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SOME 3-MANIFOLDS WHICH ADMIT KLEIN BOTTLES

BY

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ABSTRACT. Consider a closed, orientable, irreducible 3-manifold M with $|\pi_1(M)| < \infty$, in which a Klein bottle can be embedded. We present a classification of the spaces M and show that, if $\pi_1(M)$ is cyclic, then M is homeomorphic to a lens space. Note that all surfaces of even genus can be embedded in each space M. We also classify all free involutions on lens spaces whose orbit spaces contain Klein bottles.

0. Introduction. Let $M = M(K)$ be a closed, orientable, irreducible 3-manifold with $|\pi_1(M)| < \infty$, in which a Klein bottle K can be embedded. A large class of lens spaces belongs to the spaces $M(K)$ (see Bredon and Wood [1]). The goal of this paper is to classify the spaces $M(K)$, and investigate the relations between the spaces $M(K)$ and lens spaces $L(p, q)$. Especially, we extract the following result:

THEOREM 1. *If $\pi_1(M)$ is abelian, then M is homeomorphic to a lens space.*

Throughout the paper we work in the PL category. We divide the paper into five sections. In §1 we define a 3-manifold $M(p, q)$ for each pair (p, q) of relatively prime integers, and classify the spaces $M(p, q)$. In §2 we classify the spaces $M(K)$ by showing that each $M(K)$ is homeomorphic to a space $M(p, q)$ for some p, q. In §3 we prove Theorem 1 and in §4 we investigate all free involutions on lens spaces whose orbit spaces contain Klein bottles.

Our approach can be applied to an outstanding problem in the study of involutions on lens spaces. It has been asked by J. L. Tollefson whether each lens space L has a Heegaard splitting (L, F) of genus 1 (i.e., F splits L into two solid tori) such that F is invariant under a given involution h on the space L and F is in general position with respect to the fixed-point set Fix(h). All known involutions on lens spaces L have the property (that L has such a Heegaard splitting). In §5, we give new examples of involutions h with Fix(h) $\neq \varnothing$ which do not have the property.

J. H. Rubinstein (personal correspondence with the author) has obtained a classification of the spaces $M(K)$ independently and simultaneously.

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I am grateful to the referee for correcting an oversight in the proof of Proposition 2.3.

1. Definition of some 3-manifolds $M(p, q)$. Let S^1 be the set of complex numbers z with the norm $|z| = 1$, and let $D^2 = \{\rho z | 0 \leq \rho \leq 1,\ |z| = 1\}$ where ρ is a real number. We let γ always denote the involution on $S^1 \times S^1$ defined by $\gamma(z_1, z_2) = (-z_1, \bar{z}_2)$ for each $(z_1, z_2) \in S^1 \times S^1$. Then the orbit space $S^1 \times S^1/\gamma$ is a Klein bottle K. Let $M(\gamma)$ denote the twisted I-bundle over $S^1 \times S^1/\gamma$. Namely, $M(\gamma)$ is obtained from $S^1 \times S^1 \times I$ by identifying $(x, 0)$ with $(\gamma(x), 0)$ for each $x \in S^1 \times S^1$. Define a map f: $S^1 \times S^1 \to S^1 \times S^1$ by $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$, where p, q, r, s are integers with $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = 1$ and $p \geq 0$. We may identify $M(\gamma) - S^1 \times S^1/\gamma$ with $S^1 \times S^1 \times (0, 1]$ in the obvious way. We define an adjunction space $M(\gamma) \cup_f S^1 \times D^2$ by identifying $(x, 1) \in S^1 \times S^1 \times \{1\}$ with $f(x) \in \partial(S^1 \times D^2)$ for each $x \in S^1 \times S^1$. We denote this space by $M(p, q, r, s)$.

The following proposition classifies the adjunction spaces. Two lemmas will be followed by the proof.

PROPOSITION 1.1. *Two spaces* $M(p, q, r, s)$ *and* $M(p', q', r', s')$ *are homeomorphic if and only if* $p' = p$ *and* $q' = \pm q$.

According to the above proposition, the homeomorphic type of $M(p, q, r, s)$ is completely determined by the choice of p, q. In the view of this, the following definition may be justified.

DEFINITION 1.2. For each pair (p, q) of relatively prime integers, a space $M(p, q)$ is defined to be $M(p, q, r, s)$, where $p \geq 0$ and $ps - qr = 1$.

COROLLARY 1.3. *Two spaces* $M(p, q)$ *and* $M(p', q')$ *are homeomorphic if and only if* $p' = p$ *and* $q' = \pm q$.

PROOF. See Proposition 1.1.

Now we will prove Proposition 1.1.

LEMMA 1.4. *Let* $ps - qr = 1 = ps' - qr'$.

(1) $M(p, q, r, s) \approx M(p, -q, -r, s)$.

(2) $M(p, q, r, s) \approx M(p, q, r', s')$.

(*$A \approx B$ means that A and B are homeomorphic.*)

PROOF. (1) Define a map k: $S^1 \times S^1 \to S^1 \times S^1$ by $k(z_1, z_2) = (z_1, \bar{z}_2)$ for each $(z_1, z_2) \in S^1 \times S^1$, and k': $S^1 \times S^1 \times I \to S^1 \times S^1 \times I$ by $k'(x, t) = (k(x), t)$ for each $x \in S^1 \times S^1$. Since $k\gamma = \gamma k$, there exists an obvious homeomorphism $\bar{k}$ of $M(\gamma)$ induced by k' such that $\bar{k}g = gk'$, where g: $S^1 \times S^1 \times I \to M(\gamma)$ is the orbit map. Define H: $M(p, q, r, s) \to M(p, -q, -r, s)$ by $H = \bar{k}$ on $M(\gamma)$ and $H(z_1, \rho z_2) = (z_1, \rho\bar{z}_2)$ on $S^1 \times D^2$ such that

$H(M(\gamma)) = M(\gamma)$ and $H(S^1 \times D^2) = S^1 \times D^2$. It is checked that H is a well-defined homeomorphism.

(2) Since $ps - rq = 1 = ps' - r'q$, we see that $r' = r + mp$ and $s' = s + mq$ for some m. Define H: $M(p, q, r, s) \to M(p, q, r', s')$ by $H(x) = x$ on $M(\gamma)$ and $H(z_1, \rho z_2) = (z_1, \rho z_2 z_1^{-m})$ on $S^1 \times D^2$ such that $H(M(\gamma)) = M(\gamma)$ and $H(S^1 \times D^2) = S^1 \times D^2$. It is checked that H is a well-defined homeomorphism.

(1.5). Let (1, 0) and (0, 1) be the elements of $\pi_1(S^1 \times S^1)$ represented by the paths $(e^{2\pi ti}, 1)$ and $(1, e^{2\pi ti})$, $0 \leq t \leq 1$, respectively. Then the attaching map f in $M = M(p, q, r, s)$ induces an automorphism f_* on $\pi_1(S^1 \times S^1)$ such that $f_*(1, 0) = (p, r)$ and $f_*(0, 1) = (q, s)$. The matrix of f may be represented by $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Consider the orbit map g: $S^1 \times S^1 \to S^1 \times S^1/\gamma$. We may assume that $\pi_1(S^1 \times S^1/\gamma)$ is given by $\{\alpha, \beta | \alpha\beta\alpha^{-1}\beta = 1\}$ such that $g_*(1, 0) = \alpha^2$ and $g_*(0, 1) = \beta$. Then it follows from Van Kampen's theorem that

$$\pi_1(M) = \{\alpha, \beta, \delta | \alpha\beta\alpha^{-1}\beta = 1, \alpha^2 = \delta^p, \beta = \delta^q\}$$

where δ is the element of $\pi_1(S^1 \times D^2)$ represented by the center circle of $S^1 \times D^2$ with a proper orientation, or equivalently we have

$$\pi_1(M) = \{\alpha, \beta | \alpha\beta\alpha^{-1}\beta = 1, \alpha^{2q} = \beta^p\}$$

(this can be seen easily by using Van Kampen's theorem by way of f^{-1} instead of f).

(1.6). Let f be a homeomorphism of $S^1 \times S^1$. We let $L(f)$ be the space which is obtained from $S^1 \times S^1 \times I \cup S^1 \times D^2$ by identifying $(x, 1) \in S^1 \times S^1 \times I$ with $f(x) \in \partial(S^1 \times D^2)$ for each $x \in S^1 \times S^1$. Let γ' be a free involution on $S^1 \times S^1$. We define an adjunction space $L^*(\gamma', f) = L(f) \cup_{\gamma'} L(f)$, which is obtained from $L(f) \cup L(f)$ by identifying $(x, 0)$ with $(\gamma'(x), 0)$ for each $x \in S^1 \times S^1$. We let $h = h(\gamma', f)$ denote the free involution on $L^*(\gamma', f)$ such that h interchanges the two $L(f)$ by $h(x) = x$ for each $x \in L(f)$. Especially, if $\gamma' = \gamma$ and f is given by $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$ for each $(z_1, z_2) \in S^1 \times S^1$ where $p \geq 0$ and $ps - qr = 1$, we shall denote $L^*(\gamma, f)$ and $h(\gamma, f)$ by $L^*(p, q, r, s)$ and $h^*(p, q, r, s)$, respectively. One can see that the orbit space of $h^*(p, q, r, s)$ is $M(p, q, r, s)$.

Lemma 1.7. (1) *The space $L^*(p, q, r, s)$ is homeomorphic to a lens space $L(2pq, ps + rq)$.*

(2) *$L^*(p, q, r, s) \approx L^*(p', q', r', s')$ if and only if either $p' = p$ and $q' = \pm q$ or $p' = \pm q$ and $q' = \pm p$, where $ps - qr = 1 = p's' - q'r'$.*

Proof. (1) Define t': $S^1 \times D^2 \to D^2 \times S^1$ by $t'(z_1, \rho z_2) = (\rho z_2, z_1)$ for each $(z_1, \rho z_2) \in S^1 \times D^2$. Let $t = t' | S^1 \times S^1$. It is easy to see that $L^*(p, q, r, s) \approx S^1 \times D^2 \cup_{\gamma'} S^1 \times D^2 \approx D^2 \times S^1 \cup_{\gamma' t^{-1}} S^1 \times D^2$, where $\gamma' = f\gamma f^{-1}$.

We see that the matrix of $\gamma' t^{-1}$ is given by

$$\begin{pmatrix} -2pq & ps+rq \\ -(ps+rq) & 2rs \end{pmatrix},$$

and the result follows.

(2) Since the argument of the proof is elementary, we just sketch the proof. Suppose that $L^*(p, q, r, s) \approx L^*(p', q', r', s')$. By (1), we have $pq = \pm p'q'$ and $ps + qr \equiv \pm(p's' + q'r') \pmod{2pq}$ (note that $(ps + qr)^2 \equiv 1 \pmod{2pq}$). Using this relation, an elementary argument shows that either $p' = p$ or $p' = \pm q$. Now suppose that the converse holds. If we show that $L^*(p, q, r, s) \approx L^*(p, q, r', s')$ where $ps - qr = 1 = ps' - qr'$, the result follows from a combination of this and the result of (1). However, this follows from the fact that two lens spaces $L(p, q)$ and $L(p, q')$ are homeomorphic if and only if $qq' \equiv \pm 1$ or $q \equiv \pm q' \pmod{p}$ (see also the proof of Lemma 1.4(2)).

(1.8). PROOF OF PROPOSITION 1.1. If $p' = p$ and $q' = \pm q$, then the result follows from Lemma 1.4. Let $M_1 = M(p, q, r, s)$ and $M_2 = M(p', q', r', s')$. Suppose that $M_1 \approx M_2$. Abelianizing the groups of $\pi_1(M_1)$ and $\pi_1(M_2)$, we see that $H_1(M_1) = \{\alpha, \beta \mid \alpha\beta = \beta\alpha,\ \beta^2 = 1,\ \alpha^{2q} = \beta^p\}$ and $H_1(M_2) = \{\alpha, \beta \mid \alpha\beta = \beta\alpha,\ \beta^2 = 1,\ \alpha^{2q'} = \beta^{p'}\}$ (see (1.5)). Then $|H_1(M_1)| = |4q|$ and $|H_1(M_2)| = |4q'|$. Therefore, $q' = \pm q$. On the other hand, it follows from (1.6) and Lemma 1.7 that $|\pi_1(M_1)| = |4pq|$ and $|\pi_1(M_2)| = |4p'q'|$. Hence, we see that $p' = p$. This completes the proof.

2. Properties of spaces $M(p, q)$. We first state results of this section, and then give their proofs. Propositions 2.1 and 2.2 together with Corollary 1.3 classify the spaces $M(K)$.

PROPOSITION 2.1. *Let $M = M(p, q)$.*
(1) $M \approx P^3 \# P^3$ *if* $p = 0$.
(2) $M \approx S^1 \times S^2$ *if* $q = 0$.
(3) $|\pi_1(M)| = 4pq$ *if* $pq \neq 0$.

PROPOSITION 2.2. *Each space $M(K)$ is homeomorphic to a space $M(p, q)$ for some p, q.*

The proof of the following theorem will be given in §3. For convenience, we do not regard $S^1 \times S^2$ as a lens space.

THEOREM 2. *A space $M(p, q)$ is homeomorphic to a lens space if and only if $p = 1$.*

REMARK 1. It is known that a lens space $L(a, b)$ admits a Klein bottle if and only if $a = 2k$ (k even) and $b \equiv \pm(k - 1) \pmod{a}$ (see [1]). Therefore, it follows from Proposition 2.1 that $M(1, q)$ is homeomorphic to the lens space

$L(4q, 2q-1)$. Note that two lens spaces $L(a, b)$ and $L(a', b')$ are homeomorphic if and only if $b' \equiv \pm b$ or $b'b \equiv \pm 1 \pmod a$.

Two double coverings $g_1: M_1 \to M$ and $g_2: M_2 \to M$ are called equivalent if there exists a homeomorphism t of M_1 to M_2 such that $g_2 t = g_1$. The following classifies the double coverings for spaces $M(K)$.

PROPOSITION 2.3. *For each $M(p, q)$, there exists exactly one double covering if p is odd, and exactly three if p is even, up to equivalence.*

REMARK 2. Let $\tilde{M}$ be a double cover of the space $M(p, q)$. Let $L = L(2pq, ps + qr)$, where r, s are integers with $ps - qr = 1$. Then $\tilde{M} \approx L$ if p is odd, and $\tilde{M} \approx L$ or $M(p/2, q)$ if p is even. The homeomorphic type of L does not depend on the choice of r and s (see Lemma 1.7).

(2.4). PROOF OF PROPOSITION 2.1. By (1.6) and the proof of Lemma 1.7, we see that there exists a double covering $g: S^1 \times S^2 \to M$ if $pq = 0$. Therefore, in this case M is homeomorphic to $S^1 \times S^2$ or $P^3 \# P^3$ (see [11]). Since $\pi_1(M)$ is nonabelian if $p = 0$ and $\pi_1(M)$ is abelian if $q = 0$ (see (1.5)), the results of (1) and (2) follow easily. The result of (3) follows directly from (1.6) and Lemma 1.7.

LEMMA 2.5. *Let $U(K)$ be a regular neighborhood of K in $M = M(K)$, and $M' = \mathrm{cl}(M(K) - U(K))$. Then M' is homeomorphic to a solid torus.*

PROOF. Since $|\pi_1(M)| < \infty$, it follows from Van Kampen's theorem that $\ker(\pi_1(\partial M') \to \pi(M')) \neq 0$ (otherwise, $\pi_1(M)$ has a subgroup isomorphic to $Z \oplus Z$). Therefore, it follows from the loop theorem [9] that there exists a disk D properly embedded in M' such that ∂D is not contractible in $\partial M'$. Now since $M(K)$ is irreducible, it is easy to see that M' is a solid torus (otherwise, one may find a 3-cell which contains a Klein bottle).

(2.6). PROOF OF PROPOSITION 2.2. A regular neighborhood $U(K)$ may be regarded as a twisted I-bundle over $S^1 \times S^1/\gamma'$, where γ' is a free involution on $S^1 \times S^1$ (this can be done by cutting $U(K)$ along the subcomplex K). Since $\pi_1(K)$ has a unique abelian subgroup of index 2, there exists an equivalence k of $S^1 \times S^1$ such that $\gamma k = k\gamma'$ (see §1 for the map γ). Extend k to a homeomorphism k' of $S^1 \times S^1 \times I$ in the obvious way. Then since $k\gamma' = \gamma k$, there exists an obvious homeomorphism of $M(\gamma')$ to $M(\gamma)$ induced by k'. This fact together with Lemma 2.5 allows us to assume that $M(K)$ is given by $M(K) = M(\gamma) \cup_{f'} S^1 \times D^2$ for an appropriate attaching map f' of $S^1 \times S^1$. Suppose that the matrix of f' is $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ for the generators defined in (1.5). By a proper choice of orientations, we may assume that $p \geqslant 0$ and $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = 1$. Then f' is isotopic to f where f is given by $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$ for each $(z_1 z_2) \in S^1 \times S^1$ (see [7]). Therefore, one can see that $M(K)$ is homeomorphic to the space $M(p, q)$. This completes the proof.

(2.7). PROOF OF PROPOSITION 2.3. Let $M = M(p, q)$. By (1.5), we have that $\pi_1(M) = \{\alpha, \beta | \alpha\beta\alpha^{-1}\beta = 1, \alpha^{2q} = \beta^p\}$. Consider an exact sequence $\pi_1(M) \to Z_2 \to 0$, where $Z_2 = \langle a \rangle$. The sequence can be factored as:

Case 1. p is odd. Then $H_1(M) = \{\alpha, \beta | \beta^2 = 1, \alpha^{2q} = \beta\} = \{\alpha | \alpha^{4q} = 1\}$. Since k is onto, we see that $k(\alpha) = a$, and therefore $k(\beta) = 1$. This implies that $\pi_1(M)$ has a unique normal subgroup $\langle \alpha^2, \beta \rangle$ of index 2, where $\langle \alpha^2, \beta \rangle$ is the smallest normal subgroup in $\pi_1(M)$ containing α^2 and β.

Case 2. p is even. Then $H_1(M) = \{\alpha, \beta | \alpha\beta = \beta\alpha, \beta^2 = 1, \alpha^{2q} = 1\}$. Therefore, we see that k is given by (1) $k(\alpha) = 1$ and $k(\beta) = a$, (2) $k(\alpha) = a$ and $k(\beta) = 1$, or (3) $k(\alpha) = a$ and $k(\beta) = a$. Hence, ker(k) is $\langle \alpha, \beta^2 \rangle$, $\langle \alpha^2, \beta \rangle$, or $\langle \alpha\beta, \beta^2 \rangle$. Observe that $\langle \alpha, \beta^2 \rangle$, $\langle \alpha^2, \beta \rangle$ and $\langle \alpha\beta, \beta^2 \rangle$ are all distinct.

Thus, the result follows from the above two cases.

(2.8). We now define standard coverings for each $M = M(p, q)$. We may assume that $M = M(p, q, r, s)$ where $ps - qr = 1$. According to (2.7), the only subgroups of index 2 in $\pi_1(M) = \{\alpha, \beta | \alpha\beta\alpha^{-1}\beta = 1, \alpha^{2q} = \beta^p\}$ are

(1) $\langle \alpha^2, \beta \rangle$ if p is odd, and

(2) $\langle \alpha^2, \beta \rangle$, $\langle \alpha, \beta^2 \rangle$ and $\langle \alpha\beta, \beta^2 \rangle$ if p is even.

In (1.6) we have defined a free involution $h^*(p, q, r, s)$ on $L^*(p, q, r, s,)$, and $h^*(p, q, r, s)$ induces a covering g^*: $L^*(p, q, r, s) \to M(p, q, r, s)$. It is not difficult to see that this covering corresponds to the subgroup $\langle \alpha^2, \beta \rangle$ (use another representation of $\pi_1(M)$ in (1.5)).

Now suppose that p is even. Define a map v: $S^1 \times S^1 \to S^1 \times S^1$ by $v(z_1, z_2) = (z_1, -z_2)$ for each $(z_1, z_2) \in S^1 \times S^2$. Define an involution v' on $S^1 \times S^1 \times I$ by $v'(x, t) = (v(x), t)$ for each $x \in S^1 \times S^1$. Since v and γ commute (see §1 for γ), there exists an obvious map $\tilde{v}$ on $M(\gamma)$ induced by v'. One can see that $\tilde{v}$ is a free involution. Define a map $\tilde{h} = \tilde{h}(p/2, q, r, 2s)$ on $M(p/2, q, r, 2s)$ by $\tilde{h} = \tilde{v}$ on $M(\gamma)$ and $\tilde{h}(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$. It is checked that $\tilde{h}$ is a well-defined free involution (note that q is odd since p is even). Then we see that the map $\tilde{h}$ induces a covering $\tilde{g}$: $M(p/2, q, r, 2s) \to M(p, q, r, s)$, and this $\tilde{g}$ corresponds to the subgroup $\langle \alpha, \beta^2 \rangle$.

Let $\bar{g}$: $\bar{M} \to M$ be the covering projection corresponding to the subgroup $\langle \alpha\beta, \beta^2 \rangle$, and let $\bar{h}$ be the nontrivial covering transformation. In §4 (Theorem 3), we show that $\bar{M}$ is homeomorphic to $M(p/2, q, r, 2s)$, and $\bar{h}$ is equivalent to $\tilde{h}(p/2, q, r, 2s)$.

3. Proofs of Theorems 1 and 2.

(3.1). PROOF OF THEOREM 2. We assume that $q > 0$ (see Lemma 1.4 and Proposition 2.1). Suppose that $M = M(p, q)$ is homeomorphic to a lens space. Then since $\pi_1(M)$ is abelian, it follows from (1.5) that $\pi_1(M) = H_1(M) = \{\alpha, \beta | \alpha\beta = \beta\alpha, \beta^2 = 1, \alpha^{2q} = \beta^p\}$. Therefore, we see that $|\pi_1(M)| = 4q$. However, Proposition 2.1 shows that $|\pi_1(M)| = 4pq$. Therefore, we see that $p = 1$.

Now suppose that $p = 1$. Then it follows from (1.5) that $\pi_1(M) = Z_{4q}$. We regard Z_{4q} as a covering transformation group acting on S^3 in the obvious way. In the following we will find an unknotted simple closed curve in S^3 which is invariant under each element of Z_{4q}. Suppose that $M = M(p, q, r, s)$ where $ps - qr = 1$. Let $g^*: L^*(p, q, r, s) \to M(p, q, r, s)$ be the projection defined in (2.8). Let $c_1 = (e^{2\pi ti}, 1)$, $0 \leq t \leq 1$, be the simple closed curve on $S^1 \times S^1$. Note that c_1 is invariant under γ (see §1 for γ). Therefore, the simple closed curve $c = c_1 \times \{0\} \subset S^1 \times S^1 \times I$ is invariant under $h^* = h^*(p, q, r, s)$ where $S^1 \times S^1 \times I \subset L(f)$ (see (1.6) for notations). Since $p = 1$, one can see that $f(c_1 \times \{1\})$ (and therefore c) is isotopic to the center circle of $S^1 \times D^2 \subset L(f)$ (see (1.5) and (1.6)). Therefore, we see that $\pi_1(L - c) = Z$ where $L = L^*(p, q, r, s)$. Let $g: S^3 \to L$ be the projection (note that L is a lens space). Since $\pi_1(L - c) = Z$, we see that $\pi_1(S^3 - c') = Z$ where $c' = g^{-1}(c)$, and c' must be an unknotted simple closed curve (see [8]). The simple closed curve c' is invariant under k, where k is a generator of the covering transformation group of the covering $g: S^3 \to L$. Let $\tilde{h}$ be a lifting of h^* such that $h^* g = g\tilde{h}$. Since c is invariant under h, we see that c' is invariant under $\tilde{h}$. Since k and $\tilde{h}$ generate the group Z_{4q}, the simple closed curve c' is invariant under each element of Z_{4q}. This completes the proof.

(3.2). PROOF OF THEOREM 1. By Proposition 2.2, we see that M is homeomorphic to a space $M(p, q)$ for some p, q. Since $\pi_1(M)$ is abelian, we see from (3.1) that $p = 1$. Therefore, it follows from Theorem 2 that M is homeomorphic to a lens space.

4. Some involutions on lens spaces. Let $M = M(p, q)$. In $\pi_1(M) = \{\alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1, \alpha^{2q} = \beta^p\}$, we have seen that $\pi_1(M)$ has a unique subgroup of index 2 if p is odd and exactly three distinct subgroups of index 2 if p is even (see (2.8)). In the latter case, the three subgroups are $\langle \alpha^2, \beta \rangle$, $\langle \alpha\beta, \beta^2 \rangle$ and $\langle \alpha, \beta^2 \rangle$. We assume the following theorem which will be proved after the proof of Theorem 4.

THEOREM 3. (1) *There exists a homeomorphism H of $M(p, q)$ (p even) such that $H_*(\langle \alpha, \beta^2 \rangle) = \langle \alpha^2, \beta \rangle$ if and only if $p = 2$.*

(2) *There exists a homeomorphism T of $M(p, q)$ such that $T_*(\langle \alpha, \beta^2 \rangle) = \langle \alpha\beta, \beta^2 \rangle$ for each p even.*

COROLLARY 4.1. *For each $M(K)$, there exists a lens space which double-covers $M(K)$. The lens space is uniquely determined, up to homeomorphism.*

PROOF. This is a consequence of (2.8), Theorems 2 and 3.

The following theorem classifies all free involutions on lens spaces whose orbit spaces contain Klein bottles. The third part of Theorem 4 is not new (see [2] for example) but we include it here for completeness. In fact, all involutions on P^3 are known [2], [5].

THEOREM 4. (1) *A lens space* $L(p, q)$ *double-covers a space* $M(K)$ *if and only if* $p = 2ab$ *(*a, b *relatively prime) and* $q \equiv \pm(ad + bc) \pmod p$, *where* c, d *are integers with* $ad - bc = 1$.

(2) *For such* p, q $(p \neq 2)$, *there exist exactly two distinct free involutions* h *on each* $L(p, q)$ *whose orbit spaces contain Klein bottles, up to conjugation.*

(3) *There exists exactly one free involution on* P^3, *up to conjugation.*

REMARK 3. In the above theorem, the homeomorphic type of $L(p, q)$ does not depend on the choice of c, d. Indeed, two lens spaces $L(2ab, ad + bc)$ and $L(2a'b', a'd' + b'c')$ are homeomorphic if and only if either $a' = a$ and $b' = b$ or $a' = b$ and $b' = a$, where $a, b, a', b' > 0$ and $ad - bc = 1 = a'd' - b'c'$ (see Lemma 1.7).

(4.2). PROOF OF THEOREM 4. (1) This follows from Lemma 1.7, (2.8) and Corollary 4.1.

(2) Let M be an orbit space $L(p, q)/h$. It follows from [4] that h is orientation-preserving, and M is orientable. Since $L(p, q)$ is irreducible and h is a free involution, one can see that M is irreducible. Therefore, M is homeomorphic to a space $M(K)$. On the other hand, for each such orbit space M, one can see from (2.8), Theorems 2 and 3 that there exists exactly one free involution on $L(p, q)$ whose orbit space is homeomorphic to M, up to conjugation.

Now let $M_1 = M(a, b, c, d)$ and $M_2 = M(a', b', c', d')$ be spaces homeomorphic to the orbit spaces of two free involutions h on $L(p, q)$. We may assume that $b', b > 0$ (see Lemma 1.4). Then it follows from (2.8) and Corollary 4.1 that the double covers of M_1 and M_2 are homeomorphic to $L^*(a, b, c, d)$ and $L^*(a', b', c', d')$, respectively, and $L^*(a, b, c, d) \approx L^*(a', b', c', d')$. By Lemma 1.7, we have either $a = a'$ and $b = b'$ or $a = b'$ and $b = a'$. Therefore, the orbit spaces of the involutions h on $L(p, q)$ are homeomorphic to $M(a, b)$ or $M(b, a)$. On the other hand, $M(a, b) \approx M(b, a)$ iff $a = b = 1$ (see Corollary 1.3 and note that a, b are relatively prime). Therefore, if $p \neq 2$, then $M(a, b)$ and $M(b, a)$ are not homeomorphic (note that $L^*(a, b, c, d) \approx P^3$ if $a = b = 1$ (see Lemma 1.7)). Now the result follows from the first paragraph of this proof.

Now we will prove Theorem 3.

LEMMA 4.3. *The subgroup* $\langle \alpha^2, \beta \rangle$ *is abelian.*

PROOF. One can see that α^2 and β commutes in $\pi_1(M)$.

LEMMA 4.4. *The subgroup* $\langle \alpha, \beta^2 \rangle$ *is abelian if and only if* $p = 2$ *(this statement is also true for the subgroup* $\langle \alpha\beta, \beta^2 \rangle$*).*

PROOF. We only prove the case for the subgroup $\langle \alpha, \beta^2 \rangle$ since the proofs of the two cases are similar. Since $\alpha^{2q} = \beta^p$, we see that

$$\alpha^{2q} = \alpha\beta^p\alpha^{-1} = (\alpha\beta\alpha^{-1})^p$$
$$= \beta^{-p} = \alpha^{-2q}.$$

Therefore, we have $\alpha^{4q} = 1$ and $\beta^{2p} = 1$. We claim that the order m of β is $2p$. For, suppose the contrary that $m \neq 2p$. Then one can check that $m = p$, or $m < p$ (note that $m|2p$). If $m = p$, then it is easy to see that $|\pi_1(M)| < 2pq$ since $\alpha^{2q} = \beta^p = 1$ and $\alpha\beta = \beta^{-1}\alpha$. If $m < p$, then we see that $|\pi_1(M)| < 4mq < 4pq$ since $\alpha^{4q} = 1 = \beta^m$ and $\alpha\beta = \beta^{-1}\alpha$. In either case, we arrive at a contradiction to the fact that $|\pi_1(M)| = 4pq$ (see Proposition 2.1). Therefore, the order of β is $2p$.

Now suppose that $\langle \alpha, \beta^2 \rangle$ is abelian. Then since $\alpha\beta = \beta^{-1}\alpha$ and $\alpha\beta^2\alpha^{-1}\beta^{-2} = 1$, we see that $\alpha\beta^2\alpha^{-1}\beta^{-2} = \beta^{-4}$, and therefore $\beta^4 = 1$. Since the order of β is $2p$ and p is even, we see that $p = 2$.

Conversely, if $p = 2$, then we have $\beta^4 = 1$ (see the first paragraph). Since $\alpha\beta = \beta^{-1}\alpha$, we see that $\alpha\beta^2\alpha^{-1}\beta^{-2} = \beta^{-4} = 1$. Therefore, the subgroup $\langle \alpha, \beta^2 \rangle$ is abelian. This completes the proof.

(4.5). PROOF OF THEOREM 3(1). By the above two lemmas, we only need to prove that there exists a homeomorphism H of $M = M(2, q)$ such that $H_*(\langle \alpha, \beta^2 \rangle) = \langle \alpha^2, \beta \rangle$. We shall use same notations as in (1.6) and (2.8). We may assume that $M = M(2, q, r, s)$ for some r, s. Then there exist coverings $g^*: L^*(2, q, r, s) \to M$ and $\tilde{g}: M(1, q, r, 2s) \to M$, where g^* and $\tilde{g}$ correspond to $\langle \alpha^2, \beta \rangle$ and $\langle \alpha, \beta^2 \rangle$, respectively (see (2.8)). Then the involutions $h^*(2, q, r, s)$ and $\tilde{h}(1, q, r, 2s)$ are the nontrivial covering transformations corresponding to g^* and $\tilde{g}$, respectively. In order to prove Theorem 3, it is enough to show the following proposition. Two lemmas will be followed by the proof.

PROPOSITION 4.6. *The involutions $h^*(2, q, r, s)$ and $\tilde{h}(1, q, r, 2s)$ are equivalent.*

LEMMA 4.7. *The space $L^*(2, q, r, s)$ is homeomorphic to $M(1, q)$.*

PROOF. Since $2s - qr = 1$, and r is odd, one can track the following homeomorphisms (see also Lemma 1.7 and Remark 1 in §2): $L^*(2, q, r, s) \approx L(4q, 2s + qr) \approx L(4q, 2qr + 1) \approx L(4q, 2q + 1) \approx M(1, q)$.

LEMMA 4.8. *Let $\tilde{M} = M(1, q, r, 2s)$. Then $\tilde{M}$ has a Heegaard splitting $(\tilde{M}, F)$ of genus 1 such that $\tilde{h}(1, q, r, 2s)$ interchanges the sides of F.*

PROOF. Consider the double covering $g_1^*: L^*(1, q, r, 2s) \to M(1, q, r, 2s)$ defined in (2.8). Let $c_1 = (e^{2\pi ti}, 1)$ and $c_2 = (e^{2\pi ti}, -1)$, $0 \leq t \leq 1$, be the simple closed curves in $S^1 \times S^1$. Let $c_1' = c_1 \times \{0\}$ and $c_2' = c_2 \times \{0\} \subset S^1 \times S^1 \times I$ where $S^1 \times S^1 \times I \subset L(f)$ and f is defined by $f(z_1, z_2) = (z_1 z_2^q,$

$z_1^r z_2^{2s}$) for each $(z_1, z_2) \in S^1 \times S^1$. Let $L^* = L^*(1, q, r, 2s)$. Recall that $L^* = L(f) \cup_\gamma L(f)$. Let L_1 and L_2 be the two $L(f)$ in L^* (we just assign arbitrarily since γ is an involution). Since each c_i $(i = 1, 2)$ is invariant under γ, we may regard c_i' as in both L_1 and L_2. Since each c_i' is isotopic to the center circle of $S^1 \times D^2$ in L_i (see (3.1)), we see that $\pi_1(L^* - c_1' - c_2') = Z \oplus Z$. Since each c_i is invariant under γ, c_i' is invariant under $h^*(1, q, r, 2s)$. Let $U(c_i')$ be an invariant regular neighborhood of each c_i' in L^* such that $U(c_1') \cap U(c_2') = \varnothing$. Since $\pi_1(L^* - c_1' - c_2') = Z \oplus Z$, it follows from **[10]** that $X \approx S^1 \times S^1 \times I$ where $X = \mathrm{cl}(L^* - U(c_1') - U(c_2'))$. Let $\bar{c}_i = g_1^*(c_i)$ $(i = 1, 2)$. It is not difficult to show that $g_1^*(U(c_i'))$ is a regular neighborhood of $\bar{c}_i$. Furthermore, we see that $g_1^*(X) \approx S^1 \times S^1 \times I$ (see **[12]** or **[3]**). This implies that $\pi_1(\tilde{M} - \bar{c}_1 - \bar{c}_2) = Z \oplus Z$.

On the other hand, since each c_i is invariant under γ and $v(c_1) = c_2$ (see (2.8) for v), we see that $\tilde{h}(\bar{c}_1) = \bar{c}_2$ where $\tilde{h} = \tilde{h}(1, q, r, 2s)$ (note that the covering projection $g_1^*|\partial L(f)$ is essentially induced by the involution γ on $S^1 \times S^1$). Let $U(\bar{c}_i)$ $(i = 1, 2)$ be a regular neighborhood of each $\bar{c}_i$ in $\tilde{M}$ such that $U(\bar{c}_1) \cap U(\bar{c}_2) = \varnothing$ and $\tilde{h}(U(\bar{c}_1)) = U(\bar{c}_2)$. Let $\tilde{M}' = \mathrm{cl}(\tilde{M} - U(\bar{c}_1) - U(\bar{c}_2))$. Then since $\pi_1(\tilde{M} - \bar{c}_1 - \bar{c}_2) = Z \oplus Z$, it follows from **[10]** that $\tilde{M}' \approx S^1 \times S^1 \times I$. Since $\tilde{M}'$ is invariant under $\tilde{h}$ and $\tilde{h}$ interchanges the two boundary components of $\tilde{M}'$, we have an invariant surface F in $\tilde{M}'$ isotopic to each component of $\partial \tilde{M}'$ (see **[12]**). Then we see that $(\tilde{M}, F)$ is a Heegaard splitting as desired.

(4.9). PROOF OF PROPOSITION 4.6. We shall use same notations as in (1.6). Let $\tilde{M} = M(1, q, r, 2s)$ and $\tilde{h} = \tilde{h}(1, q, r, 2s)$. By Lemma 4.8, we may assume that

$$\tilde{M} = L^*(\gamma', f') \quad \text{and} \quad \tilde{h} = h(\gamma', f')$$

where γ' is a free involution on $S^1 \times S^1$ and f' is an attaching map of $S^1 \times S^1$ (parametrize one side of F, and then do the other side by means of h). Since γ' and γ are orientation-reversing free involutions on $S^1 \times S^1$, there exists a homeomorphism k of $S^1 \times S^1$ such that $k\gamma' = \gamma k$. In order to distinguish one from the other, we shall denote the two $L(f')$ of $\tilde{M}$ by $L_1(f')$ and $L_2(f')$.

Define a homeomorphism G of $L(f')$ to $L(f'k^{-1})$ by $G(x, t) = (k(x), t)$ on $S^1 \times S^1 \times I$, where $x \in S^1 \times S^1$, and $G(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$. Let $\bar{G}$ be a homeomorphism of $L^*(\gamma', f')$ to $L^*(\gamma, f'k^{-1})$ defined by $\bar{G} = G$ on each $L_i(f')$ $(i = 1, 2)$ such that $\bar{G}(L_i(f')) = L_i(f'k^{-1})$. One can check that $\bar{G}$ is a well-defined equivalence between $h(\gamma', f')$ and $h(\gamma, f'k^{-1})$.

Let $\bar{f} = f'k^{-1}$. Suppose that the matrix of $\bar{f}$ is $\binom{a\ b}{c\ d}$ for the generators defined in (1.5). By a proper choice of orientations, we may assume that $a > 0$ and $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = 1$. Define $f\colon S^1 \times S^1 \to S^1 \times S^1$ by $f(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$ for each $(z_1, z_2) \in S^1 \times S^1$. Then since $\bar{f}$ and f are isotopic, we see that there exists a homeomorphism G_1 of $L(\bar{f})$ to $L(f)$ such that $G_1(S^1 \times S^1 \times I) = S^1 \times S^1 \times I$ by $G_1(x) = x$ for each $x \in S^1 \times S^1 \times I$. Define $\bar{G}_1$ of $L^*(\gamma, \bar{f})$ to $L^*(\gamma, f)$ by $\bar{G}_1 = G_1$ on each $L_i(\bar{f})$ $(i = 1, 2)$ such that $\bar{G}_1(L_i(\bar{f})) = L_i(f)$. One can see that $\bar{G}_1$ is a well-defined equivalence between $h(\gamma, \bar{f})$ and $h(\gamma, f)$. Therefore, we may assume that

$$\tilde{M} = L^*(a, b, c, d) \quad \text{and} \quad \tilde{h} = h^*(a, b, c, d).$$

By (2.8), the orbit space of $h^*(a, b, c, d)$ is $M(a, b)$ and that of $h^*(2, q, r, s)$ is $M(2, q)$. Since $M(a, b) \approx M(2, q)$ (see (4.5)), it follows from Corollary 1.3 that $a = 2$ and $b = \pm q$. Now the proof will be completed if we show the following lemma. $h_1 \sim h_2$ means that h_1 and h_2 are equivalent.

LEMMA 4.10. *Let* $2s - qr = 1 = 2s' - qr'$.
(1) $h^*(2, q, r, s) \sim h^*(2, -q, -r, s)$.
(2) $h^*(2, q, r, s) \sim h^*(2, q, r', s')$.

PROOF. (1) Let f_1 and f_2 be homeomorphisms of $S^1 \times S^1$ given by $f_1(z_1, z_2) = (z_1^2 z_2^q, z_1^r z_2^s)$ and $f_2(z_1, z_2) = (z_1^2 z_2^{-q}, z_1^{-r} z_2^s)$ for each $(z_1, z_2) \in S^1 \times S^1$. Define G_2 of $L(f_1)$ to $L(f_2)$ by $G_2(z_1, z_2, t) = (z_1, \bar{z}_2, t)$ on $S^1 \times S^1 \times I$ and $G_2(z_1, \rho z_2) = (z_1, \rho \bar{z}_2)$ on $S^1 \times D^2$ such that $G_2(S^1 \times S^1 \times I) = S^1 \times S^1 \times I$ and $G_2(S^1 \times D^2) = S^1 \times D^2$. Define $\bar{G}_2$ of $L^*(2, q, r, s)$ to $L^*(2, -q, -r, s)$ by $\bar{G}_2 = G_2$ on each $L_i(f_1)$ $(i = 1, 2)$ such that $\bar{G}_2(L_i(f_1)) = L_i(f_2)$ (see (4.9) for $L_i(f_1)$ and $L_i(f_2)$). It is checked that $\bar{G}_2$ is an equivalence as desired.

(2) Compare the above proof with that of Lemma 1.4(1), and one may easily get the proof from that of Lemma 1.4(2).

(4.11). PROOF OF THEOREM 3(2). Let $g\colon \bar{M} \to M(p, q, r, s)$ be the double covering corresponding to the subgroup $\langle \alpha\beta, \beta^2 \rangle$ and $\bar{h}$ be the nontrivial covering transformation. In the following, we show that $\bar{h}$ is equivalent to $\tilde{h}(p/2, q, r, 2s)$ on $M(p/2, q, r, 2s)$, which suffices the proof (see (2.8)).

Since $|\pi_1(M)| < \infty$, we see that $g^{-1}(K)$ is either a Klein bottle K or a torus $S^1 \times S^1$. We divide the proof into two cases.

Case 1. $g^{-1}(K) \approx K$. Let $U(K)$ be an invariant regular neighborhood of $g^{-1}(K)$ in $\bar{M}$. Then $U(K) \approx M(\gamma)$ (see §1 for $M(\gamma)$), and the complement of $U(K)$ in $\bar{M}$ is homeomorphic to the solid torus $S^1 \times D^2$ (see Lemma 2.5). Therefore, we may assume that

$$\bar{M} = M(\gamma) \cup_f S^1 \times D^2$$

and $\bar{h}$ is given (up to equivalence) by $\bar{h} = \tilde{v}$ on $M(\gamma)$ and $\bar{h}(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$ (see (2.8) for $\tilde{v}$), where f is an appropriate equivariant attaching map. Since the orbit space of $\bar{h}$ can be given as $\bar{M}/\bar{h} = M(\gamma) \cup_{f'} S^1 \times D^2$ and $\bar{M}/\bar{h} \approx M(p, q, r, s)$, we may assume that the matrix of f' is given by $f'_* = \left(\begin{smallmatrix} p & q' \\ r' & s' \end{smallmatrix}\right)$ where $q' = \pm q$, $ps' - q'r' = 1$ and the generators of $\pi_1(S^1 \times S^1)$ are defined as in (1.5) (see Proposition 1.1 and (2.6)). Therefore, a simple computation shows that the matrix of f is given as $f_* = \begin{pmatrix} p/2 & q' \\ r' & 2s' \end{pmatrix}$. We need the following lemma.

LEMMA. *Let $M_i = M(\gamma) \cup_{f_i} S^1 \times D^2$ ($i = 1, 2$) and h_i be an involution of M_i given by $h_i = \tilde{v}$ on $M(\gamma)$ and $h_i(z_1, \rho z_2) = (-z_1, \rho z_2)$. If f_1 and f_2 are isotopic, then h_1 and h_2 are equivalent.*

PROOF. The orbit space M_i' of h_i may be given as $M_i' = M(\gamma) \cup_{f_i'} S^1 \times D^2$ for some attaching map f_i'. Then we see that f_1' and f_2' are isotopic (the isotopy class of f_i' is completely determined by the matrix of f_i'). Therefore there exists a homeomorphism t: $M_1' \to M_2'$ such that $t(x) = x$ on $M(\gamma)$ and $t(S^1 \times D^2) = S^1 \times D^2$. Now, one can use the lifting theorem to complete the proof (see (1.5) for the structures of fundamental groups of M_i and M_i').

Now we go back to Case 1. It follows from the above lemma that $\bar{h}$ is equivalent to $\tilde{h}(p/2, q', r', 2s')$, and therefore equivalent to $\tilde{h}(p/2, q, r, 2s)$ (see the proof of Lemma 1.4).

Case 2. $g^{-1}(K) \approx S^1 \times S^1$. We may assume that $\overline{M} = L^*(\gamma', f)$ and $\bar{h} = h(\gamma', f)$ for some maps γ', f (see (1.6) for notations). By following the technique used in (4.9), one can show that $\bar{h}$ is equivalent to $h^*(p, q, r, s)$ (see (1.6) for $h^*(p, q, r, s)$, and note that $M(p, q, r, s) \approx M(a, b, c, d)$ if and only if $p = a$ and $q = \pm b$). It follows from Lemma 4.4 that $g^{-1}(K) \approx S^1 \times S^1$ only if $p = 2$, and it follows from Proposition 4.6 that $h^*(2, q, r, s)$ is equivalent to $\tilde{h}(1, q, r, 2s)$. This completes the proof.

5. New examples of involutions on lens spaces. A homeomorphism t of a lens space L is called sense-preserving if t induces the identity on $H_1(L)$. In the following we give two types of new examples of non-sense-preserving involutions h on L with $\text{Fix}(h) \neq \varnothing$.

EXAMPLE 1. Let $M = M(1, q, r, s)$ where q is even and $s - qr = 1$. Then M is homeomorphic to the lens space $L(4q, 2q - 1)$ (see Remark 1 in §2). Define a homeomorphism h_1 of M by $h_1 = \tilde{v}$ on $M(\gamma)$ and $h_1(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$ where $\tilde{v}$ is the one in (2.8). It is checked that h_1 is a well-defined involution whose fixed point set is a simple closed curve.

The involution h_1 is non-sense-preserving. For, otherwise, $\text{Fix}(h_1)$ is a disjoint union of two simple closed curves (see [2]). One sees that the orbit space $M(\gamma)/\tilde{v}$ is homeomorphic to $M(\gamma)$. In fact, one can observe that the orbit space M/h_1 is homeomorphic to $M(1, q/2, 2r, s)$ ($\approx L(2q, q - 1)$).

EXAMPLE 2. Let $M = M(1, q, r, s)$ (q either odd or even). We may assume that s is odd (see Proposition 1.1). Replace v in (2.8) by $v(z_1, z_2) = (\bar{z}_1, -\bar{z}_2)$. Then since v and γ commute, we have an obvious involution $\tilde{v}'$ on $M(\gamma)$ induced be the new v (and v'). Define a homeomorphism h_2 of M by $h_2 = \tilde{v}'$ on $M(\gamma)$ and $h_2(z_1, \rho z_2) = (j\bar{z}_1, -\rho\bar{z}_2)$ where $j = 1$ or -1 according to q even or odd. It is checked that h_2 is well-defined involution with nonempty fixed-point set.

Observe that the orbit space M/h_2 is homeomorphic to the projective 3-space P^3 (the orbit space $M(\gamma)/\tilde{v}'$ is a projective 3-space minus an open 3-cell). Therefore, the involution h_2 is non-sense-preserving if $q \neq 1$ (otherwise, the orbit space cannot be a projective 3-space (see [2])). We remark that h_2 is sense-preserving if $q = 1$.

We may formulate the above observation in the following.

PROPOSITION 5.1. (1) *The space M is homeomorphic to the lens space* $L(4q, 2q - 1)$.

(2) *The involutions* h_1 *and* h_2 *are non-sense-preserving* ($q \neq 1$).

(3) *The orbit space* M/h_i ($i = 1, 2$) *is not homeomorphic to* S^3.

Let h be an involution on a lens space $L = L(p, q)$ with $\text{Fix}(h) \neq \emptyset$. A question posed by Tollefson asks whether L has a Heegaard splitting (L, F) of genus 1 such that F is invariant under h and F is in general position with respect to $\text{Fix}(h)$. If h is sense-preserving, then the property (that L has such a Heegaard splitting) holds (see [2], [5]). If h is non-sense-preserving and the property holds, then the orbit space L/h must be homeomorphic to S^3 (see [6] for involutions on $D^2 \times S^1$ and recall that every non-sense-preserving involution is orientation-preserving [4]).

Thus it follows from Proposition 5.1 that the involutions h_1 and h_2 in the examples do not give a Heegaard splitting mentioned above.

ADDED IN PROOF. In a recent paper of the author, the examples in §5 have been generalized to classify all involutions on $L(4q, 2q - 1)$. In fact, a complete classification of the involutions on spaces $M(p, q)$ (which are called Klein spaces in the paper) has been obtained by the author.

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