [4] and [5].

Since M is representable by a C^2 homeomorphism f near m, for μ sufficiently close to m we can write $\mu = f(b)$, and

$$\mu - m = f(b) - f(a) = f'(a)(b - a) + \tfrac{1}{2} f''(a)(b - a)^{(2)} + o\left(\|b - a\|^2\right).$$

Expressing (3.1) in terms of a, b, f we obtain

$$t = \frac{\left\|f'(a)(b - a) + \frac{1}{2} f''(a)(b - a)^{(2)}\right\|^2 + o\left(\|b - a\|^2\right)}{\left\langle v, f''(a)(b - a)^{(2)}\right\rangle + o\left(\|b - a\|^2\right)}.$$

We divide numerator and denominator by $\|b - a\|^2$ and get

$$\rho(m, v) = \min_{\|w\| = 1} \left\{ \frac{\|f'(a)(w)\|^2}{\left\langle v, f''(a)(w)^{(2)}\right\rangle} \,\middle|\, \left\langle v, f''(a)(w)^{(2)}\right\rangle > 0 \right\}.$$

EXAMPLE. Let M be the unit sphere in R^3 with parametric representation

$$f(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).$$

Let

$$x = r(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \qquad r < 1,$$

and

$$v = \frac{x - f(\theta, \phi)}{\|x - f(\theta, \phi)\|} = -(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta).$$

Assume also that $w = (w_1, w_2)$; then

$$\|f'(\theta, \phi)(w)\|_2^2 = w_1^2 + \sin^2\theta w_2^2$$

and

$$\left\langle v, f''(\theta, \phi)(w)^{(2)}\right\rangle = w_1^2 + \sin^2\theta w_2^2 \to \rho(m, v) = 1$$

in this case.

4. The minimum norm projection P_M. We now examine the existence of P'_M in terms of the radius of curvature. Recall that by definition A is the set where P_M is multivalued. We have

LEMMA 4.1. *Let $x \in A^c$; then*

$$x \in (\bar{A})^C = U \leftrightarrow \|x - P_M(x)\| < \rho\left(P_M(x), \frac{x - P_M(x)}{\|x - P_M(x)\|}\right).$$

PROOF. Assume $x \in (\bar{A})^C$; then by Lemma 3.2 in [1],

$$\|x - P_M(x)\| < \rho(P_M(x), v) \quad \text{where } v = (x - P_M(x))/\|x - P_M(x)\|.$$

Let $t_0 = \sup\{t | P_M(m + tv) = m = P_M(x)\}$; then by Theorem 4.8 in [3], $m + t_0 v \notin (\bar{A})^C$. Combining these 2 results we obtain $\|x - P_M(x)\| < \rho(P_M(x), v)$; then by Theorems 11–15 in [4] we have

$$x \in (\bar{A})^C = U.$$

We continue with a technical lemma:

LEMMA 4.2. *Let $A = (a_{ij})$, $B = (b_{ij})$ be $k \times k$ matrices where*

$$a_{ij} = \left\langle v, \frac{\partial^2 f(a)}{\partial t_i \partial t_j} \right\rangle, \quad b_{ij} = \left\langle \frac{\partial f(a)}{\partial t_i}, \frac{\partial f(a)}{\partial t_j} \right\rangle,$$

$$v = \frac{x - f(t_1, \ldots, t_k)}{\|x - f(t_1, \ldots, t_k)\|} \quad \text{and} \quad a = (t_1, \ldots, t_k).$$

Then we have:

(a)

$$\|f'(a)(w)\|_2^2 = \langle Bw, w \rangle,$$

(b)

$$\left\langle v, f''(a)(w)^{(2)} \right\rangle = \langle Aw, w \rangle,$$

where $w = (w_1, \ldots, w_k)$.

PROOF.

(a)

$$f'(a) = \left[\frac{\partial f}{\partial t_1} \,\middle|\, \cdots \,\middle|\, \frac{\partial f}{\partial t_i} \,\middle|\, \cdots \,\middle|\, \frac{\partial f}{\partial t_k} \right]$$

thinking of $\partial f / \partial t_i$ as a column vector. Also,

$$\|f'(a)(w)\|_2^2 = \langle f'(a)(w), f'(a)(w) \rangle = \left\langle (f'(a))^T f'(a) w, w \right\rangle.$$

But

$$(f'(a))^T f'(a) = \begin{bmatrix} \partial f/\partial t_1 \\ \hline \vdots \\ \hline \partial f/\partial t_j \\ \hline \vdots \\ \hline \partial f/\partial t_k \end{bmatrix} \left[\frac{\partial f}{\partial t_1} \,\middle|\, \cdots \,\middle|\, \frac{\partial f}{\partial t_i} \,\middle|\, \cdots \,\middle|\, \frac{\partial f}{\partial t_k} \right]$$

$$= \left(\langle \partial f/\partial t_i, \partial f/\partial t_j \rangle \right)_{i,j};$$

so (a) is proved.

(b) Write f as $(f_1, \dots, f_n)$ and $w = (w_1, \dots, w_k)$; then

$$f''(a)(w)^{(2)} = \left(\sum_{i,j} \frac{\partial^2 f}{\partial t_i \partial t_j} w_i w_j \right),$$

so that

$$\left\langle v, f''(a)(w)^{(2)} \right\rangle = \sum_{i,j} \left\langle v, \frac{\partial^2 f}{\partial t_i \partial t_j} \right\rangle w_i w_j = \langle Aw, w \rangle.$$

We now state our main theorem.

THEOREM 4.1. *Let M be a closed C^2 manifold in R^n of dimension $k < n$. Let $x \in U = (\bar{A})^C$; then P_M is Fréchet differentiable at x and the derivative is given by the formula*

$$P'_M(x) = f'(a)(B - rA)^{-1} f'(a)^T,$$

where f is the parametric representation of M, $f(a) = P_M(x)$, $r = \|x - P_M(x)\|$ and A, B are as defined in the previous lemma.

PROOF. Let $x \in U$, $y \in R^n$, and $t_0 \ni x + ty \in U$ for $|t| < t_0$. Consider a C^2 representation of M around $P_M(x)$, say $f: V \to M$, where V is open in R^k and a function

$$F(t, t_1, t_2, \dots, t_k) = \tfrac{1}{2} \|x + ty - f(t_1, \dots, t_k)\|^2.$$

F is obviously C^2.

Let

$$G(t, t_1, \dots, t_k) = (\partial F/\partial t_1, \dots, \partial F/\partial t_k).$$

If $P_M(x + ty) = f(t_1, \dots, t_k)$, then $\partial F/\partial t_i = 0$, $i = 1, \dots, k$. Say $P_M(x) = f(\bar{t}_1, \dots, \bar{t}_k)$; then we know that G is C^1 in a neighborhood of $(0, \bar{t}_1, \dots, \bar{t}_k)$.

We now investigate the invertibility of the Jacobian matrix of G with respect to $t_1, \dots, t_k$ at the point $(\bar{t}_1, \dots, \bar{t}_k)$.

$$J_G = \left(\frac{\partial^2 F}{\partial t_i \partial t_j} \right)_{i,j}.$$

By computation,

$$\frac{\partial^2 F}{\partial t_i \partial t_j} = -\left\langle x - f(\bar{t}_1, \ldots, \bar{t}_k), \frac{\partial^2 f}{\partial t_i \partial t_j} \right\rangle + \left\langle \frac{\partial f}{\partial t_i}, \frac{\partial f}{\partial t_j} \right\rangle.$$

Set

$$v = \frac{x - f(\bar{t}_1, \ldots, \bar{t}_k)}{\left\| x - f(\bar{t}_1, \ldots, \bar{t}_k) \right\|} \quad \text{and} \quad r = \left\| x - f(\bar{t}_1, \ldots, \bar{t}_k) \right\|;$$

then, according to the previous lemma,

$$J_G = B - rA.$$

Recall that

$$\rho = \rho(m, v) = \min_{\|w\|=1} \{ \langle Bw, w \rangle / \langle Aw, w \rangle | \langle Aw, w \rangle > 0 \},$$

and, by Lemma 4.1, $0 < r < \rho \leq \infty$.

If $\langle Aw, w \rangle \leq 0 \ \forall w \in \|w\| = 1$, then $B - rA$ is invertible, being positive definite.

On the other hand,

$$\begin{aligned} \langle (B - rA)w, w \rangle &= \langle Bw, w \rangle - r\langle Aw, w \rangle \\ &= (r/\rho)[\langle Bw, w \rangle - \rho\langle Aw, w \rangle] + ((\rho - r)/\rho)\langle Bw, w \rangle > 0 \end{aligned}$$

for all w in $R^k \ni \|w\| = 1$. Therefore $(B - rA)^{-1}$ exists.

By the implicit function theorem, $t_i = t_i(t)$ in a neighborhood of $(0, \bar{t}_1, \ldots, \bar{t}_k)$, and

$$\left(\frac{\partial t_1}{\partial t}, \ldots, \frac{\partial t_k}{\partial t} \right) = -\left(\frac{\partial^2 F}{\partial t_i \partial t_j} \right)_{i,j}^{-1} \cdot \left(\frac{\partial^2 F}{\partial t \partial t_i} \right)_i,$$

where $\partial^2 F / \partial t \partial t_i = -\langle y, \partial f / \partial t_i \rangle$. Since $P_M(x + ty) = f(t_1(t), \ldots, t_k(t))$, using the chain rule we obtain

$$\left. \frac{d}{dt} P_M(x + ty) \right|_{t=0} = f'(a)[(B - rA)^{-1}] f'(a)^T(y).$$

This shows P_M has directional derivatives; also the assumption that M is closed implies the continuity of P_M on U. Also $f'(a)(B - rA)^{-1}f'(a)$ depends continuously on $a = (t_1, \ldots, t_k)$ and therefore varies continuously with x so that $f'(a)(B - rA)^{-1}f'(a)^T$ is the Fréchet derivative of P_M.

We now compute the norm of $P'_M(x)$.

Corollary 4.1. *Let* $P'_M(x) = f'(a)(B - rA)^{-1}f'(a)^T$; *then*

$$\|P'_M(x)\| = \rho/(\rho - r) = -1/(1 - r/\rho),$$

where

$$1/\rho = \max_{\|w\|=1} \langle Aw, w\rangle / \langle Bw, w\rangle = 1/\rho(m, v).$$

PROOF. $P'_M(x)$ is selfadjoint and semipositive definite as we showed in the proof of Theorem 4.1. The rank of $f'(a)^T$ is k = rank of $f'(a)$ by definition. Then we have the rank of $f'(a)(B - rA)^{-1}f'(a)^T = k$. Choose k mutually orthogonal eigenvectors, $\{v_1, \ldots, v_k\}$, of $P'_M(x)$ such that

$$\|P'_M(x)\| = \lambda_1 \quad \text{where } P'_M(x)(v_1) = \lambda_1 v_1$$

for any i, $1 < i < k$.

$$f'(a)(B - rA)^{-1}f'(a)^T(v_i) = \lambda_i v_i; \tag{4.1}$$

therefore

$$f'(a)^T f'(a)(B - rA)^{-1}\big(f'(a)^T v_i\big) = \lambda_i\big(f'(a)^T v_i\big).$$

It is clear from (4.1) that $\{f'(a)^T(v_i)\}_{i=1}^k$ is a linearly independent set. Therefore, if w is an eigenvector of $f'(a)^T f'(a)(B - rA)^{-1}$, then we can write

$$w = \sum_{i=1}^k a_i f'(a)^T(v_i) \quad \text{and} \quad f'(a)^T f'(a)(B - rA)^{-1} w = \lambda w.$$

So

$$f'(a)^T f'(a)(B - rA)^{-1} \sum_{i=1}^k a_i f'(a)^T(v_i) = \sum_{i=1}^k a_i \lambda_i f'(a)^T(v_i).$$

Thus

$$\sum_{i=1}^k a_i \lambda_i f'(a)^T(v_i) = \lambda \sum_{i=1}^k a_i f'(a)^T(v_i).$$

Hence

$$a_i \lambda_i = \lambda a_i, \quad i = 1, \ldots, k, \quad \text{and} \quad \lambda_i = \lambda \quad \text{for } a_i \neq 0.$$

This shows that the maximum eigenvalue of $f'(a)^T f'(a)(B - rA)^{-1}$ is $\lambda_1 = \|P'_M(x)\|$. Recall that, by definition, $f'(a)^T f'(a) = B$. Thus

$$f'(a)^T f'(a)(B - rA)^{-1} = B(B - rA)^{-1} = (I - rAB^{-1})^{-1},$$

which implies

$$1/\lambda_1 = \text{smallest eigenvalue of } I - rAB^{-1},$$

so

$$1/\lambda_1 = 1 - r(\text{largest eigenvalue of } AB^{-1}),$$

and

$$1/\lambda_1 = 1 - r(\text{largest eigenvalue of } B^{-1}A).$$

Consider now $\max_{\|w\|=1} \langle Aw, w\rangle / \langle Bw, w\rangle = 1/\rho$. Therefore,

$$\langle Aw, w\rangle \leq \frac{1}{\rho}\langle Bw, w\rangle, \quad \text{so} \left\langle \left(\frac{1}{\rho}B - A\right)w, w\right\rangle \geq 0.$$

Since $B/\rho^{-1} - A$ is selfadjoint, the min is attained at an eigenvector, and since this min $= 0$, $(B/\rho - A)w_0 = 0$, and $w_0/\rho = B^{-1}Aw_0$. We claim $1/\rho$ is the largest eigenvalue of $B^{-1}A$; if not, set $1/\rho' > 1/\rho \ni w_1/\rho' = B^{-1}Aw_1$. Then $(B/\rho' - A)w_1 = 0$ and

$$\left\langle \left(\frac{1}{\rho'}B - A\right)w, w\right\rangle = \left\langle \left(\frac{1}{\rho}B - A\right)w, w\right\rangle + \left\langle \left(\frac{1}{\rho'} - \frac{1}{\rho}\right)Bw, w\right\rangle > 0$$

$$\forall w \ni \|w\| = 1,$$

a contradiction.

5. Examples.

EXAMPLE 5.1. Let M be the sphere of radius ρ in R^3. Let $x \in R^3 \ni \|x\| = d$. Set $r = |\rho - d|$. Then, by the previous corollary,

$$\|P'_M(x)\| = \frac{\rho_0}{\rho_0 - r} \quad \text{where } \frac{1}{\rho_0} = \max_{\|w\|=1} \frac{\langle Aw, w\rangle}{\langle Bw, w\rangle}.$$

By simple computations we obtain

$$\begin{aligned} \rho &= \rho_0 \quad \text{if } d < \rho, \\ &= -\rho_0 \quad \text{if } d > \rho, \end{aligned}$$

so that

$$\|P'_M(x)\| = \begin{cases} \rho/(\rho - r) & \text{if } d < \rho, \\ \rho/(\rho + r) & \text{if } d > \rho. \end{cases}$$

This result suggests the following observation: If M is a C^2 manifold with radius of curvature ρ at the point $m = P_M(x)$, and if $\|x - P_M(x)\| = r$, then by the previous corollary, $\|P'_M(x)\| = \rho/(\rho - r)$, which is exactly the same estimate for a sphere of radius ρ and point x whose distance from the sphere is r.

EXAMPLE 5.2. Let M be $f(x, y) = (x, y, xy)$ and $x = (0, 0, r)$ where $0 < r < 1$. Then, from

$$\begin{aligned} \|(0,0,r) - (x,y,xy)\|_2^2 &= x^2 + y^2 + (xy - r)^2 \\ &= x^2 + y^2 - 2rxy + x^2y^2 + r^2 \\ &= (rx - y)^2 + (1 - r^2)x^2 + x^2y^2 + r^2, \end{aligned}$$

it is clear that $P_M(x) = (0, 0, 0)$. Computing,

$$f'(0,0)=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\quad\text{and}\quad B=f'(0,0)^Tf'(0,0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Also $v = (0, 0, 1)$ and

$$\langle v, \partial^2 f/\partial x^2\rangle = \langle v, \partial^2 f/\partial y^2\rangle = 0,$$

and

$$\langle v, \partial^2 f/\partial x\partial y\rangle = \langle v, \partial^2 f/\partial y\partial x\rangle = 1,$$

so that $A = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Therefore,

$$P'_M(x)=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{pmatrix}1&-r\\-r&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$$

$$=\frac{1}{1-r^2}\begin{bmatrix}1&r&0\\r&1&0\\0&0&0\end{bmatrix},$$

and

$$\|P'_M(x)\| = \frac{\rho}{\rho - r},\quad\text{where } \frac{1}{\rho} = \max_{\|w\|=1}\frac{\langle Aw, w\rangle}{\langle Bw, w\rangle} = 1.$$

Thus $\|P'_M(x)\| = 1/(1-r)$.

EXAMPLE 5.3. Let

$$M=\begin{cases}(x,0), & x<0,\\ (x, 1-\sqrt{1-x^2}), & x>0, x<1;\end{cases}$$

then for $|\varepsilon| < \frac{1}{2}$ we have

$$P_M\left(\varepsilon, \tfrac{1}{2}\right) = (\varepsilon, 0)\quad\text{if } \varepsilon < 0,$$

$$= \left(\frac{2\varepsilon}{\sqrt{1+4\varepsilon^2}}, 1 - \frac{1}{\sqrt{1+4\varepsilon^2}}\right)\quad\text{if } \varepsilon > 0.$$

But $dP_M(\varepsilon, \frac{1}{2})/d\varepsilon|_{\varepsilon=0}$ does not exist. Our manifold M is C_1 at (0, 0) but not C^2.

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50011