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THE EIGENVALUE SPECTRUM AS MODULI FOR FLAT TORI

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ABSTRACT. A flat torus T carries a natural Laplace Beltrami operator. It is a conjecture that the spectrum of the Laplace Beltrami operator determines T modulo isometries. We prove that, with the exception of a subvariety in the moduli space of flat tori, this conjecture is true. A description of the subvariety is given.

A flat torus T is the Riemannian manifold that is the quotient of $\mathbf{R}^n$ by a lattice of maximal rank. T has a Laplace operator and an associated sequence of eigenvalues. The following question arises: To what extent is the geometry of T determined by the eigenvalue spectrum? J. Milnor observed that there exist two nonisometric 16-dimensional flat tori with the same eigenvalue spectrum [1], [2], [7]. We show that this phenomenon is nongeneric in the moduli space $O(n)\backslash \mathrm{GL}(n;\ \mathbf{R})/\mathrm{GL}(n;\ \mathbf{Z})$ for flat n-dimensional tori. In particular, given tori $\mathbf{R}^n/A_0\mathbf{Z}^n$ and $\mathbf{R}^n/A_1\mathbf{Z}^n$ with the same eigenvalue spectrum, they are either isometric or the quadratic forms $(A_0^tA_0)$ and $(A_1^tA_1)$ lie on a subvariety in the space of positive definite quadratic forms. The book of M. Berger, P. Gaudauchon and E. Mazet [1] and article of M. Berger [2] are suggested as general references.

A lattice is a discrete subgroup of $\mathbf{R}^n$ and can be prescribed as $A\mathbf{Z}^n$ with A a fixed matrix. An n-dimensional torus T is $\mathbf{R}^n$ factored by a lattice $L = A\mathbf{Z}^n$ with $A \in \mathrm{GL}(n;\ \mathbf{R})$. The metric structure of $\mathbf{R}^n$ projects to T such that volume$(T) = |\det A|$; T carries a Laplace Beltrami operator $\Delta = -\sum_i \partial^2/\partial x_i^2$, the projection of the Laplacian of $\mathbf{R}^n$. The set $\tilde{L} = \{\tilde{a} \in \mathbf{R}^n | \tilde{a}^t a \in \mathbf{Z},\ \forall a \in L\}$ is the dual lattice of L; $\tilde{L} = (A^{-1})^t\mathbf{Z}^n$. The eigenfunctions of T are $\exp(2\pi i\tilde{a}^t x)$ for $x \in \mathbf{R}^n$, $\tilde{a} \in \tilde{L}$. The eigenvalues of T are given as $4\pi^2\|\tilde{a}\|^2$ for $\tilde{a}$ arbitrary in $\tilde{L}$ where $\|\ \|$ is the Euclidean norm. The lengths of closed geodesics of T are given as $\|a\|$ for a aribtrary in L. The eigenvalues of T determine the dimension, volume and the lengths of closed geodesics of T [1], [2]. Tori T_0 and T_1 are called isospectral if they have the same sequence with multiplicities of eigenvalues.

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Let P be a symmetric matrix which defines a quadratic form on $\mathbf{R}^n$. The spectrum of P is defined to be the sequence with multiplicities of values $\gamma = P[N]$ where $P[N] = N^tPN$, $N \in \mathbf{Z}^n$. The sequence of squares of lengths of closed geodesics of $\mathbf{R}^n/A\mathbf{Z}^n$ is the spectrum of $A^tA = Q$; the sequence of eigenvalues is the spectrum of $4\pi^2(A^{-1})(A^{-1})^t = 4\pi^2Q^{-1}$. The Jacobi inversion formula yields for positive τ,

$$\sum_{N \in \mathbf{Z}^n} \exp(-4\pi^2\tau Q^{-1}[N]) = \frac{\text{volume}(T)}{(4\pi\tau)^{n/2}} \sum_{M \in \mathbf{Z}^n} \exp\left(\frac{-1}{4\tau} Q[M]\right).$$

We now describe the manner in which $O(n)\backslash \mathrm{GL}(n; \mathbf{R})/\mathrm{GL}(n; \mathbf{Z})$ is the moduli space of flat tori. To $A \in \mathrm{GL}(n; \mathbf{R})$ is associated the lattice $A\mathbf{Z}^n$. The tori $\mathbf{R}^n/A\mathbf{Z}^n$ and $\mathbf{R}^n/B\mathbf{Z}^n$ are isometric if and only if $A\mathbf{Z}^n$ and $B\mathbf{Z}^n$ are equivalent by multiplication on the left by an element of $O(n)$, the orthogonal group in n-dimensions. The matrices A and B are associated to the same lattice if and only if they are equivalent by multiplication on the right by an element of $\mathrm{GL}(n; \mathbf{Z})$. The tori $\mathbf{R}^n/A\mathbf{Z}^n$ and $\mathbf{R}^n/B\mathbf{Z}^n$ are isometric if and only if A and B are equivalent in $O(n)\backslash \mathrm{GL}(n; \mathbf{R})/\mathrm{GL}(n; \mathbf{Z})$. Denote the space of positive definite symmetric $n \times n$ matrices as $\mathcal{S}(n; \mathbf{R})$; we observe that the map

$$A \in \mathrm{GL}(n; \mathbf{R}) \to A^tA \in \mathcal{S}(n; \mathbf{R})$$

determines a bijection of $O(n)\backslash \mathrm{GL}(n; \mathbf{R})$ to $\mathcal{S}(n; \mathbf{R})$. Let e_i be the ith column of the identity matrix in $\mathrm{GL}(n; \mathbf{R})$ and $e_{ij} = e_i + e_j$. We consider $\mathcal{S}(n; \mathbf{R})$ to be embedded in $\mathbf{R}^m$ for $m = n(n+1)/2$. The cartesian coordinates of $P = (p_{ij}) \in \mathcal{S}(n; \mathbf{R})$ are $P[e_i] = p_{ii}$ and $(P[e_{ij}] - P[e_i] - P[e_j])/2 = p_{ij}$. For later reference we define $E \in \{e_k, e_{ij} | 1 \leq k \leq n, 1 \leq i < j \leq n\}$.

We now generalize two theorems for Riemann surfaces to n-dimensional tori [4], [6].

Theorem 1. *Let T_s be a continuous family of isospectral tori defined for $s \in [0, 1]$. The tori T_s, $s \in [0, 1]$, are isometric.*

Proof. We lift T_s a continuous curve into $O(n)\backslash \mathrm{GL}(n; \mathbf{R})/\mathrm{GL}(n; \mathbf{Z})$ to a curve $g(s)$ of $[0, 1]$ into $O(n)\backslash \mathrm{GL}(n; \mathbf{R})$. Thus $(g(s)^tg(s))$ is a curve into $\mathcal{S}(n: \mathbf{R})$. The forms $(g(s)^tg(s))$ have a spectrum independent of s. Thus for every $N \in \mathbf{Z}^n$, $(g(s)^tg(s))[N]$ is a continuous function with range contained in the spectrum of $(g(0)^tg(0))$. Since the spectrum of an element of $\mathcal{S}(n; \mathbf{R})$ is a discrete set, the functions $(g(s)^tg(s))[N]$ are constant. By the coordinate description of $\mathcal{S}(n; \mathbf{R})$, $(g(s)^tg(s))$ is constant; thus $g([0, 1])$ is a point in $O(n)\backslash \mathrm{GL}(n; \mathbf{R})$.

The following result is due to M. Kneser (unpublished) [1].

Theorem 2. *The total number of nonisometric tori with a given eigenvalue spectrum is finite.*

PROOF. By contradiction assume the existence of a sequence of distinct isospectral tori $T_1, \ldots, T_l, \ldots$. The tori each have the same dimension, volume and length of the shortest closed geodesic. Choose a lattice L_l which represents the torus T_l. By Mahler's compactness theorem a subsequence L_k exists which converges to L_0 (i.e., matrices A_k exist with $A_k\mathbf{Z}^n = L_k$ and A_k converge to A_0 where $L_0 = A_0\mathbf{Z}^n$) [3]. Let U be a neighborhood of $S_0 = A_0^tA_0$ with $\bar{U}$ compact and $\bar{U} \subset \mathfrak{S}(n; \mathbf{R})$. Define $c_1 = \max\{S[e] | e \in E, S \in U\}$. Since $S \in \mathfrak{S}(n; \mathbf{R})$ can be diagonalized by conjugation with an orthogonal matrix, we have for $\lambda_{\min}(S)$ the smallest (resp. $\lambda_{\max}(S)$ the largest) eigenvalue $\lambda_{\min}(S)\|N\|^2 \leqslant S[N] \leqslant \lambda_{\max}(S)\|N\|^2$ for $N \in \mathbf{Z}^n$. Now from the inclusion $\bar{U} \subset \mathfrak{S}(n; \mathbf{R})$ it follows that $\lambda_{\min}(S) \geqslant c_2 > 0$ for $S \in U$. In particular, for $N \in \mathbf{Z}^n$, $\|N\|^2 > c_1/c_2$ and $S \in U$ it follows that $S[N] \geqslant \lambda_{\min}(S)\|N\|^2 > c_1$. Reformulating this we have for $S \in U$, $e \in E$ and $M \in \mathbf{Z}^n$ with $S[M] = S_0[e]$ that $M \in F = \{N \in \mathbf{Z}^n | \ \|N\|^2 \leqslant c_1/c_2\}$. We now consider for $N \in F$ the finite collection of functions $S[N]$ with domain U. A neighborhood $V \subset U$ is defined as follows: $V = \{S \in U| \ |S[N] - S_0[N]| < |S[N] - S_0[M]|$ for each $N \in E$ and all M with $S_0[N] \neq S_0[M]\}$. Now for k sufficiently large, $(A_k^tA_k) \in V$. In particular, for $e \in E$, $|(A_k^tA_k)[e] - (A_0^tA_0)[e]|$ is strictly less than the distance between $(A_k^tA_k)[e]$ and any value distinct from $(A_0^tA_0)[e]$ in the spectrum of $(A_0^tA_0)$. Noting that $(A_k^tA_k)$ and $(A_0^tA_0)$ have the same spectrum we conclude $(A_k^tA_k)[e] = (A_0^tA_0)[e]$ for all $e \in E$, the desired contradiction.

The following theorem describes the structure of the equivalence relation, having the same spectrum, for forms.

THEOREM 3. *There is a properly discontinuous group G_n acting on $\mathfrak{S}(n; \mathbf{R})$ containing the transformation group induced by the* GL$(n; \mathbf{Z})$ *action $S \to S[\mathfrak{X}]$, $S \in \mathfrak{S}(n; \mathbf{R})$, $\mathfrak{X} \in$* GL$(n; \mathbf{Z})$. *Given $P, S \in \mathfrak{S}(n; \mathbf{R})$ with the same spectrum either $g(P) = S$ for some $g \in G_n$ or $P, S \in \mathbf{V}_n$ where $\mathbf{V}_n$ is a subvariety of $\mathfrak{S}(n; \mathbf{R})$. $\mathbf{V}_n = \{Q \in \mathfrak{S}(n; \mathbf{R}) | \operatorname{spec}(Q) = \operatorname{spec}(R), R \in \mathfrak{S}(n; \mathbf{R})$ with $R \neq g(Q)$ for all $g \in G_n\}$. $\mathbf{V}_n$ is the intersection of $\mathfrak{S}(n; \mathbf{R})$ and a countable union of subspaces of $\mathbf{R}^m$.*

The proof is initiated with the following lemmas.

LEMMA 4. *Let $P, S \in \mathfrak{S}(n; \mathbf{R})$ have the same spectrum. Neighborhoods U of P, V of S and a finite number of maps $g_1, \ldots, g_l$ with domain U are defined. For $Q \in U$ and $R \in V$ with the same spectrum then $R = g_j(Q)$ for some j, $1 \leqslant j \leqslant l$. The maps g_j are linear in the coordinates of $\mathbf{R}^m$ and have rational coefficients.*

PROOF. Set $c_1 = 2 \max\{S[e] | e \in E\}$. We can, noting that E is finite, choose a neighborhood V of S such that for $R \in V$, $\max\{R[e] | e \in E\} < c_1$.

A neighborhood U_1 of P is chosen with $\lambda_{\min}(Q) \geq c_2 > 0$ for $Q \in U_1$. Thus considering $\lambda_{\min}$ we have $Q[M] > c_1$ for $M \in \mathbf{Z}^n$, $Q \in U_1$ with $\|M\|^2 > c_1/c_2$. Now let $Q_0 \in U_1$ and $R_0 \in V$ be such that vectors M_k, M_{ij} exist with $Q_0[M_k] = R_0[e_k]$, $1 \leq k \leq n$ and $Q_0[M_{ij}] = R_0[e_{ij}]$, $1 \leq i < j \leq n$. A map $R = g(Q)$ linear in the coordinates of $\mathbf{R}^m$ is defined by the equations $Q[M_k] = R[e_k]$, $1 \leq k \leq n$, $Q[M_{ij}] = R[e_{ij}]$, $1 \leq i < j \leq n$. The map g has rational coefficients. Let G be the set of all maps $R = g_\alpha(Q)$, $Q \in U_1$ with (i) g_α defined by equations $R[e_k] = Q[M_k^\alpha]$, $M_k^\alpha \in \mathbf{Z}^n$, $1 \leq k \leq n$, $R[e_{ij}] = Q[M_{ij}^\alpha]$, $M_{ij}^\alpha \in \mathbf{Z}^n$, $1 \leq i < j \leq n$; (ii) $g_\alpha(U_1) \cap V \neq \varnothing$. Referring to the definitions of U_1 and V it follows that $\|M_k^\alpha\|^2, \|M_{ij}^\alpha\|^2 \leq c_1/c_2$. Thus $G = \{g_1, \ldots, g_l\}$ is finite. We restrict our consideration to those g_j, $1 \leq j \leq l$, such that a fixed neighborhood $U \subset U_1$ of P exists with $g_j(U) \subset V$. Now for $Q, R \in \mathbb{S}(n; \mathbf{R})$ with the same spectrum a bijection β of $\mathbf{Z}^n$ necessarily exists with $Q[\beta(N)] = R[N]$ for all $N \in \mathbf{Z}^n$. Consequently, for $Q \in U$ and $R \in V$ with the same spectrum, $R = g_j(Q)$ for some j, $1 \leq j \leq l$. The proof is complete.

LEMMA 5. *Let P and S have the same spectrum and β be the bijection of $\mathbf{Z}^n$ such that $P[\beta(N)] = S[N]$ for all $N \in \mathbf{Z}^n$. Let g be the map with domain U, a neighborhood of P, defined by $R = g(Q)$ where $Q[M_k] = R[e_k]$, $1 \leq k \leq n$, and $Q[M_{ij}] = R[e_{ij}]$, $1 \leq i < j \leq n$. Assume furthermore that $S = g(P)$. Then either $Q[\beta(N)] = g(Q)[N]$ for all $Q \in \mathbb{S}(n; \mathbf{R})$ or $\{Q \in \mathbb{S}(n; \mathbf{R}) | \operatorname{spec}(Q) = \operatorname{spec}(g(Q)), g(Q) \in \mathbb{S}(n; \mathbf{R})\}$ is a subvariety of $\mathbb{S}(n; \mathbf{R})$. In the latter case $\{Q \in \mathbb{S}(n; \mathbf{R}) | \operatorname{spec}(Q) = \operatorname{spec}(g(Q)), g(Q) \in \mathbb{S}(n; \mathbf{R})\}$ is the intersection of $\mathbb{S}(n; \mathbf{R})$ and a countable union of subspaces of $\mathbf{R}^m$.*

PROOF. It is clear that g is a linear map of $\mathbf{R}^m$ to $\mathbf{R}^m$. Let β be a bijection of $\mathbf{Z}^n$; then $\{Q \in \mathbf{R}^m | Q[\beta(N)] = g(Q)[N],\ N \in \mathbf{Z}^n\}$ is the intersection of countably many subspaces and thus is itself a subspace. Now either $\mathrm{V}(\beta) \overset{\text{def}}{=} \{Q \in \mathbb{S}(n; \mathbf{R}) | Q[\beta(N)] = g(Q)[N],\ N \in \mathbf{Z}^n\}$ equals $\mathbb{S}(n; \mathbf{R})$ for some bijection β, or for every bijection β of $\mathbf{Z}^n$, $\mathrm{V}(\beta)$ is the intersection of $\mathbb{S}(n; \mathbf{R})$ and a proper subspace of $\mathbf{R}^m$. Reversing the roles of Q and Q^{-1} in the Jacobi inversion formula we observe that $\operatorname{spec}(Q)$ determines $|\det Q|$. The boundary of $\mathbb{S}(n; \mathbf{R}) \subset \mathbf{R}^m$ consists of matrices of zero determinant. It is thus immediate that for $Q \in \mathbb{S}(n; \mathbf{R})$ with $\operatorname{spec}(Q) = \operatorname{spec}(g(Q))$ that $g(Q) \in \mathbb{S}(n; \mathbf{R})$. In particular, $\operatorname{spec}(Q) = \operatorname{spec}(g(Q))$ if and only if $Q \in \mathrm{V}(\beta)$ for some bijection β of $\mathbf{Z}^n$. We now consider the case that $\mathrm{V}(\beta) \neq \mathbb{S}(n; \mathbf{R})$ for all bijections β. It only remains to show that a neighborhood U_0 of P exists with

$$\{Q \in U_0 | \operatorname{spec}(Q) = \operatorname{spec}(g(Q))\} = U_0 \cap \bigcup_{n=1}^{t} \mathrm{V}(\beta_n)$$

for appropriate bijections $\beta_1, \dots, \beta_l$. Let U_0 (resp. V_0) be a relatively compact neighborhood of P (resp. S) such that $\overline{U}_0, \overline{V}_0 \subset \mathbb{S}(n; \mathbf{R})$ and $g(U_0) \subset V_0$. Now from $U_0, V_0 \subset \mathbb{S}(n; \mathbf{R})$ we have $0 < c_1 \leqslant \lambda_{\min}(Q)$, $\lambda_{\max}(Q) \leqslant c_2$ for $Q \in U_0$ and $0 < c_3 \leqslant \lambda_{\min}(R)$, $\lambda_{\max}(R) \leqslant c_4$ for $R \in V_0$. Let $\mathfrak{B}$ be the set of all bijections of $\mathbf{Z}^n$. Trivially

$$\{Q \in U_0 | \operatorname{spec}(Q) = \operatorname{spec}(g(Q))\} = U_0 \cap \bigcup_{\beta \in \mathfrak{B}} \mathbf{V}(\beta).$$

Proceeding by contradiction we assume an infinite set $\{\beta_\alpha\}$, $\alpha \in \mathfrak{A}$, of bijections exists such that (i) $\{Q \in U_0 | \operatorname{spec}(Q) = \operatorname{spec}(g(Q))\} = U_0 \cap \bigcup_{\alpha \in \mathfrak{A}} \mathbf{V}(\beta_\alpha)$, (ii) $\mathbf{V}(\beta_\alpha)$ is not properly contained in $\mathbf{V}(\beta)$, $\beta \in \mathfrak{B}$, and (iii) $\mathbf{V}(\beta_\alpha) \neq \mathbf{V}(\beta_{\alpha'})$ for $\alpha \neq \alpha'$. Let $\{\beta_l\}$ be a sequence chosen from $\{\beta_\alpha\}$, $\alpha \in \mathfrak{A}$. Given $Q_l \in U_0 \cap \mathbf{V}(\beta_l)$, then

$$c_1 \| \beta_l(N) \|^2 \leqslant Q_l[\beta_l(N)] = g(Q_l)[N] \leqslant c_4 \|N\|^2.$$

In particular, for each $N \in \mathbf{Z}^n$ there are at most finitely many possibilities for $\beta_l(N)$. By Cantor diagonalization we obtain a subsequence $\{\beta_p\}$ such that for each $N \in \mathbf{Z}^n$, $\beta_p(N)$ is independent of p for p sufficiently large. Now we define $\beta_\infty(N) = \lim_{p \to \infty} \beta_p(N)$ for each $N \in \mathbf{Z}^n$. β_∞ is an injection of $\mathbf{Z}^n$ into $\mathbf{Z}^n$. Specifically for $N \neq M \in \mathbf{Z}^n$ there is a p_0 and for $p \geqslant p_0$, $\beta_\infty(N) = \beta_p(N) \neq \beta_p(M) = \beta_\infty(M)$. β_∞ is a surjection of $\mathbf{Z}^n$ to $\mathbf{Z}^n$. Given $Q_p \in U_0 \cap \mathbf{V}(\beta_p)$ then

$$c_2 \| \beta_p(N) \|^2 \geqslant Q_l[\beta_p(N)] = g(Q_p)[N] \geqslant c_3 \|N\|^2.$$

Fix $M_0 \in \mathbf{Z}^n$; then $M_0 = \beta_p(\beta_p^{-1}(M_0))$ and thus $c_2/c_3 \|M_0\|^2 \geqslant \|\beta_p^{-1}(M_0)\|^2$. There is a p_1 and for $p \geqslant p_1$, $\beta_p(N) = \beta_\infty(N)$ for N such that $\|N\|^2 \leqslant c_2/c_3 \|M_0\|^2$. In particular, for $p \geqslant p_1$,

$$M_0 = \beta_p(\beta_p^{-1}(M_0)) = \beta_\infty(\beta_p^{-1}(M_0)).$$

The set $\{Q \in \mathbf{R}^m | Q[\beta_\infty(N)] = g(Q)[N], N \in \mathbf{Z}^n\}$ is a subspace of $\mathbf{R}^m$. Thus a constant $c_5 > 0$ exists with $\mathbf{V}(\beta_\infty) = \{Q \in \mathbb{S}(n; \mathbf{R}) | Q[\beta_\infty(N)] = g(Q)[N], \|N\| \leqslant c_5\}$. For an appropriate p_2, $\beta_p(N) = \beta_\infty(N)$ for $p \geqslant p_2$ and $\|N\| \leqslant c_5$. In particular, $\mathbf{V}(\beta_p) \subset \mathbf{V}(\beta_\infty)$, $p \geqslant p_2$. The containment $\mathbf{V}(\beta_p) \subset \mathbf{V}(\beta_\infty)$ is not proper by the maximality condition for the $\mathbf{V}(\beta_\alpha)$, $\alpha \in \mathfrak{A}$. Thus $\mathbf{V}(\beta_p) = \mathbf{V}(\beta_\infty)$ for $p \geqslant p_2$, a contradiction. The proof is complete.

PROOF OF THEOREM 3. Let g be a map defined by $g(Q) = R$ where $Q[M_k] = R[e_k]$, $1 \leqslant k \leqslant n$ and $Q[M_{ij}] = R[e_{ij}]$, $1 \leqslant i < j \leqslant n$. Let β be a bijection of $\mathbf{Z}^n$ such that

$$Q[\beta(N)] = g(Q)[N] \quad \text{for all } N \in \mathbf{Z}^n \tag{1}$$

and all Q in an open set U. The map g is defined and (1) holds throughout $\mathbf{R}^m$. We deduce from $\operatorname{spec}(Q) = \operatorname{spec}(g(Q))$ for all $Q \in \mathbb{S}(n; \mathbf{R})$ that $g(Q) \in \mathbb{S}(n; \mathbf{R})$ for all $Q \in \mathbb{S}(n; \mathbf{R})$. The fibers $g^{-1}(g(Q))$, $Q \in \mathbb{S}(n; \mathbf{R})$ are finite from Theorem 2. It now follows that g is a linear isomorphism of $\mathbf{R}^m$. Trivially the equations $g^{-1}(R)[N] = R[\beta^{-1}(N)]$ for all $N \in \mathbf{Z}^n$, all $R \in \mathbf{R}^m$ hold; g^{-1} maps $\mathbb{S}(n; \mathbf{R})$ into $\mathbb{S}(n; \mathbf{R})$. Define G_n to be the group of all linear isomorphisms g of $\mathbf{R}^m$ for which there is a β and (1) holds. Referring to Lemmas 4 and 5 the proof is complete.

DEFINITION 6. A vector $N \in \mathbf{Z}^n$ is primitive if $N \neq pM$ for $M \in \mathbf{Z}^n$ and $p \in \mathbf{Z} - \{0, \pm 1\}$.

THEOREM 7. *G_n coincides with the transformation group induced by* GL$(n; \mathbf{Z})$.

PROOF. If $Q[N_0]$ is the smallest positive value in the spectrum of Q then N_0 is primitive. Remove the sequence $\{p^2 Q[N_0]_{p=1}^{\infty}\}$ from the spectrum of Q. The smallest remaining positive value $Q[N_1]$ corresponds to a primitive vector N_1; remove the sequence $\{p^2 Q[N_1]\}_{p=1}^{\infty}$. Continuing in this manner all primitive vectors are identified, and for g and β satisfying (1), β preserves this construction.

We consider $g \in G_n$ and show that g can be transformed to the identity by conjugation with elements of GL$(n; \mathbf{Z})$. Let g be defined by the equations $Q[M_k] = g(Q)[e_k]$ and $Q[M_{ij}] = g(Q)[e_{ij}]$. M_n is a primitive vector; thus $\mathfrak{M} \in \mathrm{GL}(n; \mathbf{Z})$ exists with $\mathfrak{M} e_n = M_n$. Replacing g with the map $Q \to g(Q[\mathfrak{M}^{-1}])$ we can assume $M_n = e_n$. We now proceed by induction on the dimension n. For $n = 2$ it is classical that the eigenvalue spectrum determines the tori in $O(2) \backslash \mathrm{GL}(2; \mathbf{R})/\mathrm{GL}(2; \mathbf{Z})$ [1], [6]. Define Φ to be the projection of $\mathbf{R}^n$ onto the first $n-1$ coordinates. Let Ψ be the natural inclusion of $\mathbf{R}^{n-1}$ into $\mathbf{R}^n$ with image the first $n-1$ coordinates of $\mathbf{R}^n$. Given Q a symmetric quadratic form on $\mathbf{R}^n$, define $\tilde{Q}$ a symmetric quadratic form on $\mathbf{R}^{n-1}$ by $\tilde{Q}[x] = Q[\Psi(x)]$ for $x \in \mathbf{R}^{n-1}$. Let Q_s be a curve from [0, 1] into $\mathbf{R}^m$ such that (i) $Q_s \in \mathbb{S}(n; \mathbf{R})$ for $0 < s < 1$, (ii) $Q_1[e_n] = 0$, and (ii) $\tilde{Q}_1 \in \mathbb{S}(n - 1; \mathbf{R})$. We observe that $Q_1[\beta(N)] = g(Q_1)[N]$ for all $N \in \mathbf{Z}^n$; in particular, $g(Q_1)$ is positive semidefinite and $Q_1[M_k] = g(Q_1)[e_k]$ with $M_n = e_n$. For $R = (r_{ij})$ positive semidefinite we have by the Cauchy Schwarz inequality that $r_{ij}^2 < r_{ii} r_{jj}$; in particular, $e_i^t Q_1 e_n = e_i^t g(Q_1) e_n = 0$, $1 < i < n$. Assume that the entries q_{ij} of Q_1 with $1 < i < j < n-1$ are rationally independent. For $N, M \in \mathbf{Z}^{n-1}$ with $\tilde{Q}_1[N] = \tilde{Q}_1[M]$ it follows that $N = \pm M$. We observe for $\gamma \neq 0$ in the spectrum of Q_1 that γ has multiplicity two in the spectrum of $\tilde{Q}_1$. As $Q_1[\beta(N)] = g(Q_1)[N]$ for every $N \in \mathbf{Z}^n$ we conclude $\tilde{Q}_1$ and $\widetilde{g(Q_1)}$ have the same spectrum. The map g induces a linear map $\tilde{g}$ from a neighborhood of $\tilde{Q}_1 \in \mathbb{S}(n-1; \mathbf{R}) \subset \mathbf{R}^p$, $p = n(n-1)/2$, to a neighborhood of $\widetilde{g(Q_1)} \in \mathbb{S}(n-1; \mathbf{R})$. The map $\tilde{g}$ preserves the spectrum with the possible exception of the forms with rationally dependent entries. Those forms in $\mathbb{S}(n-1; \mathbf{R})$ with rationally dependent entries form a subset of measure zero. Referring to Theorem 3 and Lemmas 4 and 5 we conclude $\tilde{g}$ induces a spectrum preserving isomorphism of $\mathbb{S}(n-1; \mathbf{R})$ to $\mathbb{S}(n-1; \mathbf{R})$.

The map $\tilde{g}$ by the induction hypothesis corresponds to a $\mathfrak{T} \in GL(n-1; \mathbf{Z})$. Define $\mathfrak{T}_1 = \left(\begin{smallmatrix}\mathfrak{T} & 0 \\ 0 & 1\end{smallmatrix}\right)$, $R = h(Q) = g(Q)[\mathfrak{T}_1^{-1}]$ and $\alpha(N) = \beta(\mathfrak{T}_1^{-1}N)$. We observe that α is a bijection of $\mathbf{Z}^n$ with $h(Q)[N] = Q[\alpha(N)]$. It follows from the induction hypothesis that $\Phi(\alpha(N)) = \pm\Phi(N)$; for our purposes we can assume $\Phi(\alpha(N)) = \Phi(N)$. We have $\alpha(e_n) = e_n$ from the definition of α and the fact that $\beta(e_n) = e_n$. A matrix Θ is now defined by the equations $\Theta e_k = \alpha(e_k)$, $1 \leqslant k \leqslant n$. It is clear that Θ has integer entries and that $\det(\Theta) = 1$. We conclude that $\Theta \in GL(n; \mathbf{Z})$. Define the map $f \in G_n$ by $R = f(Q) = h(Q[\Theta^{-1}])$ and the bijection δ of $\mathbf{Z}^n$ by $\delta(N) = \Theta^{-1}\alpha(N)$ for all $N \in \mathbf{Z}^n$. The map f is also defined by the equations $R[e_k] = Q[\Theta^{-1}\alpha(e_k)]$ and $R[e_{ij}] = Q[\Theta^{-1}\alpha(e_{ij})]$. We conclude that $\Phi(\delta(N)) = \Phi(N)$ for all $N \in \mathbf{Z}^n$ from the definition of Θ and the corresponding fact for α. The map f modulo a choice of signs will be the identity in G_n. Noting that $\delta(e_k) = e_k$, $1 \leqslant k \leqslant n$, we conclude for $R = f(Q)$ with $R = (r_{ij})$ and $Q = (q_{ij})$ that $r_{kk} = q_{kk}$, $1 \leqslant k \leqslant n$. Now consider a particular entry r_{ij}, $1 \leqslant i < j \leqslant n$, and the defining equation

$$r_{ij} = \left(Q[\delta(e_i + e_j)] - Q[e_i] - Q[e_j]\right)/2.$$

We note from $\Phi(\delta(N)) = \Phi(N)$ for $N \in \mathbf{Z}^n$ that $\delta(e_i + e_j) = e_i + e_j + s_{ij}e_n$, $s_{ij} \in \mathbf{Z}$. The defining equation for r_{ij} becomes

$$r_{ij} = q_{ij} + s_{ij}q_{in} + s_{ij}q_{jn} + s_{ij}^2 q_{nn}/2.$$

A short computation shows that r_{ij} is independent of q_{nn} if and only if $s_{ij} = 0$ for $j < n$ or $s_{ij} = 0, -2$ for $j = n$. Now consider $Q \in \mathbb{S}(n; \mathbf{R})$ to be diagonal with q_{kk}, $k < n$, fixed. Assume r_{ij} depends on q_{nn}; r_{ij}^2 thus has quadratic growth in q_{nn} for $q_{nn} \to \infty$. Considering the inequality $r_{ij}^2 \leqslant r_{ii}r_{jj} = q_{ii}q_{jj}$ we have a contradiction since $i < n$ and q_{ii} is fixed for $q_{nn} \to \infty$. We conclude $r_{ij} = q_{ij}$ for $1 \leqslant i, j \leqslant n-1$, $r_{in} = \pm q_{in}$, $1 \leqslant i \leqslant n-1$ and $r_{nn} = q_{nn}$. Now to ascertain the signs choose $Q \in \mathbb{S}(n; \mathbf{R})$ with rationally independent entries. Assume there exist $i, k < n$ with $q_{kn} = r_{kn}$ and $q_{in} = -r_{in}$, as otherwise $R = Q[\mathfrak{Y}]$ where

$$\mathfrak{Y} = \begin{pmatrix} \mathrm{id}_{n-1} & 0 \\ 0 & \pm 1 \end{pmatrix}$$

and id_{n-1} is the identity in $GL(n-1; \mathbf{Z})$. For $Q = (q_{ab})$ and $R = (r_{ab}) = f(Q)$ there exists a vector $M = (m_1, \ldots, m_n)^t$ such that $R[M] = Q[e_k + e_i + e_n]$. By the rational independence we have $q_{nn} = m_n^2 r_{nn}$, $q_{ki} = m_k m_i r_{ki}$, $q_{in} = m_i m_n r_{in}$ and $q_{kn} = m_k m_n r_{kn}$. From the definition of f we have $q_{nn} = r_{nn}$ and $q_{ki} = r_{ki}$; thus $m_k m_i = m_n^2 = 1$. By assumption $m_k m_n = 1$ and $m_i m_n = -1$; combining these relations $1 = m_k m_i = m_k m_n m_n m_i = -1$, a contradiction. The proof is now complete.

Theorems 3 and 7 are combined in the following.

THEOREM 8. *Isospectral tori* $T_0 = \mathbf{R}^n/A_0\mathbf{Z}^n$, $T_1 = \mathbf{R}^n/A_1\mathbf{Z}^n$ *are isometric if and only if at least one of the quadratic forms* $(A_0^tA_0)$, $(A_1^tA_1)$ *is an element of* $\mathfrak{S}(n;\ \mathbf{R}) - \mathbf{V}_n$. *If* T_0 *and* T_1 *are not isometric then the entries of the matrix* $(A_1^tA_1)$ *are linear combinations with rational coefficients of the entries of the matrix* $(A_0^tA_0)$. *The set* $\mathbf{V}_n$ *is* $\mathfrak{S}(n;\ \mathbf{R})$ *intersected with a countable union of subspaces of* $\mathbf{R}^m$; *these subspaces are defined by equations with rational coefficients.*

COROLLARY 9. *Let* $\mathbf{R}^n/A\mathbf{Z}^n$ *be given such that the entries of the form* $(A^tA) \in \mathbf{V}_n \subset \mathfrak{S}(n;\ \mathbf{R})$ *satisfy at most* p *distinct linear homogeneous equations with rational coefficients. The form* (A^tA) *is contained in a subspace* W, *with* $W \cap \mathfrak{S}(n;\ \mathbf{R}) \subset \mathbf{V}_n$ *and* $m - p < \dim W < m - 1$. *If the entries of the form are rationally independent* $\mathbf{R}^n/A\mathbf{Z}^n$ *is uniquely determined by its eigenvalue spectrum.*

A form $Q \in \mathfrak{S}(n;\ \mathbf{R})$ is called semi-integral if for $Q = (q_{ij})$, $q_{kk} \in \mathbf{Z}$, $1 < k < n$ and $2q_{ij} \in \mathbf{Z}$, $1 < i < j < n$. Q semi-integral is equivalent to the statement $\operatorname{spec}(Q) \subset \mathbf{Z}$. The semi-integral forms are of particular number theoretic interest.

LEMMA 10. *Let* $\mathbf{V}_n$ *be nonempty for a particular* n. *Then semi-integral forms* $Q_0, Q_1 \in \mathbf{V}_n$ *exist.*

PROOF. $\mathbf{V}_n$ is nonempty by hypothesis. Observe that rational points are dense in subspaces defined by rational equations. In particular, $P_0, P_1 \in \mathbf{V}_n$ exist with $\operatorname{spec}(P_0) = \operatorname{spec}(P_1)$ and P_0 has rational coordinates. Since P_0 is rational a positive integer p exists with pP_0 semi-integral and thus $p \operatorname{spec}(P_0) = p \operatorname{spec}(P_1) \subset \mathbf{Z}$. In particular, pP_1 is semi-integral.

Previous results show that $\mathbf{V}_n$ is nonempty for $n > 12$ [1], [2]. In fact, an elementary construction shows that if $\mathbf{V}_n$ is nonempty then all $\mathbf{V}_m$, $m > n$, are nonempty. From Lemma 10 it suffices to consider the semi-integral forms in the cases $n = 3, \ldots, 11$. We also note that a result analogous to Theorem 8 has been obtained for the case of compact Riemann surfaces [8], [9].

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