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ANALYTICALLY DECOMPOSABLE OPERATORS[1]

BY

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ABSTRACT. The author introduces the notion of an analytically decomposable operator which generalizes the decomposable operator due to C. Foiaş in that the spectral decompositions of the underlying Banach space (1) admit a wider class of invariant subspaces called "analytically invariant" and (2) span the space only densely. It is shown that analytic decomposability is stable under the functional calculus, direct sums and restrictions to certain kinds of invariant subspaces, as well as perturbation by commuting scalar operators. It is fundamental for many of these results that every analytically decomposable operator has the single-valued extension property. An extensive investigation of analytically invariant subspaces is given. The author shows by example that this class is distinct from those of spectral maximal and hyperinvariant subspaces, but he further shows that analytically invariant subspaces have many useful spectral properties. Some applications of the general theory are made. For example, it is shown that under certain restrictions an analytically decomposable operator is decomposable.

Introduction. In their monograph on generalized spectral operators, Colojorara and Foiaş [4] begin with a development of the general theory of decomposable operators. Such operators may be defined as follows. Let T be a bounded linear operator on the complex Banach space X. A (closed) T-invariant subspace Y is called spectral maximal if for every other invariant subspace Z the spectral inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies the subspace inclusion $Z \subset Y$. The operator T is said to be decomposable if for each finite open cover $\{G_1, G_2, \ldots, G_n\}$ of $\sigma(T)$ there are invariant subspaces $Y_1, Y_2, \ldots, Y_n$ such that

(i) each Y_i is spectral maximal;

(ii) $X = Y_1 + Y_2 + \cdots + Y_n$;

(iii) $\sigma(T|Y_i) \subset G_i$ for each i.

We stress here that the sum in (ii) is a *linear* sum which is to be contrasted with the more general notion considered below.

In the present paper, we generalize the theory of decomposable operators in two directions by relaxing conditions (i) and (ii). First, we broaden the

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[1]This research is part of the author's Ph.D. thesis at Temple University, supervised by Professor I. Erdélyi.

class of admissible invariant subspaces in (i). This new type of subspace was introduced by Frunză [5], who called it "analytically invariant." The T-invariant subspace Y is called analytically invariant if for each X-valued analytic function f defined on a region V_f in the complex plane such that $(\lambda - T)f(\lambda) \in Y$ for $\lambda \in V_f$, then it follows that $f(\lambda) \in Y$ for $\lambda \in V_f$. Frunză has shown [5] that for a wide class of operators "spectral maximal" implies "analytically invariant." In §1 below we give several examples (in varying contexts) which show that the converse is false.

Second, we shall weaken condition (ii) to read "X is the *closed* linear span of $Y_1, Y_2, \ldots, Y_n$." We now formally define our principal object of study in the present paper.

DEFINITION. A bounded linear operator T on X is said to be analytically decomposable if, given any finite open cover $\{G_i\}$ of $\sigma(T)$, there are corresponding invariant subspaces Y_i such that

(i′) each Y_i is analytically invariant;

(ii′) $X = \mathrm{clm}\{Y_1, Y_2, \ldots, Y_n\}$;

(iii′) $\sigma(T|Y_i) \subset G_i$ for each i.

§1 is devoted to a systematic study of analytically invariant subspaces. Some of these results are important in later sections.

The main results concerning analytically decomposable operators are to be found in §2 of this paper where we prove that these operators share many properties with decomposable operators, especially the single-valued extension property (see §1). It will also be shown (through an example of E. Albrecht) that there are analytically decomposable operators that are not decomposable. Moreover, analytically decomposable operators have certain properties peculiar unto themselves (e.g., Theorem 2.9).

In §3 we give some applications of the general theory developed in §2. For example, we shall prove that an operator satisfying conditions (i′), (ii) and (iii) is decomposable. Theorem 3.3 states that analytic decomposability is stable under perturbation by commuting scalar-type operators.

1. Analytically invariant subspaces. Analytically invariant subspaces were first studied by Frunză [5] who characterized them in the following proposition. To state this result we must introduce some notation. An operator T has the single-valued extension property (s.v.e.p.) if the zero subspace (0) is analytically invariant under T. For an arbitrary T-invariant subspace Y we let T^Y denote the coinduced operator on the quotient Banach space X/Y (i.e., $T^Y(x + Y) = (Tx) + Y$ for $x \in X$).

1.1 PROPOSITION. *Let T be an operator with invariant subspace Y. Then Y is analytically invariant if and only if T^Y has the s.v.e.p.*

A natural question to ask about analytically invariant subspaces is how they are related to better known ones like spectral maximal or hyperinvariant subspaces. As a consequence of Proposition 1.1, Frunză showed that if T itself has the s.v.e.p. then all its spectral maximal subspaces are analytically invariant. Various examples below show that the converse is not true. We shall also see that there are analytically invariant subspaces which are not hyperinvariant, and vice-versa. On the other hand, our first theorem gives a sufficient condition under which an analytically invariant subspace is spectral maximal. For this we recall that if T has the s.v.e.p., then for each $x \in X$, the analytic function $\lambda \to (\lambda - T)^{-1}x$, defined for $\lambda \in \rho(T)$, has a unique (hence "single-valued") analytic extension into a maximal open set $\rho(x)$ in the plane. We set $\sigma(x)$ to be the complement of $\rho(x)$ in the (finite) plane. Since $\sigma(x)$ thus depends on T as well as x, in case of ambiguity we write $\sigma(x, T)$ for $\sigma(x)$.

1.2 THEOREM. *Let Y be analytically invariant under T, and suppose that for each $x \in X$, we have*

$$\sigma(x', T^Y) = \overline{\sigma(x, T) \setminus \sigma(T|Y)},$$

where x' is the coset of x in X/Y. Then Y is spectral maximal.

PROOF. Let Z be a T-invariant subspace with $\sigma(T|Z) \subset \sigma(T|Y)$. Since $y \in Z$ implies that $\sigma(y, T) \subset \sigma(T|Z)$, we have $\sigma(y', T^Y) = \varnothing$. Hence $y' = 0$ and it follows that $Z \subset Y$. Thus Y is spectral maximal.

The main property of an analytically invariant subspace Y that makes it suitable for spectral decompositions is the spectral inclusion $\sigma(T|Y) \subset \sigma(T)$. This result is a corollary to

1.3 PROPOSITION. *Let Y be T-invariant. Then $\sigma(T|Y) \subset \sigma(T)$ if and only if Y is invariant under $R(\lambda; T)$ for each $\lambda \in \rho(T)$.*

PROOF. Suppose that the spectral inclusion holds, and $\lambda \in \rho(T) \subset \rho(T|Y)$. Then $\lambda - T$ is clearly injective on Y, and this implies that for $y \in Y$ the equation $y = (\lambda - T)u$ has a unique solution $u \in Y$. Thus Y is invariant under $R(\lambda; T)$. Conversely, if $R(\lambda; T)Y \subset Y$ for all $\lambda \in \rho(T)$, then $R(\lambda; T)|Y = R(\lambda; T|Y)$. Hence $\sigma(T|Y) \subset \sigma(T)$.

1.4 COROLLARY. *If Y is analytically invariant under T, then $\sigma(T|Y) \subset \sigma(T)$.*

PROOF. It suffices to note that if $y \in Y$, then $R(\lambda; T)y$ is analytic on $\rho(T)$ and that $(\lambda - T)R(\lambda; T)y = y$. Hence $R(\lambda; T)y$ lies in Y by analytic invariance, and thus Y is invariant under $R(\cdot; T)$. By Proposition 1.3 we have $\sigma(T|Y) \subset \sigma(T)$.

By the usual methods of "vector chasing" one can prove still more.

1.5 PROPOSITION. *Let Y be a T-invariant subspace. If Y is analytically invariant, then $\sigma(T) = \sigma(T|Y) \cup \sigma(T^Y)$.*

Thus, analytically invariant subspaces share with hyperinvariant (and, in particular, with spectral maximal) subspaces the property of having well-behaved associated spectra. To see that Proposition 1.5 is a nontrivial extension, we now give an example of an analytically invariant subspace which is not spectral maximal; in fact, the following is an analytically invariant subspace which is not even hyperinvariant.

1.6 EXAMPLE. Let T be any operator with the s.v.e.p. and having an eigenspace M of dimension greater than 1. Let N be a one-dimensional subspace of M. Clearly N is not hyperinvariant but the following argument shows that it is analytically invariant.

Let $f\colon V_f \to X$ be analytic and such that $(\lambda - T)f(\lambda) \in N$ if $\lambda \in V_f$. Then any nonzero $x \in N$ must satisfy an identity of the form

$$(\lambda - T)f(\lambda) = g(\lambda)x \quad \text{on } V_f,$$

where g is a scalar-valued analytic function on V_f. Let α be the eigenvalue for N so that on V_f we have

$$(\alpha - T)(\lambda - T)f(\lambda) = g(\lambda)(\alpha - T)x = 0.$$

It follows that $(\alpha - T)f(\lambda) = 0$ on V_f. So, for $\lambda \neq \alpha$, then $(\lambda - \alpha)f(\lambda) = g(\lambda)x$ and one sees that $f(\lambda) \in N$ for all $\lambda \in V_f$.

The next example shows that a hyperinvariant subspace need not be analytically invariant.

1.7 EXAMPLE. Let T be the adjoint to the unilateral shift operator on l^2. If $|\lambda| < 1$, then λ is an eigenvalue of T with one-dimensional eigenspace M_λ generated by the vector

$$x = \sum_{n=0}^{\infty} \lambda^n e_n,$$

where $\{e_n\}$ is the standard orthonormal basis in l^2. Moreover M_λ is hyperinvariant under T. Fix λ and on the disc $|\mu| < |\lambda|$ define (inductively) a sequence of analytic scalar-valued functions t_k by

$$t_{k+1}(\mu) = \mu t_k(\mu) - \lambda^k, \qquad k \geqslant 0. \tag{1.1}$$

It follows from (1.1) that on $|\mu| < |\lambda|$, we have

$$|t_k(\mu)| \leqslant (k+1)|\lambda|^{k-1}.$$

In particular, the function f defined by the series

$$f(\mu) = \sum_{k=0}^{\infty} t_k(\mu)e_k, \qquad |\mu|<|\lambda|,$$

is an l^2-valued analytic function whose range is obviously not contained in M_λ. But for $|\mu| < |\lambda|$, we have

$$(\mu - T)f(\mu) = \sum_k \mu t_k(\mu)e_k - \sum_k t_k(\mu)Te_k$$
$$= \sum_k [\mu t_k(\mu) - t_{k+1}(\mu)]e_k = \sum_k \lambda^k e_k = x.$$

This proves that M_λ is not analytically invariant.

The next result is pivotal in §2; it is the analog of Proposition 1.3.7 in [4] for analytically invariant subspaces.

1.8 THEOREM. *Let Y be analytically invariant under T and suppose that f: $V_f \to X$ is a nonzero analytic function on the connected region V_f such that $(\lambda - T)f(\lambda) = 0$ on V_f. Then $V_f \subset \sigma(T|Y)$.*

PROOF. Since $f(\mu) \in Y$ on V_f, if $V_f \cap \rho(T|Y) \neq \varnothing$ then $f = 0$ on some open set in $V_f \cap \rho(T|Y)$. By analytic continuation it follows that $f = 0$ on V_f. The contradiction proves the theorem.

We now show that analytic invariance is stable under the functional calculus. If g is a scalar-valued analytic function on some open neighborhood B of $\sigma(T)$ and L is a closed rectifiable Jordan contour enclosing $\sigma(T)$ and contained in $B \cap \rho(T)$, then we define $g(T)$ to be the operator

$$g(T) = (2\pi i)^{-1}\int_L g(\lambda)R(\lambda; T)\,d\lambda. \tag{1.2}$$

By Cauchy's theorem, $g(T)$ is independent of L.

1.9 THEOREM. *Let T be an operator and let g be analytic on some open neighborhood of $\sigma(T)$. If Y is analytically invariant under T, then Y is analytically invariant under $g(T)$.*

PROOF. By Proposition 1.5 and (1.2), the operator $[g(T)]^Y$ is well defined, so that by Proposition 1.1 it suffices to prove that $[g(T)]^Y$ has the s.v.e.p. This results from the following computation (here, x' denotes the coset of $x \in X$ in X/Y):

$$\begin{aligned}[g(T)]^Y x' &= [g(T)x]' = \left[(2\pi i)^{-1}\int_L g(\lambda)R(\lambda; T)x\,d\lambda\right]' \\ &= (2\pi i)^{-1}\int_L g(\lambda)[R(\lambda; T)x]'\,d\lambda \\ &= (2\pi i)^{-1}\int_L g(\lambda)R(\lambda; T^Y)x'\,d\lambda \\ &= g(T^Y)x'.\end{aligned}$$

Since $g(T^Y)$ has the s.v.e.p. (see [4, p. 5]), then Y is analytically invariant under $g(T)$.

The next proposition is a partial converse to Theorem 1.9.

1.10 PROPOSITION. *Let g be a nonconstant scalar-valued function on an open neighborhood of the spectrum of the operator T. If Y is a hyperinvariant and analytically invariant subspace under $g(T)$, then Y is analytically invariant under T.*

PROOF. Let $f: V_f \to X$ be analytic on the connected domain V_f and such that $(\lambda - T)f(\lambda) \in Y$ for all $\lambda \in V_f$. We consider two cases.

Case 1. $V_f \cap \rho(T) \neq \varnothing$. Since Y is hyperinvariant under $g(T)$, we obtain $f(\lambda) \in Y$ for $\lambda \in V_f \cap \rho(T)$, hence $f(\lambda) \in Y$ for all $\lambda \in V_f$ by analytic continuation.

Case 2. $V_f \subset \sigma(T)$. Since g is nonconstant, for each fixed $\mu \in V_f$ there exists h_μ analytic on the domain of g satisfying

$$g(\mu) - g(\lambda) = (\mu - \lambda)h_\mu(\lambda);$$

hence by the functional calculus

$$g(\mu) - g(T) = (\mu - T)h_\mu(T).$$

By the hyperinvariance of Y under $g(T)$, we have

$$[g(\mu) - g(T)]f(\mu) \in Y \quad \text{for } \mu \in V_f. \tag{1.3}$$

Let $D \subset V_f$ be a disc on which g has an analytic inverse k and put $D' = g(D)$. For $\lambda \in D'$, (1.3) becomes

$$(\lambda - g(T))f(k(\lambda)) \in Y.$$

It follows from the hypothesis that $f(\mu) \in Y$ on D, so that $f(\mu) \in Y$ for $\mu \in V_f$ by analytic continuation.

1.11 COROLLARY. *Suppose T has the s.v.e.p. and let g be a nonconstant scalar-valued analytic function on some neighborhood of $\sigma(T)$. If Y is a spectral maximal subspace for $g(T)$, then Y is also spectral maximal for T.*

PROOF. Observe first that $g(T)$ has the s.v.e.p. Thus, if Y is spectral maximal for $g(T)$, it is analytically invariant and hyperinvariant under $g(T)$. Hence Y is analytically invariant under T by Proposition 1.10. Let Z be a T-invariant subspace such that $\sigma(T|Z) \subset \sigma(T|Y)$. But since $\sigma(T|Y) \subset \sigma(T)$ by Corollary 1.4, Z is invariant under $g(T)$ by Proposition 1.3 and formula (1.2). By the spectral mapping theorem, $\sigma(g(T)|Z) \subset \sigma(g(T)|Y)$, and it follows that $Z \subset Y$.

Since the following proposition is an easy consequence of the definition of analytic invariance, the proof is omitted.

1.12 PROPOSITION. *If Y is analytically invariant under T, then $\sigma(x, T|Y) = \sigma(x, T)$ for all $x \in Y$.*

Analytically invariant subspaces have transitivity properties analogous to those for spectral maximal spaces proved by Apostol in [2, Proposition 1.3.2].

1.13 PROPOSITION. *Let $Y \subset Z$ be T-invariant subspaces. Then the following hold.*

(1) *If Y is analytically invariant under T, it is also analytically invariant under $T|Z$.*

(2) *If Y is analytically invariant under $T|Z$ and Z is analytically invariant under T, then Y is analytically invariant under T.*

(3) *Z is analytically invariant under T if and only if Z/Y is analytically invariant under T^Y.*

PROOF. The proofs of (1) and (2) are straightforward; we prove only (3). Again let x' denote the coset of x in X/Y. Suppose that Z/Y is analytically invariant under T^Y and let f: $V_f \to X$ be an analytic function satisfying $(\lambda - T)f(\lambda) \in Z$ for $\lambda \in V_f$. Passing to the quotient space X/Y, we have $[f(\lambda)]' \in Z/Y$ on V_f. It follows that Z is analytically invariant under T.

Conversely, suppose that Z is analytically invariant under T and let g: $V_g \to X/Y$ be analytic and such that $(\lambda - T^Y)g(\lambda) \in Z/Y$ for $\lambda \in V_g$, where V_g is assumed to be connected. Fix $\mu \in V_g$ and let $D \subset V_g$ be a neighborhood of μ. By [5, Lemma 1], there is a disc $D' \subset D$ centered at μ and an analytic function h on D' into X such that $h(\lambda)' = g(\lambda)$ on D'. Hence $(\lambda - T)h(\lambda) \in Z$ on D'. Since Z is analytically invariant under T, we see that $h(\lambda) \in Z$ on D'. It follows that $g(\lambda) \in Z/Y$ for all $\lambda \in V_g$.

1.14 THEOREM. *Let T_i $(i = 1, 2)$ be bounded operators on Banach spaces X_i, respectively, and let Y_i be T_i-invariant subspaces of X_i. Then $Y_1 \oplus Y_2$ is analytically invariant under $T_1 \oplus T_2$ if and only if each Y_i is analytically invariant under T_i.*

PROOF. Put $Y = Y_1 \oplus Y_2$ and $T = T_1 \oplus T_2$. Then $X/Y = X_1/Y_1 \oplus X_2/Y_2$ and $T^Y = (T_1)^{Y_1} \oplus (T_2)^{Y_2}$. By [4, Proposition 1.1.3, p. 3], T^Y has the s.v.e.p. iff its components have this property. The proof is now completed by applying Proposition 1.1.

The purpose of the next example is to show that the analog of Theorem 1.14 fails for the class of spectral maximal subspaces.

1.15 EXAMPLE. Let X be the direct sum of two copies of $C[0, 1]$ and let T be the direct sum of two copies of the multiplication operator on $C[0, 1]$ defined by $(Mx)(t) = tx(t)$, $t \in [0, 1]$, $x \in C[0, 1]$. Let F_1 and F_2 be two nontrivially overlapping closed intervals in $[0, 1]$. The subspaces

$$Y_i = \{x_i \in C_i[0, 1]: \text{support}\, x_i \subset F_i\}, \qquad i = 1, 2,$$

are easily seen to be spectral maximal for the respective multipliers M_i, but it can also easily be shown that $Y_1 \oplus Y_2$ is not spectral maximal for $T = M_1 \oplus M_2$ (see, e.g., [4, p. 4]).

REMARK. By Theorem 1.14 the previous example is another example of an analytically invariant subspace that is not spectral maximal (although $Y_1 \oplus Y_2$ is hyperinvariant under T).

1.16 PROPOSITION. *If T has the s.v.e.p., then its kernel is analytically invariant under T.*

PROOF. Let $f\colon V_f \to X$ be analytic and such that $(\lambda - T)f(\lambda) \in \ker T$ for λ in V_f. Then $(\lambda - T)Tf(\lambda) = 0$ on V_f, and since T has the s.v.e.p., $Tf(\lambda) = 0$ on V_f. Hence $f(\lambda) \in \ker T$ for all λ in V_f.

1.17 PROPOSITION. *If Y is analytically invariant under T, then $\overline{TY}$ is also analytically invariant under T.*

PROOF. Obviously $\overline{TY}$ is T-invariant. Let $f\colon V_f \to X$ be analytic and satisfy $(\lambda - T)f(\lambda) \in \overline{TY}$, $\lambda \in V_f$. Since $\overline{TY} \subset Y$, it follows that $f(\lambda) \in Y$ on V_f. Hence, for $\lambda \in V_f$ we have

$$\lambda f(\lambda) \in \overline{TY} + TY \subset \overline{TY}.$$

Since $\overline{TY}$ is linear, then $f(\lambda) \in \overline{TY}$.

REMARK. It is easy to find examples that show that neither analog of Propositions 1.16 or 1.17 holds for spectral maximal subspaces. For example, let T be any quasinilpotent operator with nontrivial kernel M. Then M is not spectral maximal for T.

1.18 PROPOSITION. *Let T have the s.v.e.p. and let E be a bounded projection in X commuting with T. Then EX is analytically invariant under T.*

PROOF. Let $f\colon V_f \to X$ be analytic and such that $(\lambda - T)f(\lambda) \in EX$ for λ in V_f. Then, applying the projection $I - E$, we obtain

$$(\lambda - T)(I - E)f(\lambda) = 0, \qquad \lambda \in V_f.$$

By the single-valued extension property this implies that $(I - E)f(\lambda) = 0$ or $f(\lambda) = Ef(\lambda)$, $\lambda \in V_f$. Thus EX is analytically invariant under T.

The following result extends Proposition 1.16.

1.19 PROPOSITION. *Let M be the kernel of T. If Y is analytically invariant under T, then $\overline{Y + M}$ is also analytically invariant under T.*

PROOF. Let $f\colon V_f \to X$ be analytic and such that $(\lambda - T)f(\lambda) \in \overline{Y + M}$. Now fix $\mu \in V_f$. Then there are sequences $\{y_n\}$ and $\{u_n\}$ in Y and M, respectively, such that

$$(\mu - T)f(\mu) = \lim_n (y_n + u_n).$$

Hence, since Y is T-invariant,

$$(\mu - T)Tf(\mu) = \lim Ty_n \in Y.$$

We see that $(\lambda - T)Tf(\lambda) \in Y$ for all $\lambda \in V_f$. By analytic invariance, $Tf(\lambda)$

$\in Y$ for $\lambda \in V_f$. It follows that $f(\lambda) \in Y + \overline{Y + M}$ and $f(\lambda) \in \overline{Y + M}$ for all λ in V_f.

1.20 PROPOSITION. *Let Y_i $(i = 1, 2)$ be analytically invariant under T and suppose that $\sigma(T|Y_i)$ are disjoint. If $Y_1 + Y_2$ is closed, then it is analytically invariant under T.*

PROOF. Let $Y = Y_1 + Y_2$ and let f: $V_f \to X$ be analytic and such that $(\lambda - T)f(\lambda) \in Y$, $\lambda \in V_f$. We may suppose that V_f is connected and $V_f \subset \sigma(T)$. For in case $V_f \cap \rho(T) \neq \varnothing$, then $f(\lambda) \in Y$ on this intersection and hence on V_f by analytic continuation. Since the spectra $\sigma(T|Y_i)$ are disjoint, we conclude that V_f intersects at least one of the sets $\rho(T|Y_i)$ nonvacuously. Suppose, then, that there is some disc $D \subset V_f \cap \rho(T|Y_1)$.

We shall next prove that Y is a direct sum of the Y_i. It suffices to prove that $Z = Y_1 \cap Y_2 = (0)$. Note that Z is analytically invariant under T, so that, by Proposition 1.13(1), Z is analytically invariant under each $T|Y_i$. By Corollary 1.4, $\sigma(T|Z) \subset \sigma(T|Y_1) \cap \sigma(T|Y_2) = \varnothing$, and hence $Z = (0)$.

Thus there are complementary projections E_i: $Y \to Y_i$. Since $g(\lambda) = (\lambda - T)f(\lambda)$ is analytic, the functions $g_i(\lambda) = E_i g(\lambda)$ are also analytic. Hence $h(\lambda) = R(\lambda;\ T|Y_1)g_1(\lambda)$ is analytic on D. From this we obtain for $\lambda \in D$, that

$$(\lambda - T)f(\lambda) = g_1(\lambda) + g_2(\lambda) = (\lambda - T)h(\lambda) + g_2(\lambda),$$

or $(\lambda - T)[f(\lambda) - h(\lambda)] \in Y_2$. Moreover,

$$f(\lambda) - h(\lambda) \in Y_2 \quad \text{for } \lambda \text{ in } D$$

by the analytic invariance of Y_2 under T. It follows that $f(\lambda) \in Y$ for all $\lambda \in V_f$.

2. Analytically decomposable operators. To motivate the study of analytically decomposable operators we begin this section by indicating that there exist analytically decomposable operators that are not decomposable.

In [1] E. Albrecht produced a multiplication operator T on a certain l^1-type Banach space that has the following properties: For every open cover of $\sigma(T)$ by sets G_1, G_2 there are spectral maximal subspaces Y_1, Y_2 invariant under T such that

(a) $\sigma(T|Y_i) \subset G_i$ and

(b) $X = \overline{Y_1 + Y_2}$;

but the condition

(c) $X = Y_1 + Y_2$

fails, i.e., T is not decomposable. Albrecht showed further that the T above also has the s.v.e.p., hence every spectral maximal subspace of T is analytically invariant by [5, Theorem 2]. Since the decompositions (a) and (b) hold for arbitrary finite open covers of $\sigma(T)$, then T is analytically decomposable.

We now prove that every analytically decomposable operator has the s.v.e.p. For the proof we need the following lemma. (We note that Theorems 2.2 and 2.3 are the respective analogs of parts (i) and (ii) of Corollary 2.1.4 of [4].)

2.1 LEMMA. *Let T be analytically decomposable. If G is an open set such that $G \cap \sigma(T)$ is not empty, then there is a nonzero analytically invariant subspace Y with $\sigma(T|Y) \subset G$.*

PROOF. Let H be a second open set such that H, G cover $\sigma(T)$, but H does not contain $\sigma(T)$. By definition there are two analytically invariant subspaces Y, Z satisfying

$$X = \overline{Y + Z}, \quad \sigma(T|Y) \subset G, \quad \sigma(T|Z) \subset H.$$

Then $Y = (0)$ implies that $Z = X$ and $\sigma(T) = \sigma(T|Z) \subset H$, which is impossible by the choice of H. Hence $Y \neq (0)$ and the proof is complete.

2.2 THEOREM. *Every analytically decomposable operator has the s.v.e.p.*

PROOF. Let T be analytically decomposable and let $f\colon D \to X$ be analytic and such that $(\lambda - T)f(\lambda) = 0$ on D. If $f \neq 0$ on D then $D \subset \sigma(T)$. By Lemma 2.1 there is a nonzero analytically invariant subspace Y with $\sigma(T|Y) \subset D$. By Theorem 1.8 we also have $D \subset \sigma(T|Y)$. Since this is clearly preposterous, then $f = 0$ on D and T has the s.v.e.p.

2.3 THEOREM. *The approximate point spectrum of an analytically decomposable operator coincides with its spectrum.*

PROOF. Let A denote the approximate point spectrum of the analytically decomposable operator T, and suppose that $A \neq \sigma(T)$. Then $V = \sigma(T) \setminus A$ is relatively open, so that (by Lemma 2.1) there is a nontrivial analytically invariant subspace Y such that $\sigma(T|Y) \subset V$. Let B denote the point-set boundary of $\sigma(T|Y)$ and let A_1 be the approximate point spectrum of $\sigma(T|Y)$. There exists a $\lambda \in V \cap B$ and clearly $A_1 \subset A$. Thus we obtain the contradiction $\lambda \in A$. It follows that $A = \sigma(T)$.

2.4 THEOREM. *Let T be analytically decomposable. If f is a scalar-valued analytic function on some neighborhood of $\sigma(T)$, then $f(T)$ is also analytically decomposable.*

PROOF. Let $\{G_i\}$ be an open cover of $\sigma(f(T)) = f(\sigma(T))$. Since $\{f^{-1}(G_i)\}$ covers $\sigma(T)$, there are analytically invariant subspaces Y_i under T with $X = \mathrm{clm}\{Y_i\}$ and $\sigma(T|Y_i) \subset f^{-1}(G_i)$ for each i. By Theorem 1.9 each Y_i is also analytically invariant under $f(T)$, hence the result follows from the inclusions $\sigma(f(T)|Y_i) = f(\sigma(T|Y_i)) \subset f(f^{-1}(G_i)) \subset G_i$.

At this point we digress briefly to consider the relation of analytic decomposability to an intermediate notion of "Weak" decomposability introduced by Colojoară and Foiaş, [4, p. 217]. The operator T is called weakly decomposable if, for any finite open cover of $\sigma(T)$, the conditions (i), (ii′) and (iii) of the Introduction hold. By an argument analogous to that of Lemma 2.1 and Theorem 2.2 it can be shown that weakly decomposable operators have the s.v.e.p. From this follow the implications: "decomposable"⇒ "weakly decomposable"⇒"analytically decomposable." An open question in the present theory is whether or not the second implication is reversible. A difficult problem is the stability of weak decomposability under the functional calculus (i.e., if T is weakly decomposable is $f(T)$ also decomposable?). We are not able to answer this question now, but the above remarks and Theorem 2.4 lead immediately to

2.5 COROLLARY. *Let T be weakly decomposable. If f is analytic on a neighborhood of $\sigma(T)$, then $f(T)$ is analytically decomposable.*

Colojoară and Foiaş proved that commuting quasinilpotent perturbations of decomposable operators are also decomposable (i.e., if T is decomposable, $QT = TQ$ and $\sigma(Q) = \{0\}$, then $T + Q$ is decomposable). Such a theorem is still unknown for either weakly or analytically decomposable operators, but we do have the following.

2.6 THEOREM. *Let T be weakly decomposable and let Q be a quasinilpotent operator commuting with T. Then $T + Q$ is analytically decomposable.*

PROOF. Every spectral maximal subspace for T is Q-invariant. We prove that such a subspace is analytically invariant under $S = T + Q$. Let Y be spectral maximal for T. Then $S^Y - T^Y = Q^Y$ is quasinilpotent on X/Y, so that by [4, Theorem 1.2.3, p. 14], S^Y has the s.v.e.p. By Proposition 1.1, Y is analytically invariant under S. The rest of the proof is an easy consequence of the equalities

$$\sigma(S|Y) = \sigma(T + Q|Y) = \sigma(T|Y),$$

for each spectral maximal subspace Y of T.

2.7 PROPOSITION. *Let T be analytically decomposable and let E be a projection in X commuting with T. Then $T|EX$ is analytically decomposable.*

PROOF. Let $S = T|EX$ and let $\{G_i\}_1^n$ be an open cover of $\sigma(S)$. If we let G_0 be the complement of $\sigma(S)$, we see that $\{G_0, G_1, \ldots, G_n\}$ forms an open cover of $\sigma(T)$; hence there are subspaces $Y_0, Y_1, \ldots, Y_n$ that are analytically invariant under T such that $X = \operatorname{clm}\{Y_i\}$ and $\sigma(T|Y_i) \subset G_i$ ($i = 0, 1, \ldots, n$). By Proposition 1.18, EX is analytically invariant under T; thus the subspaces $Z_i = Y_i \cap EX$ ($i = 0, 1, \ldots, n$) are analytically invariant under S by Proposition 1.13(1). Since evidently $Z_0 = (0)$ and E is a projection, then $EX = \text{clm}\{Z_i;\ i = 1, \ldots, n\}$. Again, by Proposition 1.13(1), Z_i is analytically invariant under $T|Y_i$, so it follows from Corollary 1.4 that $\sigma(S|Z_i) = \sigma(T|Z_i) \subset \sigma(T|Y_i) \subset G_i$, $i = 1, 2, \ldots, n$. This completes the proof.

2.8 THEOREM. *Let T_j be operators on the respective Banach spaces X_j $(j = 1, 2)$ and let T be the direct sum $T_1 \oplus T_2$ on $X_1 \oplus X_2$. Then T is analytically decomposable if and only if each T_j is analytically decomposable.*

PROOF. The necessity is a direct consequence of Proposition 2.7. For the sufficiency, let $\{G_i\}$ be an open cover of $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$. Then there exist subspaces Y_{ij} analytically invariant under T_j such that

$$X_j = \text{clm}\{Y_{ij}\} \quad \text{and} \quad \sigma(T_j|Y_{ij}) \subset G_i.$$

By Theorem 1.14, $Y_i = Y_{i1} \oplus Y_{i2}$ is analytically invariant under T. Since $\sigma(T|Y_i) \subset G_i$ and $X = \text{clm}\{Y_i\}$, then T is analytically decomposable.

An analog of the next result does not seem to occur in earlier theories of spectral decompositions.

2.9 THEOREM. *If T is analytically decomposable, then $T|\overline{TX}$ is analytically decomposable.*

PROOF. Let $S = T|\overline{TX}$ and let $\{G_i\}$ be an open cover of $\sigma(S)$. By Proposition 1.17, $\overline{TX}$ is analytically invariant under T; hence, by Corollary 1.4, $\sigma(S) \subset \sigma(T)$. Without loss of generality, then, we may suppose that $\{G_i\}$ covers $\sigma(T)$. As T is analytically decomposable, we can find subspaces Y_i that are analytically invariant under T such that $X = \text{clm}\{Y_i\}$ and $\sigma(T|Y_i) \subset G_i$. Then $Z_i = \overline{TY_i}$ form a requisite system of analytically invariant subspaces under S; for the Z_i are analytically invariant under T by Proposition 1.17, and hence Z_i is analytically invariant under both S and $T|Y_i$ by Proposition 1.13(1). We thus obtain the spectral inclusions $\sigma(S|Z_i) = \sigma(T|Z_i) \subset \sigma(T|Y_i) \subset G_i$. Finally, the property $\overline{TX} = \text{clm}\{Z_i\}$ follows from the continuity of T.

2.10 THEOREM. *Let T be analytically decomposable on X. If S is an operator on a second Banach space X_1 that is similar to T, then S is analytically decomposable.*

PROOF. Let $A\colon X \to X_1$ be the bounded invertible linear mapping which gives the similarity between S and T, i.e., $AT = SA$. Since A preserves the spectral properties of T and S as well as density properties from X to X_1, it suffices to show that if Y is analytically invariant under T, then AY is analytically invariant under S. Clearly AY is an S-invariant subspace of X_1. Thus let $f\colon V_f \to X_1$ be analytic and such that $(\lambda - S)f(\lambda) \in AY$ on V_f. Hence, $A^{-1}(\lambda - S)f(\lambda) \in Y$ on V_f or $(\lambda - T)A^{-1}f(\lambda) \in Y$ on V_f. Since

$A^{-1}f(\lambda)$ is analytic on V_f, then $A^{-1}f(\lambda) \in Y$ for $\lambda \in V_f$. It follows that $f(\lambda) \in AY$, and the proof is complete.

3. Applications. In this section we give several applications of the theory developed in §§1 and 2.

Theorem 3.1 states, roughly, that if we weaken conditions (i), (ii) and (iii) on an operator to (i′), (ii) and (iii) (see the Introduction), then the latter set of hypotheses still guarantees that the operator is decomposable. In other words, our discussion at the outset of §2 shows that condition (ii) is essential for decomposability.

3.1 Theorem. *Let T be analytically decomposable and such that T also satisfies condition* (ii). *Then T is decomposable.*

Proof. Since T has the s.v.e.p. by Theorem 2.2, the spectral manifolds

$$X_T(F) = \{X \in X: \sigma(x) \subset F\}$$

are defined for each set F (for details, see [4, pp. 1–3]). Because (by Proposition 1.12) every subspace Y that is analytically invariant under T contains the range of the maximal analytic extension of $R(\cdot; T)x$ for each $x \in Y$, the proof of [4, Theorem 2.1.5, pp. 31–32] may be repeated, by replacing "spectral maximal" with "analytically invariant," to prove that $X_T(F)$ is closed whenever F is closed.

Now let $\{G_i\}$ be a finite open cover of $\sigma(T)$. Let $\{H_i\}$ be another cover of $\sigma(T)$ such that $\overline{H}_i \subset G_i$ for each i, and let Y_i be analytically invariant subspaces under T such that $\sigma(T|Y_i) \subset H_i$ and $X = Y_1 + Y_2 + \cdots + Y_n$. It follows that $Y_i \subset X_T(\overline{H}_i)$ and $\sigma(T|X_T(\overline{H}_i)) \subset G_i$ for each i. Hence T satisfies (i), (ii) and (iii) of the Introduction and is therefore decomposable.

For the next theorem we introduce the following notation. Let $L(X)$ be the algebra of all bounded linear operators on X. If $T \in L(X)$, let $H(T)$ be the family of all scalar-valued analytic functions defined on some neighborhood of $\sigma(T)$. Finally, denote by $A(T)$ the uniform closure of $\{f(T): f \in H(T)\}$ in $L(X)$.

3.2 Theorem. *Let T be analytically decomposable. Then $A(T)$ is a commutative, inverse-closed Banach subalgebra of $L(X)$ such that every operator in $A(T)$ is analytically decomposable.*

Proof. Clearly $A(T)$ is a commutative Banach algebra, since $\{f(T): f \in H(T)\}$ is a commutative linear and multiplicative system in $L(X)$. To see that $A(T)$ is inverse-closed, let $S \in A(T)$ and suppose that $S^{-1} \in L(X)$. Then there is a sequence $\{f_n\} \subset H(T)$ such $f_n(T) \to S$ in the uniform norm of $L(X)$ and $\sigma(f_n(T)) \to \sigma(S)$ in the Hausdorff metric on compact sets. It is evident that $0 \notin \sigma(f_n(T)) = f_n(\sigma(T))$ for sufficiently large n. Hence $f_n(T)^{-1}$

exists for n large enough. But for such n, $f_n(T)^{-1} = g_n(T) \in A(T)$ where $g_n(\lambda) = 1/f_n(\lambda) \in H(T)$. Thus $S^{-1} \in A(T)$ by the continuity of the map $S \to S^{-1}$ in $L(X)$.

Now let $S \in A(T)$ be arbitrary and let $\{f_n\}$ be a sequence in $H(T)$ such that $f_n(T) \to S$ uniformly. We claim that every subspace Y that is analytically invariant under T is also analytically invariant under S. Evidently Y is S-invariant, so it suffices to prove that the coinduced operator S^Y has the s.v.e.p. By Theorem 1.9, each $f_n(T)^Y$ has this property. Thus, since the spectral radius of $S^Y - f_n(T)^Y$ approaches zero, S^Y has the s.v.e.p. by [2, Theorem II.1.10, p. 1494]. This proves the claim.

Next, let $\{G_i\}_1^k$ be an open cover of $\sigma(S)$, so that $\{G_i\}$ also covers $\sigma(f_n(T))$ for sufficiently large n (because the latter spectra converge to $\sigma(S)$ in the Hausdorff metric). Fix n_0 such that the above property holds for $n > n_0$. Then $\{f_n^{-1}(G_i)\}$ covers $\sigma(T)$ for $n > n_0$ and we can thus find subspaces Y_i^n that are analytically invariant under T and such that

$$X = \mathrm{clm}\{Y_1^n, \ldots, Y_k^n\} \quad \text{and} \quad \sigma(T|Y_i^n) \subset f_n^{-1}(G_i), n > n_0.$$

Hence $\sigma(f_n(T)|Y_i^n) \subset G_i$ $(i = 1, 2, \ldots, k)$ for $n > n_0$. We may then choose n so large that $\sigma(S|Y_i^n) \subset G_i$ $(i = 1, \ldots, k)$. Since every Y_i^n is analytically invariant under S for all $n > n_0$ and all i, then S is analytically decomposable.

3.3 Theorem. *Let T be analytically decomposable and let S be a scalar-type operator in the sense of Dunford* [3]. *If T commutes with S, then TS and $T + S$ are analytically decomposable.*

Proof. Let E denote the resolution of the identity for S and let $M > 0$. Then for a suitable partition $\{b_j\}$ of $\sigma(S)$ by Borel sets and $\lambda_j \in b_j$, we have

$$\left\|S - \sum \lambda_j E(b_j)\right\| < M\|T\|^{-1}. \tag{3.1}$$

Setting $E_j = E(b_j)$, we obtain

$$\left\|TS - \sum \lambda_j TE_j\right\| \leq M. \tag{3.2}$$

Since each E_j commutes with T by [3, Theorem 5], each component $T_j = TE_j = T|E_jX$ is analytically decomposable by Proposition 2.7. Hence the direct sum operator $T' = \Sigma\lambda_j T_j$ is analytically decomposable by Theorem 2.8.

Now let Y be analytically invariant under T. Defining $Y_j = Y \cap E_jX$, we see (by Proposition 1.13(1)) that Y_j is analytically invariant under T_j because $T_jY_j = TE_jY = E_jTY \subset E_jY = Y_j$. It follows from Theorem 1.14 that $Y = \Sigma Y_j$ is analytically invariant under the sum $T' = \Sigma\lambda_j T_j$. By (3.2) and a proof similar to that of Theorem 3.2, the subspace Y is analytically invariant under TS.

Let $\{G_i\}$ be a finite open cover of $\sigma(TS)$ and let T' be as above with λ_j and

E_j chosen in such a way that $\{G_i\}$ also covers $\sigma(T')$. Since T' is analytically decomposable, there are subspaces Y_i that are analytically invariant under T' such that $\sigma(T'|Y_i) \subset G_i$ and $X = \mathrm{clm}\{Y_i\}$.

Let Z be any subspace analytically invariant under T', and write $Z_j = E_jZ$. It follows by Theorem 1.14 that $Z = \sum Z_j$ is analytically invariant under $T = \sum T_j$, hence Z is analytically invariant under TS by the argument given in the second paragraph of the proof. Since $\sigma(T'|Y_i)$ can be made arbitrarily close to $\sigma(TS|Y_i)$ in the Hausdorff metric, then $\sigma(TS|Y_i) \subset G_i$ and so TS is analytically decomposable.

To prove that $T + S$ is also analytically decomposable, note that for $\lambda \in \rho(-S)$, we have

$$T + S = (\lambda + S)\left[(T - \lambda)R(\lambda; -S) + I\right],$$

where I is the identity in $L(X)$. Now $T - \lambda$ is analytically decomposable by Theorem 2.4, and $\lambda + S$ and $R(\lambda; -S)$ are scalar-type operators by [3, Theorem 9]. By the previous part of the proof on products and Theorem 2.4 again, $T + S$ is analytically decomposable.

Theorem 3.3 is a "weak" analog of Apostol's theorem [2, Corollary II.2.11], on perturbations of decomposable operators by commuting spectral operators. However, if we restrict T, we have the following result, which generalizes Theorem 2.6.

3.4 Corollary. *Let T be weakly decomposable and let S be a spectral operator commuting with T. Then $T + S$ is analytically decomposable.*

Proof. By Dunford's fundamental theorem [3, Theorem 8], $S = A + Q$ where A is a scalar-type operator and Q is quasinilpotent; moreover, A commutes with T and Q. Then the decomposition

$$T + S = T + (A + Q) = (T + Q) + A$$

shows that $T + S$ is analytically decomposable by Theorems 2.6 and 3.3.

Corollary 3.4 has the following analog for products, but its proof is surprisingly delicate.

3.5 Theorem. *Let T be weakly decomposable and let S be a spectral operator commuting with T. Then TS is analytically decomposable.*

Proof. Let $S = A + Q$ be the canonical decomposition of S into scalar-type and quasinilpotent parts. Since $TQ = QT$ by [3, Theorem 8], then TQ is quasinilpotent. Then $TS = TA + TQ$ and TQ commutes with TA.

Let $E_0 = E(\sigma)$, where E is the resolution of the identity for S and σ is a Borel set in the plane. If Y is a spectral maximal subspace for T, we show that $Y \cap E_0X$ is analytically invariant under $TS|E_0X$. First note that Y is invariant under S, A and Q (since it is T-hyperinvariant), and $TE_0 = E_0T$ by

[3, Theorem 8]. Thus Y is invariant under TA and TQ. By a proof similar to that of Theorem 3.3, Y is analytically invariant under TA; and because TQ is quasinilpotent, Y is analytically invariant under TS. By Theorem 1.13(1), $Y \cap E_0X$ is analytically invariant under $TS|E_0X$.

Let $\{G_i\}_1^k$ be an open cover of $\sigma(TS) = \sigma(TA)$. Without loss of generality we may suppose that $\{G_i\}$ covers the whole plane. Now let $U = \Sigma\lambda_j TE_j$ be an approximation of TA as in (3.2) with each $\lambda_j \neq 0$. Since T is weakly decomposable, for j fixed there are spectral maximal subspaces $Y_i^{(j)}$ such that $X = \text{clm}\{Y_i^{(j)}\}$ and

$$\sigma(T|Y_i^{(j)}) \subset \lambda_j^{-1}G_i, \qquad i = 1, 2, \ldots, k.$$

Then the subspaces $Y_i^j = Y_i^{(j)} \cap E_jX$ are analytically invariant under

$$U|E_jX, \quad TA|E_jX \quad \text{and} \quad TS|E_jX,$$

and are such that if $Y_i = \Sigma Y_i^j$, then

$$\sigma(U|Y_i) = \bigcup_j \left(\lambda_j\sigma\left(TE_j|Y_i^j\right)\right) \subset \bigcup_j \lambda_j\left(\lambda_j^{-1}G_i\right) = \bigcup_j G_i = G_i.$$

But, by the last paragraph and Theorem 1.14, the Y_i thus defined are analytically invariant under U, TA and TS. The proof is completed by choosing U sufficiently uniformly close to TA such that $\sigma(TA|Y_i) \subset G_i$ whenever $\sigma(U|Y_i) \subset G_i$, $i = 1, 2, \ldots, k$.

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