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# ON THE EXCEPTIONAL CENTRAL SIMPLE NON-LIE MALCEV ALGEBRAS

BY

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ABSTRACT. Malcev algebras belong to the class of binary Lie algebras. Any Lie algebra is a Malcev algebra. In this paper we show that for each seven-dimensional central simple non-Lie Malcev algebra any finite dimensional Malcev module is completely reducible also for positive characteristics. This contrasts with each modular semisimple Lie algebra. As a consequence we get that the classical structure theory for characteristic zero is valid also in the modular case if semisimplicity is replaced by $\mathbf{G}_1$-separability.

The Wedderburn principal theorem is proved for Malcev algebras.

**1. Introduction.** Structures in algebra and other fields connected with an alternative Cayley algebra show exceptional features. If $C$ is an alternative algebra one recalls that the commutator algebra $C^-$ with the product defined by $a \circ b := a \cdot b - b \cdot a$ is a Malcev algebra. Let $D$ denote a Cayley algebra over a field $k$ with $\operatorname{char}(k) \neq 2, 3$, and $e$ the unit of $D$. Then any algebra $A$ isomorphic to $D^-/k \cdot e$ is a central simple and non-Lie Malcev algebra and vice versa [4]. $A$ is called an *exceptional Malcev algebra of type* $\mathbf{G}_1$, or of *type* $\mathbf{G}_1$. $A$ is said to be of *type* $\mathbf{C}_M^-$ if $A$, or equivalently $D$ is split.

Any Lie algebra is a Malcev algebra. Malcev modules are a generalization of Lie modules over Lie algebras. E. J. Taft conjectured that any finite dimensional Malcev module over a Malcev algebra of type $\mathbf{G}_1$ is completely reducible also for positive characteristics. In the following we prove the conjecture for $\operatorname{char}(k) \neq 2, 3$ (Theorem 1). As is well known the analogous statement is false for any simple Lie algebra [3]. If $\operatorname{char}(k) = 0$ the complete reducibility is shown for semisimple Malcev algebras [4]. Our proof applies the classification of irreducible Malcev modules in [1].

The Wedderburn principal theorem was recently extended by E. L. Stitzinger to Malcev algebras if $\operatorname{char}(k) = 0$, and if the radical is $\mathfrak{J}_2$-potent [7]. We prove the theorem for an arbitrary radical $R$ if $\operatorname{char}(k) = 0$, and for the modular case if $A/R$ is $\mathbf{G}_1$-separable.

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In the following we denote by $k$ a field with $\mathrm{char}(k) \neq 2, 3$. $A$ and $M$ are presumed finite dimensional $k$-vector spaces.

**2. Definitions.** Let $A$ be a binary algebra over $k$. $J$: $A \times A \times A \to A$ denotes the *Jacobi map* with $(x, y, z) \mapsto J(x, y, z)$ where $J(x, y, z) := xy \cdot z + yz \cdot x + zx \cdot y \cdot J$ *alternates* if $x^2 := x \cdot x = 0$ for all $x \in A$. Let $x, y, z, t \in A$. We recall that a *Malcev algebra $A$* is defined by

$$x^2 = 0, \tag{1}$$

and

$$J(x, y, xz) = J(x, y, z)x. \tag{2}$$

Then we have

$$(xy \cdot z)t + (yz \cdot t)x + (zt \cdot x)y + (tx \cdot y)z = ty \cdot xz, \tag{3}$$

$$2tJ(x, y, z) = J(t, x, yz) + J(t, y, zx) + J(t, z, xy) \tag{4}$$

[5]. An $A$-bimodule is called a *Malcev module* over a Malcev algebra $A$ if the *semidirect sum* or *trivial extension* $E := A \oplus M$ together with the product $(x + m) \cdot (y + m') := xy + xm' + my$ for $m, m' \in M$ is a Malcev algebra. $M$ is called a *Lie module* over $A$ if $xm = -mx$ and $J(x, y, m) = 0$. For a Malcev module $M$ the *module nucleus $N_M$* defined by $N_M := \{m \in M | \forall x, y \in A: J(x, y, m) = 0\}$ is the maximal Lie submodule. Subsequently, $A$ always denotes a Malcev algebra, and $M$ a Malcev module over $A$. We define an *$A$-module homomorphism $f$* of $M$ into a second $A$-Malcev module $M'$ by $f(xm) = xf(m)$; if moreover $f$ is injective, $f$ is a *monomorphism* of the $A$-modules etc.

$M$ is *irreducible* over $A$ if $M \neq \{0\}$, and $\{0\}$ and $M$ are the only submodules over $A$. If $M = \bigoplus M_i$, $1 < i < s$, $s \in \mathrm{N}$, each $M_i$ an irreducible submodule, then $M$ is called *completely reducible* over $A$. Then equivalently for any submodule $P$ there is a submodule $N$ so that $M = P \oplus N$ [3]. If $A$ is canonically considered as an $A$-bimodule, and $M$ isomorphic to $A$ then $M$ is called *regular*. For $A$ of type $\mathrm{C}_M^-$ the irreducible Malcev modules are up to isomorphism the regular module and the one-dimensional zero module [1].

Let $\rho$: $A \to \mathrm{End}_k(M)$ denote the canonical representation with $\rho(x)$: $m \mapsto mx$. $k[Y]$ denotes the ring of polynomials in the indeterminate $Y$. A map $\varphi$ of $A$ into the subset of irreducible polynomials with $x \mapsto \varphi_x$ is called a *primary function*, a map of $A$ in $k$ a *root*. Then

$$M_\varphi := \left\{m \in M | \ \forall x \in A \ \exists r \in \mathrm{N}: \left(\varphi_x(\rho(x))\right)^r(m) = 0\right\}.$$

If $\varphi_x = Y - \gamma(x)$ for any $x$ then $M_\varphi$ is designated by $M_\gamma$ or $M_\gamma(A)$. Set ${}_1(M_\gamma) := \{m \in M | \forall x \in A: \ \rho(x)(m) = \gamma(x) \cdot m\}$. $M_\varphi$ is called a *primary component*, and $M_\gamma$ a *root space*. If $M_\varphi \neq \{0\}$ then $\varphi$ is called *characteristic* or *essential* for $M$, and similarly for roots. $A$ *splits* over $M$ if for any $\rho(x)$ the roots of its minimum polynomial $m_x$ in $k[Y]$ are in $k$. $M$ is *smooth* if moreover those roots are distinct that is if any $m_x$ is separable over $k$. A *splitting subalgebra* is defined in the obvious way.

We recall that for any nilpotent splitting subalgebra $H$ there is a *root space decomposition* $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$ [4, Lemma 5]; then

$$A_\beta A_\gamma \subset A_{\beta+\gamma} \quad \text{if } \beta \neq \gamma, \tag{5}$$

$$A_\beta^2 \subset A_{2\beta} + A_{-\beta}, \tag{6}$$

$$J(A_0, A_\beta, A_\beta) \subset A_{-\beta}, \tag{7}$$

$$J(A_0, A_\beta, A_\gamma) = \{0\} \quad \text{if } \beta \neq \gamma, \tag{8}$$

$$J(A_\beta, A_\gamma, A_\delta) = \{0\} \quad \text{if } \beta \neq \gamma \neq \delta \neq \beta, \tag{9}$$

$$J(A_\beta, A_\gamma, A_\gamma) = \{0\} \quad \text{if } \beta \neq 0, \gamma, -\gamma, \tag{10}$$

for $\beta, \gamma, \delta \in \Delta$. A nilpotent subalgebra $H$ of $A$ is called a *Cartan subalgebra* if $H = A_0$ [4]. $A$ is *split* if it has a splitting Cartan subalgebra.

Let $\mathbf{Z}_3$ denote the integers modulo 3, and let the elements of $\mathbf{Z}_3$ be represented by 1, 2, 3. Choose $\nu \in \mathbf{Z}_3$. If $A$ is of type $\mathbf{C}_M^-$ and $\varepsilon \in k \setminus \{0\}$ then $A$ has a basis $T_\varepsilon = \{h, x_\nu, x'_\nu | \ \nu \in \mathbf{Z}_3\}$ with $x_\nu h = \varepsilon x_\nu$, $x'_\nu h = -\varepsilon x'_\nu$, $x_\nu x_{\nu+1} = 2x'_{\nu+2}$, $x'_\nu x'_{\nu+1} = \varepsilon x_{\nu+2}$, $x_\nu x'_\nu = h$, and $x_\nu x'_{\nu+1} = x'_\nu x_{\nu+1} = 0$ [4], [5]. Hence for any $\nu$, $\{h, x_\nu, x'_\nu\}$ is a basis of a split simple three-dimensional Lie (Malcev) algebra $B$ of *type* $\mathbf{A}_1$. Then $\{x_{\nu+1}, x'_{\nu+2}\}$ is the basis of a non-Lie Malcev module of *type* $\mathbf{M}_2$ over $B$. If $H$ is a splitting Cartan subalgebra of $A$, and $A_\alpha \oplus H \oplus A_{-\alpha}$ the corresponding root space decomposition, then we may choose $T_\varepsilon$ with $H = \langle h \rangle$ and $x_\nu \in A_\alpha$ [4]. The module of type $\mathbf{M}_2$ is up to isomorphism the only non-Lie Malcev module over the Lie algebra of type $\mathbf{A}_1$ [1].

For two algebras $B$, $C$ over $k$, $B \oplus. C$ denotes their direct product. Similarly we designate the direct product of two A-submodules $M_1$ and $M_2$ by $M_1 \oplus. M_2$. If $X$ is a vector space over $k$, $x_i \in X$ with $1 < i < r$, $r \in \mathbf{N}$, let $\langle x_1, \ldots, x_r \rangle$ denote the subspace generated by the $x_i$. For a map $f: X \to Y$, $Y$ a set, let $X^f := f(X)$. For further definitions see [1], [2], [4].

**3. The exceptional decomposition of a module.** Theorem 1 is preceded by four lemmas.

LEMMA 1. *Let $A$ be a Malcev algebra, $H$ a nilpotent subalgebra, and $M$ a Malcev module over $A$. If $A = J(A, A, A)$, and $A = \bigoplus A_\pi$ with $\pi \in \Phi$ the primary decomposition over $H$ then for $M$ over $H$ we have*

$$M = \bigoplus M_\pi \quad \textit{for } \pi \in \Phi.$$

PROOF. By base field extension we may consider roots instead of characteristic primary functions. Thus let $A = \bigoplus A_\gamma$, $\gamma \in \Delta$, be a $H$-root space decomposition. Assume $M \neq \bigoplus M_\gamma$, $\gamma \in \Delta$. Then there exists $M_\beta \neq \{0\}$ with $\beta \notin \Delta$. From (8)–(10) then $J(M_\beta, A, A) = \{0\}$. By (4)

$$M_\beta \subset M_\beta A = M_\beta J(A, A, A) \subset J(M_\beta, A, A) = \{0\}.$$

Thus $M_\beta = \{0\}$, proving the lemma. □

Let $h \in A$, $h \neq 0$. If $H = \langle h \rangle$ and $\alpha$: $H \to k$ a $k$-linear map we may identify $\alpha$ with $\alpha(h)$. We have

LEMMA 2. *Let $A$ be a Malcev algebra, $h \in A$ with $h \neq 0$, and $H = \langle h \rangle$, $M$ a Malcev module over $A$. Suppose that $A$ and $M$ are smooth over $H$. The root spaces are taken over $H$. Let $A = A_\alpha \oplus H \oplus A_{-\alpha}$ with $\alpha \neq 0$. For $\beta \in \{\alpha, -\alpha\}$ let $M_{2\beta} = \{0\}$. Then for $m \in M_\beta$, $n \in M_0$, and $x, y \in A_\beta$, $y' \in A_{-\beta}$ with $xy' = \delta h$, $\delta \in k$ we get*

$$mx \cdot y' = -2my' \cdot x - 2\beta\delta m, \tag{11}$$

$$mx \cdot y = -m \cdot xy, \tag{12}$$

$$nx \cdot y = -n \cdot xy, \tag{13}$$

$$nx \cdot y' + \beta\delta n \in N_M. \tag{14}$$

PROOF. By (6) $M_\gamma A_\gamma \subset M_{-\gamma}$. For (11) observe

$$\begin{aligned}\beta my' \cdot x &= (mh \cdot y')x \quad \text{and by (3)}\\ &= -(hy' \cdot x)m - (y'x \cdot m)h - (xm \cdot h)y' + xh \cdot my'\\ &= \beta\delta hm + \delta hm \cdot h + \beta xm \cdot y' + \beta x \cdot my'\\ &= -2\beta^2\delta m - \beta mx \cdot y' - \beta my' \cdot x.\end{aligned}$$

Thus $mx \cdot y' = -2my' \cdot x - 2\beta\delta m$.

To obtain (12), consider

$$\begin{aligned}\beta mx \cdot x &= (mh \cdot x)x \quad \text{and again by (3)}\\ &= -(xm \cdot h)x + xh \cdot mx = -2\beta mx \cdot x.\end{aligned}$$

Hence $3\beta mx \cdot x = 0$, therefore, by $\operatorname{char}(k) \neq 3$ and $\beta \neq 0$, $mx \cdot x = 0$. Linearization gives $mx \cdot y = -my \cdot x$. Again applying (3)

$$\begin{aligned}\beta mx \cdot y = (mh \cdot x)y &= -(hx \cdot y)m - (xy \cdot m)h - (ym \cdot h)x + yh \cdot mx\\ &= -\beta m \cdot xy - \beta my \cdot x - \beta mx \cdot y,\end{aligned}$$

and hence $mx \cdot y = -m \cdot xy$, which is (12).

To establish (13), from (3)

$$(nx \cdot x)h = -(xh \cdot n)x + hx \cdot nx = -\beta xn \cdot x - \beta x \cdot nx = 2\beta nx \cdot x.$$

Since $M_{2\beta} = \{0\}$ then $nx \cdot x = 0$. By means of linearization $nx \cdot y = -ny \cdot x$. Through further application of (3)

$$\begin{aligned}nx \cdot y = \beta^{-1} nx \cdot yh &= \beta^{-1}\{(yx \cdot h)n + (xh \cdot n)y + (hn \cdot y)x + (ny \cdot x)h\}\\ &= -n \cdot xy - nx \cdot y - ny \cdot x = -n \cdot xy.\end{aligned}$$

Let $w \in A_\beta$, and $w' \in A_{-\beta}$. For (14) we obtain by (3) and (13)

$$\begin{aligned}(nx \cdot y')w &= -(xy' \cdot w)n - (y'w \cdot n)x - (wn \cdot x)y' + wx \cdot ny' \\ &= -\beta\delta nw - (n \cdot wx)y' - ny' \cdot wx = -\beta\delta nw.\end{aligned}$$

Noting (8) then $(nx \cdot y')w' = (ny' \cdot x)w' = -\beta\delta nw'$. Therefore $(nx \cdot y' + \beta\delta n) \cdot A_\gamma = \{0\}$ for $\gamma = 0, \alpha, -\alpha$ which proves (14). □

COROLLARY 1. *Let $A$ be split of type* $\mathrm{A}_1$, *$M$ a Malcev module over $A$, $H$ a splitting Cartan subalgebra of $A$, $H = \langle h \rangle$, and $M$ smooth for $H$. For any root $\beta$ of $H$ with $\beta \neq 0$ and $A_\beta \neq \{0\}$ let $M_{2\beta} = \{0\}$.*

*Then*

$$M = N_M \oplus . J(M, A, A).$$

*$J(M, A, A)$ is completely reducible over $A$.*

PROOF. Let $M \neq N_M$. Take a basis $\{x_\alpha, x_{-\alpha}, h\}$ for $A$ with $\alpha \in k \setminus \{0\}$, $x_\alpha x_{-\alpha} = h$ and $x_\beta h = \beta x_\beta$ for $\beta \in \{\alpha, -\alpha\}$. By smoothness, $M$ is split over $H$. Since $J$ alternates and (7)–(9) then $M = N_M + (M_\alpha \oplus M_{-\alpha})$. Let $m \in M_\beta$ with $mx_\beta \neq 0$. By $M_{2\beta} = \{0\}$ from (11) and (12)

$$(mx_\beta \cdot x_{-\beta})x_\beta = -2\beta\delta m x_\beta$$

with $x_\beta x_{-\beta} = \delta h \neq 0$. Hence $\langle mx_\beta \rangle \oplus \langle mx_\beta \cdot x_{-\beta} \rangle$ is an irreducible non-Lie submodule of type $\mathrm{M}_2[1]$. Thus $P := M_\alpha \cdot x_\alpha \oplus M_{-\alpha} \cdot x_{-\alpha}$ is a sum of submodules of type $\mathrm{M}_2$. From (11) we have $M = P + N_M$. Since $J(mx_\beta, x_\beta, x_{-\beta}) = 3\beta\delta m x_\beta$ this sum is direct. Therefore $M = N_M \oplus J(M, A, A)$. The complete reducibility of $J(M, A, A)$ is trivial. □

LEMMA 3. *Let $A$ be of type $\mathrm{C}_M^-$, $H$ a splitting Cartan subalgebra, and $M$ an $A$-Malcev module.*

*Then $M$ is smooth over $H$.*

PROOF. Since $N_M \cdot A = \{0\}$ the assertion is trivial for $M = N_M$. Let $M \neq N_M$, $E$ the semidirect sum of $A$ and $M$, and $H = \langle h \rangle$. We consider the root spaces over $H$. By Lemma 1, $M_\gamma \neq \{0\}$ implies $A_\gamma \neq \{0\}$. Now $E_0 = H \oplus M_0$. Since $J(H, M_0, E) = \{0\}$, from [5, Lemma 5.12] then $HM_0 \subset N_M$.

Thus $HM_0 \cdot A = \{0\}$. By this and (8) it follows that $M_0 A_\gamma \subset {}_1(M_\gamma)$. Observing $A_\beta = A_{-\beta} A_{-\beta}$ for $\beta \neq 0$ and (3) one gets ${}_1(M_\beta)A_\beta \subset {}_1(M_{-\beta})$. Hence the sum of $H$-eigen spaces of $M$ is a submodule.

Let $n \in M_0$, $x \in A_\beta$, $x' \in A_{-\beta}$, $\beta \neq 0$, and $xx' = h$. With (14) and observing $nh = nx \cdot x' - nx' \cdot x$ together with $nx' \cdot x \in {}_1(M_0)$ then

$$\beta nx \cdot x' = -(nx \cdot x')x \cdot x' = -(nx' \cdot x)x' \cdot x = \beta nx' \cdot x.$$

Thus $nh = 0$, therefore $M_0 = {}_1(M_0)$. Consider now $m \in M_\beta$. We show $m \in {}_1(M_\beta)$. Assume that $mh \neq \beta m$, and set $\hat{m} := mh - \beta m$. By (8) we then have $\hat{m}x' = mh \cdot x' - \beta mx' = mx' \cdot h = 0$, hence $\hat{m} \cdot A_{-\beta} = \{0\}$. Without restriction let $\hat{m} \in {}_1(M_\beta)$, and $h$, $x_\nu$, $x'_\nu$ with $\nu \in \mathbf{Z}_3$ constitute a basis $T_\beta$. For $\mu, \nu \in \mathbf{Z}_3$ with $\mu \neq \nu$ set $x := x_\mu$, $x' := x'_\mu$, $y := x_\nu$. From (11) together with $\hat{m} \cdot A_{-\beta} = \{0\}$, and (3), (12) one derives $2\beta\hat{m}y = -(\hat{m}x \cdot x')y = \beta\hat{m}y$. Thus $\langle \hat{m} \rangle$ is irreducible over $A$, implying $\beta = 0$ in contradiction to $\beta \neq 0$. Therefore $M_\beta = {}_1(M_\beta)$. □

LEMMA 4. *Let $A$ be split of type $\mathbf{G}_1$, $H$ and $M$ as in Lemma 3, $H = \langle h \rangle$, and the root spaces taken over $H$.*

*If $\beta \neq 0$, $m \in M_\beta$, and $T_\beta = \{x_\nu, x'_\nu, h | \nu \in \mathbf{Z}_3\}$ then*

$$\sum mx'_\nu \cdot x_\nu = -\beta m \quad \text{for } \nu \in \mathbf{Z}_3. \tag{15}$$

PROOF. Set $x := x_1, y := x_2, z := x_3, x' := x'_1$, and similarly for $y'$, $z'$. We get

$$\begin{aligned} mx' \cdot x &= -\tfrac{1}{2}(yz \cdot m)x \quad \text{and by (3)} \\ &= \tfrac{1}{2}\{(zm \cdot x)y + (mx \cdot y)z + (xy \cdot z)m - xz \cdot ym\} \\ &= my' \cdot y - mz' \cdot z + \beta m + my \cdot y' \quad \text{with (12),} \\ &= -my' \cdot y - mz' \cdot z - \beta m \quad \text{by (11).} \end{aligned}$$

Hence $mx' \cdot x + my' \cdot y + mz' \cdot z = -\beta m$ which is (15). □

For a $k$-vector space and an extension field $K$ of $k$ we define $X_K := X \otimes_k K$. We get

THEOREM 1. *Let $A$ be a Malcev algebra of type $\mathbf{G}_1$ and $M$ a Malcev module over $A$.*

*Then $M$ is completely reducible over $A$.*

PROOF. For $M = N_M$ the theorem is trivial. Suppose $M \neq N_M$.

(1) Let $A$ be of type $\mathbf{C}_M^-$, and $H$ a splitting Cartan subalgebra, $M = M_\alpha \oplus M_0 \oplus M_{-\alpha}$ the root space decomposition of $M$ over $H$ with $\alpha \neq 0$ according to Lemma 1. If $M = M_0$ then $M = N_M$ by Lemma 3. Thus let $M_\alpha \neq \{0\}$, and $m \in M_\alpha$, $m \neq 0$. From (15) there exists $z' \in A_{-\alpha}$ with $n := mz' \neq 0$. We show that $n$ generates a regular submodule by an argument similar to that in [1]. Take the $k$-linear map $f: A \to M$ defined by

$$f(h) := n, \qquad f(x) := \beta^{-1}xn \quad \text{if } x \in A_\beta$$

when $\beta \neq 0$. We claim that $f$ is a module homomorphism over $A$. For $x, y \in A_\beta$, $\beta \neq 0$, we obviously have $xf(h) = f(xh)$, $hf(x) = f(hx)$, and by (13), $xf(y) = f(xy)$. It remains to show for $y' \in A_{-\beta}$ that $y'f(x) = f(y'x)$, equivalently

$$nx \cdot y' = -\beta\delta n \tag{16}$$

where $xy' = \delta h$, $\delta \in k$. We may restrict ourselves to a basis $T_\beta$ of $A$ corresponding to $H$ with $x_\nu \in A_\beta$. Let $x_\mu x'_\nu = \delta h$, and $x_\mu x'_\lambda = \eta h$ with $\delta$, $\eta \in k$, where $\lambda, \mu, \nu \in \mathbf{Z}_3$. Then by (11) and (12)

$$(mx'_\lambda \cdot x_\mu)x'_\nu = -\tfrac{1}{2}(mx_\mu \cdot x'_\lambda)x'_\nu - \beta\eta m x'_\nu$$
$$= -\tfrac{1}{2}m(x_\mu \cdot x'_\lambda x'_\nu) - \beta\eta m x'_\nu = -\beta\delta m x'_\lambda.$$

We derive the last equality from the multiplication relations for $T_\beta$. Hence we have (16). Since $f(h) = n \neq 0$, $f$ is an $A$-module monomorphism of $A$ in $M$.

Thus $P := M_\alpha \cdot x'_1 \oplus M_\alpha \oplus M_{-\alpha}$ obviously is a direct sum of regular submodules by (15). For $n' \in M_0$ from (14) then $p := n'x_1 \cdot x'_1 + \beta n' \in N_M$. Hence $M_0 = N_M + P_0$. By $N_M A = \{0\}$ and (16) the sum is direct, and we have $M = N_M \oplus . P$. $N_M$ and $P$ are completely reducible over $A$, hence $M$ too.

(2) Suppose that $A$ is not split. By [4] $A$ has a Cartan subalgebra $H$ with $H = \langle h \rangle$. Now $A = H \oplus A_\pi$ over $H$ with $\pi_h = Y^2 - c(h)$ and $c(h) \in k \setminus k^2$. Then $M = M_0 \oplus M_\pi$ by Lemma 1. Let $K$ denote a splitting field of $\pi_h$.

Set $P := J(M, A, A)$. Then $M = N_M \oplus . P$ by the corresponding decomposition for $M_K$. For $P_0$ we choose a basis $\{n_i | 1 < i < s, s \in \mathrm{N}\}$. Let $P(i) := k \cdot n_i \oplus A \cdot n_i$. Then $P(i)_K = K \cdot n_i \oplus A_K \cdot n_i$. $P(i)_K$ is regular over $A_K$ by (1). If $f$ denotes the $A_K$-module monomorphism of $A_K$ in $M_K$ with $f(h) = n_i$ then $f(A)$ is regular over $A$. Further

$$f(A) = f(k \cdot h) \oplus f(A \cdot h) = k \cdot f(h) \oplus A \cdot f(h) = P(i).$$

$P(i)$ is irreducible, and $P = \bigoplus . P(i)$ for $1 < i < s$. Hence $M$ is completely reducible over $A$.

The theorem is proved. □

The following propositions are well known if $\mathrm{char}(k) = 0$ for the semi-simple case [2], [4]. They extend the classical structure theory to positive characteristics for the exceptional case.

PROPOSITION 2. *Let $A = \bigoplus . A_i$ with $1 < i < r$, $r \in \mathrm{N}$, where each $A_i$ is of type* $\mathrm{G}_1$. *Let $M$ be a Malcev module over $A$.*

*Then $M$ is completely reducible over $A$.*

*Moreover $M = N_M \oplus . (\bigoplus . P_j)$ with $1 < j < s$, $s \in \mathrm{N}_0$, $N_M A = \{0\}$, where for any index $j$ there is an index $i$ so that $P_j$ is regular over $A_i$, and $P_j A_l = \{0\}$ if $l \neq i$ for $1 < l < r$.*

PROOF. If $r = 1$ the first part of the statement is Theorem 1, and the second is a corollary of the proof of Theorem 1. We proceed by induction on the number of simple direct factors of $A$, and assume that the statement is valid for $r \in \mathrm{N}$. Let now $A = \bigoplus . A_i$ with $1 < i < r + 1$. Set $A' = \bigoplus . A_i$, $2 < i < r + 1$. We choose a Cartan subalgebra $H$ of $A$. Then $H = H_1 \oplus . H_2$, $H_1$ a Cartan subalgebra of $A_1$ and $H_2$ a Cartan subalgebra of $A'$.

If $N(1)$ designates the nucleus of $M$ over $A_1$ then

$$M = N(1) \oplus J(M, A_1, A_1)$$

is a sum of completely reducible $A_1$-submodules of $M$. $J(M, A_1, A_1)$ decomposes into a direct sum of regular $A_1$-submodules. We show that $N(1)$ and $J(M, A_1, A_1)$ are submodules over $A$. For this we may assume that $H$ is splitting over $A$. The root spaces of $A$ for $H$ unequal to $H$ are just those of $A_1$ for $H_1$ and of $A'$ for $H_2$ unequal to $H_1$ and $H_2$. The corresponding characteristic roots $\gamma$ are obvious. When $\gamma(H_1) \neq \{0\}$ then $\gamma(H_2) = \{0\}$ thus $A_\gamma \subset A_1$, and vice versa. Applying (4) and Lemma 3 together with (8)–(10) we get

$$J(M, A_1, A_1)A' \subset J(M, A_1, A') = \{0\}.$$

For example if $\beta(H_1) \neq \{0\}$ then $\beta(H_2) = \{0\}$ hence $J(M_\beta, A_\beta, H_2) = \{0\}$ by smoothness of $A$ and $M$ over $H_2$. If $\Delta_1$ is the set of the characteristic roots of $H_1$ in $A_1$, let $(A_1)^1 := \bigoplus (A_1)_\delta(H_1)$ with $\delta \in \Delta_1 \setminus \{0\}$. From (8) for $H_1$ one has $N(1)A' \cdot (A_1)^1 = \{0\}$. Noting Lemma 3 then $N(1)A' \cdot A_1 = \{0\}$. Thus $N(1)A' \subset N(1)$.

Hence the above yields a direct sum of $A$-modules. By the induction hypothesis $N(1)$ decomposes as asserted over $A'$. The proposition is evident. □

The *radical* $R$ of $A$ is by definition the unique maximal solvable ideal. $A$ is called *semisimple* if $R = \{0\}$. *Separability* is defined as usual. In case of $\operatorname{char}(k) = 0$ any semisimple Malcev algebra is separable by the nondegeneracy of the Killing form. $A$ is called $\mathbf{G}_1$-*separable* if there is a base field extension $K$ of $k$ so that the base field extension $A_K$ decomposes into a direct sum of algebras of type $\mathbf{C}_M^-$.

Since the hypothesis of characteristic 0 in the proof of [2, Theorem 2] is only used to establish that $M$ is reducible that proof actually gives the following slightly stronger result

PROPOSITION 3. *Let $A$ be a Malcev algebra and $M$ a Malcev module over $A$. If $A$ is $\mathbf{G}_1$-separable, then any derivation of $A$ in $M$ is inner.* □

COROLLARY 2. *Let $A$ be a Malcev algebra, and $C$ a $\mathbf{G}_1$-separable subalgebra. Then any derivation of $C$ in $A$ can be extended to an inner derivation of $A$.* □

For an ideal $I$ of $A$ let $\mathcal{H}^0(I) := I$ and $\mathcal{H}^r(I) := \mathcal{H}^{r-1}(I) \cdot I + (\mathcal{H}^{r-1}(I) \cdot I) \cdot A$ if $r \in \mathbf{N}$. $I$ is called $\mathcal{H}$-*nilpotent* if $\mathcal{H}^r(I) = \{0\}$ for some $r \in \mathbf{N}$ [2]. The *index* $n_{\mathcal{H}}$ of $\mathcal{H}$-nilpotency is the minimal $n_{\mathcal{H}} \in \mathbf{N}$ with $\mathcal{H}^{n_{\mathcal{H}}}(I) = \{0\}$. Nilpotency and $\mathcal{H}$-nilpotency of $I$ are equivalent. The *nilradical* $N$ of $A$ is by definition the maximal nilpotent ideal, hence $N \subset R$. We recall, if $B$ is a subalgebra of $A$, and $A = B \oplus R$ then this decomposition is called a *Wedderburn* or *Levi decomposition*, and $B$ a *Wedderburn* or *Levi factor* of $A$.

Similarly as in [2, Theorem 3] we get as a further consequence of Theorem 1

PROPOSITION 4. *Let $A$ be a Malcev algebra with radical $R$. $n_{\mathcal{K}}$ denotes the index of $\mathcal{K}$-nilpotency of the nilradical of $A$. Suppose that* $\mathrm{char}(k) > 2n_{\mathcal{K}} - 1$. *Let $B$ be a Levi factor, and $C$ a $\mathbf{G}_1$-separable subalgebra of $A$.*

*Then there is an inner automorphism $\alpha$ of $A$ with*

$$C^{\alpha} \subset B. \quad \square$$

COROLLARY 3. *Let $A$ be as in Proposition 4, and $A/R$ $\mathbf{G}_1$-separable. Then any two Levi factors are conjugate by an inner automorphism of $A$.* $\square$

**4. The Wedderburn splitting.** Let $S$ be an ideal of $A$, and $S^2 = \{0\}$. Let $\varphi$: $A \to A/S$ with $x \mapsto \underline{x} := x + S$ denote the canonical map. If $H$ is a nilpotent subalgebra of $A$ and $\gamma$ a linear root of $H$ we define $\gamma$: $H^{\varphi} \to k$ by $\gamma(\underline{h}) := \gamma(h)$ if $h \in H$. Then obviously

$$(A_{\gamma}(H))^{\varphi} = (A^{\varphi})_{\gamma}(H^{\varphi}) \quad \text{and} \quad (A_{\gamma}(H))^{\varphi} = A_{\gamma}(H)/S_{\gamma}(H). \qquad (17)$$

$S$ is a Malcev module over $A^{\varphi}$ in the canonical way. If $C \subset A^{\varphi}$ denote $C^{\varphi^{-1}} := \varphi^{-1}(C)$. Let $M_{\gamma}(h) := M_{\gamma}(\langle h \rangle)$.

LEMMA 5. *Let $S$ be an ideal of $A$ with $S^2 = \{0\}$, and $L$ an abelian subalgebra of $A/S$. Furthermore, let $S$ be smooth over $L$. Then $A$ contains a subalgebra $H$ with $H^{\varphi} = L$ and $H^3 = \{0\}$.*

PROOF. If $\dim(L) = 0$ the assertion is trivial. We use induction on the dimension of $L$ and assume the statement of the lemma for some $n \in \mathbf{N}_0$. Suppose $\dim(L) = n + 1$ and $c \in L$, $c \neq 0$. By the hypothesis of the induction there exists a subalgebra $T$ of $A$ with $T^3 = \{0\}$, $T^{\varphi} \subset L$, $\dim(T^{\varphi}) = n$, and $c \notin T^{\varphi}$. Then $T \subset A_0(T)$, and $L \subset A_0(T)^{\varphi}$ by (17).

We choose $h \in A_0(T)$ with $\underline{h} = c$. Further $S = \bigoplus S_{\gamma}(h)$ for $\gamma \in \Delta$ denotes the root space decomposition over $h$. Let $h_i \in T$ for $i = 1, \ldots, n$, the $\underline{h_i}$ linearly independent. Then $h_i h = \sum {}^i r_{\gamma}$ with $\gamma \in \Delta$ and ${}^i r_{\gamma} \in S_{\gamma}(h) \cap A_0(T)$. Set $h_i^* := h_i - \sum \beta^{-1}\, {}^i r_{\beta}$ for $\beta \in \Delta \setminus \{0\}$. Note $h_i^* \in A_0(T) \cap A_0(h)$. Let $H$ be the subalgebra of $A$ generated by $h$ and the $h_i^*$ for $i = 1, \ldots, n$. Hence $H \subset A_0(T) \cap A_0(h)$, and $H^2 \subset S_0(L)$. Thus $H^3 = \{0\}$. $\square$

We prove

THEOREM 5. *Let $A$ be a Malcev algebra over $k$, $R$ the radical of $A$, and* $\mathrm{char}(k) = 0$, *or* $\mathrm{char}(k) > 3$. *If* $\mathrm{char}(k) > 3$ *let $A/R$ be $\mathbf{G}_1$-separable.*

*Then $A$ decomposes*

$$A = B \oplus R$$

*where $B$ is a semisimple subalgebra of $A$ with $B \cong A/R$.*

PROOF. If $A/R = \{0\}$ or $R = \{0\}$ then the theorem is trivial. Assume that $A/R \neq \{0\}$ and $R \neq \{0\}$. By standard reduction we may assume $R^2 = \{0\}$, and $R$ an irreducible $A$-Malcev module. Further we may suppose that $k$ is algebraically closed. So $A/R = \oplus . C_i$ with $1 \leq i \leq n$, $n \in \mathbf{N}$, any $C_i$ a simple split subalgebra. In the course of proof we will distinguish different cases. Let $\underline{A} := A/R$, $\underline{x} := x + R$, and $\varphi: x \mapsto \underline{x}$.

(1) Let char$(k) = 0$, $\underline{A}$ a Lie algebra, and $R$ a Lie module over $\underline{A}$. Then $A$ is a Lie algebra: If $J(A, A, A) = \{0\}$ there is nothing to show. Otherwise $J(A, A, A) = R$. By Lemma 5 there is obviously a Cartan subalgebra $H$ of $A$ so that $H^{\varphi}$ is a Cartan subalgebra of $\underline{A}$, and $H^3 = \{0\}$. Decompose $A$ into $H$-root spaces. Since $H = A_0$, then $J(A_0, A_0, A_0) = \{0\}$. From $\dim((A_\beta)^{\varphi}) \leq 1$ for $\beta \neq 0$ and $R$ Lie we then have $J(A, A, A) = \{0\}$. Hence $A$ is a Lie algebra for which the theorem is known.

(2) It remains to treat the case that $R$ is not a Lie module over $\underline{A}$, or $\underline{A}$ is not a Lie algebra, or char$(k) > 3$ with $\underline{A}$ $\mathrm{G}_1$-separable. We proceed by induction on the number $n$ of the simple ideals of $\underline{A}$.

Let $n = 1$. Suppose that $\underline{A}$ is a Lie algebra of type $\mathbf{A}_1$. Let $h \in A$ so that $\langle \underline{h} \rangle$ is a Cartan subalgebra of $\underline{A}$. Decompose $A$ and $R$ over $h$. Let us consider three cases for $R$. If $R$ is the one-dimensional zero module then $R = \langle r_0 \rangle$ and $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$, $\alpha \neq 0$, with $R \subset A_0$. We choose $h' \in A_\alpha A_{-\alpha}$ with $\underline{h'} = \underline{h}$. Then obviously $A_\alpha \oplus \langle h' \rangle \oplus A_{-\alpha}$ is a Levi factor of $A$.

If $R$ is non-Lie then $R$ is necessarily of type $\mathbf{M}_2$ over $\underline{A}$, and $R = R_\alpha \oplus R_{-\alpha}$ with $R_\beta = \langle r_\beta \rangle$ where $R_\beta \subset A_\beta$ for $\beta \in \{\alpha, -\alpha\}$. Let $x_\beta \in A_\beta$ with $\underline{x_\beta} \neq 0$. Then $J(x_\alpha, x_{-\alpha}, h) = 0$. Any Lie triple of elements generates a Lie subalgebra. Hence $\langle h, x_\alpha, x_{-\alpha} \rangle$ is a Levi factor of $A$.

Assume third that $R$ is regular over $A$. Hence $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$ with $A_0 = \langle h, r_0 \rangle$, $A_\beta = \langle x_\beta, r_\beta \rangle$ with $\beta \in \{\alpha, -\alpha\}$, and $r_\beta := \beta^{-1} x_\beta r_0$. Note that a canonic $A$-module isomorphism is induced by $h \mapsto r_0$, and $x_\beta \mapsto r_\beta$. Suppose that $\{\underline{x_\alpha}, \underline{x_{-\alpha}}, h\}$ is a standard basis for $\underline{A}$. After eventual substitutions $h - \gamma r_0/\alpha$, or $x_{-\alpha} - \delta r_{-\alpha}$ with $\gamma, \delta \in k$, for $h$ or $x_{-\alpha}$ if necessary then $\langle x_\alpha, x_{-\alpha}, h \rangle$ is a Levi factor of $A$.

Now let $\underline{A}$ be of type $\mathbf{C}_M^-$ and $R$ regular over $\underline{A}$. Take a basis $T_\alpha$ of $\underline{A}$, $T_\alpha = \{y_\nu, y'_\nu, u \mid \nu \in \mathbf{Z}_3\}$ and set $C := \langle u, y_1, y'_1 \rangle$. $C$ is a subalgebra of type $\mathbf{A}_1$. $R$ has a $C$-decomposition

$$R = B_{1R} \oplus . N_{1R} \oplus . N_{2R},$$

with $B_{1R}$ regular and $N_{1R}$, $N_{2R}$ of type $\mathbf{M}_2$ over $C$ [1]. In view of the minimal solvable ideals of $C^{\varphi^{-1}}$, and its completely reducible radical, $C^{\varphi^{-1}}$ contains a Levi factor $B_1$. Let $x, x', h \in B_1$ with $\underline{x} = y_1$, $\underline{x'} = y'_1$, $\underline{h} = u$, and $H := \langle h \rangle$. We decompose $A$ over $H$ into root spaces, $A = A_\alpha \oplus A_0 \oplus A_{-\alpha}$.

We claim $A_\gamma = {}_1(A_\gamma)$. For $\nu \in \{2, 3\}$ choose $x_\nu \in A_\alpha$, $x'_\nu \in A_{-\alpha}$ with $\underline{x_\nu} = y_\nu$, $\underline{x'_\nu} = y'_\nu$. Let $r_0 \in R_0$, $r_0 \neq 0$. If $\beta \neq 0$ and $z \in A_\beta$ set $r_z := \beta^{-1} z r_0$.

Since a Lie triple $x_\nu, x'_\nu, h$ generates a Lie subalgebra, $x_\nu h = \alpha x_\nu + \delta_\nu r_{x_\nu}$, with $\delta_\nu \in k$. We show $\delta_\nu = 0$. For

$$\begin{aligned}\alpha x_\nu x \cdot x' &= xx_\nu \cdot x'h \\ &= (x'x_\nu \cdot h)x + (x_\nu h \cdot x)x' + (hx \cdot x')x_\nu + (xx' \cdot x_\nu)h \quad \text{by (3)} \\ &= \alpha x_\nu x \cdot x' + \delta_\nu (r_{x_\nu} \cdot x)x' + \alpha x_\nu h - \alpha x_\nu h - \delta_\nu r_{x_\nu} h \\ &= \alpha x_\nu x \cdot x' - 3\alpha\delta_\nu r_{x_\nu},\end{aligned}$$

hence $\delta_\nu = 0$.

Therefore $A_\alpha = {}_1(A_\alpha)$, and equally for $-\alpha$. Thus $A$ is smooth for $H$.

By Corollary 1, $A$ is completely reducible over $B_1$. Hence

$$A = B_{1R} \oplus . B_1 \oplus . N_{1R} \oplus . N_{2R} \oplus . N_1 \oplus . N_2$$

with $N_1, N_2$ of type $\mathbf{M}_2$ over $B_1$. We may assume $x_2, x_3' \in N_1$ and $x_3, x_2' \in N_2$. If $x_2 x_2' = h + \eta r_0$ with $\eta \in k \setminus \{0\}$, replace $x_2$ by $x_2^* := x_2 - \eta r_{x_2}$. Hence we may suppose $x_2 x_2' = h$.

We assert that $B := B_1 \oplus N_1 \oplus N_2$ is an algebra of type $\mathbf{C}_M^-$. We let $y := x_2$, $y' := x_2'$, $z := x_3$, $z' := x_3'$. Then

$$\begin{aligned}yz' &= (2\alpha)^{-1} yh \cdot xy = (2\alpha)^{-1}\{(xh \cdot y)y + (yx \cdot h)y\} \quad \text{by (3)} \\ &= xy \cdot y = -2y \cdot z'.\end{aligned}$$

Thus $yz' = 0$. Similarly $zy' = 0$. Further

$$\begin{aligned}zz' &= (2\alpha)^{-1} x'y' \cdot xy, \quad \text{and with (3)} \\ &= \tfrac{1}{2}\{xx' + yy'\} = h.\end{aligned}$$

From this with (3)

$$yz = \alpha^{-2} z'x' \cdot x'y' = 2x'$$

and similarly $y'z' = \alpha x$. Therefore $B^2 \subset B$. Hence $B$ is a Levi factor of $A$.

If $R$ is the one-dimensional zero module, take $B_1$ as before. Similarly $A$ has a $B_1$-module decomposition

$$A = B_1 \oplus . R \oplus . N_1 \oplus . N_2.$$

By a similar argument one derives that $B := B_1 \oplus N_1 \oplus N_2$ is a Levi factor. Thus the theorem is shown if $\underline{A}$ is of type $\mathbf{A}_1$ or of type $\mathbf{C}_M^-$ when $\mathrm{char}(k) \neq 2, 3$. Let $\mathrm{char}(k) = 0$. Then by [1, Satz 11] we know if $\underline{A}$ is a simple Lie algebra not of type $\mathbf{A}_1$ then $R$ is a Lie module over $\underline{A}$, and the decomposition exists by (1). Hence we have shown the theorem for $n = 1$.

We assume as induction hypothesis that the theorem is valid if $\underline{A}$ has exactly $n$ simple direct factors, $n \in \mathbf{N}$. Let $\underline{A} = \oplus . C_i$, $1 < i < n + 1$. By (1) and [1, Satz 11] the remaining part of the proof is obviously reduced to the case that $C_1$ is either of type $\mathbf{C}_M^-$, or $C_1$ is of type $\mathbf{A}_1$ with $R$ non-Lie over $C_1$. In the latter case by the classification $R$ is a module of type $\mathbf{M}_2$ over $C_1$. Set $G := \oplus . C_i$ with $2 < i < n + 1$. In view of [2, Theorem 1] or of Proposition 2 respectively, we have either $RC_1 = R$ and $RG = \{0\}$, or $RC_1 = \{0\}$.

Let $B_1$ be a Levi factor of $C_1^{\varphi^{-1}}$, existing by the preceding argument. $H_1$ denotes a Cartan subalgebra of $B_1$. Now $RB_1 = R$, or $RB_1 = \{0\}$. In the first case let $\hat{A}_0 := (G^{\varphi^{-1}})_0(H_1)$. Hence $(\hat{A}_0)^\varphi = G$ by (17). By the induction hypothesis $\hat{A}_0$ contains a Levi factor $B_2$. Take a Cartan subalgebra $H_2$ of $B_2$, and set $H := H_1 \oplus. H_2$. Then $H^2 = \{0\}$ by smoothness. We decompose $A$ into $H$-root spaces $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$, $\Delta$ the set of characteristic roots of $H$ in $A$.

If $\Delta_i$ is the set of the characteristic roots of $H_i$ in $B_i$ for $i \in \{1, 2\}$ and $\gamma \in \Delta_i$, let $\gamma^*\colon H \to k$ be the trivial linear extension with $\gamma^*(h) := \gamma(h)$ if $h \in H_i$, and $\gamma^*(h) = 0$ for $h \in H_j$ if $j \in \{1, 2\}$ and $j \neq i$. Set $\Delta_i^* := \{\gamma^* | \gamma \in \Delta_i\}$. Then $\Delta = \Delta_1^* \cup \Delta_2^*$. Hence $B_2 \subset H_2 \oplus (\bigoplus A_\beta)$, $\beta \in \Delta_2^* \setminus \{0\}$. Observing (17), $B_1 \subset (H_1 \oplus (\bigoplus A_\alpha)) + R$ with $\alpha \in \Delta_1^* \setminus \{0\}$. Because of $RB_2 = \{0\}$ and the composition of the root spaces with (5), $B_1B_2 = \{0\}$. Thus $B := B_1 \oplus. B_2$ is a Levi factor of $A$.

Finally suppose $RB_1 = \{0\}$. Decompose $A$ as a $B_1$-module, $A = B_1 \oplus. R \oplus. V$ with $V^\varphi = G$. Since $B_1V \subset R \cap V$ then $B_1V = \{0\}$. Further $V \oplus R = G^{\varphi^{-1}}$ is a subalgebra. It contains a Levi factor $B_2$ by the hypothesis of the induction. Thus $B := B_1 \oplus. B_2$ is a Levi factor of $A$.

This proves the theorem. □

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