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# TWO DIMENSIONAL $\varepsilon$-ISOMETRIES

BY

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ABSTRACT. An affirmative answer to the antipodal $\varepsilon$-isometry conjecture is established for 2-dimensional Banach spaces.

1. The $\varepsilon$-isometry problem is concerned with a Banach space $E$ and a surjective transformation $A$, $A(0) = 0$, for which

$$\big|\|Ax - Ay\| - \|x - y\|\big| < \varepsilon. \tag{1.1}$$

The question is whether there is a constant $k$ depending only on $E$ (independent of $A$ and $\varepsilon$) such that for each $A$ there is a linear isometry $U$ satisfying

$$\|Ax - Ux\| < k\varepsilon. \tag{1.11}$$

The question has been answered in the affirmative for $L_p$, $1 < p < \infty$, and $C(0, 1)$ [1]–[5]. It seems likely that an affirmative answer attaches to all finite dimensional Banach spaces and not only to finite dimensional sections, though in [6] this is established, among other things, solely for $l_1^n$, the Banach space of $n$-sequences with an $l_1$ norm. Our main purpose is to close the gap for the 2-dimensional case. Other aspects of the general problem for $n \geqslant 2$ are considered also.

We list nomenclature meriting special attention. We shall write $E_2$ for a 2-dimensional Banach space with unit sphere $S = S(0)$. Assume an angle parameter, $s$, locates points on $S$ with $0 \leqslant s < 2\pi$. Points on $S$ are denoted either by their parameter value or by $x$, $y$ or $p$, $q$ etc. The set of extreme points of $S$ is denoted by Ex. A *line of support* $l_{(p_0)}$ at $p_0 \in S$ is a straight line passing through $p_0$ (and possibly through other points of $S$) such that $S$ lies entirely in one of the half planes into which $l_{(p_0)}$ divides $E_2$. If a line of support intersects $S$ in a line segment $L$, then $L$ is an *edge* and the end points of $L$ are extreme points referred to as *edge points* (nonexposed points in the standard terminology). Let $\tau(s)$ be the angle measured from a support line at $x(s) \in S$ to a fixed line. The support function $\tau$ defined by $\{\tau(s)|0 \leqslant s < 2\pi\}$ is an upper semicontinuous possibly multiple valued function. A *corner point* is a point for which there are several lines of support. A *short arc* is a

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connected piece of $S$ which does not include two antipodal points. We use $P_2$ when $S$ contains an edge bounded by corner points. Write $N$ for the positive integers. Let $F_2$ invariably denote a section of $l_1$ by a 2-dimensional plane. Let $v_0, v_1, -v_0, -v_1$ be extreme points of $S$. We often write $v_2 = -v_0$, $v_3 = -v_1$ and call $\{v_i|0 \leqslant 1 \leqslant 3\}$ *base points*. We shall be particularly interested in the case that the base points are edge points with $v_0, v_1$ the end points of $L$ (and $-v_0, -v_1$ end points of $-L$). Denote by $S(w, t)$ or $w + S(0, t)$ the sphere of radius $t$ about $w$. For arbitrary points $p$, $q$ on $S$ write

$$r_i = \|p - v_i\|, \qquad s_i = \|q - v_i\|.$$

Write

$$d_i(p, q) = |r_i - s_i|, \qquad d(p, q) = \sup_{0 \leqslant i \leqslant 3} d_i(p, q).$$

In Example 3, this notation is extended to the case of $2n$ base points in $l_1^n$.

**2. Preliminary results.** A crucial test of the universal validity of a result like (1.11) would be $l_1$. It is therefore of particular interest that extremely general types of unit spheres, $S$, are obtained as sections of $l_1$ by skewed 2-hyperplanes. These comprise cases not included in the class $\mathfrak{S}$ of [6]. In our first example a simple criterion is used to locate discrete extreme points and so produce an $S$ with a finite set for the limit points of the extreme points. The second example is convincing evidence that no obvious qualitative restriction on the extreme points of $S$ will discriminate against the 2-sections of $l_1$.

EXAMPLE 1. *The unit sphere of $F_2$ (with the $l_1$ induced norm) with a discrete nonfinite set of extreme points.* Let $a_i$ and $b_i$, $i \geqslant 2$, be nonnegative with $a_i > b_i$, $i \geqslant 2$. Suppose $a_0 = 0$, $a_1 = 1 - \Sigma_2^\infty a_i > 0$, $b_0 = 1 - \Sigma_2^\infty b_i > 0$, $b_1 = 0$. Evidently the points $x = \{a_i\}$, $y = \{b_i\}$ are on $S \in P_2$. Moreover the segment $[x, y]$ is on $S$. To see that $x$ and $y$ are extreme points, note that for $t > 0$

$$|-tx + (1+t)y| = 1 + 2t - t\sum_2^\infty a_i - (1+t)\sum_2^\infty b_i + \sum_2^\infty |-ta_i + (1+t)b_i|.$$

Evidently

$$\sum (1+t)b_i \leqslant t\sum a_i + \sum |-ta_i + (1+t)b_i|.$$

Hence

$$|-tx + (1+t)y| \geqslant 1 + 2ta_1 > 1.$$

This shows $y$ is an extreme point. Similarly $x$ is an extreme point.

Suppose in fact that

$$a_i = i^{-3}, \qquad b_i = i^{-6}, \qquad i \geqslant 2.$$

To locate other extreme points, let

$$z_m = (x - m^3y)/\|x - m^3y\| = \{c_i(m)\}$$

where

$$\begin{aligned} c_j(m) &< 0, \qquad j < m, \\ c_j(j) &= 0, \\ c_j(m) &> 0, \qquad j > m. \end{aligned}$$

Since $c_j(m)$, $c_j(m+1) \geqslant 0$, the segment $[z_m, z_{m+1}]$ is on $S$. By the type of argument used above for $x$ and $y$, $z_m$ and $z_{m+1}$ are extreme points of $S$. By symmetry $-z_m$ is an extreme point also and as $m \to \infty$, $z_m \to -y$.

EXAMPLE 2. *A 2-dimensional section of $l_1$ whose unit sphere consists entirely of extreme points.* The construction involves extraction of four disjunct nonfinite sequences of the integers. For convenience we take these as follows:

$N_o$ denotes the odd positive integers, $N_e$ the even integers, $N_o^1$ the odd primes, $N_e^1$ the multiples of 4 and $N_o^2 = N_o - N_o^1$, $N_e^2 = N_e - N_e^1$. We denote by $z_1$ and $z_2$ certain sequences $\{a_i | i = 0, 1, \ldots\}$ and $\{b_i | i = 0, 1, \ldots\}$ which will be defined in successive stages below, over the ranges, $N_0^1$, $N_o^2$, $N_e^1$, $N_e^2$. Pick a positive sequence $a_i$, $i \in N_o^2$, so that $\Sigma a_i < \frac{1}{4}$. Pick a sequence of negative numbers $b_i$, $i \in N_o^1$, so that $\Sigma |b_i| < \frac{1}{4}$. Choose positive numbers $a_i$, $i \in N_e^2$, so that $\Sigma a_i < \frac{1}{4}$ and positive numbers $b_i$, $i \in N_e^1$, so that $\Sigma b_i < \frac{1}{4}$.

Use $Q$ for the rationals between 0 and 1. Let $H$ denote the usual order correspondence of the rationals, that is to say

$$Hn = r_n \in Q.$$

Consider the order preserving epimorphisms

$$\begin{aligned} N_0^1 &\xrightarrow{A^1} N, \quad N_e^1 \xrightarrow{B^1} N, \\ N_0^2 &\xrightarrow{A^2} N, \quad N_e^2 \xrightarrow{B^2} N. \end{aligned} \tag{2.1}$$

Define

$$\begin{aligned} D^1 &= HA^1, \quad E^1 = HB^1, \\ D^2 &= HA^2, \quad E^2 = HB^2. \end{aligned} \tag{2.2}$$

Let

$$\begin{aligned} w_s &= k(s)((1 - s)z_1 + sz_2), \quad 0 \leqslant s \leqslant 1, \\ &= k(s)(1 - s)(z_1 + \sigma z_2), \quad s < 1, \end{aligned} \tag{2.3}$$

where

$$k(s) = \|(1-s)z_1 + sz_2\|^{-1}.$$

If $w_s$ were not an extreme point, then for some $\Delta$, the segment $[w_s, w_{s+\Delta}]$ would be on the unit sphere. Write

$$w_s \sim \{x_i\}, \qquad w_{s+\Delta} \sim \{y_i\}. \tag{2.4}$$

A point on the segment joining them would have norm 1 iff

$$1 = \sum |tx_i + (1-t)y_i| = t\sum |x_i| + (1-t)\sum |y_i|, \qquad 0 < t < 1,$$

or equivalently

$$x_i y_i \geqslant 0, \qquad i = 0, 1. \tag{2.5}$$

We fill out the definition of $z_1$ and $z_2$ by assigning values to $a_i$ for $i \in N - (N_o^2 \cup N_e^2)$ and to $b_i$ for $i \in N - (N_o^1 \cup N_e^1)$. Specifically for $i \in N_o^1$ write $r(i) = D^1 i$ (note $r(i)$ is not the $i$th rational in the ordering $H$). Define $a_i$ as $r(i)|b_i|$.

If $i \in N_o^2$ write $\rho(i) = D^2 i$ and define $b_i$ as $-\rho(i)a_i$.

Define

$$e(i) = E^1(i), \qquad i \in N_e^1,$$
$$f(i) = E^2(i), \qquad i \in N_e^2.$$

Then define

$$a_i = e(i)b_i, \qquad i \in N_e^1,$$
$$b_i = f(i)a_i, \qquad i \in N_e^2.$$

Finally take

$$a_0 = 1 - \sum_{i \neq 0} a_i, \qquad b_0 = 1 - \sum_{i \neq 0} |b_i|.$$

We now show that with $x_i = a_i + \sigma b_i$, $y_i = a_i + (\sigma + \delta)b_i$, (2.5) cannot be satisfied for any $\delta$. Suppose first $\sigma$ is irrational and inferior to 1. Then $\sigma$ determines a Dedekind cut on $Q$. Since $\{r(i) | i \in N_o^1\} = Q$,

$$\sigma = \inf_{N_o^1} r(i) \quad \text{for } \{i | r(i) > \sigma\},$$
$$= \inf_{N_o^1} a_i/|b_i| \quad \text{for } \{i | a_i/|b_i| > \sigma\}. \tag{2.6}$$

Thus (2.5) in our present notation, would require for $\delta > 0$

$$(a_i - \sigma|b_i|)(a_i - \sigma|b_i| - \delta|b_i|) > 0, \qquad i \in N_o^1. \tag{2.7}$$

Since $\sigma$ is irrational

$$a_i - \sigma|b_i| \neq 0, \quad i \in N_o^1,$$

and from (2.6) for assigned positive $\delta$, and some $i \in N_o^1$

$$\sigma < a_i/|b_i| < \sigma + \delta \tag{2.8}$$

in contradiction to (2.7). If $\Delta$ and, hence $\delta$, is negative, then the contradiction

$$\sigma - \delta < a_i/|b_i| < \sigma, \qquad i \in N_o^1, \tag{2.9}$$

arises from

$$\begin{aligned}\sigma &= \sup r(i) \quad \text{for } \{i | r(i) < \sigma\} \cap N_o^1 \\ &= \sup a_i/|b_i| \quad \text{for } \{i | a_i/|b_i| < \sigma\} \cap N_o^1.\end{aligned}$$

If now $\sigma$ is irrational and greater than 1, the contradictions of the type of (2.8) and (2.9) arise from restriction of $i$ to $N_o^2$

$$\begin{aligned}\sigma &= \inf\{\rho(i)^{-1} | \rho(i) < \sigma^{-1}\} \\ &= \inf\{|b_i|/a_i \,|\, |b_i|/a_i > \sigma\}^{-1}.\end{aligned} \tag{2.10}$$

Therefore the extreme points on the part of $S$ between $z_1$ and $z_2$ are dense. Similar reasoning applies, using $N_e^1$ and $N_e^2$, to the portion of $S$ between $z_1$ and $-z_2$. By symmetry we conclude the extreme points are dense on $S$. Though density of extreme points on $S^n$, $n > 1$, does not guarantee every point of $S^n$ is an extreme point, this latter conclusion does follow for $n = 1$ by an easy argument.

**3. Geometry of the unit sphere.** In the case of $l_1^n$, the symmetric distribution of the $2n$ extreme points results in elegant algorithms. For instance if $v_0, \ldots, v_n, v_{n+1}, \ldots, v_{2n-1}$ where $v_{i+n} = -v_{i-1}$, $i \geqslant 1$, are the base points of $l_1^n$ and the definitions of $d_i(p, q)$ and $d(p, q)$ are extended in the obvious way to the case of $2n$ base points instead of 4, then $d_i(p, q) \leqslant M$ uniformly in $i$ implies $\|p - q\| < cM$, $c = n$ [6]. Our next example shows that even for $n = 2$ no such universal bound $c$ exists for $E_n$.

EXAMPLE 3. *An $S$ for which $\|p - q\| \geqslant (k - 1)d(p, q)$, where $k$ is arbitrarily assigned.* We define a polygonal unit sphere $S$ about (1, 0). To help in visualization we use $e$ for edge, $r$ for radius or ray from the origin to $S$, and $l$ for line. $S$ contains the edge $\{(0, y) \,|\, |y| \leqslant a\}$ and the edge points $p = (0, a)$ and $q = (0, 0)$ and also the edge points $v_0 = (1 - \lambda, ka)$, $v_1 = (1 + \lambda, ka)$. Let $\rho a$ be the slope of the edge $e(1)$ starting at $(0, a)$. (The vector $v(3)$ from $-v_1$ to $p$ has slope $-(k + 1)/(1 - \lambda)$.) Let $r(2)$ be the parallel radius (through the origin (1, 0)) and let $l(5)$ be the line of the same slope, $\rho a$, through (0, 0). The intersection of $e(I)$ and $r(2)$, that is to say the end point of $r(2)$, is at

$$\begin{aligned}\bar{x}, \bar{y} = \big(&k + \lambda/\left(\rho(1 - \lambda) + k + 1\right), \\ &\rho a\big((k + \lambda)/\left(\rho(1 - \lambda) + k + 1\right)\big) + a\big).\end{aligned} \tag{3.1}$$

The intersection coordinates of $l(5)$ and $v(3)$ are

$$X, Y = ((1 - \lambda)(1 + k\lambda)/(k + 1)(k + \lambda), (k - 1)(1 - \lambda)a/(k + \lambda)).$$

The vector $v(7)$ along $v(3)$ to $X, Y$ has the same length as the vector $v(4)$ from $-v_1$ to 0, 0. The radius $r(2)$ from 1, 0 to $\bar{x}, \bar{y}$ has length 1 of course, and is used as a gauge for the lengths of $v(3)$, $v(7)$ and $v(4)$. Thus

$$|\|v(3)\| - \|v(4)\|| = \left| \frac{\|v(3)\| - \|v(7)\|}{\|r(2)\|} \right|.$$

Since $r(2)$, $r(3)$ and $v(7)$ are parallel the ratio above is the same as that of the $x$ or the $y$ coordinates, i.e. $d_3(p, q) = (X - 0)/(\bar{x} - 1)$.

Choose

$$\rho < (k^2 - 1)/(1 + k\lambda).$$

This assures $\bar{y} < ka$. The value of $d_3(p, q)$ is then

$$d_3(p, q) = (1 + k\lambda)/k(k + \lambda).$$

Since the radius to $(1, ka)$ from $(1, 0)$ has unit length,

$$\|p - q\| = k^{-1},$$

Thus for small $\lambda$

$$\|p - q\| > (k - 1)d_3(p, q).$$

Assume $S$ is symmetric with respect to $y = 0$. Evidently $d_2(p, q) < d_3(p, q)$ and $d_0(p, q)$ and $d_1(p, q)$ are of the order of $d_3(p, q)$ whence

$$\|p - q\| > (k - 1)d(p, q).$$

Although no universal $c$ exists according to Example 3, it is essential to our developments that for each $E_2$ there is an associated bound $c$. The validity of the assertion depends on the convexity of $S$. The germane consequence of this convexity requirement may be formulated succinctly as

Co: Let $\tau'(s)$ be an arbitrary angle in the set $\{\tau(s)\}$ for fixed $s$. That is to say, $\tau$ dominates the single valued function $\tau' = \{\tau'(s) | 0 < s < 2\pi\}$. Then $\tau'$ is monotonic in $s$.

**4. Metric relations.** We shall make use of the following simple lemma:

LEMMA 4.1. *If $S(v, t)$ passes through $z \in S$, $z \neq cv$, the lines of support at $z$ one to each 1-sphere, coincide iff $S(v, t) \cap S(0)$ includes an edge containing $z$.*

The hypothesis implies the angle of support is the same at two points on $S$, one of which is $z$, since $z$, 0, $v$ are not collinear. Moreover these points are joined by a short arc. By monotoneity (cf. Co) this angle remains constant on this short arc. That is to say this short arc is a line segment.

The relevant theorem is

THEOREM 4.2. *For each $E_2$ and choice of base points, there is a positive bound*

*$M$ such that for every positive $\delta$ and $d(p, q) < \delta$ where $p$ and $q$ are on $S$, $\|p - q\| < M\delta$.*

Suppose the assertion false. Then for each $\delta_n$ in a sequence converging monotonically to 0, there is a positive $M_n$, monotonically increasing to $\infty$ with $n$, for a pair $p_n$, $q_n$ on $S$ with

$$d(p_n, q_n) < \delta_n, \qquad \|p_n - q_n\| > M_n\delta_n.$$

No generality is lost by the assumption that $p_n$ and $q_n$ are on the short arc from $-v_1$ to $v_0$. Suppose first that $M_n\delta_n > \delta > 0$ for all $n$. By compactness of $S$, $\bar{p}$ and $\bar{q}$ exist with

$$\|\bar{p} - \bar{q}\| \geq \delta, \qquad d(\bar{p}, \bar{q}) = 0.$$

It is easy to see $[\bar{p}, \bar{q}]$ must lie on $S$. Take as base unit vector, $i$, the line from 0 through the midpoint of $[\bar{p}, \bar{q}]$ and let the other base vector $j$ be parallel to $[\bar{p}, \bar{q}]$. Then with $\lambda_1 < 1$,

$$v_0 = \lambda_0 i + k_0 aj, \quad \bar{p} = -i + aj,$$
$$v_1 = \lambda_1 i + k_1 aj, \quad \bar{q} = -i - aj.$$

That $[\bar{p}, \bar{q}]$ is on $S(v_3, \|v_3 - \bar{p}\|)$ implies a parallel line segment is on $S$, but since $\bar{p}$, $\bar{q}$ is on $S$, also, this parallel line segment is an extension $E$ of $[\bar{p}, \bar{q}]$. Hence the extension of $\bar{p} + v_1$ $(= p - v_3)$ meets $E$ in a point whose ordinate is

$$K_1 a = (k_1 + 1)a/(1 - \lambda_1).$$

Similarly the extension of $v_0 - \bar{q}$ meets $E$ in a point with ordinate

$$-K_0 a = -(1 + k_0)a/(1 + \lambda_0).$$

Hence the line segment $[(-1, -K_0a), (-1, K_1a)]$ is part of the line segment $E$ and is on $S$.

Define the two unit vectors $u$, $v$

$$u = -i + K_1 aj, \qquad v = -i - K_0 aj.$$

The line through the upper end point of $E$ and $v_0$ and that through the lower end point of $E$ and $V_2$ must intersect according to Co to the left of $E$, that is to say the vector to the intersection point must have a negative $i$ component (and a fortiori, this must be true when other points on $E$ are used). Specifically then the vector to the intersection of the lines of the vectors

$$u + t(v_0 - u), \qquad |t| < \infty,$$
$$w + s(-v_3 - w), \qquad |s| < \infty,$$

must have a negative $i$ component. This component is $N/(2 - \alpha)$ where

$$N = (k_1 + k_0) + \lambda_0(1 + 2k_1) - \lambda_1(1 + 2k_0)$$

and $\alpha = \lambda_0 k_1 - \lambda_1 k_0$ so

$$\frac{N}{2-\alpha} = \frac{\lambda_0 - \lambda_1 + 2\alpha + k_0 + k_1}{2-\alpha}. \tag{4.21}$$

Assume $\lambda_1 > \lambda_0 > 0$. The requirement that (4.21) be negative can then be met only if

$$\begin{aligned} &\text{(a)} \quad \alpha > 2, \\ &\text{(b)} \quad 2\alpha + k_0 + k_1 > \lambda_1 - \lambda_0 \end{aligned} \tag{4.22}$$

or

$$\begin{aligned} &\text{(a)} \quad \lambda_1 - \lambda_0 \geqslant 2\alpha + k_0 + k_1, \\ &\text{(b)} \quad 2 \geqslant \alpha > 0 \end{aligned} \tag{4.23}$$

or

$$\alpha < 0. \tag{4.24}$$

Since $\lambda_1 - \lambda_0 < 2$, (4.22a) implies (4.22b). The $j$ coordinate, labelled $J$, of the intersection of the line through $v_0$ and $v_1$ and the line through $E$ is

$$J = \frac{(-\alpha + k_0 - k_1)a}{\lambda_1 - \lambda_0}. \tag{4.25}$$

According to (4.22), $k_1 > 2 + \lambda_1 k_0/\lambda_0$ (where $\lambda_0 > 0$) whence $J \leqslant 0$ and so $J < K_1 a$. This would flout the required convexity of $S$.

For (4.23) the convexity requirement on $S$ in view of (4.25) is

$$J \geqslant K_1 a, \qquad \lambda_1 - \lambda_0 + k_1 - k_0 \leqslant \lambda_1 \alpha. \tag{4.26}$$

The numerator $N$ of (4.21) is negative and so (4.26) implies

$$22k_1 + 2\alpha \leqslant \lambda_1 \alpha \tag{4.27}$$

a contradiction since $\lambda_1 < 1$.

For (4.24) the intersection of the line through $-v_1$, $-v_0$ and that through $\bar{p}, \bar{q}$ has $j$ component $J'$

$$J' = \alpha + (k_0 - k_1)/(\lambda_1 - \lambda_0). \tag{4.28}$$

The convexity condition is $J' < -K_0 a$ or

$$k_0 - k_1 + \lambda_1 - \lambda_0 \leqslant -\lambda_0 \alpha. \tag{4.29}$$

Combine this with (4.23a) to get $-(\lambda_0 + 2)\alpha \geqslant 2k_0$ in contradiction with (4.23b).

We now consider the case that $\lambda_0 < 0 < \lambda_1$ so $\alpha < 0$, cf. (4.24). Then (4.26) and (4.29) yield

$$\lambda_1 - \lambda_0 + \lambda_0 \alpha \leqslant k_1 - k_0 \leqslant \lambda_1 \alpha - (\lambda_1 - \lambda_0).$$

Since $\alpha < 0$, this implies the contradiction

$$0 < (\lambda_1 - \lambda_0)(2 - \alpha) \leqslant 0.$$

Suppose now that $\bar{p} = \bar{q}$. Evidently a line of support $L$ at $\bar{p}$ has the slope of a line of support at $(-1, -K_0' = -(v_0 + i)/\|v_0 + i\|)$ and at $(-1, K_1' = (v_1 - i)/\|v_1 - i\|)$. By convexity this implies that $S$ includes the part of $L$ between $K_1'$ and $-K_0'$ at least. The type of argument used for $\bar{p}$, $\bar{q}$; $K_1 a$, $K_0 a$ carries over with $\bar{p} = \bar{q}$, $K_0'$, $K_1'$ since only the relative values of $K_0$, $K_1$, $k_0$, $k_1$, $\lambda_0$ and $\lambda_1$ were used to get our contradictions.

**5. Restrictions and limits.** It is convenient to modify $A$ somewhat. Observe first that $A$ is within $2\varepsilon$ of a continuous $4\varepsilon$ transformation and hence we may as well assume that $A$ is continuous and $A(0) = 0$. Next in order to minimize computational complexities we require that $A$ be antisymmetric within $\varepsilon$, that is to say

$$\|Ax + A(-x)\| < 2\varepsilon$$

but then we may as well assume exact antisymmetry since with the antisymmetric transformation

$$Bx = (Ax - A(-x))/2,$$
$$2\|Bx - Ax\| = \|Ax + A(-x)\| < 2\varepsilon.$$

Define

$$Tx = \|x\| \, \|Bx\|^{-1} Bx.$$

We summarize the relevant properties in a lemma, but in applications we replace both $\varepsilon'$ and $\varepsilon''$ by the notation $\varepsilon$.

LEMMA 5.1. (a) $\|Tx - Bx\| < \varepsilon'$.
(b) $\|Tx\| = \|x\|$, $T0 = 0$.
(c) $|\|Tx - Ty\| - \|x - y\|| < \varepsilon''$.
(d) *$T$ is an epimorphism.*

For (a) note

$$\big\| \|x\| \, \|Bx\|^{-1} Bx - Bx \big\| = |\|x\| - \|Bx\|| < |\|x\| - \|Ax\|| + \|Ax - Bx\| < 3\varepsilon.$$

Relation (c) is a consequence of

$$\|Tx - Ty\| < \|Tx - Bx\| + \|Ty - By\| + \|Bx - By\| < \|x - y\| + 11\varepsilon,$$
$$11\varepsilon = \varepsilon''.$$

Finally (d) is a consequence of the Borsuk-Ulam theorem [6].

Subsequences of $R$, denoted by $R' = \{r'\}$, $R''$, $R_1$, etc., are understood to admit no finite limit point. The associated set of cluster points of $\{Tr'x/r' | r' \in R'\}$ is denoted by $C'(x)$. We may view $C$ as a possibly multivalued transformation on $rS$ to $rS$ represented by $C: x \to Cx$.

For convenience we collect some useful known results in

LEMMA 5.2. (a) *$C$ takes* Ex *into* Ex.

(b) *C is closed and connected.*

(c) *Extended Mazur-Ulam theorem: If $x_1$ and $x_2$ are distinct points, and $C'$ is single valued on $x_1$ and $x_2$, then*

$$C'\frac{x_1+x_2}{2}=\frac{1}{2}(C'x_1+C'x_2).$$

These assertions are established in [6].

LEMMA 5.3. *If $C'$ is single valued it is continuous.*

Let $x_n \to x$ and let $y = C'x, y_n = C'x_n$. For any $r'' \in R'' \subset R'$

$$\|y-y_n\| < \left\|y-\frac{Tr''x}{r''}\right\| + \left\|y_n-\frac{Tr''x_n}{r''}\right\| + \left\|\frac{Tr''x_n}{r''}-\frac{Tr''x}{r''}\right\|. \tag{5.31}$$

For arbitrary positive $\delta$, choose $n$ so that $x_n$ is within $\delta/4$ of $x$ in norm. Next choose $r''$ so that the first and the second norms on the right are inferior to $\delta/4$ for this $n$ and so that $\varepsilon/r''$ is less than $\delta/4$. The left-hand side of (5.31) is therefore dominated by $\delta$ which establishes the assertion of the lemma.

LEMMA 5.4. *If $C'x_1$ is a singleton for some $R'$ then $C'x$ is a singleton for all $x \in S$ and hence $U'x = C'(x)$ defines a linear isometry.*

Assume then that $y_1 = C'(x_1)$ and that $C'(x_0)$ is not a singleton with $x_0$ chosen as an extreme point. Assume (A): both $y_0$ and $y_0'$ are in $C'(x_0)$ and precede $y_1$ (or both follow $y_1$) in clockwise order on $S$. (Note that if $C'(x)$ contains more than two points, then (A) is certainly satisfied.) Suppose $\bar{x} = \lambda x_0 + \mu x_1$, $\mu > 0$, $\lambda > 0$, is an arbitrary point on the short arc $x_0, x_1$. It follows trivially from (5.2c) that $\lambda y_0 + \mu y_1 = C'(\lambda x_0 + \mu x_1) \in S$. Clearly $[y_0, y_0']$ is parallel to $[\lambda y_0 + \mu y_1, \lambda y_0' + \mu y_1]$ where $y_0, y_0'$ are in Ex $S$. This is possible only if the arc from $y_0$ to $y_0'$ contains (or is contained in) the arc from $\lambda y_0 + \mu y_1$ to $\lambda y_0' + \mu y_1$, and this possibility is ruled out by (A). That $y_0' = y_1$ is ruled out since then for $R'' \subset R'$, $C''x_0 = y_1$ in contradiction with $\|x_0 - x_1\| = \|C''x_0 - C''x_1\|$.

Suppose (B) : $y_0$ and $y_0'$ box $y_1$ in between, with $\|y_1 - y_0\| = \|y_1 - y_0'\|$. Let the subsequence $R''$ give $C''(x_0) = y_1$ and let $R'''$ yield $C'''(x_0) = y_0'$. Let $R' = R'' \cup R'''$. The generic point $\bar{x}$ on the short arc $[-x_1, x_0]$ has the representation $\bar{x} = \lambda x_0 + \mu(-x_1)$, $\lambda, \mu > 0$. Hence

$$\begin{aligned} C'''(\bar{x}) &= \lambda C'''(x_0) + \mu C'''(-x_1) \\ &= \lambda y_0' + \mu(-y_1). \end{aligned}$$

Accordingly as $x$ traverses the arc $-x_1$ to $x_1$ clockwise, $C'''(x)$ moves from $-y_1$ to $y_1$ counterclockwise. Similarly $C''(x)$ moves from $-y_1$ to $y_1$ clockwise.

In view of (5.2b) and (5.3) and since the connectivity of $R$ guarantees the connectivity of the graph of $Trx/r$ for fixed $x$ there is a sequence $R_1$ for which $L_{R_1}Tr_1z/r_1 = y$ and a sequence $R_2$ for which $L_{R_2}Tr_2z/r_2 = -y$. Then

$$|r_1 - r_2| + \varepsilon > \|Tr_1z - Tr_2z\| > r_1 + r_2 - \delta, \qquad r_1, r_2 > M.$$

Such an inequality is impossible. The Mazur-Ulam theorem implies the second conclusion in the lemma. (Of course this does not rule out the possibility that $C(x)$ is not a singleton. We shall show, however, that in the cases we are concerned with, this is indeed the fact.)

LEMMA 5.5. *If $x_0$ is a corner point (or an edge point) so is $C(x_0)$ and $C(x_0)$ is a singleton.*

The previous lemma asserts that for suitable $R'$, $C' = U'$ is a linear isometry. This implies that distinct support lines (or support lines on $S$) through $x_0$ go into distinct support lines (or support lines on $S$) through $y_0 = U'(x_0)$. The first assertion of the lemma is then immediate. Suppose $x_0$ is a corner point. Then $C(x_0)$ is a connected set consisting exclusively of corner points. The set of corner and edge points is patently denumerable. Since $C(x_0)$ is connected, $C(x_0)$ is a singleton.

COROLLARY 5.6. *If $S$ contains a corner point or an edge point $C(x)$ is a singleton for all $x$ and $U(x) = L_R Trx/r$ defines a linear isometry.*

The result is a direct consequence of the preceding two lemmas and the Mazur-Ulam theorem [7].

A consequence is, for instance, that if $S$ is a lens i.e. has two antipodal corner points $x_0$, $-x_0$ and all other extreme points are exposed points with unique tangents, then $Cx_0$ is either $x_0$ or $-x_0$ and the only isometries possible are the identity or the obvious involutions. (In connection with (5.5) and (5.6) the plausible extension to curvature preservation at a point under $U$ is without interest, since curvature is geometrically defined in terms of osculating spheres, so the curvature at every point of $S$ is 1.)

The corollary is a substantial advance on the results available in [6] where the requirement $\mathfrak{S}$ is imposed that all the components in Ex be singletons.

**6. Validation for $P_2$.** To simplify the statement of the central results in this section we observe that where (5.6) obtains, we can define $T' = U^{-1}T$ so that the associated $U'$ is now the identity isometry. We assume this has been done but drop the prime on $T'$ in Theorem 6.1.

THEOREM 6.1. *In $P_2$ let $v_0$ and $v_1$ be base corner points for the edge $L$ on $S$. Then for some $m$ and all large $r$.*

$$\|Trv_i - rv_i\| < m\varepsilon, \qquad i = 0, 1, 2, 3.$$

Since $U'$ is the identity isometry, for $r > \bar{r}$ the line of the vector $T'rv_0/r + T'rv_1/r$ cuts both $L$ and $-L$. Write $T$ for $\bar{r}^{-1}T'\bar{r}$. All segments with endpoints on $L$ and $-L$ respectively are of length 2. Let $a$ be the length of $L$. Denote the line segment joining $Tv_1$ and $T - v_0$ by $M$.

Several cases need consideration:

(6.1a) $Tv_0$ lies to the left of $v_0$ and $Tv_1$ is on $L$.

(6.1b) $Tv_0$ lies to the left of $v_0$ and $Tv_1$ lies to the right of $L$.

(6.1c) $Tv_0$ and $Tv_1$ lie on $L$.

Denote $\|v_i - Tv_i\|$ by $\Delta_i$, $i = 0, 1$. Then for (6.1a)

$$a - \varepsilon < \|Tv_1 - Tv_0\| < a + \Delta_0 - \Delta_1$$

or

$$\varepsilon > \Delta_1 - \Delta_0.$$

Let $\lambda\Delta_0$ be the length of the projection parallel to $L$ of the short arc from $Tv_0$ to $v_0$, on $M$. Then it is easily seen that

$$2 - \lambda\Delta_0 > 2 - \varepsilon$$

or

$$\varepsilon/\lambda > \Delta_0, \qquad \varepsilon(1 + 1/\lambda) > \Delta_1.$$

The lines of support at $v_0$ and the diagonal through $[v_0, -v_0]$ intersect at angles greater than some positive number $r_0$. This is then certainly still true when $[v_0, -v_0]$ is replaced by $M$. Hence there is positive lower bound $\lambda_0$ for $\lambda$ as one approaches $-v_0$ in the clockwise direction.

For (6.1b)

$$2 - \mu\Delta_1 - \lambda\Delta_0 > 2 - \varepsilon$$

where $\mu\Delta_1$ is the length of the projection of the short arc from $v_1$ to $Tv_1$ on $M$. Then

$$\varepsilon/\mu > \Delta_1, \qquad \varepsilon/\lambda > \Delta_0.$$

Again the fact that $v_1$ is a corner point implies $\mu > \mu_0 > 0$.

In the case of (6.1c)

$$a - \varepsilon < a - \Delta_0 - \Delta_1$$

whence

$$\varepsilon > \Delta_0, \qquad \varepsilon > \Delta_1.$$

In all cases, then,

$$\|Tv_i - v_i\| < m\varepsilon \tag{6.11}$$

where $m = \sup((1 + \lambda_0^{-1}), \mu_0^{-1}, 1)$.

If $v_i$ and $Tv_i$ are replaced by $rv_i$ and $Trv_i$ then $a$, $\Delta_0$, $\Delta_1$ are replaced by $ra$, $\Delta_0(r)$, $\Delta_1(r)$ but $\lambda_0$, $\mu_0$ and $\varepsilon$ are unaffected. Hence (6.11) is independent of $r$.

THEOREM 6.2. *For $P_2$ restricted as in* (6.1), $\|Tx - Ux\| < K\varepsilon$ *for all $x$.*

We assume as above that $T$ is really $U^{-1}T$ so that $U$ is the identity isometry. For $x$ inside the unit disk compactness assures a bound of the form asserted. Denote it by $k\varepsilon$.

For $\|x\| \geqslant 1$ the asserted bound is a consequence of (4.2) and (5.6). Specifically

$$\big|\|Tx - Trv_j\| - \|Tx - rv_j\|\big| \leqslant \|Trv_j - rv_j\|, \qquad j = 0, 1, 2, 3.$$

By (5.1c)

$$\big|\|Tx - Trv_j\| - \|x - rv_j\|\big| \leqslant \varepsilon.$$

Hence

$$\big|\|Tx - rv_j\| - \|x - rv_j\|\big| \leqslant \varepsilon + \|Trv_j - rv_j\|.$$

In view of (6.1) the right-hand side is inferior to $(m + 1)\varepsilon$. The hypothesis of (4.2) is satisfied with the assignment $\delta = (m + 1)\varepsilon$. The conclusion of (4.2) is then

$$\|Tx - x\| \leqslant M(m + 1)\varepsilon.$$

Let $K = \max(k, M(m + 1))$.

**7. Extensions.** Besides expecting (1.11) to be valid for finite dimensional Banach spaces, one may conjecture its validity for finite field $\varepsilon$ isometries (that is to say for

$$f = I - F : E \to E$$

where $F$ takes $E$ into a finite dimensional subspace) since $E_N$ associates a complementary space $E^N$, i.e. $E = E_N \oplus E^N$. Even the case of the general $E_2$ is open. We present some results in the spirit of the earlier developments of this paper.

THEOREM 7.1. *Let $f = I - F$, where $I$ is the identity transformation and $F$ is continuous and closed on $E$ to $E_N$, $N < \infty$, be an $\varepsilon$ isometry with $F(0) = 0$. Then*

(a) *$C(x) = L_R Frx/r$ is compact and connected.*
(b) *$f$ is surjective.*
(c) *$\bar{f} = I - C$ is upper semicontinuous.*
*For $N = 2$*
(d) *$\bar{f}$ is surjective.*
(e) $\deg \bar{f} = \deg f = 1$.

To establish (7.1a) suppose $y_n \to y$, where $y_n \in Cx$. Let $\delta_N \downarrow 0$ be a sequence of positive constants. For $\delta_N$ select $y_n$ and $r(n)$ depending on $N$ so that

$$\|y_n - y\| < \delta_N/2 \quad \text{and} \quad \left\|y_n - \frac{Tr(n)x}{r(n)}\right\| < \delta_N/2.$$

Hence

$$\left\| y - \frac{Tr(n)x}{r(n)} \right\| < \delta_N.$$

Define the sequence $R' = \{r(n)\}$. Then

$$L_{R'} \frac{Tr'x}{r'} = y.$$

That $C(x)$ is connected is a reflection of the preservation of connectedness by a map (of the connected set $R$).

To establish (7.1b), suppose

$$f(x) = x - Fx = y$$

has no solution, $x$, for some $y$. Write

$$x = x_N \oplus x^N, \qquad y = y_N \oplus y^N.$$

Then

$$x^N = y^N, \qquad x_N - F(x_N \oplus x^N) = y_N$$

has no solution $x_N \in E_N$, that is to say the map of $E_N$ to $E_N$ defined by

$$f_{x^N}(x_N) = x_N - F(x_N \oplus x^N)$$

does not cover $E_N$. According to [6] this implies the existence for arbitrary $M > 0$ of a pair $x'_N$, $x''_N$ with

$$f_{x^N}(x'_N) = f_{x^N}(x''_N), \qquad \|x'_N - x''_N\| > M$$

in contradiction to

$$\begin{aligned} \|x'_N - x''_N\| - \varepsilon &< \|f(x'_N \oplus x^N) - f(x''_N \oplus x^N)\| \\ &= \|f_{x^N}(x'_N) - f_{x^N}(x''_N)\| = 0. \end{aligned}$$

We establish (7.1c) by showing $C$ is upper semicontinuous. Thus let $x_n \to x$ and $y_n \to y$, $y_n \in Cx_n$. To see that $y \in Cx$, pick a positive $\delta$. Choose $n$ so that

$$\|x - x_n\| < \delta/4, \qquad \|y - y_n\| < \delta/4.$$

Pick $r(n)$ so that

$$\|y_n - Tr(n)x_n/r(n)\| < \delta/4, \qquad \varepsilon < r(n)\delta/4.$$

Then

$$\begin{aligned} \left\| y - \frac{Tr(n)x}{r(n)} \right\| &< \|y - y_n\| + \left\| y_n - \frac{Tr(n)x_n}{r(n)} \right\| \\ &\quad + r(n)^{-1}\|Tr(n)x_n - Tr(n)x\| \\ &< \delta. \end{aligned}$$

Hence with $R'$ the sequence with elements $\{r(n)\}$, $y = L_{R'}\, Tr(n)x/r(n)$.

We turn now to (7.1d). Suppose for some $y \in E$ that

$$\bar{f}(x) \supset y$$

has no solution. Since from

$$E = E_2 \oplus E^2,$$

$x \in E$ has the representation $x = x_2 \oplus x^2$, we are supposing in particular that

$$\bar{f}_2(x_2) = x_2 - C(x_2) \supset y_2$$

admits no solution $x_2$. However, by an immediate inference from (5.3) and (5.4), for some subsequence $R'$, $C'(x_2)$ is a singleton for all $x_2 \in E_2$ and $C'$ is continuous. Thus $\bar{f}_2'$ is a homogeneous isometry of $E_2$ into $E_2$ and $\bar{f}_2'(0) = 0$ whence $\bar{f}_2'$ is surjective as a consequence of the Mazur-Ulam theorem **[6]**.

Let $D$ be the unit open ball about 0 in $E$ with boundary $D'$ and let $\varepsilon < 1$. In view of (7.1a) and (7.1b),

$$\bar{f}(x) = x - C(x)$$

is an acyclic subset of $x + S$ so $Cx \neq x$ on $D'$. Accordingly a degree is defined **[8]**. Suppose

$$\bar{f}_\lambda = I - \lambda C, \qquad 0 \leqslant \lambda \leqslant 1.$$

This defines an admissible multivalued homotopy **[8]** wherefore since $\bar{f}_1 = \bar{f}$, $\deg \bar{f} = \deg \bar{f}_0 = 1$ [**8**, Lemma 4.3]. The multivalued homotopy, $h$, between $\bar{f}$ and $f$ is defined by

$$\begin{aligned} h(t, x) &= x - tF(t^{-1}x), \qquad 0 < t \leqslant 1, \\ &= x - C(x), \qquad t = 0. \end{aligned}$$

Thus $h$ is an upper semicontinuous admissible homotopy. Accordingly

$$\deg f = \deg h(1) = \deg h(0) = \deg \bar{f} = 1.$$

(We note that (7.1d) may be used to give an immediate alternative proof of (7.1b) at least in the case $N = 2$. If $y \notin \operatorname{Im} f$, $\|y\| = 1 > \varepsilon$ for instance, then, since $\deg f$ is defined by $\deg f_N$ where $f_N = f | E_N \cap \bar{D}$, for $y \in E_N \supset E_2$, one is led to the contradiction that $f_N$ takes $S^{N-1}$ into the punctured sphere $S^{N-1} - y$ and yet $\deg f_N = 1$.)

In order to gain immediate access to the results of §6 it is natural to restrict the ε-isometry $f$ to be a sort of layer map. Thus

THEOREM 7.2. *Let $S \subset E_2$ contain an edge. Let $E = E_2 \oplus E^2$ and let $\pi$ be the projection of $E$ onto $E_2$. Suppose $F$ is a (continuous) map defined by $Fx = F\pi x = -F(-x)$. Then there is a true isometry $U$ defined by*

$$Ux = x - L_R Frx/r$$

*with* $\|Ux - x + Fx\| \leqslant m\varepsilon$ *for some positive* $m$ *and all* $x$.

Evidently $f\pi x = \pi f x$. Since $E_2$ is a $P_2$, §§5 and 6 can be applied. Thus the limits exist in

$$U\pi x = L_R f r \pi x / r = \pi x - L_R \, F r \pi x / r.$$

For $x = \pi x \oplus x^2$ define $Ux = U\pi x + x^2$. Then

$$\begin{aligned} \|Ux - fx\| &= \|U\pi x + x^2 - \pi x - x^2 + F\pi x\| \\ &= \|U\pi x - f\pi x\| \leqslant m\varepsilon \end{aligned}$$

by (6.2).

**Added in proof (January 27, 1978).** We are indebted to Professor R. D. Bourgin for (a) a reference to A. J. Lazar, Pacific J. Math. **33** (1970), 337–344, which bears on our Examples 1 and 2, and (b) for questions on the relevance of the last section of our paper [3].

Indeed by combining certain natural modifications of [3] with results in the earlier sections of this paper, we are able to completely settle the 2-dimensional case. (Advances in the general finite dimensional situation seem likely also in this way.) Though knowledge of [3] is not required below, comparison with somewhat similar developments in that paper is facilitated by largely taking over the notation.

THEOREM 8. *For each* $E_2$ *and some* $K$, $\|Tx - Ux\| < K\varepsilon$ *uniformly in* $x$.

In view of this result, the words "contain an edge" may be elided in Theorem 7.2.

The results in §6 required the presence of corner points. When there are no corner points, we need to establish the existence of a pair of exposed points. The question is moot only if no edge point $e$ is a corner point. Consider, then, a connected neighborhood $N(e)$ on $S$. Then the curve $N(e)$ admits second derivatives except for a countable set of points. Indeed, convexity ensures the existence of a first derivative except on a countable subset. This first derivative is, however, a monotone nonincreasing function and so admits a derivative except at a countable subset. Since $e$ is not a corner point, $N(e)$ contains a pair, $y_0, \bar{y}_0$ of disjunct points at which the second derivative exists and is not 0, so $y_0, \bar{y}_0$ are exposed points.

In view of 5.4, $R' = \{r_n\}$ exists for which $C'(x)$ is single valued for all $x \in S$ and $U'$ is a linear isometry. By compactness

$$U'x = L_{R'} T r_n x r_n^{-1}$$

and so for some bounded ultimately monotone nonincreasing function $k(r)$ which approaches 0 as $r \to \infty$

$$\|Tx - U'x\| \leqslant k(\|x\|)\|x\|. \tag{8.11}$$

Let $L$ be the tangent line to $S$ at $y_0$. The ray from the origin parallel to $L$ intersects $S(r)$ in $y_1$. Let $U'^{-1}y_1 = x_1$, $U'^{-1}y_0 = x_0$, $Tx_1 = y_2$, where $\|y_0\| = \|x_0\| = 1$, $\|y_1\| = \|y_2\| = r$. Evidently then for $r$ large $y_2 = Tx_1 = uy_1 + wy_0$ where $w > 0$. We make tacit use of (5.1) and (5.2), particularly (5.2c).

It is easy to see from (5.1b) and (5.1c) that

$$\|Tr_nx_0\| - \|Tr_nx_0 - Tx_1\| - \varepsilon \leqslant r_n\big(\|y_0\| - \|y_0 - y_1r_n^{-1}\|\big). \tag{8.12}$$

As $r_n \to \infty$, the right-hand side of (8.12) defines the derivative taken in the direction of the tangent line at $y_0$. Since the second derivative exists at $y_0$, Taylor's formula implies that

$$r_n\big(\|y_0\| - \|y_0 - r_n^{-1}y_1\|\big) = o(1). \tag{8.13}$$

Write

$$Tr_nx_0 = a_ny_0 + Z_ny_1$$

where $a_n = r_n + l_n$. ($l_n$ is negative, though this fact plays no role.) Then (8.11), with $x$ taken as $r_nx_0$ becomes

$$r_n^{-1}\|a_ny_0 + Z_ny_1 - r_ny_0\| = r_n^{-1}\|l_ny_0 + Z_ny_1\| \leqslant k(r_n). \tag{8.14}$$

Since this measures the approach to $y_0$ by $r_n^{-1}Tr_nx_0$ along the convex $S$ with the components taken along the fixed direction of $y_0$ and of $y_1$, (8.13) implies

$$r_n^{-1}l_n = o(1), \quad r_n^{-1}Z_n = o(1). \tag{8.15}$$

In particular $a_n \to \infty$.

Next the left-hand side of (8.12) can be written

$$\left\{a_n\big(\|y_0 + a_n^{-1}Z_ny_1\| - \|y_0 + (Z_n - u)y_1(a_n - w)^{-1}\|\big)\right\}$$
$$+ w\|y_0 + (Z_n - u)y_1(a_n - w)^{-1}\| - \varepsilon. \tag{8.16}$$

The curly bracket in (8.16) is $o(1)$ as a consequence of (8.13). Because of (8.15), (8.16) is

$$w + o(1) - \varepsilon$$

whence

$$w \leqslant \varepsilon. \tag{8.17}$$

Two immediate consequences ensue: the first is

$$\|Tx_1 - Ux_1\| \leqslant |u - 1|\,\|y_1\| + w\|y_0\| \leqslant |u - 1|\,\|y_1\| + \varepsilon$$

and the second results from

$$|u|\,\|y_1\| - \varepsilon \leqslant \|uy_1 + wy_0\| \leqslant |u|\,\|y_1\| + \varepsilon$$

in combination with $\|y_2\| = \|y_1\|$ to give

$$\big||u| - 1\big|\,\|y_1\| \leqslant \varepsilon.$$

Accordingly for $u > 0$

$$\|y_2 - y_1\| \leq 2\varepsilon. \tag{8.18}$$

On the other hand, (8.11) implies

$$\|(r^{-1}u - 1)y_1 + r^{-1}wy_0\| \leq k(r).$$

The type of justification given for (8.15) establishes that $r^{-1}u \to 1$, $r^{-1}w \to 0$, but then $u$ must be positive for large $r$, say $r > N$ and (8.18) applies. Similar results hold for $\bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{x}_1, \bar{x}_0$. Specifically, for $r > \bar{N}$

$$\|\bar{y}_2 - \bar{y}_1\| \leq 2\varepsilon. \tag{8.19}$$

Evidently $x_1$ and $\bar{x}_1$ are the counterparts of $rv_0$ and $rv_1$ in (6.1). For $\|x\| \leq N + \bar{N}$

$$\|Tx - U'x\| \leq 2(N + \bar{N}) \leq k\varepsilon.$$

For $\|x\| \geq N + \bar{N}$ we again replace $T$ by $U'^{-1}T$ and invoke (4.2) in conjunction with (8.18) and (8.19), just as in the proof of (6.2) to conclude the demonstration of Theorem 8.

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