TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 244, October 1978

APPLICATION OF THE SECTOR CONDITION TO THE CLASSIFICATION OF SUBMARKOVIAN SEMIGROUPS[1]

BY

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ABSTRACT. Let P_t, $t > 0$, be a strongly continuous submarkovian semigroup on a real Hilbert space $L^2(\mathbf{X}, m)$. The measure m is assumed to be excessive and the L^2 generator A is assumed to satisfy an estimate (the sector condition) which permits the application of Dirichlet spaces (not necessarily symmetric). Other submarkovian semigroups $P_t^{\sim}$ with the same local generator and cogenerator and relative to which m is again excessive are classified in terms of generators for processes which live on a suitable boundary.

Introduction. Let P_t, $t > 0$, be a strongly continuous, contractive submarkovian semigroup on $L^2(\mathbf{X}, m)$. This means that each P_t is a bounded linear contraction on the real Hilbert space $L^2(\mathbf{X}, m)$ and that $0 < P_tf < 1$ whenever $0 < f < 1$, of course in the [a.e.m.] sense. The state space $\mathbf{X}$ is a separable locally compact Hausdorff space and the reference measure m is Radon. We are interested in studying, and to some extent classifying other such semigroups $P_t^{\sim}$ which have the same local generator and dual local generator as P_t.

At present we cannot even formulate this precisely without first imposing two conditions. The first condition involves the L^2 generator A defined by

$$Af = \operatorname*{Lim}_{t\downarrow 0} (1/t)\{P_tf - f\} \tag{0.1}$$

with the understanding that the limit must exist strongly in the L^2 sense, and also the associated Dirichlet form

$$E(g,f) = -(g, Af). \tag{0.2}$$

Of course (,) denotes the standard inner product on $L^2(\mathbf{X})$. The conditions are

0.1.1. *Sector condition*. There exists a constant $M > 0$ such that for f, $g \in \text{domain}(A)$

$$|E(g,f)|^2 < M^2E(g,g)E(f,f). \quad \square \tag{0.3}$$

Received by the editors July 20, 1976 and, in revised form, February 7, 1977.

AMS (MOS) subject classifications (1970). Primary 60J25, 60J50; Secondary 60J35.

Key words and phrases. Submarkovian, Dirichlet spaces, sector condition, boundary generation.

[1] This research was supported in part by the Institute for Advanced Study at Princeton, New Jersey, and in part by the National Science Foundation under Grant MPS75-01872.

0.1.2. The reference measure m is *excessive*. That is, if $f \geq 0$ is in $L^2(X) \cap L^1(X)$, then for all $t > 0$

$$\int m(dx)P_tf(x) \leq \int m(dx)f(x). \quad \square \tag{0.4}$$

The sector condition is fundamental for the techniques used in this paper (and also in [1]). It can be replaced by the weaker condition that for some $\alpha > 0$

$$|E_\alpha(g,f)|^2 \leq M^2E_\alpha(g,g)E_\alpha(f,f), \tag{0.3'}$$

where $E_\alpha(g,f) = E(g,f) + \alpha(g,f)$. It may be that the second condition is primarily a matter of convenience and that in practice it can be suppressed if necessary.

Notice that (0.4) together with the submarkov property implies that each P_t is contractive on $L^2(X)$, which we have already postulated above. More important (0.4) implies that the dual semigroup P_t^* defined by

$$(P_t^*g,f) = (g,P_tf) \tag{0.5}$$

is also submarkovian. Indeed our conditions are symmetric in P_t and P_t^*. As a general rule we will use the symbol *, the prefix "co" and/or the adjective dual for objects associated with P_t^*.

Before describing our results for other submarkovian semigroups $P_t^\sim$, we discuss certain extensions of the domain of the Dirichlet form E.

The first extension is to the *extended Dirichlet space* $\mathbf{F}_{(e)}$. This is the natural completion of domain(A) relative to E alone. To simplify matters we assume throughout that P_t is transient, and then $\mathbf{F}_{(e)}$ is an "honest" Hilbert space. Except for certain technical points, the pattern is the same as in [2, §1] for the symmetric case.

The bilinear form $E(g,f)$ is well defined when $g \in \mathbf{F}_{(e)}$ is bounded with compact support and when f belongs to $\mathbf{F}_{\text{loc}}$, the set of functions which are locally in $\mathbf{F}_{(e)}$. The local generator $\mathcal{Q}$ is defined by

$$E(g,f) = -\int m(dx)g(x)\mathcal{Q}f(x) \tag{0.6}$$

when there exists a locally integrable function $\mathcal{Q}f$ such that (0.6) is valid. Similarly the dual local generator is defined by

$$E(g,f) = -\int m(dx)\mathcal{Q}^*g(x)f(x). \tag{0.6*}$$

Just as for the symmetric case [2, Part II], the quadratic part $E(f,f)$ of the Dirichlet form has a Beurling-Deny decomposition

$$E(f,f) = D(f,f) + \tfrac{1}{2}J\langle f,f\rangle + \tfrac{1}{2}\int \{\kappa(dx) + \kappa^*(dx)\}f^2(x) \tag{0.7}$$

where $J(dy, dz)$, the Levy measure, measures the intensity for jumping within the state space **X** and where $\kappa(dx)$, the killing measure, measures the intensity for jumping to the death point. Of course κ^* is the killing measure for the dual semigroup. The remainder $D(f,f)$, the diffusion form, is a local form as defined in [2, §11]. The symmetrized reflected space $\mathbf{F}_{\text{ref}}^{\text{symm}}$ is the set of $f \in \mathbf{F}_{\text{loc}}$ for which (0.7) converges. Every $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ has a local decomposition

$$f = f_D + H^M f \tag{0.8}$$

where f_D belongs to the extended Dirichlet spade $\mathbf{F}_{(e)}^D$ which corresponds to the absorbed process for D and where $H^M f$ is harmonic on D. Here D is any open set with compact closure. (A function is understood to be harmonic if it is fixed by the hitting operators for the complement of any compact set.) We showed in [2, §14] that in the symmetric case it is always possible to pass to the limit in D and get a global decomposition

$$f = f_0 + h \tag{0.9}$$

with $f_0 \in \mathbf{F}_{(e)}$ and with h harmonic on all of **X**. The argument used in [2] is not applicable here and indeed we have not been able to decide whether or not such a decomposition is always possible. Fortunately we are able to get around this gap in our knowledge by focusing our attention on the *reflected space* $\mathbf{F}_{\text{ref}}$, the set of $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ for which such a decomposition is possible. It turns out that $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ has a decomposition (0.9) if and only if it has a dual decomposition

$$f = f_0^* + h^*. \tag{0.9*}$$

For technical reasons we restrict our attention further to the active reflected space $\mathbf{F}_{\text{ref},a}$ which is actually contained in $L^2(\mathbf{X})$. This space is used to define a boundary Δ and a reference measure ν on Δ. Each $f \in \mathbf{F}_{\text{ref},a}$ determines a "boundary function" $\gamma f \in L^2(\Delta, \nu)$ and the decompositions (0.9), (0.9*) can be written

$$f = f_0 + H\gamma f; \qquad f = f_0^* + H^*\gamma f$$

where H, H^* are the hitting operator and dual hitting operator for the boundary Δ. Also for $\alpha > 0$ there are decompositions

$$f = f_\alpha + H_\alpha \gamma f; \qquad f = f_\alpha^* + H_\alpha^* \gamma f \tag{0.10}$$

where now f_α, f_α^* belong to the Dirichlet space $\mathbf{F} = \mathbf{F}_{(e)} \cap L^2(\mathbf{X})$.

For $f \in \mathbf{F}_{\text{ref},a}$ we can write

$$E(f,f) = E(f_0^*, f_0) + N(\gamma f, \gamma f) \tag{0.11}$$

where in general for φ defined on Δ we have

$$N(\varphi, \varphi) = \tfrac{1}{2}\int_\Delta \nu(d\eta)\int_\Delta \nu(d\xi)u_{0,\infty}(\eta, \xi)\{\varphi(\eta) - \varphi(\xi)\}^2$$
$$+ \tfrac{1}{2}\int_{\mathbf{X}} \kappa^*(dx)\, H\varphi^2(x) + \tfrac{1}{2}\int_{\mathbf{X}} \kappa(dx)H^*\varphi^2(x)$$
$$+ \tfrac{1}{2}\int_\Delta \nu(d\xi)\{\pi_\infty p^*(\xi) + \pi^*_\infty p(\xi)\}\varphi^2(\xi). \tag{0.12}$$

The individual terms in (0.12) can be defined directly in terms of the approximate Markov process $\mathcal{P}$ which in constructed exactly as for the symmetric case in [2, §5]. The Feller density $u_{0,\infty}(\eta, \xi)$ is the intensity for entering from η on Δ and exiting at ξ on Δ. The function $\pi_\infty p^*(\xi)$ is the intensity for entering at time $-\infty$ or from the passive point δ (but not jumping in) and exiting at ξ on Δ. The function $\pi^*_\infty p(\xi)$ has a dual meaning. The second term corresponds to the possibility of entering by jumping in from the death point ∂ and exiting at Δ, and the third to the dual possiblity. (See the analogous description in [3, §3].)

Now we are ready to describe our results for submarkovian semigroups $P_t^\sim$ with generator $A^\sim$ and adjoint generator $A^{\sim *}$ contained in $\mathcal{Q}$ and $\mathcal{Q}^*$ respectively. At present we can obtain the results in a general setting only if we assume that also the adjoint semigroup $P_t^{\sim *}$ is submarkovian. However we do not assume that the generator $A^\sim$ satisfies the sector condition 0.1.1. Both domain($A^\sim$) and domain($A^{\sim *}$) are contained in the active reflected space $\mathbf{F}_{\mathrm{ref},\alpha}$. The resolvent $G_\alpha^\sim$, $\alpha > 0$, can be represented

$$G_\alpha^\sim g = G_\alpha g + H_\alpha R_{(\alpha)}\pi^*_\alpha g \tag{0.13}$$

where G_α, $\alpha > 0$, is the resolvent for the original semigroup P_t, where H_α is the α-order hitting operator mentioned above and is bounded from $L^2(\Delta, \nu)$ to $L^2(\mathbf{X}, m)$, and where π^*_α is the adjoint to the dual hitting operator H^*_α and is bounded from $L^2(\mathbf{X})$ to $L^2(\Delta)$. The operator $R_{(\alpha)}$ is bounded on $L^2(\Delta)$ and must contain all of the "new information". In fact this information can be expressed neatly in terms of a single operator B, the "boundary generator." The operators $R_{(\alpha)}$ are determined by B in a canonical manner, exactly as in the symmetric case. (See p. 20.13 in [2].) The operator B on $L^2(\Delta)$ generates a contractive submarkovian semigroup which is strongly continuous on a subspace of $L^2(\Delta)$ and B together with its adjoint B^* satisfy estimates

$$- ([\varphi - c]^+, B\varphi)_\Delta$$
$$\geqslant \int \nu(d\eta)\int \nu(d\xi)u_{0,\infty}(\eta, \xi)[\varphi - c]^+(\eta)\{c - \varphi \wedge c(\xi)\}$$
$$+ \int \nu(d\xi)\pi^*_\infty p(\xi)c[\varphi - c]^+(\xi) + \int \kappa(dx)H^*c[\varphi - c]^+(x), \tag{0.14}$$

$$-(B^*\psi, [\psi - c]^+)$$

$$> \int \nu(d\eta) \int \nu(d\xi) u_{0,\infty}(\eta, \xi)\{c - \psi \wedge c(\eta)\}[\psi - c^+](\xi)$$

$$+ \int \nu(d\xi)\pi_\infty p^*(\xi) c[\psi - c]^+(\xi) + \int \kappa^*(dx) Hc[\psi - c]^+(x), \tag{0.14*}$$

where $\varphi \in \text{domain}(B)$, $\psi \in \text{domain}(B^*)$ and $c > 0$. Conversely any such B determines a unique semigroup $P_t^\sim$ as above.

The two estimates (0.14), (0.14*) are easily understood if one thinks in terms of constructing the Markov process which corresponds to $P_t^\sim$ by starting with the process on Δ which corresponds to the semigroup generated by B and then inserting excursions into **X** and finally collapsing the time scale. (See Chapter I in [3] for a detailed description of this construction on the symmetric case.) Our assumption that $A^\sim$ and $A^{\sim *}$ are contained in the local generators $\mathfrak{A}$ and $\mathfrak{A}^*$ implies that the excursions into **X** must have a particular conditional distribution. The complete excursions into **X** force the process on Δ to have at least a certain intensity for jumping within Δ, the incomplete excursions force the process and dual process to have at least a certain intensity for "jumping to a death point." This is the meaning of (0.14) and (0.14*).

To our knowledge it was M. Motoo [4] who first discovered this method for constructing the process which corresponds to $P_t^\sim$. It plays a fundamental role in [3], but in the present paper we view it only as a heuristic tool. In actually formulating our proof for the estimates (0.14), (0.14*) we were strongly influenced by a paper of H. Kunita [5] which deals with diffusions. However Kunita gets a weaker estimate than (0.14), (0.14*) and he is able to prove a converse only after imposing additional restrictions on $P_t^\sim$.

We can say considerably more if we restrict ourselves to the case when both P_t and $P_t^\sim$ are symmetric. That is, we assume that each P_t is a symmetric operator on $L^2(\mathbf{X}, m)$ and then 0.1.1 and 0.1.2 are automatic. We consider only submarkovian semigroups $P_t^\sim$ which are also symmetric on the same Hilbert space $L^2(\mathbf{X}, m)$. Then the results are more naturally formulated in terms of the Dirichlet space $(\mathbf{H}^\sim, Q^\sim)$ on $L^2(\Delta, \nu)$ which corresponds to the boundary generator B. (See §1 in [2].) If any exist at all then there is always the *excursion space* $(\mathbf{N}, N)$ with N defined by (0.12) and with $\mathbf{N}$ the totality of functions $\varphi \in L^2(\Delta)$ such that (0.12) converges. The associated semigroup P_t^{ref} acting on $L^2(\mathbf{X}, m)$ is called the *reflected semigroup* and corresponds precisely to the Dirichlet space $(\mathbf{F}_{\text{ref},a}, E)$ described above. (It is easy to check that for smooth diffusions this is consistent with standard terminology.) Moreover $\mathbf{F}_{\text{ref},a} = \mathbf{F} \cap L^2(\mathbf{X}, m)$ and $E(g, f)$, since it is symmetric, is automatically well defined for all $g, f \in \mathbf{F}_{\text{ref}}$. Sometimes this gives direct control over the Dirichlet space $(\mathbf{N}, N)$ on $L^2(\Delta)$. Before continuing, we illustrate this with a simple example.

Let $\mathbf{X}$ be the upper half-plane $\mathbf{R}^{2,+} = \{(x,y): x \in \mathbf{R} \text{ and } y > 0\}$ and let P_t be the absorbing barrier diffusion generated by $\mathcal{C} = D_1D_1 + D_2D_2 + (1 - \alpha)y^{-1}D_2$ where D_1, D_2 denote partial derivatives with respect to x, y respectively and $\alpha > 0$ is constant. Since also

$$\mathcal{C} = y^{\alpha-1}D_1y^{1-\alpha}D_1 + y^{\alpha-1}D_2y^{1-\alpha}D_2$$

it is clear that $\mathcal{C}$ and therefore P_t is symmetric with respect to the measure $dm = y^{1-\alpha}dxdy$. The associated Dirichlet form is

$$E(f,f) = -\int dm\, f\mathcal{C}f = \int_{-\infty}^{+\infty} dx \int_0^{\infty} dy\, y^{1-\alpha}\{(D_1f)^2 + (D_2f)^2\}. \quad (0.15)$$

If $\alpha \geq 2$, then it can be shown by direct arguments that (0.15) can converge only if f is constant on the boundary $\mathbf{R}$. Thus the reflected diffusion can be defined only after the boundary $\mathbf{R}$ is collapsed to a point. If $0 < \alpha < 2$ then there exist functions f with nontrivial boundary behavior such that (0.15) converges and so there exists a reflected diffusion which distinguishes points on $\mathbf{R}$. It is known that if we normalize the reference boundary measure to be Lebesgue measure, then the associated process on the boundary is the symmetric stable process with index α. This was first proved by F. Spitzer [24] for the case $\alpha = 1$ and by S. A. Molchanov and E. Ostrovskii [22] for the full range $0 < \alpha < 2$. In fact this can be deduced directly from the identity (0.11) by studying the action of the transformations

$$\delta_a(x,y) = (ax, ay); \qquad \rho_b(x,y) = (x + b, y)$$

for $a > 0$ and b real. The global hitting operator H must commute with both transformations and

$$E(f \circ \rho_b, f \circ \rho_b) = E(f,f); \qquad E(f \circ \delta_a, f \circ \delta_a) = a^{\alpha-1}E(f,f).$$

This is enough to determine the process on the boundary modulo a scale factor.

We return now to our general discussion of the symmetric case. The two basic estimates (0.14), (0.14*) are replaced by the condition that $\mathbf{H}^{\sim}$ is contained in $\mathbf{H}^{\text{ref}}$ and the difference $Q^{\sim} - N$ is contractive on $\mathbf{H}^{\sim}$. The latter means that we can write $Q^{\sim} = N + Q_0^{\sim}$ where $Q_0^{\sim}$ is itself contractive. Thus symmetric submarkovian semigroups $P_t^{\sim}$ with generator $A^{\sim}$ contained in $\mathcal{C}$ are completely classified by such pairs $(\mathbf{H}^{\sim}, Q^{\sim})$. However a pair $(\mathbf{H}^{\sim}, Q^{\sim})$ actually corresponds to a $P_t^{\sim}$ only if it is closable in the sense of [25]. Deciding whether a particular pair actually is closable can be an interesting technical problem about which little is known at present. (Perhaps the deepest results can be found in a recent preprint of M. Fukushima [23]. Also this is discussed for diffusions in §§15 and 17 of [3].) Nevertheless the general result does provide an effective technique for actually constructing examples. Also boundary conditions for the generator $A^{\sim}$ can be formulated very simply. For a given function f the normal derivative is defined as a linear functional $(\partial f/\partial n)(\psi)$ acting on bounded functions ψ in $\mathbf{H}^{\text{ref}}$ by means of difference quotients on an appropriate sample space. It is not hard to show that for smooth diffusions this corresponds to the classical inner normal derivative. A given $f \in \mathbf{F}_{\text{ref},a}$ belongs to domain$(A^{\sim})$ if and only if the restriction $\gamma f \in \mathbf{H}^{\sim}$ and f satisfies the boundary condition

$$(\partial f/\partial n)(\psi) = Q_0(\gamma f, \psi) \tag{0.16}$$

for all bounded $\psi \in \mathbf{H}^{\sim}$. (All these results for the symmetric case can be found in §§14, 15 of [2] and §9 of [3].)

At present we know very little about all this in any nonsymmetric setting. A natural guess is that if $\mathbf{F}_{\text{ref},a}$ properly contains $\mathbf{F} = \mathbf{F}_{(e)} \cap L^2(\mathbf{X}, m)$ then the reflected semigroup P_t^{ref} exists and its boundary generator B is given by

$$B\varphi(\xi) = \int u_{0,\infty}(\xi, \eta)\{\varphi(\eta) - \varphi(\xi)\}\nu(d\eta) - \pi_\infty^* p(\xi)\varphi(\xi) - \pi^*\kappa(\xi)\varphi(\xi) \tag{0.17}$$

with the first term on the right being defined by an appropriate limiting procedure. In fact our results for the symmetric case suggest that the "correct" limiting procedure would lead to the formula

$$B\varphi = (\partial/\partial n)H\varphi \tag{0.18}$$

where $H\varphi$ is the unique harmonic extension of φ.

However we do not even know how to define P_t^{ref} in a general setting. The simplest approach would be to first define the reflected Dirichlet form $E^{\text{ref}}(g, f)$ for all $g, f \in \mathbf{F}_{\text{ref},a}$ and then use E_α^{ref} to define the resolvent G^{ref}, at least for α sufficiently large, by the formula

$$E_\alpha^{\text{ref}}(g, G_\alpha f) = (g, f). \tag{0.19}$$

We doubt that this will work in general. In the Appendix we consider an example where $\mathbf{F}_{\text{ref},a}$ properly contains $\mathbf{F}$ but there does not seem to be any natural way to define $E^{\text{ref}}(g, f)$. This approach does work for the special case of diffusions with a smooth uniformly elliptic local generator.

The estimates for the boundary generator B are established in §7. In §8 we prove a converse result which shows in particular that (0.14), (0.14*) are the "correct estimates."

In §1 we use elementary techniques from Ergodic Theory to distinguish the transient and recurrent cases, just as in [2] for the symmetric case. To simplify matters we assume once and for all in the remainder of the paper that P_t is transient.

The extended Dirichlet space $\mathbf{F}_{(e)}$ is studied in some detail in §2. It follows from an estimate of H. Kunita [10, Lemma 3.1] that the resolvent operators G_α, $\alpha > 0$, are bounded on $\mathbf{F}_{(e)}$. Indeed αG_α is bounded uniformly in α and converges strongly to the identity 1 as $\alpha \uparrow \infty$. In the symmetric case the spectral theorem implies this and also the corresponding result for the transition operators P_t. We do not know if the latter is valid in general. However we do use a simple result from the theory of holomorphic semigroups to show that each P_t maps $L^2(\mathbf{X})$ into the Dirichlet space $\mathbf{F}$. This will enable us in §3 to extend to the present setting a result first established by M. Fukushima [9] for the symmetric case.

Our main purpose in §§3–5 is to extend certain results which will be needed in later sections. Often the real point is to fix notations, but we will also include some proofs which are significantly different from their symmetric counterparts. In particular we establish in §5 the Beurling-Deny decomposition for the quadratic form $E(f, f)$ and this is fundamental for studying the reflected space in §6. We have indicated above that the results in [3] do not automatically carry over. The main point seems to be that the sectorial estimate (0.3), unlike symmetry, is not automatically inherited by the reflected space. This is illustrated by the example in the Appendix.

As a general rule we will state and prove results only for P_t, the dual results being taken for granted. General notational conventions are the same as in [2] and [3]. For example all functions are understood to be measurable with respect to the obvious sigma algebra and specified up to appropriate null sets.

While the manuscript was being prepared we received several preprints from Yves le Jan [21], [26], [27] which contain closely related results. Le Jan restricts his attention to the case when also $A^\sim$ satisfies the sector condition. Then, just as for the symmetric case [2], [3] the generator on the boundary corresponds to a "Dirichlet space on the boundary." An interesting new feature is that (0.14) and (0.14*) must be supplemented by a direct estimate on Dirichlet norms. (See condition (b) of Theorem III.2 in [21].)

I have already mentioned the influence of M. Motoo and H. Kunita on this paper. In addition I again acknowledge my debt to M. Fukushima whose pioneering work [6], [7], [8] inspired my own research in this area.

I thank the staff and faculty at the Institute for Advanced Study in Princton for their hospitality and support while I was doing this research.

1. Transience and recurrence. We begin with

LEMMA 1.1. *The P_t, $t > 0$, form a strongly continuous contraction semigroup on $L^1(\mathbf{X})$.* □

PROOF. It follows directly from (0.4) that each P_t is a contraction on L^1. That is,

$$\int dx\,|P_tf(x)| \le \int dx\,|f(x)|. \tag{1.1}$$

Thus we need only establish for $f \in L^1$

$$\operatorname*{Lim}_{t\downarrow 0} \int m(dx)|f(x) - P_tf(x)| = 0. \tag{1.2}$$

Because of (0.4) it suffices to consider $f \in L^1 \cap L^2$ and then (1.2) would follow immediately from strong continuity of the P_t on L^2 if m were bounded. To handle the case of unbounded m, we use the notion of almost uniform integrability introduced by Hunt in his monograph [11]. It suffices to consider $f \geq 0$ in (1.2). Since $P_tf \to f$ in L^2, also $P_tf \to f$ in measure and so by the dominated convergence theorem

$$\operatorname*{Lim}_{t\downarrow 0} \int m(dx)\{f(x) \wedge P_tf(x)\} = \int m(dx)f(x).$$

But for all $t > 0$ we have $\int m(dx)f(x) \geq \int m(dx)P_tf(x)$ and it follows that

$$\operatorname*{Lim}_{t\downarrow 0} \int m(dx)\{P_tf(x) - f \wedge P_tf(x)\} = 0.$$

Thus for every $\alpha > 0$

$$\begin{aligned}
&\int_{[P_tf > \alpha f]} m(dx)P_tf(x)\\
&\quad\le \int_{[P_tf > \alpha f]} m(dx)f(x) + \int m(dx)\{P_tf(x) - f \wedge P_tf(x)\}\\
&\quad\le (1/\alpha)\int m(dx)P_tf(x) + \int m(dx)\{P_tf(x) - f \wedge P_tf(x)\}
\end{aligned}$$

which is enough to guarantee that the functions P_tf, $t > 0$, are almost uniformly integrable in the sense of [11] and (1.2) follows from Lemma 6.8 in [11]. □

From now on it is understood that the P_t, as well as any other order preserving operators, are extended to act on general functions as on p. 1.6 in [2].

Next we define for $\alpha > 0$ the *resolvent operators*

$$G_\alpha f = \int_0^\infty dt\, e^{-\alpha t}P_t \tag{1.3}$$

with the integral interpreted as a limit of Riemann sums. Also we define the *Green's operator*

$$Gf = \operatorname*{Lim}_{n\uparrow\infty} \int_0^n dt\, P_tf. \tag{1.4}$$

Arguing exactly as in §1 of [2], we prove

LEMMA 1.2. *Let* $f, g \geqslant 0$ *in* L^1 *and define*

$$A = \{x: Gf(x) = +\infty, Gg(x) < +\infty\}.$$

Then $g = 0$ [*a.e.m.*] *on* A *and* $Gg = 0$ [*a.e.m.*] *on* A. □

Now fix $f_0 \in L^1$ with $f_0 > 0$ [a.e.m.] and define

$$C = \{x: Gf_0(x) = +\infty\}; \quad C^* = \{x: G^*f_0(x) = +\infty\},$$
$$D = \{x: Gf_0(x) < +\infty\}; \quad D^* = \{x: G^*f_0(x) < +\infty\}.$$

By Lemma 1.2 these are well defined modulo m-null sets independent of the choice of f_0. The point of all this is

THEOREM 1.3. (i) $C = C^*$ *and* $D = D^*$ *modulo m-null sets.*

(ii) $$\int_C m(dx)Gl_D(x) = \int_D m(dx)Gl_C(x) = 0.$$

(iii) *If* $f \in L^1$ *and* $f \geqslant 0$, *then* $Gf < +\infty$ [*a.e.m.*] *on* D *and* $Gf = 0$ *or* $+\infty$ [*a.e.m.*] *on* C. □

PROOF. (iii) is an immediate consequence of Lemma 1.2. To prove the rest of the theorem choose $\psi \in L^1$ such that $\psi > 0$ [a.e.m.] on D^* and such that $\int m(dx)\psi(x)G^*f_0(x) < +\infty$. This implies that $G\psi < +\infty$ [a.e.m.] and, by Lemma 1.2, $\psi = 0$ [a.e.m.] on C. Thus $C \subset C^*$ [a.e.m.]. Similarly $C^* \subset C$ [a.e.m.] and (i) follows. The vanishing of the first integral in (ii) follows since $G\psi = 0$ [a.e.m.] on C by Lemma 1.2 and the vanishing of the second integral follows by duality. □

1.2. DEFINITION. The semigroup P_t is *transient* if $Gf < +\infty$ [a.e.m.] whenever $f \geqslant 0, f \in L^1$ and *recurrent* if for all such f the set $[0 < Gf < +\infty]$ is m-null. □

It follows from Theorem 1.3 that there is no real loss of generality in assuming that the semigroup is either transient or recurrent. To simplify matters we *assume* once and for all in the remainder of the paper *that* P_t *is transient.*

2. The extended Dirichlet space. Following H. Kunita [10], we begin by introducing for $\alpha > 0$ the approximating form

$$E^\alpha(g, f) = \alpha(g, f - \alpha G_\alpha f). \tag{2.1}$$

This makes sense at least for $f, g \in L^2$. Moreover

$$E^\alpha(g, f) = E(g, \alpha G_\alpha f). \tag{2.2}$$

The estimates

$$|E^\alpha(g, f)|^2 \leqslant M^2 E^\alpha(g, g)E(f, f), \tag{2.3}$$

$$E^\alpha(f, f) \leqslant M^2 E(f, f) \tag{2.4}$$

are established by Kunita. (For a proof see [10] or Lemma 1.1 in [1].)

For $\alpha > 0$ let $G_\alpha(dx, dy)$ be the unique measure on $\mathbf{X} \times \mathbf{X}$ such that for $g, f \in L^2$

$$\iint G_\alpha(dx, dy)\, g(x) f(y) = (g, G_\alpha f). \tag{2.5}$$

Then

$$\begin{aligned} E^\alpha(g,f) &= \frac{1}{2}\alpha\int m(dx)\{1 - \alpha G_\alpha 1(x)\}\, g(x) f(x) \\ &+ \frac{1}{2}\alpha\int m(dx)\{1 - \alpha G_\alpha^* 1(x)\}\, g(x) f(x) \\ &+ \frac{1}{2}\alpha^2 \iint G_\alpha(dx, dy)\big(g(x)\{f(x) - f(y)\} + \{g(y) - g(x)\} f(y)\big) \end{aligned} \tag{2.6}$$

and in particular

$$\begin{aligned} E^\alpha(f,f) = \frac{1}{2}\ &\alpha\int m(dx)\{1 - \alpha G_\alpha 1(x)\} f^2(x) \\ &+ \frac{1}{2}\alpha\int m(dx)\{1 - \alpha G_\alpha^* 1(x)\} f^2(x) \\ &+ \frac{1}{2}\alpha^2 \int G_\alpha(dx, dy)\{f(x) - f(y)\}^2. \end{aligned} \tag{2.7}$$

From now on we define $E^\alpha(f, f)$ by (2.7) whenever this converges. Of course it follows from (2.7) that if T is a normalized contraction (that is, satisfies (1.12) in [2]), then

$$E^\alpha(Tf, Tf) < E^\alpha(f, f). \tag{2.8}$$

2.1. Definition. Let f be defined and finite [a.e.m.] on $\mathbf{X}$. Then f belongs to the *extended Dirichlet space* $\mathbf{F}_{(e)}$ if there exists a sequence f_n, $n > 1$, in domain(A) such that

2.1.1. $E(f_n - f_m, f_n - f_m) \to 0$ as $m, n \to \infty$.

2.1.2. $f_n \to f$ [a.e.m.]. □

2.2. Definition. The *Dirichlet space* $\mathbf{F} = \mathbf{F}_{(e)} \cap L^2$. □

Just as §1 of [2] for the symmetric case, the extended Dirichlet space $\mathbf{F}_{(e)}$ is a Hilbert space relative to the symmetrized Dirichlet norm

$$\{g, f\} = \tfrac{1}{2}E(g, f) + \tfrac{1}{2}E(f, g). \tag{2.9}$$

(Actually Lemma 2.2 below is needed for the argument in [2].) The Dirichlet norm E itself extends to a bounded bilinear form on $\mathbf{F}_{(e)}$ and

$$E(f, f) = \operatorname*{Lim}_{\alpha\uparrow\infty} E^\alpha(f, f). \tag{2.10}$$

Of course (0.3) and also (2.3) and (2.4) are preserved by this extension. The Dirichlet space $\mathbf{F}$ is a Hilbert space relative to

$$\{g,f\}_\alpha = \tfrac{1}{2}E_\alpha(g,f) + \tfrac{1}{2}E_\alpha(f,g) \tag{2.11}$$

where for $\alpha > 0$

$$E_\alpha(g,f) = E(g,f) + \alpha(g,f). \tag{2.12}$$

Moreover $f \in L^2$ belongs to **F** if and only if

$$\sup_\alpha E^\alpha(f,f) < +\infty \tag{2.13}$$

and in this case the approximating f_n in Definition 2.1 can be chosen so that also $f_n \to f$ in L^2. Thus our present definition of **F** is consistent with [1].

It follows easily from (2.8) that if T is a normalized contraction and if $f \in \mathbf{F}_{(e)}$, then also $Tf \in \mathbf{F}_{(e)}$ and

$$E(Tf, Tf) \leqslant E(f,f). \tag{2.14}$$

This property plays a fundamental role in the symmetric case. However, the following two inequalities, proved by Kunita in [10], are crucial in general.

$$\begin{aligned} E(f - \{f^+ \wedge 1\}, f + \{f^+ \wedge 1\}) \geqslant 0, \\ E(f + \{f^+ \wedge 1\}, f - \{f^+ \wedge 1\}) \geqslant 0. \end{aligned} \tag{2.15}$$

We refer also to Lemma 1.6 in [1] for a proof.

Lemma 2.1. (i) *Each G_α, $\alpha > 0$, maps L^2 into* **F**. *Also G_α extends naturally to an everywhere defined operator on $\mathbf{F}_{(e)}$ satisfying*

$$E(\alpha G_\alpha f, \alpha G_\alpha f) \leqslant M^2 E(f,f) \tag{2.16}$$

and $\alpha G_\alpha \to 1$ in the strong operator topology as $\alpha \uparrow \infty$.

(ii) *Each P_t, $t > 0$, maps L^2 into* **F** *and indeed for some $M' > 0$ depending only on M*

$$E(P_t f, P_t f) \leqslant (M'/t)(f,f). \quad \square \tag{2.17}$$

Proof. (i) follows directly from (2.4) and (2.2). (Also see Lemma 1.4 in [1].) To prove (ii) we invoke an elementary result from the theory of holomorphic semigroups. For present purposes only consider the complex Hilbert space L_c^2 with inner product given by

$$(f^\sim + ig^\sim, f + ig)_c = (f^\sim, g) + (g^\sim, g) - i(g^\sim, f) + i(f^\sim, g)$$

and extend A to L_c^2 in the obvious way. Since

$$((f + ig), A(f + ig))_c = (f, Af) + (g, Ag) - i(g, Af) + i(f, Ag)$$

we conclude that for general $F \in \text{domain}(A)$

$$\text{Re}(F, AF)_c \leqslant 0; \qquad |\text{Im}(F, AF)_c| \leqslant -M\,\text{Re}(F, AF)_c. \tag{2.18}$$

It follows from Theorem 1.24 on p. 490 in [13] that the P_t have a holomorphic extension to the region $\text{Re}\, t \geqslant 0$; $M|\text{Im}\, t| \leqslant \text{Re}\, t$ where they continue to satisfy the contraction estimate $\|P_t F\|_{L_c^2} \leqslant \|F\|_{L_c^2}$. Finally the Cauchy integral formula applied directly to the operators P_t shows that for $t > 0$

$$\|(d/dt)P_tF\|_{L^2_t} \leq (M'/t)\|F\|_{L^2_t} \tag{2.19}$$

and this immediately implies (2.17). □

REMARK. If A is a normal operator on L^2 (that is, A commutes with its adjoint A^*), then the spectral theorem immediately implies that for $f \in$ domain A

$$E(P_tf, P_tf) \leq E(f,f) \tag{2.20}$$

which is enough to guarantee that P_t extends to an everywhere defined operator on the extended space $\mathbf{F}_{(e)}$ and also that $\mathrm{Lim}_{t\downarrow 0}E(f - P_tf, f - P_tf) = 0$. In particular (2.16) is true with $M = 1$ when A is normal. We know that (2.20) is not true in the general case. A counterexample can be constructed already when $\mathbf{X}$ contains two points. However, we do not know if there is a weaker estimate

$$E(P_tf, P_tf) \leq M''E(f,f). \tag{2.20$'$}$$

Notice that the Hille-Yosida theorem cannot be applied since we have no control over the iterates $\{\alpha G_\alpha\}^n$. □

We finish this section with

LEMMA 2.2. *$P_t \to 0$ in the strong operator topology on L^2 as $t\uparrow\infty$.*

PROOF. It follows from (2.19) that

$$(AP_tf, AP_tf) \leq (M'/t)^2(f,f) \tag{2.19$'$}$$

which implies that $P_tg \to 0$ in L^2 whenever $g \in$ range(A). Therefore we need only show that range(A) is dense. But if g is orthogonal to range(A), then $A^*g = 0$ and therefore $P_t^*g = g$ for all $t > 0$. But this is impossible unless $g = 0$ since by transience

$$\int_T^{T+1} dt\, P_t^*g \to 0$$

[a.e.m.] as $T\uparrow\infty$ and therefore weakly in L^2 as $T\uparrow\infty$ for $g \in L^2 \cap L^1$ and therefore for all $g \in L^2$. □

3. Hitting operators and the absorbed process. Let $C_{\mathrm{com}}(\mathbf{X})$ or C_{com} be the collection of continuous functions on $\mathbf{X}$ with compact support. In the remainder of the paper we assume that $\mathbf{F}_{(e)} \cap C_{\mathrm{com}} = \mathbf{F} \cap C_{\mathrm{com}}$ is uniformly dense in C_{com} and also dense in the Hilbert space $\mathbf{F}_{(e)}$. Just as for the symmetric case [2, §2], this can always be guaranteed if we are willing to replace $\mathbf{X}$ by an appropriate maximal ideal space.

We take for granted the potential theory developed in §3 of [2] for the symmetric case and in §2 of [1] for the general situation. The notation of [1] will be used consistently except that we will generally work with the form E as in [2] rather that with E_1 as in [1]. In particular a set of capacity zero will be called quasi-polar rather than polar as in [2]. All functions in $\mathbf{F}_{(e)}$ will be understood to be represented by quasi-continuous versions. Often functions specified up to quasi-equivalence will be treated as if they are defined everywhere and relations which are valid quasi-everywhere will be stated as if valid everywhere.

We begin with a technical result which follows directly from the spectral theorem in the symmetric case but seems to require indirect approximation techniques in general.

LEMMA 3.1. (i) *As* $\alpha\downarrow 0$ *the operators* $\alpha G_\alpha \to 0$ *in the strong operator topology on the Hilbert space* $\mathbf{F}_{(e)}$.

(ii) *If* $N\nu$ *is a potential, then so is* $P_t N\nu$ *for* $t > 0$. *Also*

$$E(P_t N\nu, P_t N\nu) \leq M^2 E(N\nu, N\nu), \tag{3.1}$$

$$P_t N\nu \to N\nu \quad \text{strongly in } \mathbf{F}_{(e)} \text{ as } t\downarrow 0, \tag{3.2}$$

$$P_t N\nu \to 0 \quad \text{weakly in } \mathbf{F}_{(e)} \text{ as } t\uparrow\infty. \tag{3.3}$$

(iii) *Let* $\nu \in \mathfrak{M}$ *and for* $t > 0$ *define*

$$\nu_t(x) = (1/t)\{N\nu(x) - P_t N\nu(x)\}. \tag{3.4}$$

Then $G\nu_t \to N\nu$ *strongly in* $\mathbf{F}_{(e)}$ *as* $t\downarrow 0$ *and also the measures* $\nu_t \cdot m \to \nu$ *vaguely as* $t\downarrow 0$. □

PROOF. The estimate (2.17) implies that $\alpha G_\alpha f \to 0$ as $\alpha\downarrow 0$ for $f \in \mathbf{F}$ and the extension to general $f \in \mathbf{F}_{(e)}$ follows with the help of the estimate (2.16), proving (i). For (ii) we must proceed indirectly since we have no correspondent to (2.16) for the P_t. We consider first the case when $\nu = \varphi \cdot m$ with φ bounded and integrable and with $G\varphi$ bounded. (The latter is easily achieved with the help of the maximum principle.) For $\varepsilon > 0$, Lemma 2.1(ii) guarantees that $P_t G_\varepsilon \varphi \in \mathbf{F}$ and the estimate

$$E(P_t G_\varepsilon \varphi, P_t G_\varepsilon \varphi) = \int m(dx) P_t G_\varepsilon \varphi(x)\{P_t\varphi(x) - \varepsilon G_\varepsilon P_t \varphi(x)\}$$

$$\leq \{\sup G\varphi(x)\} \int m(dx)\varphi(x)$$

is enough to show that $P_t G\varphi \in \mathbf{F}_{(e)}$. Also since $P_t G\varphi$ is a potential and since $P_t G\varphi \leq G\varphi$

$$E(P_t G\varphi, P_t G\varphi) \leq E(G\varphi, P_t G\varphi)$$

and (3.1) follows from the sectorial estimate (0.3). Now it is routine to extend this much to a general potential $N\nu$ since for every $\alpha > 0$ clearly $\alpha G_\alpha N\nu = G\{N\nu - \alpha G_\alpha N\nu\}$. We observe next that by the estimate (3.1) and by Lemma 3.1(v) in [2], there exists $f_0 \in \mathbf{F}_{(e)}$ such that $P_t N\nu \to f_0$ weakly in $\mathbf{F}_{(e)}$ as $t\downarrow 0$ and also $P_{t_n} N\nu \to f_0$ quasi-everywhere for every sequence $t_n\downarrow 0$. Since $\alpha G_\alpha N\nu$

$\to N\nu$ strongly as $\alpha\uparrow\infty$ it must be that $f_0 = N\nu$ and therefore (3.2) follows from the estimate

$$E(N\nu - P_tN\nu, N\nu - P_tN\nu) \leqslant \int \nu(dx)\{N\nu(x) - P_tN\nu(x)\}.$$

Similarly there exists $f_\infty \in \mathbf{F}_{(e)}$ such that $P_tN\nu \to f_\infty$ weakly in $\mathbf{F}_{(e)}$ as $t\uparrow\infty$ and (i) guarantees that $f_\infty = 0$, which completes the proof of (ii). Strong convergence of $G\nu_t$ to $N\nu$ follows directly from (3.2) and (3.3) since for $T > 0$

$$\int_0^T ds\ P_s\nu_t = (1/t)\int_0^T ds\ P_sN\nu - (1/t)\int_T^{T+t} ds\ P_sN\nu$$

and therefore

$$G\nu_t = (1/t)\int_0^t ds\ P_sN\nu. \tag{3.5}$$

Of course vague convergence of the measures then follows directly from regularity of $\mathbf{F}_{(e)}$. □

REMARK. Weak convergence can be replaced by strong convergence in (3.3) when the generator A is normal. (See the remark following Lemma 2.1.) We do not know if this is true in general. □

The techniques of §4 in [2] can be used to establish the existence of a process $\mathcal{P}_x$, $x \in \mathbf{X} \setminus N$, and also a coprocess $\mathcal{P}_x^*$, $x \in \mathbf{X} \setminus N$. In the remainder of the paper we take for granted the results and also the notation of §4 in [2]. (We indicate in §4 in [1] how to supress the assumption that the reference measure m is excessive. Also the techniques originally used by M. Fukushima [8] can be applied, as is shown in a recent paper by S. Carillo Menendez [14]. Alternatively the reader may prefer to assume that the given semigroup is Feller and invoke the "classical construction" [12, II], thus avoiding any reference to [2], [14], or [8].) Also we will consistently use the notation of [2] and also [3] for hitting times $\sigma(A)$, hitting operators H_α^A and last exit times $\sigma^*(A)$.

The *absorbed semigroup* P_t^D, $t > 0$, and *resolvent* G_α^D, $\alpha > 0$, are defined exactly as §7 of [2]. The arguments of [2] can easily be adapted to show that the sector condition (0.3) is valid also for the absorbed generator and therefore the *absorbed extended Dirichlet space* $\mathbf{F}_{(e)}^D$, E^D is well defined. Moreover, just as in [2]

$$\mathbf{F}_{(e)}^D = \{f \in \mathbf{F}_{(e)}: f = 0 \text{ quasi-everywhere on } M\},$$
$$E^D(g,f) = E(g,f) \quad \text{for } g,f \in \mathbf{F}_{(e)} \tag{3.6}$$

where $M = \mathbf{X} \setminus D$. (We will use this notation in the rest of the paper.) Furthermore the hitting operator H^M maps $\mathbf{F}_{(e)}$ into $\mathbf{F}_{(e)}$ and

$$\mathbf{F}_{(e)} = \mathbf{F}^{D}_{(e)} \oplus H^{M}\mathbf{F}_{(e)},$$
$$E(g, H^{M}f) = 0 \quad \text{for } g \in \mathbf{F}^{D}_{(e)}, f \in \mathbf{F}_{(e)}. \tag{3.7}$$

Thus for general $g, f \in \mathbf{F}_{(e)}$

$$E(g,f) = E(H^{*M}g, H^{M}f) + E(g - H^{*M}g, f - H^{M}f) \tag{3.8}$$

and also

$$E(g, H^{M}f) = E(H^{*M}g, f) = E(H^{*M}g, H^{M}f) = E(H^{M}g, H^{M}f)$$
$$= E(H^{*M}g, H^{*M}f). \tag{3.9}$$

The *balayage operators* π_u^M are defined as in [2]

$$H_u^M N_u \mu = N_u \pi_u^M \mu. \tag{3.10}$$

We finish this section by applying Lemma 2.1(ii) to extend a remarkable result proved by M. Fukushima in the symmetric case [9].

THEOREM 3.2. *Suppose that there exists a quasi-polar set N such thatfor all $x \in \mathbf{X} \setminus N$ the measure $G_1(x; d\cdot)$ is absolutely continuous with respect to m. Then N can be chosen so that also for every $t > 0$ and every $x \in \mathbf{X} \setminus N$ the measure $P_t(x, d\cdot)$ is absolutely continuous with respect to m.* □

PROOF. Just as in [9] a single quasi-polar set N can be chosen so that

$$\mathcal{P}_*[\sigma^+(B) = +\infty] = 1$$

whenever B is quasi-polar and $x \in \mathbf{X} \setminus N$. The point is that Lemma 2.1(ii) guarantees that if A is m-null then $P_t(y, A) = 0$ for quasi-every y and therefore for every $x \in \mathbf{X} \setminus N$

$$P_{2t}(x, A) = \int P_t(x, dy) P_t(y, A) = 0. \quad \square$$

4. An approximate Markov process. Here and in the remainder of the paper D_k, $k \geqslant 1$, is an increasing sequence of open sets which have compact closure and which increase to $\mathbf{X}$. The complements are denoted $M_k = \mathbf{X} \setminus D_k$. For typographic convenience k and $\sim k$ will often be used as labels in place of D_k and M_k. Thus the equilibrium measures μ_k and coequilibrium measures μ_k^* are defined by

$$H^k 1 = N\mu_k; \qquad H^{*k} 1 = N^* \mu_k^*.$$

Just as in [2] the time reversal operator ρ_k and the truncation operator τ_k are defined on the set $\Omega \cap [\sigma(D_k) < +\infty]$ by

$$\rho_k \omega(t) = \begin{cases} \omega(\sigma^*(D_k) - t - 0) & \text{for } 0 \leqslant t < \sigma^*(D_k), \\ \partial & \text{for } t \geqslant \sigma^*(D_k); \end{cases}$$

$$\tau_k \omega(t) = \begin{cases} \omega(t) & \text{for } 0 \leqslant t < \sigma^*(D_k), \\ \partial & \text{for } t \geqslant \sigma^*(D_k). \end{cases}$$

We begin with a technical result which is fundamental for our construction of the approximate Markov process. A proof is included since it is much more involved than the one for the symmetric case. (It is not hard to see that $P_x[\sigma^*(D_k) = +\infty] = 0$ for each k and for quasi-every x. This will be used implicitly below.)

LEMMA 4.1. *For $\xi > 0$ on Ω and for $k, l > 1$*

$$\int \mu_k^*(dx)\mathcal{E}_x[\sigma(D_l) < +\infty; \xi \circ \rho_l] = \int \mu_l(dy)\mathcal{E}_y^*[\sigma(D_k) < +\infty; \xi \circ \tau_k]. \quad \square \tag{4.1}$$

PROOF. We use a superscript or subscript α for the sample space probabilities, etc., when the original semigroup P_t is replaced by $P_t^\alpha = e^{-t\alpha}P_t$. Certainly it suffices to consider the special case when $\xi = f_0(t_0) \cdots f_n(X_{t_n})$ with $0 < t_0 < t_1 < \cdots < t_n$ and with $f_0, \ldots, f_n \in \mathbf{F} \cap C_{\text{com}}$. For $\varphi > 0$ bounded and integrable and for $\alpha > 0$

$$\int m(dx)\varphi(x)\ \mathcal{E}_x^\alpha[\sigma(D_l) < \infty; \xi \circ \rho_l]$$

$$= \int m(dx)\varphi(x)\mathcal{E}_x^\alpha[t_n < \sigma^*(D_l); f_0(X_{\sigma^*(D_l)-t_0}-0) \cdots f_n(X_{\sigma^*(D_l)-t_n}-0)]$$

$$= \operatorname*{Lim}_{p\uparrow\infty} \int m(dx)\varphi(x) \sum_{k=0}^{\infty} P_{k/p}^\alpha f_n \cdots P_{t_1-t_0}^\alpha f_0 P_{t_0}^\alpha (1 - P_{1/p}^\alpha) H_\alpha^l 1(x)$$

$$= \operatorname*{Lim}_{p\uparrow\infty} \int \mu_l^\alpha(dy) \int_0^{1/p} ds\, P_s^{*\alpha} P_{t_0}^{*\alpha} f_0 \cdots f_n \sum_{k=0}^{\infty} P_{k/p}^{*\alpha}\, \varphi(y).$$

As $p\uparrow\infty$ certainly $(1/p)\Sigma_{k=0}^\infty P_{k/p}^{*\alpha}\varphi \to G_\alpha^*\varphi$ in L^2 and therefore

$$f_0 P_{t_1-t_0}^{*\alpha} \cdots f_n (1/p) \sum_{k=0}^{\infty} P_{k/p}^{*\alpha}\varphi \to f_0 P_{t_1-t_0}^{*\alpha} \cdots f_n G_\alpha^* \varphi$$

in L^2. The operators $\{p\int_0^{1/p} ds\, P_s^{*\alpha}\}P_{t_0}^{*\alpha}$ are uniformly bounded from L^2 to $\mathbf{F}$ and therefore converge to $P_{t_0}^{*\alpha}$ in the strong operator topology. Thus

$$\int_0^{1/p} ds\, P_s^{*\alpha} P_{t_0}^{*\alpha} f_0 \cdots f_n \sum_{k=0}^{\infty} P_{k/p}^{*\alpha}\varphi \to P_{t_0}^{*\alpha} f_0 \cdots f_n G_\alpha^* \varphi$$

strongly in $\mathbf{F}$ and since $\mu_l^\alpha \in \mathfrak{M}_\alpha$ we can conclude that

$$\int m(dx)\varphi(x)\mathcal{E}_x^\alpha[\sigma(D_k) < +\infty; \xi \circ \rho_l] = \int \mu_l^\alpha(dy) P_{t_0}^{*\alpha} f_0 \cdots f_n G_\alpha^* \varphi(y)$$

which is enough to imply

$$\mathcal{E}_x^\alpha[\sigma(D_l) < +\infty; \xi \circ \rho_l] = N_\alpha \mu_l^{\alpha\sim}(x) \tag{4.2}$$

where $\mu_l^{\alpha\sim}$ is the unique measure in $\mathfrak{M}_\alpha$ determined by

$$\int \mu_l^{\alpha\sim}(dx)\psi(x) = \int \mu_l^{\alpha}(dx)P_{t_0}^{*\alpha}f_0 \cdots f_n\psi(y).$$

Next we pass to the limit $\alpha\downarrow 0$ in (4.2). It is easy to see that for quasi-every x the left side converges to the corresponding expression with α suppressed. Since $H_\alpha^l 1 = (1 - \alpha G_\alpha)H^l 1$, it follows from Lemma 2.1(i) that the functions $H_\alpha^l 1$ are uniformly bounded in E norm and since $H_\alpha^l 1 \to H^l 1$ quasi-everywhere as $\alpha\downarrow 0$ it must be that $H_\alpha^l 1 \to H^l 1$ weakly in the Hilbert space $\mathbf{F}_{(e)}$. But then for $g \in \mathbf{F} \cap C_{\text{com}}$

$$\int \mu_l^\alpha(dx)g(x) = E(g, H_\alpha^l 1) + \alpha(g, H_\alpha^l 1) \to E(g, H^l 1) = \int \mu_l(dx)g(x). \tag{4.3}$$

Thus $\mu_l^\alpha \to \mu_l$ vaguely and the norms $E_\alpha(H_\alpha^l 1,\ H_\alpha^l 1) = \mu_l^\alpha(D_l^-)$ remain bounded. This implies that also the norms $E_\alpha(N_\alpha\mu_l^{\alpha\sim},\ N_\alpha\mu_l^{\alpha\sim})$ and therefore also $E(N_\alpha\mu_l^{\alpha\sim},\ N_\alpha\mu_l^{\alpha\sim})$ remain bounded. Since $\mu_l^{\alpha\sim} \to \mu_l^\sim$ vaguely we can conclude from this that $N_\alpha\mu_l^{\alpha\sim} \to N\mu_l^\sim$ weakly in $\mathbf{F}_{(e)}$. This permits us to pass to the limit $\alpha\downarrow 0$ in (4.2) and get

$$\mathcal{E}_x[\sigma(D_l) < +\infty; \xi \circ \rho_l] = N\mu_l^\sim(x). \tag{4.4}$$

Finally, the left side of (4.1)

$$\begin{aligned} &= \int \mu_k^*(dx)N\mu_l^\sim(x) \\ &= \int \mu_l^\sim(dx)N^*\mu_k(x) \\ &= \int \mu_l(dx)P_{t_0}^* f_0 \cdots f_n H^{*k}1(x) \end{aligned}$$

which equals the right side. □

Now it is routine to carry over the construction in §5 of [2]. Let the extended sample space Ω_∞, the trajectory variables X_t, first hitting time $\sigma(A)$ and last exit time $\sigma^*(A)$, birth time ζ^* and death time ζ and time reversal operator ρ be defined exactly as in [2]. Then with the shift $\theta_{\sigma(k)}$ mapping Ω_∞ into Ω as on p. 6.5 in [2], we have

THEOREM 4.2. *There exist unique measures $\mathcal{P}$, $\mathcal{P}^*$ on Ω_∞ determined by*

$$\mathcal{E}[\sigma(D_k) < +\infty; \xi \circ \theta_{\sigma(k)}] = \int \mu_k^*(dx)\ \mathcal{E}_x\xi,$$

$$\mathcal{E}^*[\sigma(D_k) < +\infty; \xi \circ \theta_{\sigma(k)}] = \int \mu_k(dy)\mathcal{E}_y^*\xi \tag{4.5}$$

for $\xi \geq 0$ on Ω and for every k. Moreover for $\xi \geq 0$ on Ω_∞

$$\mathcal{E}\xi \circ \rho = \mathcal{E}^*\xi \tag{4.6}$$

and for $\varphi \geq 0$ on $\mathbf{X}$

$$\mathcal{E}\int_{\zeta^*}^{\zeta} dt\ \varphi(X_t) = \mathcal{E}^*\int_{\zeta^*}^{\zeta} dt\ \varphi(X_t) = \int m(dx)\varphi(x). \tag{4.7}$$

Also for $A \subset \mathbf{X}$ *Borel*

$$C_{ap}(A) = \mathcal{P}[\sigma(A) < +\infty] = \mathcal{P}^*[\sigma(A) < +\infty]. \quad \square \tag{4.8}$$

The continuous increasing additive functionals $a(\nu; t)$ are defined just as in §6 in [2] for the symmetric case and again it is true that

$$\mathcal{E}a(\nu; \zeta) = \mathcal{E}^*a(\nu; \zeta) = \int \nu(dx), \tag{4.9}$$

$$\tfrac{1}{2}\mathcal{E}a(\nu; \zeta)^2 = \tfrac{1}{2}\mathcal{E}^*a(\nu; \zeta)^2 = \mathcal{E}(\nu). \tag{4.10}$$

The martingale functionals Mf together with the square functionals $\langle Mf\rangle$ and $\langle M_c f\rangle$ are defined as in [2] so that

$$E(f,f) = \tfrac{1}{2}\ \mathcal{E}Mf(\zeta)^2 = \tfrac{1}{2}\ \mathcal{E}\ \int_{\zeta^*}^{\zeta} \langle Mf\rangle(dt). \tag{4.11}$$

Also the comartingale functionals M^*f and $\langle M^*f\rangle$, $\langle M_c f^*\rangle$ are defined so that

$$E(f,f) = \tfrac{1}{2}\mathcal{E}^*\{M^*f(\zeta)\}^2 = \tfrac{1}{2}\mathcal{E}^*\int_{\zeta^*}^{\zeta} \langle M^*f\rangle(dt). \tag{4.12}$$

We do not know if there are corresponding expressions for $E(g,f)$ when $g \neq f$.

5. Beurling-Deny decomposition. The *Killing measure* κ and *cokilling measure* κ^* are the unique Radon measures on $\mathbf{X}$ such that

$$\mathcal{E}[X_{\zeta-0} \in \mathbf{X};\ \varphi(X_{\zeta-0})] = \int \kappa(dx)\varphi(x),$$

$$\mathcal{E}[X_{\zeta^*} \in \mathbf{X};\ \varphi(X_{\zeta^*})] = \int \kappa^*(dx)\varphi(x). \tag{5.1}$$

It is routine to show also that for quasi-every x

$$\mathcal{E}_x[X_{\zeta-0} \in \mathbf{X};\ \varphi(X_{\zeta-0})] = N\varphi\cdot\kappa(x),$$

$$\mathcal{E}_x^*[X_{\zeta-0} \in \mathbf{X};\ \varphi(X_{\zeta-0})] = N^*\varphi\cdot\kappa^*(x). \tag{5.2}$$

More generally, if we identify the subset $[X_{\zeta^*} \in X]$ of Ω_∞ with Ω in the obvious way, then for $\xi \geqslant 0$ on Ω_∞

$$\mathcal{E}[X_{\zeta^*} \in \mathbf{X};\ \xi] = \int \kappa^*(dx)\mathcal{E}_x\xi,$$

$$\mathcal{E}^*[X_{\zeta^*} \in \mathbf{X};\ \xi] = \int \kappa(dx)\mathcal{E}_x^*\xi. \tag{5.3}$$

Finally if D is open and if $M = \mathbf{X}\setminus D$, then for quasi-every x in D

$$\mathcal{E}\left[\sigma(M) = +\infty; X_{\zeta-0} \in D; \varphi(X_{\zeta-0})\right] = N^D\varphi \cdot \kappa(x),$$
$$\mathcal{E}^*\left[\sigma(M) = +\infty; X_{\zeta-0} \in D; \varphi(X_{\zeta-0})\right] = N^{*D}\varphi \cdot \kappa^*(x). \tag{5.4}$$

It follows from (4.11) and from Meyer's results on decomposing square-integrable martingales [15] that for $f \in \mathbf{F}_{(e)}$

$$\tfrac{1}{2}\mathcal{E}\sum_t I(X_{t-0} \neq X_t; X_{t-0}, X_t \in \mathbf{X})\{f(X_t) - f(X_{t-0})\}^2$$
$$< \tfrac{1}{2}\mathcal{E}Mf(\zeta)^2 = E(f,f). \tag{5.5}$$

(This depends on the fact that the processes $Mf(t)$ and $f(X_t)$ have exactly the same discontinuities. See Theorem 6.4(i) in [2].) This estimate establishes the existence of a unique Radon measure $J(dx, dy)$ on $\mathbf{X} \times \mathbf{X} \setminus$ (diagonal) such that for $F > 0$ on $\mathbf{X} \times \mathbf{X}$

$$\mathcal{E}\sum_t I(X_{t-0} \neq X_t; X_{t-0}, X_t \in \mathbf{X})F(X_{t-0}, X_t) = \iint J(dx, dy)F(x,y). \tag{5.6}$$

Also for any $f \in \mathbf{F}_{(e)}$

$$\frac{1}{2}\iint J(dx, dy)\{f(x) - f(y)\}^2 < E(f,f). \tag{5.7}$$

Of course we will call J the Lévy measure. Exactly as in §10 in [2] we have for quasi-every $x \in D$

$$\mathcal{E}_x\left[\sigma(M) < \zeta; X_{\sigma(M)-0} \in D; \varphi(X_{\sigma(M)-0})\psi(X_{\sigma(M)})\right]$$
$$= N^D\left\{\varphi \cdot \int J(\cdot, dy)\psi(y)\right\}(x) \tag{5.8}$$

and for quasi-every $x \in \mathbf{X}$

$$\mathcal{E}_x\sum_t I(X_{t-0} \neq X_t; X_{t-0}, X_t \in \mathbf{X})\varphi(X_{t-0})\psi(X_t)$$
$$= N\left\{\varphi \cdot \int J(\cdot, dy)\psi(y)\right\}(x). \tag{5.9}$$

The killing and cokilling measures can be recovered from

$$\int \kappa(dx)f(x) = \operatorname*{Lim}_{\alpha\uparrow\infty} \alpha(1 - \alpha G_\alpha 1, f),$$
$$\int \kappa^*(dx)f(x) = \operatorname*{Lim}_{\alpha\uparrow\infty} \alpha(f, 1 - \alpha G_\alpha^* 1), \tag{5.10}$$

which is valid if f is quasi-continuous and bounded with compact support or if f is the square of a function in $\mathbf{F}_{(e)}$. (See §11 in [2].) The Lévy measure can be recovered from

$$E(g,f) = -\iint J(dx,dy)g(x)f(y) \tag{5.11}$$

which is valid whenever f, $g \in \mathbf{F}_{(e)}$ have disjoint supports. To prove (5.11) choose D open and containing the support of g such that the closure D^- is disjoint from the set where $f > 0$. Then $H^M f = N^D \nu$ on D where $\nu = \int J(\cdot, dy)f(y)$ and therefore

$$E(g,f) = E(g, f - H^M f) = -E^D(g, N^D \nu) = -\int \nu(dx)g(x)$$

which is equivalent to (5.11).

Let $f \in \mathbf{F}_{(e)}$ be bounded and let $g \in \mathbf{F} \cap C_{\text{com}}$. For $\alpha > 0$

$$\begin{aligned} E^\alpha(gf,f) - \tfrac{1}{2}E^\alpha(g,f^2) &= \alpha(gf, f - \alpha G_\alpha f) - \tfrac{1}{2}\alpha(g, f^2 - \alpha G_\alpha f^2) \\ &= \tfrac{1}{2}\alpha(g, f^2\alpha G_\alpha 1 + \alpha G_\alpha f^2 - 2fG_\alpha f) + \tfrac{1}{2}\alpha(gf^2, 1 - \alpha G_\alpha 1) \\ &= \frac{1}{2}\int m(dx)g(x)\langle A_\alpha f\rangle(x) + \frac{1}{2}\alpha\int m(dx)g(x)f^2(x)\{1 - \alpha G_\alpha 1(x)\}, \end{aligned} \tag{5.12}$$

where we have defined

$$\langle A_\alpha f\rangle(x) = \alpha^2\int G_\alpha(x,dy)\{f(x) - f(y)\}^2. \tag{5.13}$$

Thus there exists a unique Radon measure $\langle Af\rangle(dx)$ such that

$$\langle A_\alpha f\rangle(x)m(dx) \to \langle Af\rangle(dx) \quad \text{vaguely}, \tag{5.14}$$

$$\frac{1}{2}\int \langle Af\rangle(dx)g(x) + \frac{1}{2}\int \kappa(dx)g(x)f^2(x) = E(gf,f) - \frac{1}{2}E(g,f^2). \tag{5.15}$$

Just as in §11 in [2] it can be shown that $\langle Af\rangle$ charges no quasi-polar set and that

$$\int m(dx)\langle A_\alpha f\rangle(x)g(x) \to \int \langle Af\rangle(dx)g(x) \tag{5.14'}$$

for any bounded $g \in \mathbf{F}_{(e)}$. Also

$$\begin{aligned} \langle Mf\rangle(dt) &= a(f^2\cdot\kappa + \langle Af\rangle), \\ \langle M^*f\rangle(dt) &= a(f^2\cdot\kappa^* + \langle A^*f\rangle), \end{aligned}$$

and if ν is a bounded Radon measure charging no polar set such that $N^*\nu$ is bounded, then

$$\int \nu(dx)\mathcal{E}_x\{Mf(\zeta) - Mf(0)\}^2$$

$$= \operatorname*{Lim}_{\alpha\uparrow\infty} \int \nu(dx)\int_0^\infty dt\, \alpha^2 e^{-\alpha t}\mathcal{E}_x\int_0^\zeta ds\{f(X_{t+s}) - f(X_s)\}^2$$

$$= \lim_{\alpha\uparrow\infty} \int \nu(dx)G\{f^2\alpha(1-\alpha G_\alpha 1) + \langle A_\alpha f\rangle\}(x). \tag{5.16}$$

If $N\nu$ is bounded then the dual version of (5.16) is valid. (Our notation here is inconsistent with [2] since in [2] we worked with the semigroup rather than the resolvent. But the necessary adaptations are obvious.) Of course (5.15) immediately implies

$$E(f,f) = \frac{1}{2}\int \langle Af\rangle(dx) + \frac{1}{2}\int \kappa^*(dx)f^2(x) + \frac{1}{2}\int \kappa(dx)f^2(x)$$

$$= \frac{1}{2}\int\langle A^*f\rangle(dx) + \frac{1}{2}\int \kappa^*(dx)f^2(x) + \frac{1}{2}\int \kappa(dx)f^2(x) \tag{5.17}$$

and this enables us to extend the definition of $\langle Af\rangle$ and $\langle A^*f\rangle$ to general $f \in \mathbf{F}_{(e)}$.

In general it is not true that $\langle Af\rangle = \langle A^*f\rangle$, although the total mass must be the same. This can be seen very easily by considering the case when $\mathbf{X}$ is a finite set. However, it turns out that the continuous parts $\langle A_c f\rangle$ and $\langle A_c^* f\rangle$, defined by

$$\int \langle Af\rangle(dx)\psi(x) = \int \langle A_c f\rangle(dx)\psi(x) + \iint J(dx,dy)\psi(x)\{f(x)-f(y)\}^2,$$

$$\int \langle A^*f\rangle(dy)\psi(y) = \int \langle A_c^*f\rangle(dy)\psi(y)$$

$$+\iint J(dx,dy)\{f(x)-f(y)\}^2\psi(y) \tag{5.18}$$

are equal. The technique of §13 in [2] can be used to establish for $f \in \mathbf{F}_{(e)}$ the identity

$$\frac{1}{2}\int_D \langle A_c f\rangle(dx) + \frac{1}{2}\int_D \kappa^*(dx)f^2(x) + \frac{1}{2}\int_D \kappa(dx)f^2(x)$$

$$+\frac{1}{2}\iint_{(D\times D\cup D\times M\cup M\times D)} J(dx,dy)\{f(x)-f(y)\}^2$$

$$= E(f - H^{*M}f, f - H^M f) + \tfrac{1}{2}\mathcal{E}\sum_i \{f(X_{r(i)}) - f(X_{e(i)-0})\}^2$$

$$+\tfrac{1}{2}\mathcal{E} I(X_{\zeta^*\in D})f^2(X_{\sigma(M)}) \tag{5.19}$$

where entrance times $e(i)$ and return times $r(i)$ for excursions into D are defined as in [2]. The identity

$$\langle A_c f\rangle(dx) = \langle A_c^* f\rangle(dx) \tag{5.20}$$

follows from (5.19) after applying time reversal and varying D. Also it can be shown that

$$\langle M_c f\rangle(dt) = \langle M_c^* f\rangle(dt) = a(\langle A_c f\rangle; dt) \tag{5.21}$$

and if f' is a normalized contraction of f, then

$$\langle A_c f'\rangle(dx) \leq \langle A_c f\rangle(dx). \tag{5.22}$$

We define the *diffusion form*

$$D(f,f) = \frac{1}{2} \int \langle A_c f\rangle(dx) \tag{5.23}$$

and just as for the symmetric case we have the *Beurling-Deny* decomposition

$$E(f,f) = D(f,f) + \frac{1}{2} \iint J(dx,dy)\{f(x) - f(y)\}^2$$
$$+ \frac{1}{2} \int \kappa^*(dx) f^2(x) + \frac{1}{2} \int \kappa(dx) f^2(x). \tag{5.24}$$

Also the appropriate analogue to Theorem 11.10 in [2] is valid. In general there is no such decomposition for $E(g,f)$ when $g \neq f$. This can be seen already by considering the case of infinitely divisible processes on the line.

6. The reflected space. We say that *bounded* f belongs to the *local Dirichlet space* $\mathbf{F}_{\text{loc}}$ if each $x \in \mathbf{X}$ has a neighborhood U for which there exists $f_u \in \mathbf{F}$ such that $f = f_u$ quasi-everywhere on U. From the appropriate analogue of (11.23) in [2] it follows that for any such f there exists a unique Radon measure $\langle A_c f\rangle(dx)$ such that $\langle A_c f\rangle(dx) = \langle A_c f_u\rangle(dx)$ on U. For unbounded f we proceed indirectly. Let the truncation $\tau_n f$ be defined by

$$\tau_n f(x) = \begin{cases} f(x) & \text{if } |f(x)| < n, \\ n \operatorname{sgn} f(x) & \text{if } |f(x)| \geq n. \end{cases} \tag{6.1}$$

If each $\tau_n f \in \mathbf{F}_{\text{loc}}$, then by (5.22) the measures $\langle A_c \tau_n f\rangle(dx)$ increase with n and therefore

$$\langle A_c f\rangle(dx) = \operatorname*{Lim}_{n\uparrow\infty} \langle A_c \tau_n f\rangle(dx) \tag{6.2}$$

exists. We say that general (possibly unbounded) f belongs to $\mathbf{F}_{\text{loc}}$ if each truncation $\tau_n f \in \mathbf{F}_{\text{loc}}$, if $\langle A_c f\rangle(dx)$ is Radon, if f is locally in $L^2(\kappa)$ and if

$$\iint J(dx,dy)\{f(x) - f(y)\}^2 < +\infty$$

whenever U has compact closure.

Here and in the remainder of the paper we work under

6.1. TECHNICAL ASSUMPTION. If $f \in \mathbf{F}_{\text{loc}}$ and if U is open with compact closure, then there exists $f_u \in \mathbf{F}_{(e)}$ such that $f = f_u$ quasi-everywhere on U. □

REMARK. The point of 6.1 is this. If f is locally excessive in the obvious sense and if we have control over the terms of the Beurling-Deny decomposition, then we want to be sure that f can be approximated by $\mathbf{F}$ in the sense of 6.1. Lemma 14.1 in [2] shows that 6.1 is vacuous for bounded f. We do not believe that this is also the case for unbounded f (although we do believe that any counterexample would be pathological). Thus the treatment in §14 of [2] is incorrect as it stands. However everything can be fixed by defining $\mathbf{F}_{\mathrm{loc}}$ as above and postulating 6.1. □

6.2. DEFINITION. $f \in \mathbf{F}_{\mathrm{loc}}$ belongs to the *symmetrized reflected space* $\mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$ if

$$E(f,f) = \frac{1}{2} \int \langle A_c f\rangle(dx) + \frac{1}{2}\int \{\kappa^*(dx) + \kappa(dx)\} f^2(x) + \frac{1}{2}\iint J(dx, dy)\{f(x) - f(y)\}^2 \tag{6.3}$$

converges. □

The martingale and comartingale functionals $Mf(t)$ and $M^*f(t)$ can be defined for $f \in F_{\mathrm{ref}}^{\mathrm{symm}}$ just as in the symmetric case (p. 14.2 in [2]) and we take this for granted below.

Fix once and for all a metric d on $\mathbf{X}$ and for $\varepsilon > 0$ let

$$J_\varepsilon(dx, dy) = J(dx, dy) I(d(x,y) > \varepsilon). \tag{6.4}$$

Clearly

$$J_\varepsilon(D, \mathbf{X}) + J_\varepsilon(\mathbf{X}, D) < +\infty \tag{6.5}$$

whenever D is open with compact closure. Consider one such D, let $g \in \mathbf{F}_{(e)}^{D}$ (the extended absorbed space for D), let $f \in \mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$ and let $f^\# \in \mathbf{F}$ be such that $f = f^\#$ quasi-everywhere in an ε-neighborhood on D. Then

$$E(g,f) = E(g,f^\#) + \iint J_\varepsilon(dx, dy) g(x)\{f(x) - f(y) - f^\#(x) + f^\#(y)\} \tag{6.6}$$

is well defined independent of the choice of $f^\#$ and satisfies the two estimates

$$|E(g,f)| < E(g,g)^{1/2}\left\{ME(f^\#,f^\#)^{1/2} l + \|f - f^\#\|_\infty J_\varepsilon(D,\mathbf{X})\right\}, \tag{6.7}$$

$$|E(g,f)| < 3ME(g,g)^{1/2}\left\{E(f^\#,f^\#)^{1/2} + E(f,f)^{1/2}\right\}. \tag{6.8}$$

The two estimates are obvious. That $E(g,f)$ is well defined follows since if $f_1^\#, f_2^\#$ are two such functions, then by (5.11)

$$\begin{aligned} E(g,f_1^\#) - E(g,f_2^\#) &= \iint J(dx, dy) g(x)\{f_2^\#(y) - f_1^\#(y)\} \\ &= \iint J_\varepsilon(dx, dy) g(x)\{f_1^\#(x) - f_1^\#(y) - f_2^\#(x) + f_2^\#(y)\}. \end{aligned}$$

Now asume that f is bounded and choose uniformly bounded f_n in $\mathbf{F}_{(e)}$ such that each $f_n = f$ in an ε-neighborhood of D and such that $f_n \to f$ quasi-everywhere. Then $f_n - H^M f_n$ belongs to $\mathbf{F}^D$ and (6.7) is valid with $g = f_n - H^M f_n$ and f replaced by f_n. Since $E(f_n - H^M f_n, f_n) = E(f_n - H^M f_n, f_n - H^M f_n)$, we get an estimate

$$E(f_n - H^M f_n, f_n - H^M f_n) < E(f_n - H^M f_n, f_n - H^M f_n)^{1/2}$$
$$\times \left\{ ME(f^\#, f^\#)^{1/2} + \|f_n - f^\#\|_\infty J_\varepsilon(D, \mathbf{X}) \right\}$$

which is enough to guarantee that the left side remains bounded as $n \uparrow \infty$ and therefore $f - H^M f \in \mathbf{F}^D$. Also it is clear that $H^M f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ and that

$$E(g, H^M f) = 0 \quad \text{for } g \in \mathbf{F}_{(e)}^D. \tag{6.9}$$

Denoting $f - H^M f$ by f^D, we can apply (6.8) to get

$$E(f^D, f^D) < 3ME(f^D, f^D)^{1/2} \left\{ E(f^\#, f^\#)^{1/2} + E(f, f)^{1/2} \right\} \tag{6.10}$$

and this enables us to remove the *a priori* restriction that f be bounded, by taking truncations and passing to the limit. We summarize in

THEOREM 6.1. *Let D be open with compact closure, let $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ and let $f^\# \in \mathbf{F}_{(e)}$ be such that $f = f^\#$ quasi-everywhere on D.*

(i) *If $g \in \mathbf{F}_{(e)}^D$, then $E(g, f)$ is well defined by* (6.6).

(ii) *f has a local decomposition*

$$f = f^D + H^M f \tag{6.11}$$

with $f^D \in \mathbf{F}_{(e)}^D$ and with $H^M f$ satisfying (6.9). *Moreover, f^D satisfies the estimate* (6.10). □

Combining Theorem 6.1 with the local dual decomposition, we get

$$E(f, f) = E(f_D^*, f_D) + E(H^{*M} f, H^M f). \tag{6.12}$$

Since also $E(H^{*M} f, H^M f) = E(H^M f, H^M f)$ which is nonnegative, (6.12) implies the inequality

$$E(f_D^*, f_D) < E(f, f). \tag{6.13}$$

In the symmetric case $f_D^* = f_D$ and (6.13) allows us to pass to the limit in D and get a global decomposition. (Also see Theorem 14.4 in [2].) We do not know if this can be done in general. To get around this gap in our knowledge we single out those $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ for which a global decomposition does exist.

6.3. DEFINITION. $f \in \mathbf{F}_{\text{ref}}^{\text{symm}}$ belongs to the *reflected Dirichlet space* $\mathbf{F}_{\text{ref}}$ if the norms $E(H^{*M} f, H^M f)$ remain bounded as D runs over the open subsets of $\mathbf{X}$ with compact closure. □

6.4. DEFINITION. A function h is *harmonic* if it is specified and finite quasi-everywhere and if

$$H^M f(x) = f(x) \tag{6.14}$$

for quasi-every $x \in D$ whenever D is open with compact closure. (It is understood that the left side of (6.14) must converge absolutely for quasi-every $x \in D$.) □

THEOREM 6.2. *Every $f \in \mathbf{F}_{\mathrm{ref}}$ has unique decompositions*

$$f = f_0 + h; \qquad f = f_0^* + h^* \tag{6.15}$$

with $f_0, f_0^* \in \mathbf{F}$ and h, h^* harmonic and coharmonic. □

PROOF. If $f \in \mathbf{F}_{\mathrm{ref}}$, then by the triangle inequality

$$E(f^D, f^D)^{1/2} < E(f,f)^{1/2} + E(H^M f, H^M f)^{1/2}$$
$$= E(f,f)^{1/2} + E(H^{*M} f, H^M f)^{1/2}$$

and therefore f has a global decomposition $f = f_0 + h$ where $f_0 \in \mathbf{F}_{(e)}$ and where $h \in \mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$ satisfies

$$E(g, h) = 0 \tag{6.16}$$

whenever $g \in \mathbf{F}_{(e)}^D$ for some D as above. If f is bounded then it is also clear that h is harmonic and we are done. In order to treat unbounded f, we establish an identity which will also play an important role in the remainder of the paper. Fix bounded $f \in \mathbf{F}_{\mathrm{ref}}$. Just as for the symmetric case (Proposition 4.5 in [2]), $f_0(X_t) \to 0$ and $f_0^*(X_t) \to 0$ as $t \uparrow \zeta$ when $X_{\zeta-0} = \partial$ and therefore by the martingale convergence theorem $\Phi = \mathrm{Lim}_{t\uparrow\zeta} f(X_t)$ is well defined on $[X_{\zeta-0} = \partial]$ and we get the same result if f is replaced by h or h^*. (Of course such statements are understood to be true [a.e. $\mathcal{P}_x$] and also [a.e. $\mathcal{P}_x^*$] for quasi-every x.) By convention we set $\Phi = 0$ on the set $[X_{\zeta-0} \in \mathbf{X}]$. It is clear that $Mh(t) = h(X_t)$ when $t < \zeta$ and that $Mh(\zeta) = \Phi$. Thus for each k

$$\tfrac{1}{2}\mathcal{E} I[\sigma(D_k) < +\infty]\{Mh(\zeta) - Mh(X_{\sigma(D_k)})\}^2$$
$$= \tfrac{1}{2}\mathcal{E} I[\sigma(D_k) < +\infty]\{\Phi - h(X_{\sigma(D_k)})\}^2$$

and after passing to the limit in k we get

$$\tfrac{1}{2}\mathcal{E}\{Mh(\zeta) - Mh(\zeta^*)\}^2 = \tfrac{1}{2}\mathcal{E} I(X_{\zeta^*} = \partial)\{\Phi - \Phi \circ \rho\}^2$$
$$+ \tfrac{1}{2}\mathcal{E} I(X_{\zeta^*} \in \mathbf{X})\{\Phi - h(X_{\zeta^*})\}^2. \tag{6.17}$$

On the one hand

$$\frac{1}{2}\,\mathcal{E} I(X_{\zeta^*} \in \mathbf{X})\Phi^2 = \frac{1}{2}\,\mathcal{E} I(X_{\zeta^*} \in \mathbf{X})\{\Phi - h(X_{\zeta^*})\}^2 + \frac{1}{2}\int \kappa^*(dx) h(x)^2$$

and on the other hand it is clear from (5.15), (5.18) and (6.3) that

$$\frac{1}{2}\ \mathcal{E}\left[Mh(\zeta) - Mh(\zeta^*)\right]^2 = \frac{1}{2}\ \mathcal{E}\int_{\zeta^*}^{\zeta}\langle Mh\rangle(dt)$$
$$= E(h, h) - \frac{1}{2}\int \kappa^*(dx)h(x)^2$$

and so (6.17) is equivalent to

$$E(h, h) = \tfrac{1}{2}\mathcal{E}\{\Phi - \Phi \circ \rho\}^2 \tag{6.18}$$

which is the desired identity. It follows in particular from (6.18) that if f' is a normalized contraction of f, then

$$E(h', h') \leq E(h, h). \tag{6.19}$$

A similar argument working instead with the absorbed process for D establishes for general $f \in \mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$

$$E(H^{*M}f', H^{M}f') \leq E(H^{*M}f, H^{M}f). \tag{6.20}$$

Now we are ready to complete the proof of the theorem. First (6.20) implies that if $f \in \mathbf{F}_{\mathrm{ref}}$, then also $f^+ \in \mathbf{F}_{\mathrm{ref}}$ and we can assume $f \geq 0$. Again by (6.20) the truncations $\tau_n f$ belong to $\mathbf{F}_{\mathrm{ref}}$ and the limits $f_0^\sim$-Lim$(\tau_n f)_0$, $h^\sim = \mathrm{Lim}\{\tau_n f - (\tau_n f)_0\}$ exist in $\mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$. Since $h^\sim$ is harmonic, we will be done if we can show that $h = h^\sim$ or equivalently if $f_0 - f_0^\sim = 0$. But this is true since on the one hand $f_0 - f_0^\sim \in \mathbf{F}_{(e)}$ and so there exists a sequence g_n, $n \geq 1$, with each $g_n \in \mathbf{F}^D$ for some D and with $g_n \to f_0 - f_0^\sim$ relative to the E norm while on the other hand $f_0 - f_0^\sim = h^\sim - h$ and therefore $E(g_n, f_0 - f_0^\sim) = 0$ for every n. □

We will only be interested in those $f \in \mathbf{F}_{\mathrm{ref}}$ for which the variable Φ introduced above vanishes on the set $[\zeta = +\infty]$ and satisfies

$$\int dx\left\{\mathcal{E}_x e^{-\zeta}\Phi^2 + \mathcal{E}_x^* e^{-\zeta}\Phi^2\right\} < +\infty. \tag{6.21}$$

Therefore we explicitly introduce the collection of all such Φ for which (6.21) does converge. It is easy to see that there exists a countable subcollection $\mathcal{H}_0$ of bounded functions in $\mathcal{H}$ satisfying the following two conditions.

6.5.1. $\mathcal{H}_0$ is an algebra over the rationals.

6.5.2. If $\Phi \in \mathcal{H}$, then there exists a sequence Φ_n, $n \geq 1$, in $\mathcal{H}_0$ such that

$$\int dx\ \left\{\mathcal{E}_x e^{-\zeta}(\Phi - \Phi_n)^2 + \mathcal{E}_x^* e^{-\zeta}(\Phi - \Phi_n)^2\right\} \to 0,$$
$$\mathcal{E}\left\{(\Phi - \Phi_n) - (\Phi - \Phi_n) \circ \rho\right\}^2 \to 0. \qquad \square$$

Both in condition 6.5.2 and in the proof of Theorem 6.2 we have viewed the variables Φ as being simultaneously defined on Ω and Ω_∞. Just as for the symmetric case (see p. 3.9 in [3] and p. 20.1 in [2]) there is no real loss of generality in assuming that also $\mathbf{H}_0$ can be chosen to satisfy

6.5.3. If $\Phi \in \mathcal{H}_0$, then the functions $\mathcal{E}_{\cdot}\Phi$ and $\mathcal{E}_{\cdot}^{*}\Phi$ agree quasi-everywhere with continuous functions on $\mathbf{X}$. □

Now we are ready to introduce our boundary. Let Δ be the collection of continuous nontrivial homomorphisms on $\mathcal{H}_0$ and let δ be the trivial one. $\mathcal{H}_0$ is viewed also as a function space on $\mathbf{X} \cup \Delta \cup \{\delta\}$ in the obvious way and then $\mathbf{X} \cup \Delta \cup \{\delta\}$ is given the coarsest topology which contains the given topology on $\mathbf{X}$ together with the topology generated by $\mathcal{H}_0$. Because of 6.5.3 the subspace topology on $\mathbf{X}$ agrees with the original topology. In general $\mathbf{X} \cup \Delta \cup \{\delta\}$ need not be compact or even locally compact, but this makes no difference. From the martingale convergence theorem it follows that on the set $[X_{\zeta-0} = \partial]$ the limit $X_\zeta^{\text{ref}} = \operatorname{Lim}_{t\uparrow\zeta} X_t$ is well defined as a point in $\Delta \cup \{\delta\}$ (modulo the usual exceptional set of sample paths) by the identity

$$\Phi(X_\zeta^{\text{ref}}) = \Phi, \qquad \Phi \in \mathcal{H}_0, \tag{6.22}$$

where on the right side Φ is viewed as a variable on Ω or on Ω_∞ and on the left Φ is viewed as a function on Δ. By convention we put $X_\zeta^{\text{ref}} = \delta$ on the set where $X_{\zeta-0} \in \mathbf{X}$. Of course $X_{\zeta^*}^{\text{ref}}$ is defined in a similar way on Ω_∞.

The boundary reference measure ν is defined by

$$\int_\Delta \nu(d\xi)\varphi(\xi) = \frac{1}{2} \int dx \left\{ \mathcal{E}_x e^{-\zeta} \varphi(X_\zeta^{\text{ref}}) + \mathcal{E}_x^* e^{-\zeta} \varphi(X_\zeta^{\text{ref}}) \right\}. \tag{6.23}$$

The global hitting operators H, H_α are defined by

$$H\varphi(x) = \mathcal{E}_x \varphi(X_\zeta^{\text{ref}}),$$
$$H_\alpha \varphi(x) = \mathcal{E}_x e^{-\alpha\zeta} \varphi(X_\zeta^{\text{ref}}). \tag{6.24}$$

By convention any φ on Δ satisfies $\varphi(\delta) = 0$ so that in particular

$$H1(x) = \mathcal{P}_x\left[X_\zeta^{\text{ref}} \in \Delta\right]. \tag{6.25}$$

Notice also that automatically

$$\mathcal{P}_x\left[\zeta = +\infty; X_{\zeta-0}^{\text{ref}} \in \Delta\right] = 0. \tag{6.26}$$

We say that $f \in \mathbf{F}_{\text{ref}}$ is *admissible* if the variable Φ of Theorem 6.2 vanishes on $[\zeta > +\infty]$ and satisfies (6.21). It is clear then from 6.5.2 that $f \in \mathbf{F}_{\text{ref}}$ is *admissible* if and only if (6.15) can be replaced by

$$f = f_0 + H\gamma f; \qquad f = f_0 + H^*\gamma f \tag{6.15$'$}$$

with $\gamma f \in L^2(\Delta, \nu)$ (necessarily unique). Moreover for such f the identity (6.18) can be written

$$E(H\gamma f, H\gamma f) = \tfrac{1}{2} \mathcal{E}\left\{\gamma f(X_\zeta^{\text{ref}}) - \gamma f(X_{\zeta^*}^{\text{ref}})\right\}^2 \tag{6.18$'$}$$

and (6.19) is equivalent to

$$E(HT\gamma f, HT\gamma f) \leqslant E(H\gamma f, H\gamma f) \tag{6.19$'$}$$

where T is a normalized contraction in the sense of (1.12) in [2].

Now we introduce some additional machinery as in §20 of [2] and §3 of [3]. It follows from (6.23) that H_1 as well as H_α for $\alpha > 0$ is bounded from $L^2(\Delta, \nu)$ to $L^2(\mathbf{X}, dx)$. Therefore the global balayage operators π_α, $\alpha > 0$, defined by

$$\int_\Delta \nu(d\xi)\varphi(\xi)\pi_\alpha f(\xi) = \int_{\mathbf{X}} dx\, f(x) H_\alpha \varphi(x) \tag{6.27}$$

are bounded from $L^2(\mathbf{X}, dx)$ to $L^2(\Delta, \nu)$. For $\alpha < \beta$ we introduce the Feller operator

$$U_{\alpha,\beta} = (\beta - \alpha)\pi_\alpha^* H_\beta = (\beta - \alpha)\pi_\beta^* H_\alpha \tag{6.28}$$

together with the density $u_{\alpha,\beta}(\xi, \eta)$ determined by

$$U_{\alpha,\beta}\varphi(\xi) = \int u_{\alpha,\beta}(\xi, \eta)\varphi(\eta)\nu(d\eta)$$

and the bilinear form

$$U_{\alpha,\beta}(\varphi, \psi) = \int \nu(d\xi)\varphi(\xi) U_{\alpha,\beta}\psi(\xi)$$

and the quadratic form

$$U_{\alpha,\beta}\langle\varphi, \varphi\rangle = \int \nu(d\xi) \int \nu(d\eta) u_{\alpha,\beta}(\xi, \eta)\{\varphi(\xi) - \varphi(\eta)\}^2.$$

These satisfy

$$U_{\alpha,\kappa} = U_{\alpha,\beta} + U_{\beta,\kappa} \quad \text{for } \alpha < \beta < \kappa, \tag{6.29}$$

$$U_{\alpha,\beta}(\varphi, \psi) = \mathcal{E}\{e^{-\alpha(\zeta - \zeta^*)} - e^{-\beta(\zeta - \zeta^*)}\}\varphi(X^{\mathrm{ref}}_{\zeta^* - 0})\psi(X^{\mathrm{ref}}_\zeta). \tag{6.30}$$

In particular $U_{\alpha,\infty} = \mathrm{Lim}_{\beta\uparrow\infty} U_{\alpha,\beta}$ is well defined and

$$\int \nu(d\xi)\varphi(\xi)U_{0,\infty}\varphi(\xi) = \mathcal{E}\varphi(X^{\mathrm{ref}}_{\zeta^*-0})\psi(X^{\mathrm{ref}}_\zeta). \tag{6.31}$$

Let

$$p(x) = \mathcal{P}_x[X^{\mathrm{ref}}_\zeta = \delta; X_{\zeta-0} = \partial],$$
$$p^*(x) = \mathcal{P}^*_x[X^{\mathrm{ref}}_\zeta = \delta; X_{\zeta-0} = \partial]. \tag{6.32}$$

Then $\alpha\pi_\alpha p_\alpha^*$, $\alpha\pi_\alpha^* p$ increases as $\alpha\uparrow\infty$ to functions $\pi_\infty p^*$, $\pi_\infty^* p$ on Δ such that

$$\int \nu(d\xi)\varphi(\xi)\pi_\infty^* p(\xi) = \mathcal{E}\varphi(X^{\mathrm{ref}}_{\zeta^*-0}) I(X^{\mathrm{ref}}_\zeta = \delta, X_{\zeta-0} = \partial),$$

$$\int \nu(d\eta)\varphi(\eta)\pi_\infty p^*(\eta) = \mathcal{E}\varphi(X^{\mathrm{ref}}_\zeta) I(X^{\mathrm{ref}}_{\zeta^*-0} = \delta, X_{\zeta^*} = \partial). \tag{6.32'}$$

Finally, for φ on Δ define

$$N(\varphi,\varphi) = \frac{1}{2} U_{0,\infty}\langle\varphi,\varphi\rangle + \frac{1}{2}\int \nu(d\eta)\{\pi_\infty p^*(\eta) + \pi_\infty^* p(\eta)\}\varphi^2(y)$$
$$+ \frac{1}{2}\int \{\kappa^*(dx)H\varphi^2(x) + \kappa(dy)H^*\varphi^2(y)\}. \tag{6.33}$$

Then for admissible $f \in \mathbf{F}_{\text{ref}}$ such that $f_0 \in \mathbf{F}_{(e)}^D$ for some D with compact closure, we have by (6.18′) and (6.16)

$$E(f,f) = E(f_0,f_0) + N(\gamma f,\gamma f) \tag{6.34}$$

and the restriction on f_0 can easily be removed by passage to the limit.

6.6. DEFINITION. $f \in \mathbf{F}_{\text{ref}}$ belongs to the *active reflected space* $\mathbf{F}_{\text{ref},a}$ if f is admissible and if $f \in L^2(\mathbf{X})$. □

The next theorem shows that $\mathbf{F}_{\text{ref},a}$ plays the role of $\mathbf{F}_{\text{ref}}$ for E_α, $\alpha > 0$.

THEOREM 6.3. *If $f \in \mathbf{F}_{\text{ref},a}$ then for $\alpha > 0$ there are unique decompositions*

$$f = f_\alpha + H_\alpha\gamma f; \qquad f = f_\alpha^* + H_\alpha^*\gamma f \tag{6.35}$$

with $f_\alpha, f_\alpha^* \in \mathbf{F}$. Conversely if $f \in \mathbf{F}_{\text{ref}}^{\text{symm}} \cap L^2$ is α-harmonic for some $\alpha > 0$, then $f \in \mathbf{F}_{\text{ref},a}$ and therefore $f = H_\alpha\gamma f$. □

PROOF. If $f \in \mathbf{F}_{\text{ref},a}$ then $H_\alpha\gamma f \in L^2(\mathbf{X})$ and therefore $f_\alpha = f - H_\alpha\gamma f \in L^2(\mathbf{X})$. Also $f_\alpha = f_0 + \alpha G_\alpha f - \alpha G_0 f_0$ must belong to $\mathbf{F}_{(e)}$ since $f_0 - \alpha G_\alpha f_0 \in \mathbf{F}_{(e)}$ and $\alpha G_\alpha f \in \mathbf{F}$. Thus $f_\alpha \in \mathbf{F}$ and the direct part of the theorem is proved. Also for $f \in \mathbf{F}_{\text{ref},a}$ the difference $H\gamma f - H_\alpha\gamma f \in \mathbf{F}_{(e)}$ and since $H\gamma f - H_\alpha\gamma f = \alpha G H_\alpha\gamma f$, it must be that $E(g, H\gamma f - H_\alpha\gamma f) = \alpha(g, H_\alpha\gamma f)$ for $g \in \mathbf{F}_{(e)}$. Thus by (6.34)

$$E(H_\alpha\gamma f, H_\alpha\gamma f) = E(H\gamma f, H\gamma f) + E(H_\alpha\gamma f - H\gamma f, H_\alpha\gamma f - H\gamma f)$$
$$= N(\gamma f,\gamma f) + \alpha(H\gamma f - H_\alpha\gamma f, H_\alpha\gamma f)$$

or equivalently

$$E_\alpha(H_\alpha\gamma f, H_\alpha\gamma f) = N(\gamma f,\gamma f) + U_{0,\alpha}(\gamma f,\gamma f). \tag{6.36}$$

This is our tool for proving the converse. Note first that for any $\varphi \in L^2(\Delta,\nu)$

$$U_{0,\alpha}(\varphi,\varphi) = \lim_{\beta\downarrow 0} U_{\beta,\alpha}(\varphi,\varphi) = \lim_{\beta\downarrow 0}(\alpha-\beta)\big(H_\beta\varphi, \{1-(\alpha-\beta)G_\alpha\}H_\beta\varphi\big)$$

is nonnegative and so (6.36) implies

$$E(H\gamma f, H\gamma f) \leq E_\alpha(H_\alpha\gamma f, H_\alpha\gamma f). \tag{6.37}$$

An appropriate version of (6.36) for the absorbed process implies that for any $f \in \mathbf{F}_{\text{ref}}^{\text{symm}} \cap L^2(\mathbf{X})$

$$E(H^M f, H^M f) \leq E_\alpha(H_\alpha^M f, H_\alpha^M f) \tag{6.37′}$$

and in particular any such f which is α-harmonic must belong to $\mathbf{F}_{\text{ref}}$. Next observe that

$$
\begin{aligned}
N(\varphi, \varphi) + U_{0,\alpha}(\varphi, \varphi) &\geqslant \tfrac{1}{2} U_{0,\alpha}\langle \varphi, \varphi \rangle + U_{0,\alpha}(\varphi, \varphi) \\
&\quad + \frac{1}{2} \int \left\{ \kappa^*(dx)\alpha G H_\alpha \varphi^2(x) + \kappa(dx)\alpha G^* H_\alpha^* \varphi^2(x) \right\} \\
&\quad + \frac{1}{2} \int dt \left\{ p^*(x)\alpha H_\alpha \varphi^2(x) + p(x)\alpha H_\alpha^* \varphi^2(x) \right\} \\
&= \frac{1}{2} \int dx (H^* 1 + N^* \kappa^* + p^*)\alpha H_\alpha \varphi^2(x) \\
&\quad + \frac{1}{2} \int dx (H 1 + N\kappa + p)\alpha H_\alpha^* \varphi^2(x) \\
&= \frac{1}{2} \int dx \left\{ \alpha H_\alpha \varphi^2(x) + \alpha H_\alpha^* \varphi^2(x) \right\}
\end{aligned}
$$

and since $U_{0,\alpha}$ increases with α, the identity (6.36) implies an estimate

$$E_\alpha(H_\alpha \gamma f, H_\alpha \gamma f) \geqslant (\alpha \wedge 1) \int \nu(d\xi)\gamma f^2(\xi). \tag{6.38}$$

But it is easy to see that this estimate does not really depend on our assumption that $f \in \mathbf{F}_{\mathrm{ref},a}$, and so the converse is completely proved. □

It follows from (6.16) that if $f \in \mathbf{F}_{\mathrm{ref},a}$, then

$$E(g, H\gamma f) = 0; \qquad E(H^*\gamma f, g) = 0 \tag{6.16$'$}$$

whenever $g \in \mathbf{F}_{(e)}^D$ for some g. From now on we make (6.16′) true by definition for general $g \in \mathbf{F}_{(e)}$. Then we have

$$E(H\gamma f, H\gamma f) = E(H^*\gamma f, H\gamma f) = E(H^*\gamma f, H^*\gamma f). \tag{6.39}$$

As mentioned in the Introduction it is not possible in general to define $E(g, f)$ for arbitrary $g, f \in \mathbf{F}_{\mathrm{ref}}$. A simple counterexample is worked out in the Appendix.

It is clear from the proof of Theorem 6.3 that if $\varphi \in L^2(\Delta)$ satisfies

$$N(\varphi, \varphi) < +\infty, \tag{6.40}$$

then $H_\alpha \varphi$, $H_\alpha^* \varphi \in \mathbf{F}_{\mathrm{ref},a}$ for all $\alpha > 0$. Thus $\gamma \mathbf{F}_{\mathrm{ref},a}$ is precisely the set of $\varphi \in L^2(\Delta)$ satisfying (6.40).

7. Kunita's classification.

7.1. Definition. A function f belongs to the domain of the *local generator* $\mathcal{Q}$ if $f \in \mathbf{F}_{\mathrm{loc}}$, if $E(g, f)$ can be defined by (6.6) for all $g \in \mathbf{F} \cap C_{\mathrm{com}}$ (of course $f^{\#} = f$ in a neighborhood of the support of g), and if there exists a locally integrable function $\mathcal{Q}f$ such that

$$E(g, f) = -\int m(dx)\, g(x) \mathcal{Q} f(x) \tag{7.1}$$

for such g. □

It is clear from (6.7) that the left side of (7.1) can always be defined for $g \in \mathbf{F} \cap C_{\mathrm{com}}$ when $f \in \mathbf{F}_{\mathrm{loc}}$ is bounded.

Now let $G_\alpha^{\sim}$, $\alpha > 0$, be a submarkovian resolvent on $L^2(\mathbf{X}, dx)$ such that also the adjoint resolvent $G_\alpha^{\sim *}$, $\alpha > 0$, is submarkovian and such that the L^2 generator $A^\sim$ is contained in the local generator $\mathcal{Q}$ and also the adjoint generator $A^{\sim *}$ is contained in the dual local generator $\mathcal{Q}^*$. Let $f = G_\alpha^\sim \varphi$ with $\varphi \in L^2(\mathbf{X})$ bounded. Then $h = G_\alpha^\sim \varphi - G_\alpha \varphi$ belongs to $\mathbf{F}_{\text{loc}}$ and $E_\alpha(g, h) = 0$ for $g \in \mathbf{F} \cap C_{\text{com}}$. By the local decomposition at the beginning of §6, this is enough to guarantee that h is α-harmonic and therefore by the martingale convergence theorem

$$G_\alpha^\sim \varphi(x) = G_\alpha \varphi(x) + \mathcal{E}_x e^{-\alpha\zeta} I(X_{\zeta-0} = \partial) \operatorname*{Lim}_{t \uparrow \zeta} G_\alpha^\sim \varphi(X_t). \tag{7.2}$$

In particular

$$G_\alpha 1(x) \leq G_\alpha^\sim 1(x) \leq G_\alpha 1(x) + \mathcal{E}_x I(X_{\zeta-0} = \partial) e^{-\alpha\zeta}.$$

Lemma 15.1 in [2] is also valid here and therefore

$$\operatorname*{Lim}_{\alpha \uparrow \infty} \int dx\, g(x) \mathcal{E}_x \alpha e^{-\alpha\zeta} I(X_{\zeta-0} = \partial) = 0 \tag{7.3}$$

whenever g is bounded with compact support. This together with (5.10) implies that

$$\operatorname*{Lim}_{\alpha \uparrow \infty} \alpha(g, 1 - \alpha G_\alpha^\sim 1) = \int \kappa(dx) g(x) \tag{7.4}$$

whenever $g \in \mathbf{F}$ is bounded with compact support. Similarly

$$\operatorname*{Lim}_{\alpha \uparrow \infty} \alpha(1 - \alpha G_\alpha^{\sim *} 1, g) = \int \kappa^*(dx) g(x). \tag{7.4*}$$

Now let $f \in \text{domain } A^\sim$ be bounded and choose $g \in \mathbf{F} \cap C_{\text{com}}$ with $0 \leq g \leq 1$. Then

$$\begin{aligned}
-(f, A^\sim f) &= \operatorname*{Lim}_{\alpha \uparrow \infty} \alpha(f, f - \alpha G_\alpha^\sim f) \\
&= \lim_{\alpha \uparrow \infty} \Big\{ \tfrac{1}{2}\alpha\big(f^2, 1 - \alpha G_\alpha^\sim 1\big) + \tfrac{1}{2}\alpha\big(1 - \alpha G_\alpha^{\sim *} 1, f^2\big) \\
&\qquad + \frac{1}{2}\alpha^2 \int m(dx) \int G_\alpha^\sim(x, dy)(f(x) - f(y))^2 \Big\} \\
&\geq \operatorname*{Lim}_{\alpha \uparrow \infty} \Big\{ \tfrac{1}{2}\alpha\big(f^2 g, 1 - \alpha G_\alpha^\sim 1\big) + \tfrac{1}{2}\alpha\big(1 - \alpha G_\alpha^{\sim *} 1, f^2 g\big) \\
&\qquad + \frac{1}{2}\alpha^2 \int m(dx) g(x) \int G_\alpha^\sim(x, dy)(f(x) - f(y))^2 \Big\} \\
&= \frac{1}{2} \int \kappa(dx) g(x) f^2 + \frac{1}{2} \int \kappa^*(dx) g(x) f(x)^2 \\
&\qquad + \operatorname*{Lim}_{\alpha \uparrow \infty} \frac{1}{2} \alpha^2 \int m(dx) g(x) \int G_\alpha^\sim(x, dy) \{f(x) - f(y)\}^2
\end{aligned}$$

$$\geq \frac{1}{2}\int \kappa(dx)g(x)f(x)^2 + \frac{1}{2}\int \kappa^*(dx)g(x)f(x)^2$$
$$+ \lim_{\alpha\uparrow\infty} \frac{1}{2}\alpha^2\int m(dx)g(x)\int G_\alpha(x,dy)\{f(x)-f(y)\}^2. \tag{7.5}$$

This implies that

$$-(f, A^{\sim}f) \geq E(f,f) \tag{7.6}$$

and in particular $f \in \mathbf{F}_{\mathrm{ref}}^{\mathrm{symm}}$. To see why note first that $f \in \mathbf{F}_{\mathrm{loc}}$ because of (7.2) and therefore we can choose bounded $f^{\#} \in \mathbf{F}$ such that $f = f^{\#}$ quasi-everywhere in a neighborhood of the support of g. Then

$$\frac{1}{2}\alpha^2\int m(dx)g(x)\int G_\alpha(x,dy)\{f(x)-f(y)\}^2$$
$$= \frac{1}{2}\alpha^2\int m(dx)g(x)\int G_\alpha(x,dy)\{f^{\#}(x)-f^{\#}(y)\}^2$$
$$+ \frac{1}{2}\alpha^2\left(g, G_\alpha\{f^2 - f^{\#2}\}\right) + \alpha^2\left(gf, G_\alpha\{f^{\#}-f\}\right). \tag{7.7}$$

By (5.14′) the first term on the right converges to

$$\frac{1}{2}\int \langle Af^{\#}\rangle(dx)g(x)$$
$$= \frac{1}{2}\int \langle A_c f\rangle(dx)g(x) + \frac{1}{2}\iint J(dx,dy)g(x)\{f^{\#}(x)-f^{\#}(y)\}^2.$$

Suppose we can show that

$$\lim_{\alpha\uparrow\infty}\alpha^2\left(g, G_\alpha\{f^2-f^{\#2}\}\right) = \iint J(dx,dy)g(x)\{f(y)^2 - f^{\#}(y)^2\},$$
$$\lim_{\alpha\uparrow\infty}\alpha^2\left(gf, G_\alpha\{f-f^{\#}\}\right) = \iint J(dx,dy)g(x)f(x)\{f(y)-f^{\#}(y)\}. \tag{7.8}$$

Then (7.5) becomes

$$-(f, A^{\sim}f) \geq \frac{1}{2}\int \kappa^*(dx)g(x)f(x)^2 + \frac{1}{2}\int \kappa(dx)g(x)f(x)^2$$
$$+ \frac{1}{2}\int \langle A_c f\rangle(dx) + \frac{1}{2}\iint J(dx,dy)g(x)\{f^{\#}(x)-f^{\#}(y)\}^2$$
$$+ \frac{1}{2}\iint J(dx,dy)g(x)\{f(y)^2 - f^{\#}(y)^2 + 2f(x)f^{\#}(y) - 2f(x)f^{\#}(y)\}$$

and since the last two terms combine to give $\frac{1}{2}\iint J(dx,dy)g(x)\{f(x) - f(y)\}^2$, we can pass to the limit in g and establish (7.6). Thus we have reduced (7.6) to (7.8). Now (5.11) implies that

$$\lim_{\alpha\uparrow\infty} \alpha^2(g, G_\alpha\varphi) = \iint J(dx, dy)g(x)\varphi(y) \tag{7.9}$$

whenever $\varphi \in \mathbf{F}_{(e)}$ has support disjoint from g. Also (7.9) is true if $\varphi = H1 - \psi$ where $\psi \in \mathbf{F}_{(e)}$ agrees with $H1$ in a neighborhood of the support of g. This follows because

$$\alpha^2(g, G_\alpha\{H1 - \psi\}) = -\alpha(g, H1 - \alpha G_\alpha H1) + \alpha(g, \psi - \alpha G_\alpha\psi)$$

converges to $E(g, \psi)$ by (7.3) and because

$$E(g, \psi) = E(g, \psi - H1) = \iint J(dx, dy)g(x)\{\psi(y) - H1(y)\}$$

as is clear from (5.11) and (6.6). The first line in (7.8) now follows since $f(y)^2 - f^{\#}(y)^2$ can be approximated from below by functions φ for which (7.9) is valid and also $c\{H1(y) - \psi(y)\} - \{f(y)^2 - f^{\#}(y)^2\}$ can be so approximated for some ψ as above and some $c > 0$. The second line follows in a similar way, after working with gf instead of g.

The technique used above to establish the estimate (7.6) is a simple modification of the one first used by M. Fukushima [6] for multidimensional Brownian motion. It is routine to extend (7.6) to general $f \in$ domain $A^\sim$ and of course the analogue is also true for $f \in$ domain $A^{\sim *}$. Combining this with the converse part of Theorem 6.3, we get

THEOREM 7.1. *Both* domain($A^\sim$) *and* domain($A^{\sim *}$) *are contained in the active reflected space* $\mathbf{F}_{\mathrm{ref},a}$. □

One pleasant consequence of Theorem 7.1 is that (7.2) can be replaced by the more compact formula

$$G_\alpha^\sim\varphi(x) = G_\alpha\varphi(x) + H_\alpha\gamma G_\alpha^\sim\varphi(x) \tag{7.2'}$$

at least for $\varphi \in L^2(\mathbf{X})$.

Of course the above argument can be refined to replace (7.6) by an analogue to (15.14) in [2]. But this is pointless in the nonsymmetric setting. To get the "correct result" we must adapt the techniques used by H. Kunita in [5]. First we observe that if $f \in \mathbf{F}_{\mathrm{ref}}$ is bounded, then

$$\lim_{\alpha\uparrow\infty} \alpha(g, f - \alpha G_\alpha^\sim f) = E(g, f). \tag{7.10}$$

In fact this follows from (7.3). Since

$$\alpha^2(g, H_\alpha\gamma G_\alpha^\sim f) = \alpha^2(H_\alpha^*\gamma G_\alpha^{\sim *}g, f) \leq \alpha\|g\|_\infty(H_\alpha^* 1, f)$$

we can replace $G_\alpha^\sim$ by G_α in (7.10) and then we need only check the separate cases when $g \in \mathbf{F}_{(e)}$ and when $g = H\gamma g$. (We have already observed on p. 15.3 in [2] that (7.10) need not be true for general bounded $f \in \mathbf{F}_{(e)}$, even in the symmetric case. Notice that (7.8) is actually a special case of (7.10) although at the time we had to use a slightly different argument since we did not know that $f \in \mathbf{F}_{\mathrm{ref}}$.)

Now we are ready to adapt the argument on p. 323 in [5]. Let $f \in$ domain$(A^{\sim})$ be bounded and write $f = f_\alpha + H_\alpha \gamma f$. Choose $f_n \in \mathbf{F} \cap C_{\text{com}}$ uniformly bounded such that $f_n \to f_\alpha$ relative to E_α. For each n, for $c > 0$, and for all $\beta > \alpha$

$$\beta\big(\{f - f_n - c\}^+, (1 - \beta G^{\sim}_{\alpha+\beta})\{f - f_n\}\big) \geqslant 0. \tag{7.11}$$

This can be verified by checking cases, exactly as in the proof of Lemma 1.6 in [1]. By the $\alpha > 0$ version of (7.10) we can pass to the limit $\beta \uparrow \infty$ and get

$$\big(\{f - f_n - c\}^+, (\alpha - A^{\sim})f\big) - E_\alpha\big(\{f - f_n - c\}^+, f_n\big) \geqslant 0$$

and therefore after a passage to the limit in n

$$\big(\{H_\alpha \gamma f - c\}^+, (\alpha - A^{\sim})f\big) - E_\alpha\big(\{H_\alpha \gamma f - c\}^+, f_\alpha\big) \geqslant 0. \tag{7.12}$$

But

$$\begin{aligned} E_\alpha\big(\{H_\alpha \gamma f - c\}^+, f_\alpha\big) &= E_\alpha\big(\{H_\alpha \gamma f - c\}^+ - H^*_\alpha(\gamma f - c)^+, f_\alpha\big) \\ &= \big(\{H_\alpha \gamma f - c\}^+ - H^*_\alpha(\gamma f - c)^+, (\alpha - \mathcal{Q})f_\alpha\big) \\ &= \big(\{H_\alpha \gamma f - c\}^+ - H^*_\alpha(\gamma f - c)^+, (\alpha - A^{\sim})f\big) \end{aligned}$$

and so (7.12) is equivalent to

$$\big(H^*_\alpha(\gamma f - c)^+, (\alpha - A^{\sim})f\big) \geqslant 0. \tag{7.13}$$

This corresponds to (6.4) in [5]. Next

$$\big(H^*_\alpha \gamma f, (\alpha - A^{\sim})f\big) = \big(f, (\alpha - A^{\sim})f\big) - \big(f^*_\alpha, (\alpha - A^{\sim})f\big)$$

which by (7.6) is $\geqslant E_\alpha(f, f) - (f^*_\alpha, (\alpha - A^{\sim})f) = E_\alpha(f, f) - E_\alpha(f - H^*_\alpha \gamma f, f - H_\alpha \gamma f) = E_\alpha(H^*_\alpha \gamma f, H_\alpha \gamma f)$ and therefore by (6.36)

$$\big(H^*_\alpha \gamma f, (\alpha - A^{\sim})f\big) \geqslant N(\gamma f, \gamma f) + U_{0,\alpha}(\gamma f, \gamma f). \tag{7.14}$$

This corresponds to the estimate at the bottom of p. 232 in [5]. Now define an operator $R_{(\alpha)}$ on $\pi^*_\alpha L^2(\mathbf{X})$ by

$$R^*_{(\alpha)}\pi_\alpha g = \gamma G^{\sim *}_\alpha g \tag{7.15}$$

With $f = G^{\sim}_\alpha g$ and $\varphi = \pi^*_\alpha g$ so that $\gamma f = R_{(\alpha)}\varphi$, the estimate (7.14) becomes

$$(R_{(\alpha)}\varphi, \varphi)_\Delta \geqslant N(R_{(\alpha)}\varphi, R_{(\alpha)}\varphi) + U_{0,\alpha}(R_{(\alpha)}\varphi, R_{(\alpha)}\varphi) \tag{7.14$'$}$$

and by (6.38) and (6.36) the operator $R_{(\alpha)}$ is bounded relative to the $L^2(\Delta)$ norm. (Of course $(\ ,\ )_\Delta$ denotes the standard inner product on $L^2(\Delta)$.)Since $\pi^*_\alpha L^2(\mathbf{X})$ is certainly dense, $R_{(\alpha)}$ extends uniquely to a bounded everywhere defined operator on $L^2(\Delta)$. Now (7.13) is equivalent to

$$\big([R_\alpha\varphi - c]^+, \varphi\big)_\Delta \geqslant 0. \tag{7.13$'$}$$

This together with the argument on p. 342 in [10] is enough to guarantee that $R_{(\alpha)}$ is the Green's operator for a submarkovian semigroup on $L^2(\Delta)$.

REMARK. The operator $R_{(\alpha)}$ here corresponds to the kernel $M_\alpha(\eta, \xi)$ on pp. 320–322 in [5]. It appears that the existence argument for $M_\alpha(\eta, \xi)$ given in [5] has a gap and indeed that Proposition 6.2 in [5] is incorrect as stated. □

Of course the dual operator $R^*_{(\alpha)}$ defined by

$$R^*_{(\alpha)}\pi_\alpha g = \gamma G_\alpha^{\sim *} g \tag{7.15*}$$

is also the Green's operator for a submarkovian semigroup. Moreover

$$\int \nu(d\xi)\pi_\alpha f(\xi) R_{(\alpha)}\pi_\alpha^* g(\xi) = (f, H_\alpha \gamma G_\alpha^\sim g) = (H_\alpha^* \gamma G_\alpha^{\sim *} f, g)$$

$$= \int \nu(d\eta) R^*_{(\alpha)}\pi_\alpha f(\eta)\pi_\alpha^* g(\eta)$$

and therefore

$$(\varphi, R_{(\alpha)}\psi)_\Delta = (R^*_{(\alpha)}\varphi, \psi)_\Delta \tag{7.16}$$

which is enough to guarantee that the associated submarkovian semigroups are adjoint relative to $L^2(\Delta)$. It is clear also that

$$G_\alpha^\sim = G_\alpha + H_\alpha R_{(\alpha)}\pi_\alpha^*,$$
$$G_\alpha^{\sim *} = G_\alpha^* + H_\alpha^* R^*_{(\alpha)}\pi_\alpha. \tag{7.17}$$

By Lemma 6.3 in [5] we have for $0 < \alpha < \beta$

$$R_{(\alpha)} = R_{(\beta)} + R_{(\alpha)} U_{\alpha,\beta} R_{(\beta)} = R_{(\beta)} + R_{(\beta)} U_{\alpha,\beta} R_{(\alpha)}. \tag{7.18}$$

Now let P be the orthogonal projector of $L^2(\Delta)$ onto the range $R_{(\alpha)}L^2(\Delta)$. Of course P is independent of $\alpha > 0$ by (7.19). If $R_{(\alpha)\lambda}$, $\lambda > 0$, is the resolvent corresponding to $R_{(\alpha)}$, then $\varphi \in L^2(\Delta)$ satisfies $P\varphi = \varphi$ if and only if

$$\lim_{\lambda \uparrow \infty} (\varphi, \{1 - \lambda R_{(\alpha)\lambda}\}\varphi)_\Delta = 0. \tag{7.19}$$

(The point is that if $P\varphi \neq \varphi$ then for all $\lambda > 0$ we have $(\varphi, \lambda R_{(\alpha)\lambda}\varphi)_\Delta <$ $(P\varphi, P\varphi)_\Delta < (\varphi, \varphi)_\Delta$.) This implies that also P is the orthogonal projector of $L^2(\Delta)$ onto the range $R^*_{(\alpha)}L^2(\Delta)$. In fact $PL^2(\Delta)$ is a subspace of a special type. There exists a subset of Δ and a sub-sigma-algebra of the Borel algebra on this subset such that $PL^2(\Delta)$ consists precisely of those functions in $L^2(\Delta)$ which vanish in the complement of the subset and are measurable with respect to the sub-sigma-algebra. To show this we need only check that if $\varphi \in PL^2(\Delta)$, then for any $c > 0$ also $(\varphi - c)^+ \in PL^2(\Delta)$. But this follows from the estimate

$$\left([\varphi - c]^+, [\varphi - c]^+ - \lambda R_{(\alpha)\lambda}[\varphi - c]^+\right)_\Delta = \left([\varphi - c]^+, \varphi - \lambda R_{(\alpha)\lambda}\varphi\right)_\Delta$$
$$- \left([\varphi - c]^+, c - \lambda R_{(\alpha)\lambda}\varphi\right)_\Delta < \left([\varphi - c]^+, \varphi - \lambda R_{(\alpha)\lambda}\varphi\right)_\Delta.$$

(All this is stated in Proposition 6.4 in [5] and Theorem 4 in [10].) The *generator* B_α corresponding to $R_{(\alpha)}$ satisfies

$$B_\alpha R_{(\alpha)} = -P. \tag{7.20}$$

It is clear from (7.18) that domain(B_α) is independent of $\alpha > 0$. Also for $0 < \alpha < \beta$ and for $\varphi \in L^2(\Delta)$

$$\begin{aligned} B_\beta R_{(\alpha)}\varphi &= B_\beta R_{(\beta)}\{\varphi + U_{\alpha,\beta}R_{(\alpha)}\varphi\} = -P\varphi - PU_{\alpha,\beta}R_{(\alpha)}\varphi \\ &= B_\alpha R_{(\alpha)}\varphi - PU_{\alpha,\beta}R_{(\alpha)}\varphi \end{aligned}$$

and therefore

$$B_\alpha = B_\beta + PU_{\alpha,\beta}. \tag{7.21}$$

By Lemma 6.1 in [5] the operator $U_{0,\alpha}$ is bounded on $L^2(\Delta)$ and therefore we can define an operator B by

$$B = B_\alpha + PU_{0,\alpha} \tag{7.22}$$

with the understanding that domain(B) = domain(B_α). The relation (7.18) guarantees that for $\lambda > 0$ and for $0 < \alpha < \beta$

$$R_{(\alpha)\lambda} = R_{(\beta)\lambda} + R_{(\alpha)\lambda}U_{\alpha,\beta}R_{(\beta)\lambda}. \tag{7.23}$$

Thus we can pass to the limit $\alpha \downarrow 0$ and establish the existence of a submarkovian resolvent R_λ, $\lambda > 0$, satisfying for $\beta > 0$

$$R_\lambda = R_{(\beta)\lambda} + R_\lambda U_{0,\beta}R_{(\beta)\lambda} = R_{(\beta)\lambda} + R_{(\beta)\lambda}U_{0,\beta}R_\lambda \tag{7.23'}$$

and it is easy to check that B is the generator. Similarly the adjoint B^* is defined by

$$B^* = B_\alpha^* + PU_{0,\alpha}^* \tag{7.22*}$$

and generates a submarkovian resolvent R_λ^*, $\lambda > 0$. Now we are ready to establish the estimates

$$\begin{aligned} -([\varphi - c]^+, B\varphi)_\Delta &\geq \iint \nu(d\eta)\nu(d\xi)u_{0,\infty}(\eta,\xi)[\varphi - c]^+(\eta)\{c - \varphi \wedge c(\xi)\} \\ &\quad + \int \nu(d\xi)\pi_\infty^* p(\xi)c[\varphi - c]^+(\xi) + \int \kappa(dx)H^*c[\varphi - c]^+(x), \end{aligned} \tag{7.24}$$

$$\begin{aligned} &-(B^*\psi, [\psi - c]^+)_\Delta \\ &\quad\geq \iint \nu(d\eta)\nu(d\xi)u_{0,\infty}(\eta,\xi)\{c - \psi \wedge c(\eta)\}[\psi - c]^+(\xi) \\ &\quad + \int \nu(d\xi)\pi_\infty p^*(\xi)c[\psi - c]^+(\xi) + \int \kappa^*(dx)Hc[\psi - c]^+(x) \end{aligned} \tag{7.24*}$$

for $\varphi \in$ domain(B) and $\psi \in$ domain(B^*). It follows from the converse result to be established in §8 that these two estimates together characterize the possibilities for B.

We prove (7.24) by a series of reductions. It suffices to establish the estimate

$$R_{(\alpha)\lambda}\{\alpha\pi_\alpha^* 1 + \lambda\} \leq 1 \tag{7.25}$$

for all $\alpha, \lambda > 0$. The function $\alpha\pi_\alpha^* 1$ is bounded and so the perturbed generator $B_\alpha^\sim = B_\alpha + (\alpha\pi_\alpha^* 1)$ also generates a nonnegative semigroup on $PL^2(\Delta)$ with resolvents $R_{(\alpha)\lambda}^\sim$ determined by the iteration procedure

$$R_{(\alpha)\lambda}^0 = R_{(\alpha)\lambda}; \qquad R_{(\alpha)\lambda}^{n+1} = R_{(\alpha)\lambda} + R_{(\alpha)\lambda}(\alpha\pi_\alpha^* 1)R_{(\alpha)\lambda}^n$$

$$R_{(\alpha)\lambda}^\sim = \operatorname{Lim} R_{(\alpha)\lambda}^n.$$

Certainly (7.25) implies that $\lambda R_{(\alpha)\lambda}^0 1 \leq 1$ and if $\lambda R_{(\alpha)\lambda}^n 1 \leq 1$ then also

$$\begin{aligned}\lambda R_{(\alpha)\lambda}^{n+1} 1 &= \lambda R_{(\alpha)\lambda} 1 + \lambda R_{(\alpha)\lambda}(\alpha\pi_\alpha^* 1) R_{(\alpha)\lambda}^n 1 \\ &\leq 1 - \lambda R_{(\alpha)\lambda}(\alpha\pi_\alpha^* 1) + \lambda R_{(\alpha)\lambda}(\alpha\pi_\alpha^* 1) \leq 1\end{aligned}$$

and after passage to the limit $n\uparrow\infty$ we conclude that $R_{(\alpha)\lambda}^\sim$, $\lambda > 0$, is submarkovian. Thus, with the obvious notation for the standard inner product of $L^2(\Delta)$, we have for $\varphi \in \text{domain}(B)$

$$\begin{aligned}\lambda\big([\varphi - c]^+, \{1 - \lambda R_\lambda\}\varphi\big)_\Delta &\geq \lambda\big([\varphi - c]^+, \{1 - \lambda R_\lambda\}\varphi \wedge c\big)_\Delta \\ &= \lambda\big([\varphi - c]^+, \{1 - \lambda R_{(\alpha)\lambda}^\sim + \lambda R_{(\alpha)\lambda}^\sim((\alpha\pi_\alpha^* 1) - U_{0,\alpha})R_\lambda\}\varphi \wedge c\big)_\Delta \\ &\geq \lambda^2\big([\varphi - c]^+, R_{(\alpha)\lambda}^\sim((\alpha\pi_\alpha^* 1) - U_{0,\alpha})R_\lambda\varphi \wedge c\big)_\Delta\end{aligned}$$

and after passage to the limit $\lambda\uparrow\infty$

$$\begin{aligned}&-\big([\varphi - c]^+, B\varphi\big)_\Delta \\ &\quad\geq \int \nu(d\xi)(\alpha\pi_\alpha^* 1(\xi))[\varphi - c]^+(\xi)c - U_{0,\alpha}\big([\varphi - c]^+, \varphi \wedge c\big) \\ &\quad= \int \nu(d\xi)\int \nu(d\eta)u_{0,\alpha}(\xi,\eta)[\varphi - c]^+(\xi)\{c - \varphi \wedge c(\eta)\} \\ &\qquad+ \int \nu(d\xi)\{\alpha\pi_\alpha^* 1(\xi) - U_{0,\alpha}1(\xi)\}[\varphi - c]^+(\xi)c\end{aligned} \tag{7.26}$$

and (7.24) follows after passage to the limit $\alpha\uparrow\infty$. Thus (7.25) implies (7.24). To prove (7.25) fix $\beta > \alpha$ and note that (7.23) immediately implies

$$R_{(\alpha)\lambda} = R_{(\beta)} + R_{(\alpha)\lambda}\{U_{\alpha,\beta} - \lambda\}R_{(\beta)}$$

and since $R_{(\beta)}\beta\pi_\beta^* 1 = \gamma\beta G_\beta^\sim 1 \leq 1$, this yields the estimate

$$R_{(\alpha)\lambda}\beta\pi_\beta^* 1 \leq 1 + R_{(\alpha)\lambda}U_{\alpha,\beta}1 - \lambda R_{(\alpha)\lambda}R_{(\beta)}\beta\pi_\beta^* 1$$

and since $\beta\pi_\beta^* 1 \geq \alpha\pi_\alpha^* 1 + U_{\alpha,\beta}1$, this implies

$$R_{(\alpha)\lambda}\{\alpha\pi_\alpha^* 1 + \lambda R_{(\beta)}\beta\pi_\beta^* 1\} \leq 1. \tag{7.27}$$

Thus (7.25) will be proved if we can show that as $\beta\uparrow\infty$

$$R_{(\beta)}\beta\pi_\beta^* 1 \uparrow P1 \qquad [\text{a.e. } \nu]. \tag{7.28}$$

That is, (7.28) implies (7.24). Suppose we have shown that as $\beta\uparrow\infty$

$$R_{(\alpha)}\beta U_{\alpha,\beta} \to 1 \tag{7.29}$$

in the strong operator topology on $PL^2(\Delta)$. Clearly $R_{(\beta)}\beta\pi_\beta^* 1 = \gamma\beta G_\beta^{\sim} 1$ increases to some limit which we denote by θ. If $0 \leqslant \varphi \leqslant 1$ belongs to $PL^2(\Delta)$ and if $\psi \geqslant 0$ in $PL^2(\Delta)$ then certainly

$$(\psi, R_{(\beta)}U_{\alpha,\beta}\varphi)_\Delta \to (\psi, \varphi)_\Delta. \tag{7.30}$$

But also $R_{(\beta)}U_{\alpha,\beta}\varphi \leqslant R_{(\beta)}U_{\alpha,\beta}1 \leqslant R_{(\beta)}\beta\pi_\beta^* 1 \leqslant \theta$ and therefore

$$(\psi, R_{(\beta)}U_{\alpha,\beta}\varphi)_\Delta \leqslant (\psi, \theta)_\Delta. \tag{7.31}$$

Of course (7.30) and (7.31) together imply that $\theta = P1$. Thus (7.29) implies (7.24). The estimate (7.14′) guarantees that the operators $R_{(\beta)}$, $\beta \geqslant \alpha$, are uniformly bounded on $L^2(\Delta)$. Since $R_{(\alpha)}L^2(\Delta)$ is dense in $PL^2(\Delta)$, the identity (7.18) guarantees that the operators $R_{(\beta)}U_{\alpha,\beta}$ are uniformly bounded on $L^2(\Delta)$ and also that (7.29) will be proved if we can show that

$$\operatorname*{Lim}_{\beta\uparrow\infty} R_{(\beta)} = 0 \tag{7.32}$$

in the strong operator topology on $PL^2(\Delta)$. Thus (7.32) implies (7.24). But (7.32) can be proved by arguing as on pp. 20.15, 20.16 in [2]. If (7.32) fails then there exists nontrivial $\varphi \in L^2(\Delta)$ with $R_{(\alpha)}\varphi \leqslant 1$ such that $\psi = \operatorname{Lim}_{\beta\uparrow\infty} R_{(\beta)}\varphi$ is nontrivial. (Of course $R_{(\beta)}\varphi$ decreases as $\beta\uparrow\infty$.) But then for every $\beta > \alpha$

$$(\varphi, R_{(\alpha)}\varphi)_\Delta \geqslant (\varphi, R_{(\beta)}\varphi)_\Delta$$

which by (7.14′) is

$$\geqslant N(R_{(\beta)}\varphi, R_{(\beta)}\varphi) + U_{0,\beta}(R_{(\beta)}\varphi, R_{(\beta)}\varphi) \geqslant U_{\alpha,\beta}(\psi, \psi)$$

and therefore

$$U_{\alpha,\infty}(\psi,\psi) = \operatorname*{Lim}_{\beta\uparrow\infty} (\beta-\alpha)\int dx\, H_\alpha\psi(x)\{1-(\beta-\alpha)G_\beta\}H_\alpha\psi(x)$$

is finite which implies that $H_\alpha\psi \in \mathbf{F}$. But Proposition 4.5 in [2] can also be applied here (see also the proof of Theorem 6.2) and we conclude that this is impossible for nontrivial ψ. This contradiction finally proves (7.24). We summarize in

THEOREM 7.1. *Let $G_\alpha^{\sim}$, $\alpha > 0$, be a submarkovian resolvent on $L^2(\mathbf{X}, dx)$ such that the generator $A^{\sim}$ is contained in the local generator $\mathfrak{A}$. Suppose also that the adjoint resolvent $G_\alpha^{\sim *}$, $\alpha > 0$, is submarkovian and that the adjoint generator $A^{\sim *}$ is contained in the dual local generator $\mathfrak{A}^*$.*

(i) *Both* domain($A^{\sim}$) *and* domain($A^{\sim *}$) *are contained in the active reflected space $\mathbf{F}_{\text{ref},a}$ so that $\gamma G_\alpha^{\sim} g$, $\gamma G_\alpha^{\sim *} g$ are well defined functions in $L^2(\Delta, \nu)$ for $g \in L^2(\mathbf{X}, m)$.*

(ii) *The operators $G_\alpha^{\sim}$, $G_\alpha^{\sim *}$ can be represented* (7.17) *where the operators*

$R_{(\alpha)}$, $R^*_{(\alpha)}$ are bounded on $L^2(\Delta)$ and determined as follows. There exists a closed subspace $PL^2(\Delta)$ of the special type described above and a closed densely defined operator B on $PL^2(\Delta)$ such that if $f \in$ domain$(A^\sim)$, then $\gamma f \in$ domain(B) and for all $g \in \mathbf{F}_{\mathrm{ref},a}$

$$-\int m(dx)\, g(x) A^\sim f(x) = E(g, f - H\gamma f) - (\gamma g, B\gamma f)_\Delta. \quad (7.33)$$

Both B and B^* satisfy the estimates (7.24), (7.24) and both generate submarkovian semigroups on $L^2(\Delta)$ which are strongly continuous on $PL^2(\Delta)$. Also for $\alpha > 0$ the operators B_α, B^*_α defined by (7.22), (7.22*) generate such semigroups and $R_{(\alpha)}$, $R^*_{(\alpha)}$ are the associated Green's operators.* □

The correct analogue to the basic estimate (6.9) in [5] is

$$-(\varphi^+, B\varphi)_\Delta \geq N(\varphi^+, \varphi^+) + U_{0,\infty}(\varphi^+, \varphi^-). \quad (7.34)$$

We show now that (7.24) and (7.24*) together imply (7.34). For $\alpha > 0$ certainly (7.24) implies (7.26) and therefore

$$-([\varphi - c]^+, B_\alpha^\sim \varphi)_\Delta = -([\varphi - c]^+, B\varphi)_\Delta + U_{0,\alpha}([\varphi - c]^+, \varphi)$$

$$-\int \nu(d\xi)\alpha\pi^*_\alpha 1(\xi)[\varphi - c]^+(\xi)\varphi(\xi)$$

$$\geq -\int \nu(d\xi)\alpha\pi^*_\alpha 1(\xi)\{[\varphi - c]^+(\xi)\}^2 + U_{0,\alpha}([\varphi - c]^+, [\varphi - c]^+)$$

and since $\alpha\pi^*_\alpha 1$ is bounded, this is enough to imply that $B_\alpha^\sim$ generates a submarkovian semigroup. (See Theorem 1 in [10].) But this implies for every $\lambda > 0$ the estimate

$$R_{(\alpha)\lambda}\{\lambda + \alpha\pi^*_\alpha 1\} \leq 1. \quad (7.35)$$

To prove (7.35) we need only show that for $\psi \geq 0$ in $PL^2(\Delta)$

$$(\psi, R_{(\alpha)\lambda}\{\lambda + \alpha\pi^*_\alpha 1\})_\Delta \leq (\psi, 1)_\Delta. \quad (7.36)$$

The left side of (7.36)

$$= \lim_{\mu\uparrow\infty} (\{\alpha\pi^*_\alpha 1 + \lambda\}R^*_{(\alpha)\lambda}\psi, \mu R^\sim_{(\alpha)\mu+\lambda}1)_\Delta$$

$$= \lim_{\mu\uparrow\infty} \mu(R^{\sim *}_{(\alpha)\mu+\lambda}\{\alpha\pi^*_\alpha 1 + \lambda\}R^*_{(\alpha)\lambda}\psi, 1)_\Delta$$

$$= \lim_{\mu\uparrow\infty} \mu(\{1 - \mu R^{\sim *}_{(\alpha)\mu+\lambda}\}R^{\sim *}_{(\alpha)\lambda}\{\alpha\pi^*_\alpha 1 + \lambda\}R^*_{(\alpha)\lambda}\psi, 1)_\Delta$$

$$= \lim_{\mu\uparrow\infty} \mu(\{1 - \mu R^{\sim *}_{(\alpha)\mu+\lambda}\}R^{\sim *}_{(\alpha)\lambda}\psi, 1)_\Delta$$

$$= \lim_{\mu\uparrow\infty} \mu(\{1 - (\mu + \lambda)R^{\sim *}_{(\alpha)\mu+\lambda}\}R^*_{(\alpha)\lambda}\psi, 1)_\Delta$$

$$= \lim_{\mu\uparrow\infty} \mu(\psi, R^\sim_{(\alpha)\mu+\lambda}1)_\Delta$$

$$- \lim_{\mu\uparrow\infty} \mu(R^*_{(\alpha)\lambda}\psi, 1 - (\mu + \lambda)R^\sim_{(\alpha)\mu+\lambda}1)_\Delta.$$

The first limit is the right side of (7.36) and the second is nonnegative because $R_{(\tilde{\alpha})\mu}$, $\mu > 0$, is submarkovian and so (7.36) and therefore (7.35) is proved. In a similar way (7.24*) implies

$$R^*_{(\alpha)\lambda}\{\lambda + \alpha\pi_\alpha 1\} \le 1. \tag{7.35*}$$

Now for $\varphi \in \text{domain}(B)$

$$\begin{aligned} -(\varphi, B_{(\alpha)}\varphi)_\Delta &= \lim_{\lambda\uparrow\infty} \lambda(\varphi, \{1 - \lambda R_{(\alpha)\lambda}\}\varphi)_\Delta \\ &= \lim_{\lambda\uparrow\infty} \Big[\tfrac{1}{2}\lambda(\varphi^2, \{1 - \lambda R_{(\alpha)\lambda}1\})_\Delta + \tfrac{1}{2}\lambda(\{1 - \lambda R^*_{(\alpha)\lambda}1\}, \varphi^2)_\Delta \\ &\qquad + \frac{1}{2}\lambda^2 \int \nu(d\xi) \int R_{(\alpha)\lambda}(\xi, d\eta)\{\varphi(\xi) - \varphi(\eta)\}^2 \Big] \\ &\ge \lim_{\lambda\uparrow\infty} \tfrac{1}{2}\{(\varphi^2, \lambda R_{(\alpha)\lambda}\alpha\pi^*_\alpha 1) + (\lambda R^*_{(\alpha)\lambda}\alpha\pi_\alpha 1, \varphi^2)\} \\ &= \frac{1}{2} \int \nu(d\xi)\{\alpha\pi^*_\alpha 1(\xi) + \alpha\pi_\alpha 1(\xi)\}\varphi^2(\xi) \end{aligned}$$

which implies that $B_{(\alpha)} + \frac{1}{2}(\alpha\pi^*_\alpha 1) + \frac{1}{2}(\alpha\pi_\alpha 1)$ generates a contractive semigroup on $L^2(\Delta)$. Since this semigroup certainly preserves positive functions, we can apply Theorem 2.1 in [16] and conclude that

$$-(\varphi^+, B_{(\alpha)}\varphi)_\Delta \ge \frac{1}{2} \int \nu(d\xi)\{\alpha\pi^*_\alpha 1(\xi) + \alpha\pi_\alpha 1(\xi)\}\varphi^+(\xi)^2$$

or equivalently

$$\begin{aligned} -(\varphi^+, B\varphi)_\Delta \ge \frac{1}{2} \int \nu(d\xi)\{\alpha\pi^*_\alpha 1(\xi) + \alpha\pi_\alpha 1(\xi)\}\varphi^+(\xi)^2 \\ - U_{0,\alpha}(\varphi^+, \varphi^+) + U_{0,\alpha}(\varphi^+, \varphi^-) \end{aligned}$$

and (7.34) follows after passage to the limit $\alpha\uparrow\infty$. Thus (7.24) and (7.24*) together do imply (7.34). However (7.34) is weaker and Kunita is able to establish a converse using (7.34) only because he is dealing with the special case when $1 = H1 = H^*1$ and $\alpha G^\sim_\alpha 1 = 1$.

It is clear that the resolvent $G^\sim_\alpha$, $\alpha > 0$, is completely determined by the boundary generator B. In fact Kunita shows that B (which corresponds to Q in [5]) can be used to formulate precise boundary conditions for the generator $A^\sim$ and it is easy to see that this carries over to the present context. In the next section we establish a converse. This will show in particular that the operator B can be viewed as classifying possible resolvents $G^\sim_\alpha$, $\alpha > 0$, which satisfy the hypotheses of Theorem 7.1.

8. A converse result. Let $PL^2(\Delta)$ be a special subspace of $L^2(\Delta)$ as described in §7 and let B be a closed and densely defined operator on $PL^2(\Delta)$ which generates a strongly continuous semigroup on $PL^2(\Delta)$ and which satisfies

$$- ([\varphi - c]^+, B\varphi)_\Delta + b([\varphi - c]^+, [\varphi - c]^+)_\Delta$$

$$\geq \int \nu(d\eta) \int \nu(d\xi) u_{0,\infty}(\eta, \xi)[\varphi - c]^+(\eta)\{c - \varphi \wedge c(\xi)\}$$

$$+ \int \nu(d\xi)\pi_\infty^* p(\xi) c[\varphi - c]^+(\xi) + \int \kappa(dx) H^* c[\varphi - c]^+(\xi) \quad (8.1)$$

for some $b \geq 0$. This implies in particular that both B and $B_\alpha = B - PU_{0,\alpha}$ generate submarkovian semigroups which are strongly continuous (but not necessarily contractive). For B this follows directly from Theorem 1 in **[10]**. For B_α we first note that

$$- ([\varphi - c]^+, B_\alpha\varphi) + b([\varphi - c]^+, [\varphi - c]^+)$$

$$\geq - U_{0,\alpha}([\varphi - c]^+, \varphi \wedge c) + U_{0,\alpha}([\varphi - c]^+, \varphi)$$

$$= U_{0,\alpha}([\varphi - c]^+, [\varphi - c]^+) \geq 0.$$

However the real point of (8.1) is that our proof of (7.35) can be applied here. Thus if we define $R_{(\alpha)}\varphi = \mathrm{Lim}_{\lambda\downarrow 0} R_{(\alpha)\lambda}\varphi$ for $\varphi \geq 0$, then

$$\alpha R_{(\alpha)} \pi_\alpha^* 1 \leq 1. \quad (8.2)$$

Now it is routine to check that the operators $G_\alpha^\sim$ defined by (7.17) form a submarkovian semigroup. It is clear that the generators $A^\sim$, $A^{\sim *}$ are contained in the local generators $\mathcal{A}$, $\mathcal{A}^*$ in some formal sense, but we will not pursue this further at the present level of generality. We only point out that the converse of Theorem 7.1 is valid. In particular (7.24) and (7.24*) together characterize possible boundary generators B (assuming of course that B generates a semigroup).

Appendix We discuss an example which indicates that sometimes there is no natural way to extend the Dirichlet form $E(g, f)$ to the full reflected space $\mathbf{F}_{\text{ref}}$.

Let $\mathbf{X}$ be the upper half-plane

$$\mathbf{R}^{2,+} = \{(x, y) : x \in \mathbf{R}, y \geq 0\}.$$

The reference measure is Lebesgue measure $dxdy$ and the local generator $\mathcal{A}$ is defined by

$$\mathcal{A} = D_1 D_1 + D_2 D_2 + w(y) D_1$$

where

$$w(y) = \begin{cases} y^{-1}(1 - \log y)^{-1} & \text{for } 0 < y < 1, \\ 0 & \text{for } y \geq 1. \end{cases}$$

The semigroup P_t is the absorbing barrier diffusion generated by $\mathcal{A}$. Certainly Lebesgue measure is excessive and the sector condition will be established if we can get an estimate

$$\left|\int_{-\infty}^{+\infty} dx \int_0^1 dy\, y^{-1}(1-\log y)^{-1} g D_1 f\right|$$
$$\leq M\left\{\int_{-\infty}^{+\infty} dx \int_0^1 dy (D_1 f)^2\right\}^{1/2}\left\{\int_{-\infty}^{+\infty} dx \int_0^1 dy (D_2 g)^2\right\}^{1/2} \quad \text{(A.1)}$$

for f, $g \in C^2_{\text{com}}(\mathbf{R}^{2,+})$. But this follows from the Cauchy-Schwarz inequality since

$$\int_{-\infty}^{+\infty} dx \int_0^1 dy\, y^{-2}(1-\log y)^{-2} g^2$$
$$= \int_{-\infty}^{+\infty} dx \int_0^1 dy\, y^{-2}(1-\log y)^{-2}\left\{\int_0^y dt\, D_2 g(x,t)\right\}^2$$
$$\leq \int_{-\infty}^{+\infty} dx \int_0^1 dy\, y^{-1}(1-\log y)^{-2}\int_0^y dt (D_2 g(x,t))^2$$
$$\leq \left\{\int_{-\infty}^{+\infty} dx \int_0^1 dt (D_2 g(x,t))^2\right\}\int_0^1 dy\, y^{-1}(1-\log y)^{-2}.$$

The symmetrized reflected space $\mathbf{F}_{\text{ref}}^{\text{symm}}$ is the classical Sobolev space and $\mathbf{F}_{(e)}$ is the completion in this space of $C^2_{\text{com}}(\mathbf{R}^{2,+})$. In fact, (A.1) is valid if only $g \in \mathbf{F}_{(e)}$ and this is enough to guarantee that actually $\mathbf{F}_{\text{ref}} = \mathbf{F}_{\text{ref}}^{\text{symm}}$. However, since $w(y)$ is not integrable, no such estimate can be valid for arbitrary f, $g \in \mathbf{F}_{\text{ref}}$. This seems to preclude any natural definition of $E(g, f)$ for general $g, f \in \mathbf{F}_{\text{ref}}$. Notice that the "problem term" is simply ignored when $g = f$.

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