TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 245, November 1978

# EXPLOSIONS IN COMPLETELY UNSTABLE FLOWS. II
# SOME EXAMPLES

BY

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ABSTRACT. A dynamical system with all points wandering is "explosive" if some $C^0$ perturbation has nonwandering points. It is known that the plane admits no explosive cascades or flows; in this paper, examples are constructed to show that all open manifolds except $R^1$ and $R^2$ admit explosive flows (and hence cascades).

A dynamical system is *completely unstable* if its nonwandering set is empty–that is, if every point has a neighborhood whose images under the action of the dynamical system are eventually disjoint from it. Several authors ([5], [7], [12]) have observed the phenomenon of "explosiveness"–that a completely unstable dynamical system can have perturbations which are not completely unstable. The easiest examples arise from systems on compact manifolds which have "$\Omega$-explosions", by deleting the nonwandering set from the phase space. This kind of construction, however, results in examples on a restricted class of phase spaces. Mendes [7] has shown, as a consequence of the Brouwer translation theorem, that such "explosive" examples do not occur in the plane, $R^2$; but he has also shown how the "deletion" idea can be used to construct an explosive diffeomorphism of $R^3$. Takens and White [12] find, on any manifold, a generic set of flows, among which the completely unstable flows are nonexplosive. Yet the question remains, which manifolds offer a topological obstruction to explosiveness–that is, where besides the plane is the "generic" situation actually the "general" situation?

In this paper, we show that the plane is atypical in this respect. We construct examples that prove the following result, announced in the note [8].

DEFINITION. A flow $\phi$ is *$C^r$-explosive* if $\Omega(\phi) = \varnothing$ but every neighborhood of $\phi$ (in the strong $C^r$ topology on flows) contains flows with $\Omega(\psi) \neq \varnothing$.

THEOREM. *If $M$ is an open manifold (noncompact, connected, finite-dimensional, paracompact, without boundary) not homeomorphic to either $R^1$ or $R^2$, then there exists a $C^r$-explosive flow on $M$.*

The proof treats three distinct cases:

Received by the editors March 10, 1977.
*AMS (MOS) subject classifications* (1970). Primary 58F10; Secondary 34C35, 57D99, 54H20.
*Key words and phrases*. Completely unstable flow, $\Omega$-explosion, open manifold.
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(i) $\dim M \geqslant 3$,
(ii) $M = X - K$, where $X$ is a closed surface and $K$ is a closed subset,
(iii) $\dim M = 2$ and $M$ does not embed in any closed surface.

We will treat these cases in the order above, but will begin with a specific example which enters the construction.

The author would like to thank the referee for helpful comments on this paper and its companion [9], and to acknowledge with pleasure several conversations with R. Devaney on this material.

**1. A cylindrical example.** An example much like the following is sketched by Takens and White [**12**]; we shall treat it in some detail before using it in our general constructions.

The phase space of this example is the infinite cylinder

$$M = R \times S^1$$

which we can picture embedded in $R^3$ as the surface

$$x = \cos\theta, \quad y = \sin\theta, \quad z = z.$$

The number "$z$" and the angle "$\theta$" serve as a kind of global coordinate system on $M$.

Consider the flow defined by the system of equations

$$\dot{z} = \cos\theta, \qquad \dot{\theta} = \sin^2\theta. \tag{1}$$

There are no equilibria; the integral curves are given by the formula

$$z + \csc\theta = \text{constant} \tag{2a}$$

together with the two vertical lines

$$\theta = 0, \tag{2b}$$

$$\theta = \pi. \tag{2c}$$

The flow is upward ($z$ increasing) along (2b) and downward along (2c).

These lines divide the cylinder into two invariant strips, on each of which the foliation by integral curves forms a "Reeb component". We note that every orbit in the strip $0 < \theta < \pi$ is negatively asymptotic to (2b) and positively asymptotic to (2c)

$$\lim_{t\to-\infty}\theta(t) = 0, \quad \lim_{t\to+\infty}\theta(t) = \pi, \qquad 0 < \theta(0) < \pi,$$

while in the strip $\pi < \theta < 2\pi$, orbits are positively asymptotic to (2b) and negatively asymptotic to (2c):

$$\lim_{t\to-\infty}\theta(t) = \pi, \quad \lim_{t\to+\infty}\theta(t) = 2\pi, \qquad \pi < \theta(0) < 2\pi.$$

Thus, the interior of each strip is wandering; on the other hand, the two vertical lines are wandering because any orbit leaving a neighborhood of one of the lines eventually flows near the other line, and near these lines the function "$z$" is strictly monotone along orbits. The phase portrait is sketched in Figure 1.

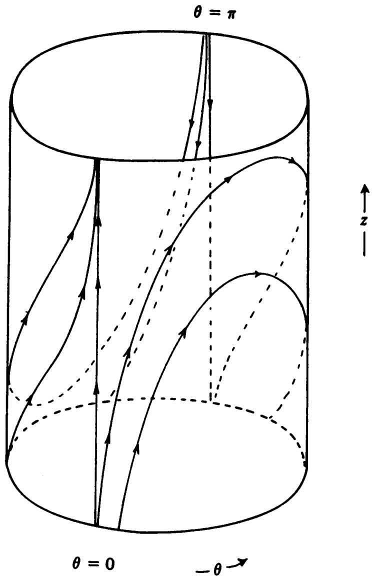


FIGURE 1
System (1)

It is clear, however, that a slight "push" in the direction of increasing $\theta$ can produce periodic orbits. For example, if $\varepsilon \neq 0$ is a small constant, the perturbed flow

$$\dot{z} = \cos\theta, \qquad \dot{\theta} = \sin^2\theta + \varepsilon^2, \tag{3$\varepsilon$}$$

has integral curves of the form

$$\varepsilon \tan[\varepsilon(z - z_0)] = \sin\theta.$$

We claim these represent periodic orbits. On one hand, $\dot{\theta} > \varepsilon^2$ everywhere, so that $\theta$ ranges over all real values along any orbit. On the other hand, since $|\sin\theta| < 1$, we have

$$\tan|\varepsilon(z - z_0)| < 1/\varepsilon$$

so that the quantity inside the absolute value is confined, along any orbit, to a single branch of the arc tangent. Thus, as $\theta$ goes from 0 to $2\pi$, $z$ must return to its original value, and the orbit is periodic.

Of course, ($3\varepsilon$) is only a uniform ($C^r$) perturbation of (1). But if the constant "$\varepsilon$" is replaced by a function $\varepsilon(\theta, z)$ which is positive near the two lines, then perturbations in any ($C^r$) Whitney neighborhood of (1) can be tailored to have periodic orbits. While the analytic formulae are more involved in this case, the geometric behavior, sketched in Figure 2, is clear.

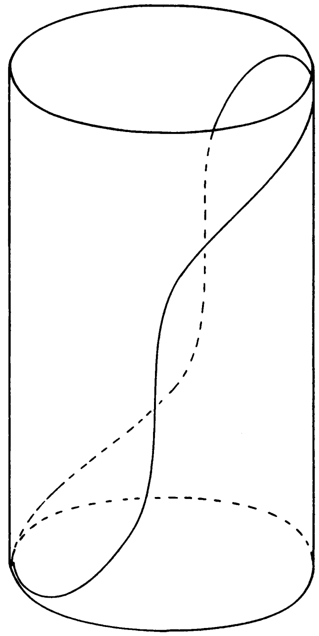

FIGURE 2

Perturbation of (1)

For the purposes of our construction in the next section, we prove the following bifurcation theorem for our example:

LEMMA 1. *The flow* (1) *is homotopic, through completely unstable flows, to the parallel flow*

$$\dot{z} = 1, \qquad \dot{\theta} = 0. \tag{4}$$

PROOF. Our basic observation is that the vectorfield (1) is really a function of just $\cos\theta$, so a homotopy of $\cos\theta$ to the constant function "1", keeping invariant the points $\theta = 0$ where $\cos\theta = 1$, will give us the desired bifurcation.

For example, consider the "convex combination" homotopy for $\cos\theta$. This yields the one-parameter family of vectorfields defined by

$$\begin{aligned} \dot{z} &= \lambda + (1-\lambda)\cos\theta, \\ \dot{\theta} &= 1 - [\lambda + (1-\lambda)\cos\theta]^2 \\ &= (1-\lambda)[\sin^2\theta + \lambda(1-\cos\theta)^2]. \end{aligned} \tag{5$\lambda$}$$

Note that when $\lambda = 0$, (5$\lambda$) is (1), while when $\lambda = 1$, (5$\lambda$) is (4). We need, then, to show that for $0 < \lambda < 1$, (5$\lambda$) is a completely unstable flow.

We note first that there are no equilibria, because for each $\lambda$,

$$(\dot{z})^2 + \dot{\theta} = 1.$$

Furthermore, independently of $\lambda$, the vectorfield for $\theta = 0$ is

$$\dot{z} = 1, \qquad \dot{\theta} = 0$$

so that the line (2b) remains an orbit for all $\lambda$.

Now, consider the strip $0 < \theta < 2\pi$ and any intermediate $\lambda$, $0 < \lambda < 1$. Since

$$|\dot{z}| < 1$$

we have (for $0 < \theta < 2\pi$ and $0 < \lambda < 1$)

$$\dot{\theta} = 1 - (\dot{z})^2 > 0.$$

Since $\dot{\theta}$ is independent of $z$, this implies that all orbits in this strip satisfy

$$\lim_{t \to -\infty} \theta(t) = 0, \qquad \lim_{t \to +\infty} \theta(t) = 2\pi$$

and every orbit in the open strip wanders. Furthermore, since $\dot{z}$ is near 1 for points near $\theta = 0$, every orbit satisfies

$$\lim_{t \to -\infty} z(t) = -\infty, \qquad \lim_{t \to +\infty} z(t) = +\infty \qquad (0 < \lambda)$$

and thus every orbit separates the strip into two invariant open sets. We will show that the point $P$: $z = 0$, $\theta = 0$ wanders: pick any orbit $\mathcal{O}$ in the open strip. Since it is a closed subset of $M$, and $P$ does not belong to it, we can take $0 < \varepsilon < \operatorname{dist}(P, \mathcal{O})$ and consider the disc

$$B_\varepsilon = \{(z, \theta) \in M \mid |z| < \varepsilon, |\theta| < \varepsilon\},$$

which is disjoint from $\mathcal{O}$. The line $\theta = 0$ separates $B_\varepsilon$ into two sets,

$$\begin{aligned} B_\varepsilon^+ &= \{(z, \theta) \in M \mid |z| < \varepsilon, 0 < \theta < \varepsilon\}, \\ B_\varepsilon^- &= \{(z, \theta) \in M \mid |z| < \varepsilon, -\varepsilon < \theta < 0\} \\ &= \{(z, \theta) \in M \mid |z| < \varepsilon, 2\pi - \varepsilon < \theta < 2\pi\}. \end{aligned}$$

The orbits of $B_\varepsilon^+$ and $B_\varepsilon^-$ are separated from each other by the line $\theta = 0$ and the orbit $\mathcal{O}$; on the other hand, since $z$ is strictly increasing near $\theta = 0$, every point of $B_\varepsilon$ leaves $B_\varepsilon$ and, by the limit behavior of $\theta$ and $z$, never reenters $B_\varepsilon$.

Thus, $B_\varepsilon$ is a wandering neighborhood of $P$, and for each $\lambda$, (5$\lambda$) is completely unstable. This gives us the desired bifurcation.

**2. Case 1.** $\dim M \geqslant 3$. Using (5$\lambda$), we will now construct explosive flows on all open manifolds of dimension 3 or more. The idea is to use Lemma 1 to embed the example (1) in a flow on a "rod" which is parallel on the boundary, and then to glue this rod into any flow with an orbit that is unbounded in forward and backward time. Most of the technical work is contained in two lemmas.

For convenience, we establish some notation. Suppose $n \geqslant 3$. Let $U$ denote the closed set in $R^n$

$$U = \{x = (x_1, \ldots, x_n) | x_1^2 + \cdots + x_{n-1}^2 \leq 1\}.$$

Note that $U$ is diffeomorphic to $D^{n-1} \times R$, where $D^{n-1}$ denotes the closed unit disc in $R^{n-1}$; its "core" is the curve

$$\Gamma = \{x = (x_1, \ldots, x_n) | x_1 = \cdots = x_{n-1} = 0\}.$$

A "thickened core" is the closed neighborhood of $\Gamma$

$$G = \{x = (x_1, \ldots, x_n) | x_1^2 + \cdots + x_n^2 \leq \tfrac{1}{2}\}.$$

Finally, consider the cylinder embedded in $G$,

$$C = \left\{x = (x_1, \ldots, x_n) | x_1^2 + x_2^2 = \frac{1}{4}, x_3 = \cdots = x_{n-1} = 0\right\}.$$

Lemma 2. *There exists a completely unstable $C^\infty$ flow on $U$ such that*

(i) *If $x \in U - G$, $\dot{x}_i = 0$ for $i = 1, \ldots, n-1$ and $\dot{x}_n = 1$.*

(ii) *$C$ is an invariant set on which the flow is conjugate to example* (1) *above.*

Proof. We will find it easier to work in "cylindrical" coordinates: let $\psi$: $R^n \to R^n$ be the transformation

$$\psi(r, \theta, z, u_1, \ldots, u_{n-3}) = (r \cos \theta, r \sin \theta, u_1, \ldots, u_{n-3}, z).$$

Note that

$$U = \psi\{0 \leq \theta \leq 2\pi, 0 \leq r, r^2 + u_1^2 + \cdots + u_{n-3}^2 \leq 1\},$$

$$G = \psi\{0 \leq \theta \leq 2\pi, 0 \leq r, r^2 + u_1^2 + \cdots + u_{n-3}^2 \leq \tfrac{1}{2}\},$$

$$C = \psi\{0 \leq \theta \leq 2\pi, r = \tfrac{1}{2}, u_1 = \cdots = u_{n-3} = 0\},$$

$$\Gamma = \psi\{0 \leq \theta \leq 2\pi, r = 0, u_1 = \cdots = u_{n-3} = 0\}.$$

Now, pick a $C^\infty$ function

$$\lambda = \lambda(r, u_1, \ldots, u_{n-3})$$

such that

$$\begin{aligned}
&0 \leq \lambda \leq 1,\\
&\lambda = 1 \quad \text{for } r^2 + u_1^2 + \cdots + u_{n-3}^2 \leq \tfrac{1}{8},\\
&\lambda = 1 \quad \text{for } r^2 + u_1^2 + \cdots + u_{n-3}^2 \geq \tfrac{1}{2},\\
&\lambda = 0 \quad \text{for } r = \tfrac{1}{2}, u_1 = \cdots = u_{n-3} = 0.
\end{aligned}$$

We note that $\lambda$ is independent of $\theta$ and $z$, by definition.

We define a vectorfield $\bar{V}$ on the domain of $\psi$ by

$$\begin{aligned}
\dot{r} &= 0,\\
\dot{\theta} &= (1-\lambda)\left[\sin^2\theta + \lambda(1-\cos\theta)^2\right],\\
\dot{u}_i &= 0, \qquad i = 1, \ldots, n-3,\\
\dot{z} &= \lambda - (1-\lambda)\cos\theta.
\end{aligned}$$

We note that this defines a $C^\infty$ vectorfield $V$ on the range of $\psi$; the only possible problem is at $r = 0$, but near $r = 0$, $\overline{V}$ is constant. $V$ certainly satisfies the requirements of our lemma.

LEMMA 3. *Given any $C^\infty$ vectorfield $Y$, defined on $U$, for which $\Gamma$ is a single integral curve, there exists a new $C^\infty$ vectorfield $Z$, defined on $U$, satisfying the following*:

(i) *Near the boundary of $U$, $Y = Z$.*

(ii) *The restriction of $Z$ to $U - G$ is conjugate to $Y|(U - \Gamma)$, and the conjugating map is the identity near the boundary of $U$.*

(iii) *$Z|G$ agrees with the vectorfield $V$ in Lemma 2.*

PROOF. If $Y$ happens to be parallel to $\Gamma$ on $G$ (i.e., if $\dot{x}_i = 0$, $i = 1, \ldots, n - 1$, and $\dot{x}_n = 1$ on $G$), then we can simply define $Z$ to equal $Y$ on $U - G$ and $V$ on $G$; since $Y$ agrees with $V$ slightly inside $G$, this makes $Z$ $C^\infty$. Similarly, if $\Gamma$ has an invariant closed neighborhood $P \subset G$ on which the flow is conjugate to the parallel flow on $G$, then we can find a diffeomorphism of $U$ which equals the identity near the boundary of $U$, preserves $\Gamma$, and maps $P$ onto $G$ via the conjugacy; then we can apply the previous sentence to the vectorfield induced by this diffeomorphism.

Thus, our problem reduces to the

*Claim.* There exists a $C^\infty$ vectorfield $\overline{Y}$ on $U$ such that

(a) $\Gamma$ has a neighborhood $P \subset G$ on which the flow of $\overline{Y}$ is conjugate to a parallel flow on $G$.

(b) $\overline{Y}|(U - P)$ is conjugate to $Y|(U - \Gamma)$, and the conjugacy is the identity near the boundary of $U$.

To construct $\overline{Y}$, we note that $\Gamma$ can be covered by "flowboxes" of $Y$, and so it is possible to find a closed neighborhood $Q$ of $\Gamma$, not necessarily invariant under the flow of $Y$, but on which the vectorfield $Y$ is conjugate to a constant vectorfield. Thus, there is a diffeomorphism $h$ of $Q$ onto a neighborhood of $\{\vec{0}\} \times R$ in $R^{n-1} \times R$ taking $\Gamma$ to $\{\vec{0}\} \times R$, and taking $Y$ to the vectorfield

$$\begin{aligned}
\dot{h}_i &= 0 \qquad (i = 1, \ldots, n-1)\\
\dot{h}_n &= 1.
\end{aligned}$$

We adopt "polar" coordinates in $R^{n-1}$ by defining

$$\begin{aligned}
r^2 &= h_1^2 + \cdots + h_{n-1}^2,\\
\theta &= (h_1/r, \ldots, h_{n-1}/r) \in S^{n-2} \qquad (r \neq 0).
\end{aligned}$$

By cutting down $Q$, we can assume that the image of $h$ has the form

$$Q_1 = \{(r, \theta, h_n) \in R \times S^{n-2} \times R \,|\, r \leqslant \alpha(h_n), \theta \in S^{n-1}\}$$

where $\alpha(h_n) > 0$ is a $C^\infty$ function. Now, pick a $C^\infty$ "bump function" $\beta(r, h_n)$ such that

(a) $\beta = 0$ for $r \geqslant (3/4)\alpha(h_n)$,
(b) $\beta = 1$ for $r \leqslant (1/2)\alpha(h_n)$,
(c) $\beta' < 0$ for $(1/2)\alpha(h_n) < r < (3/4)\alpha(h_n)$.

We remark that the vectorfield $\bar{Y}_1$ defined in polar coordinates by

$$\dot{r} = r\alpha'(h_n)\beta(r, h_n)/\alpha(h_n),$$
$$\dot{\theta} = 0, \qquad \dot{h}_n = 1,$$

has the following properties:

(a) For $r > (3/4)\alpha(h_n)$, $\bar{Y}_1$ is parallel to the $h_n$-axis (so agrees with the image of $Y$).

(b) For $r \leqslant (1/2)\alpha(h_n)$, the integral curves of $\bar{Y}_1$ have the form

$$r(h_n) = (\text{constant}) \times \alpha(h_n), \qquad \theta = \text{constant}.$$

(c) $\bar{Y}_1$ is well defined as a $C^\infty$ vectorfield on $Q_1$.

The first property is clear; to see (b), we note that in the region $r \leqslant (3/4)\alpha$, we have $\beta = 1$, so

$$r'/r = \alpha'/\alpha.$$

To see (c), we rewrite $\bar{Y}_1$ in cartesian coordinates as

$$\dot{h}_i = \alpha'(h_n)\beta(r, h_n)h_i/\alpha(h_n), \qquad \dot{h}_n = 1.$$

Now a consequence of (b) is that the set

$$P_1 = \{r < \alpha/2\}$$

is a neighborhood of $\Gamma$, inside $Q_1$, invariant under the flow of $\bar{Y}_1$. Thus if we pull $\bar{Y}_1$ on $Q_1$ back to a vectorfield $\bar{Y}$ on $Q \subset G$, its germ at the boundary of $Q$ agrees with the original vectorfield $Y$, so we can define $\bar{Y}$ to equal $Y$ off $Q$. The flow of $\bar{Y}$ then satisfies statement (a) of our claim, but it is not clear that our new vectorfield is conjugate on the complement of this neighborhood to $Y$, because a $Y$-orbit entering $Q$ may be connected by $\bar{Y}$ to a different $Y$-orbit upon leaving $Q$. Unless $\Gamma$ is separated from the Auslander recurrent set (in which case the claim is unnecessary), this can change the conjugacy type of the flow.

However, we can insure that an orbit entering $Q$ leaves $Q$ at the same point under both $Y$ and $\bar{Y}$ by making a further assumption on the functions $\alpha$ and $\beta$ above: it is clear that by further shrinking $Q$, we can assume that

(i) $\alpha(-h) = \alpha(h)$ for all $h$,
(ii) $\alpha'(h) < 0$ for $h > 0$,

and then we can pick $\beta$ so that

(iii) $\beta(r, -h) = \beta(r, h)$ for all $h$.
In this case, an examination of the formulas reveals that the vectorfield $\bar{Y}_1$ is carried into its negative by the involution

$$r \to r, \qquad \theta \to \theta, \qquad h_n \to -h_n.$$

This means that the integral curves of $\bar{Y}_1$ are preserved by the involution; only their orientation is changed. Since $Q_1$ is now also symmetric (by (i)), this means that the $\bar{Y}_1$-orbit entering $Q_1$ at $(\alpha(h), \theta, h)$ $(h \leqslant 0)$ leaves $Q_1$ at $(\alpha(h), \theta, -h)$. But the same is true of $Y$-orbits. Hence, our "surgery" in $Q$ does not alter the connections between orbits entering and leaving $Q$. We see, then, that on the complement of the orbits that stay in $Q$ for all (positive and negative) time, $Y$ and $\bar{Y}$ are conjugate. $\Gamma$ is the only $Y$-orbit entirely contained in $Q$, but we know that the $\bar{Y}$-invariant set in $Q$ includes the neighborhood of $\Gamma$ corresponding to $P_1$. It is easy to see that (since $\theta$ is constant along orbits) this invariant set is homeomorphic to $G$ and parallelizable. Thus, $\bar{Y}$ satisfies all the conditions of the claim, and so Lemma 3 is proven.

Given these lemmas, we can easily establish

Proposition 1. *If $M$ is an open manifold and* $\dim M \geqslant 3$, *then there exists an explosive $C^\infty$ flow on $M$.*

Proof. We note with Krych [5] that, by [3], there exists a smooth function $L$: $M \to R$ with no critical points. If we pick a metric on $M$ and take the gradient of $L$, its flow will be completely unstable but, by [9], not explosive. However, let $\gamma$ be some orbit of the gradient flow. By complete instability, $\gamma$ is a closed embedding of $R$ in $M$, and so $\gamma$ has a closed tubular neighborhood $N$ in $M$ [2]. Of course, $N$ need not be invariant under the flow, but it is diffeomorphic to $D^{n-1} \times R$, and hence to $U$ (of Lemmas 2 and 3) by a diffeomorphism taking $\gamma$ to $\Gamma$. Now, grad $L$ transfers to a vectorfield on $U$, which we take as $Y$ in Lemma 3; the conclusion of Lemma 3 gives a vectorfield $Z$ on $U$ which transfers back to $N$ so as to agree with grad $L$ on the boundary. By construction, the flow of the vectorfield

$$X = \begin{cases} \operatorname{grad} L & \text{off } N, \\ Z & \text{on } N, \end{cases}$$

is completely unstable, and, on the image of the cylinder $C$, contains a copy of (1); so it is explosive. This establishes Proposition 1.

It is easy to see that all our constructions could be achieved by means of isotopies from the identity. Thus, we can strengthen the statement of Proposition 1 to a bifurcation theorem:

Corollary 1. *If $M$ is an open manifold,* $\dim M \geqslant 3$, *and $\phi$ is any* *completely unstable flow on $M$, then $\phi$ is homotopic, through smooth completely unstable flows, to an explosive flow.*

Actually, this construction has implications beyond the situation of completely unstable systems, for it shows that, when $M$ is not compact, $\Omega$-explosions can occur in regions that have no dynamic connection with the nonwandering set:

COROLLARY 2. *There exist flows (on open manifolds) satisfying axiom* A *and the no-cycle condition which are not $\Omega$-stable.*

PROOF. For example, the construction of Proposition 1 could equally well be applied to the linear hyperbolic flow in $R^3$,

$$\dot{x} = -x, \quad \dot{y} = -y, \quad \dot{z} = z,$$

near any orbit that does not belong to any of the coordinate planes. Then the original flow has $\Omega = \{0\}$, and there are, in the classical sense, "no cycles". But with a copy of (1) glued in somewhere far from the origin, there exist perturbations with periodic orbits in addition to the hyperbolic equilibrium.

**3. Case 2. Punctured closed surfaces.** By a *punctured closed surface*, we mean any open 2-dimensional manifold $M$ which can be obtained from a closed surface $X$ by deleting a closed subset $K$. It is clear that each component of $M$ can be obtained in this way be deleting (from, possibly, a different closed surface $X$) a set whose components are discs or points; thus we will think of $M$ as $X - K$, where $X$ is a closed surface and $K \subset X$ is a totally disconnected closed set.

We now turn to constructing our examples on all appropriate punctured closed surfaces.

PROPOSITION 2. *If $M$ is a punctured closed surface*

$$M = X - K$$

*($X$ a closed surface, $K \subset M$ closed and totally disconnected) and $M$ is not homeomorphic to $S^2$-{(one point)}, then there exists an explosive flow on $M$.*

PROOF. We will distinguish three subcases:

(i) $X = S^2$,

(ii) $X = P^2$ (projective plane),

(iii) $\chi(X) < 0$ ($\chi$ = Euler characteristic).

(i) $X = S^2$. In this case, by hypothesis, $K$ has at least two points. When $K$ has precisely two points, $M$ is the cylinder, and the desired example has already been constructed earlier, as (1).

If $K$ has more than two points, we are looking at a "punctured cylinder". Since $K$ is closed and totally disconnected, it is a subset of a Cantor set, and we can assume, up to homeomorphism, that the extra points of $K$, as a subset of the cylinder, are a subset of the line $\theta = 0$. Now, the vectorfield given by (1) would still be an example, except that it is not complete and hence does not generate a flow. But we can multiply it by an appropriate smooth real-valued function vanishing precisely on $K$ so that the resulting vectorfield is complete on the punctured cylinder, and since the complement of $K$ on the line $\theta = 0$ is nonempty, we can still perturb to create periodic orbits, as before.

(ii) $X = P^2$. If $K$ is a single point, then $X - K$ is (the interior of) a Möbius band, which is double-covered by the cylinder. The flow (1) on the cylinder does not project to the Möbius band, but the flow

$$\dot{z} = \cos 3\theta, \qquad \dot{\theta} = \sin^2 3\theta,$$

which has six instead of two "Reeb components", does project to an explosive flow on the Möbius band. The details are an easy variation on (1); the projected flow is sketched in Figure 3a.

There is another way to arrive at this example, which illustrates a pattern we will follow in case (iii). $P^2$ can be represented as a plane "polygon" whose boundary consists of two curves, which are identified as a single nonbounding cycle in $P^2$ (a generator of $H_1(P^2) = Z_2$); the two "vertices" of the polygon where the two edges touch are identified in $P^2$ as a single point, which we take as $K$. One can pick an orientation on this cycle which transfers to a clockwise orientation of the boundary of the polygon. One can then define a flow in the polygon by filling the interior with loops based at a single vertex and circulating clockwise, and making each edge an orbit (Figure 3b). The corresponding flow on $P^2 - K$ (Figure 3c) has no $\alpha$- or $\omega$-limit points,

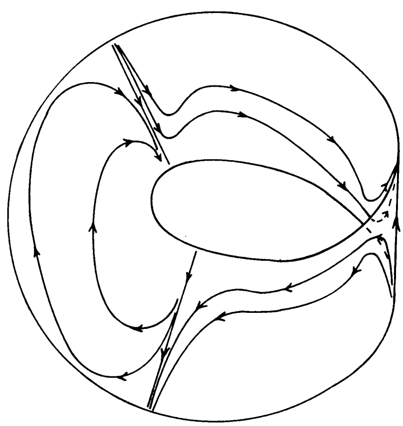

FIGURE 3

The Case $X = P^2$

(a)

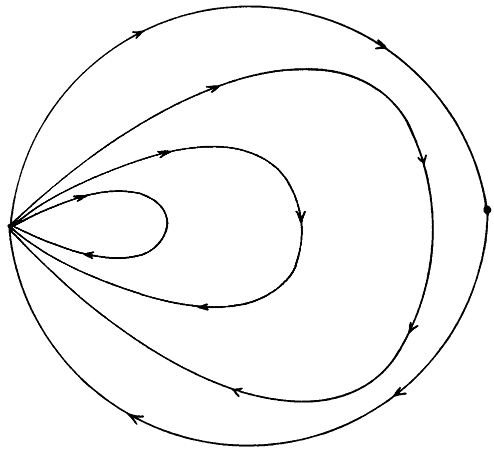

FIGURE 3
The Case $X = P^2$
(b)

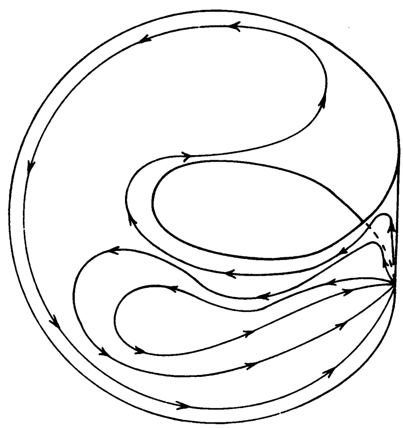

FIGURE 3
The Case $X = P^2$
(c)

but the single cycle corresponding to the edges is nonwandering, because orbits starting near the edge on one "side" return later, on the other "side".

To remedy this, we need to separate the two "sides" of the cycle; we cut $P^2$ along the cycle and insert a strip, which can be represented in the plane as a strip added to one edge of the polygon. The closure of the strip in $P^2 - K$ is homeomorphic to $[0, 2\pi] \times R$, and we can define a flow on the strip conjugate to the one defined by equations (1), with $\theta$ regarded as a real number, $0 < \theta < 2\pi$, instead of an element of the circle. There are actually two ways this flow can be put into the strip, depending on which "side" corresponds to $\{0\} \times R$. Now the orbits defined by the loops interior to the polygon distinguish an "earlier" and a "later" side of the cycle: orbits starting near the "earlier" side afterwards pass near the "later" side. (In the polygon, the "later" edge occurs clockwise from the "earlier" one, starting from the distinguished vertex.) On the other hand, perturbations of the flow (1) result in orbits for which $\theta$ increases from 0 to $2\pi$. Thus, we pick a homeomorphism of the strip with $[0, 2\pi] \times R$ so that $\{0\} \times R$ identifies with the "later" edge, $\{2\pi\} \times R$ identifies with the "earlier" edge, and the orientation of the cycle agrees with the standard orientation in the "$R$" factor (Figure 3d). The flow

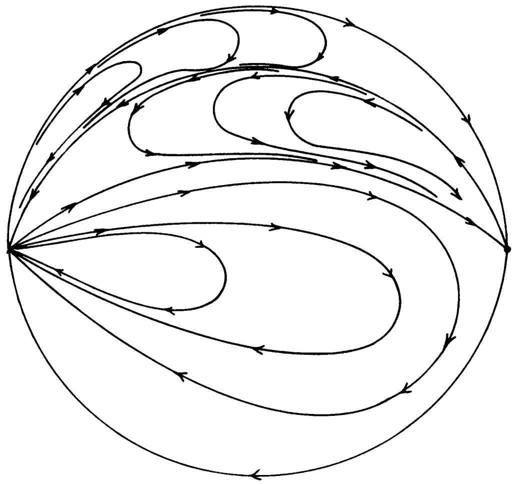

FIGURE 3
The Case $X = P^2$
(d)

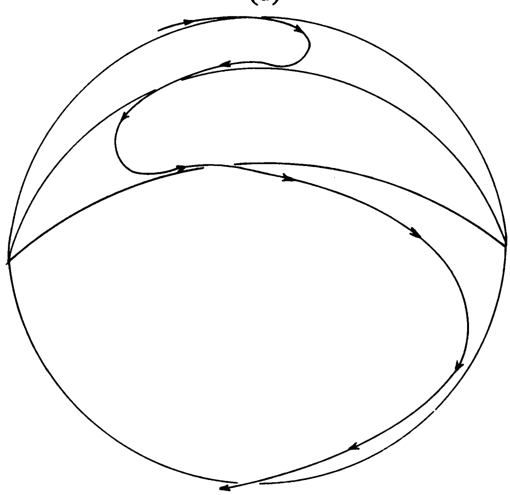

FIGURE 3
The Case $X = P^2$
(e)

is then completely unstable by arguments similar to those in (1), and explosive because by a small perturbation it is possible to have an orbit cross the strip from $\{0\} \times R$ to $\{2\pi\} \times R$, then to follow a part of one of the loops interior to the polygon from the "earlier" edge ($\{2\pi\} \times R$) to the "later" edge ($\{0\} \times R$), and close up into a periodic orbit (Figure 3e). This flow is, of course, conjugate to the version defined earlier.

If $K$ has more than one point, we again place all of $K$ on one edge of the strip and multiply by a function to make the flow complete. As before, the flow remains explosive.

(iii) $\chi(X) \leqslant 0$. This is simply an elaboration of the construction sketched above for $X = P^2$. Once again, we start with $K$ a single point. By the standard classification theory for closed surfaces (see, e.g., [**6**, pp. 72–80]), $X$ can be represented as a polygon in the plane, with certain pairwise identifications of the edges, and all vertices identified to a single point in $X$, which we take as $K$.

Each identified pair of edges represents a nonbounding cycle in $X$; an orientation of each cycle induces orientations of the edges of the polygon. If it is possible to orient the cycles in $X$ so that the edges of the polygon are all oriented clockwise, then we mimic the procedure outlined for $X = P^2$: fill the interior of the polygon with clockwise loops at one vertex, and replace the cycles by strips on which the flow is conjugate to (1), with $\{0\} \times R$ corresponding to the "later" side and $\{2\pi\} \times R$ corresponding to the "earlier" side. As before, one can create periodic orbits crossing any one of the cycles.

On the other hand, if a clockwise orientation of the edges consistent with orientations of the cycles in $X$ is not possible, we modify the construction slightly. Again we replace each cycle with a strip, and again fill the interior of the polygon with clockwise loops from one vertex. This forces an orientation of each edge of each strip in $X$; for those strips whose two edges are oriented the same way, we proceed as before, gluing in a copy of (1). However, some strips will now have edges oriented in opposite directions; these are homeomorphic to $[0, \pi] \times R$, and we can pick the homeomorphism so that the "later" edge is identified with $\{0\} \times R$, its orientation agreeing with that of $R$. We note, then, that the restriction of equations (1) to $0 \leqslant \theta \leqslant \pi$ gives a flow on the strip for which $\theta = \pi$ is an orbit with $\theta$ decreasing; thus, the orientation of the other edge of the strip is consistent with the orientation coming from the loops. Again, it is possible to perturb so as to create periodic orbits which, starting from $\{0\} \times R$, increase $\theta$ to $\pi$, then follow part of a loop interior to the polygon back to $\{0\} \times R$.

In either case, we have an explosive flow: in the language of [**9**], the "later" edge $\{0\} \times R$ of each strip has $\{\pi\} \times R$ in its first positive prolongation, $J_1^+$, and if the edges of the strip are oriented the same way, then the "earlier" edge $\{2\pi\} \times R$ is in the next prolongation, $(J_1^+)^2$, of $\{0\} \times R$. On the other hand, looking at the loops interior to the polygon, the "later" edge is in the first prolongation of the "earlier" edge, so that each edge is in its own second prolongation $J_2^+$ (which contains the union $(J_1^+)^n$, $n = 1, \ldots$) and so is recurrent in the sense of Auslander **[1]**. Q.E.D.

In effect, our construction in this case, at least for $X \neq S^2$, has been to create a flow on $X$ whose nonwandering set is a single equilibrium, which we put in $K$, but with "cycles" of prolongational limits (Auslander recurrence) which can be achieved without passing through $K$. We then deleted $K$. Thus we were really elaborating the idea of **[5]** and **[7]**.

However, not all surfaces can be obtained by deleting points from a closed surface; for example the "infinite ladder"–two cylinders joined by a countable number of tubes–has an infinite number of "handles", and so cannot be embedded in any closed surface. On the other hand, every such open surface can be formed by "gluing together" a countable family of punctured closed surfaces (**[4]**, **[11]**). In the next section, we will construct examples on these more complicated surfaces. However, it will be useful, first, to establish the following elaboration of our technique above. We state it as a theorem about closed surfaces; its relevance to our problem will become clear during the construction in the next section.

LEMMA 4. *Let $X$ be a closed surface, $X \neq S^2$, $P^2$, and let $K = \{x_0, x_1, \ldots, x_n\}$ be a finite nonempty set of points in $X$. If $n \geqslant 1$, suppose we have specified that certain of the $x_i$ $(i \geqslant 1)$ are to be "sinks" and certain others are to be "sources". Then there exists a flow $\phi$ on $X$ with the following properties*:

(1) $\Omega(\phi) = K$.

(2) *Each $x_i$, $i \geqslant 1$, is a fixed sink or source of $\phi$, as specified.*

(3) *Every nonequilibrium orbit has either an $\alpha$-limit or an $\omega$-limit* (*or both*) *at* $x_0$.

(4) *The flow has "cycles"–that is, its Auslander recurrent set* **[9]** *intersects* $X - K$.

PROOF. As before, $X$ can be represented as a polygon with identifications, and all vertices identified with a single point, which we take to be $x_0$. We wish to repeat the construction of the preceding proof, but this time we have to take account of our specified sinks and sources.

We can assume that $x_1, \ldots, x_n$ lie in the interior of our polygon; thus we can enclose them in a loop based at a single vertex, and separate them from each other by loops, all based at this vertex. Thus, each $x_i$, $i \geqslant 1$, is enclosed in a "crescent" with ends at $x_0$.

We can orient each of the loops we have drawn so far at will. Now, in each crescent, we draw an additional loop which hits the point $x_i$ in its interior. This loop will consist of two orbits, separated by $x_i$, and their orientation is determined by whether $x_i$ is a sink or source. Finally, we need to fill in the orbits between the loops.

Note that the various loops and their orientations divide each crescent into two triangles, whose vertices are $x_i$ and the two ends of the crescent. Since $x_i$ is either a sink or a source, the orientation of the edges of each triangle is not that of the boundary of a simplex; thus, the edge path formed by the outer edge of the crescent and one of the two orbits divided by $x_i$ is homotopic, preserving endpoints, to the other orbit (see Figure 4). This gives a foliation

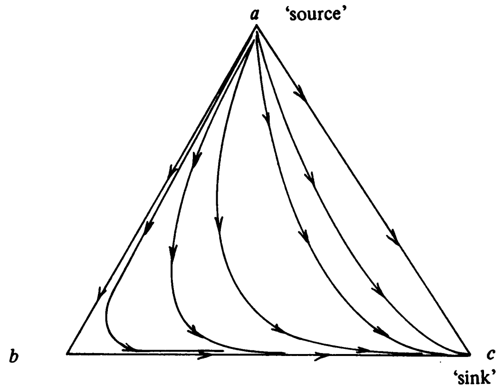


FIGURE 4
Lemma 4
(a)

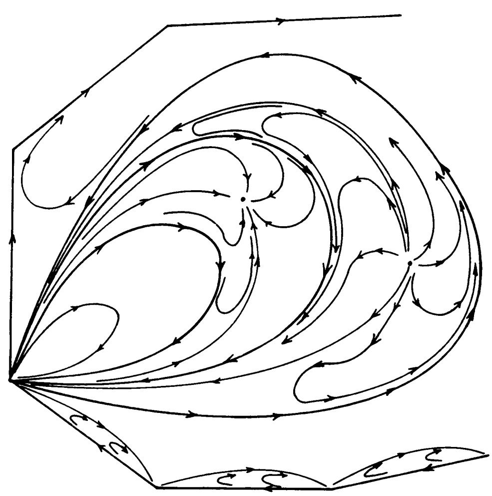

FIGURE 4
Lemma 4
(b)

of the interior of the triangle by lines oriented consistently with the edges, and gives us the flow lines inside the crescents.

We are left then with filling in the interior of the innermost loop, and the part of the polygon exterior to the loops. The interior of the innermost loop is simply filled with loops. The exterior of the loops can be regarded as a new polygon, with a single extra edge coming from the outermost loop. We can therefore apply the construction of Proposition 2 to this, and obtain the desired flow.

**4. Case 3. General surfaces.** We turn now to the final case of our theorem, when $M$ is an open surface which cannot be embedded in a closed surface. A representation of any open surface as a sphere with (possibly an infinite number of) holes, handles, and crosscaps was given by von Kerekjarto ([**4**, Chapter 5]). Some questions concerning von Kerekjarto's original arguments have been raised and treated more recently by Richards [**11**]. We shall make use of one simple consequence of this theory, which we state formally, but without proof:

LEMMA 5 (SEE [**4**], [**11**]). *Suppose $M$ is an open surface which does not embed in any closed surface. Then there exists a compact, connected submanifold-with-boundary $M_0$ in $M$, with the following properties*:

(1) *Distinct boundary components of $M_0$ bound distinct components of $M - M_0$.*

(2) *The interior of $M_0$ is homeomorphic to $X - K$, where $X$ is a closed surface of Euler characteristic $\leqslant 0$ and $K$ is a finite set of points in $X$.*

The general representation of any open surface, $M$, is in fact a union of surfaces like $M_0$, except that $X$ may be $S^2$ or $P^2$. If more than one of these compact pieces is a punctured projective plane, we can take a union of pieces to obtain $M_0$ satisfying (2); on the other hand, if all but a finite number of the compact pieces are punctured spheres, then $M$ itself has, in von Kerekjarto's terminology, only "ends of type 1", and so embeds in a closed surface.

Let us fix $M$ and $M_0$ as above, and fix $X$, $K = \{x_0, \ldots, x_n\}$. The boundary components of $M_0$ are circles corresponding to the points of $K$; we number these boundaries consistently with the numbering of the $x_i$:

$$\partial M_0 = B_0 \cup \cdots \cup B_n.$$

Let us, for convenience, denote the component of $M - M_0$ adjoining $B_i$ by $C_i$.

Our general plan is to start, as in the higher-dimensional case, with a gradient flow on $M$. We will replace the flow inside $M_0$ by one coming from Lemma 4, so as to create an explosive situation. Our main difficulty will be in "patching together" the gradient flow on $M - M_0$ with the explosive flow inside $M_0$. The two flows have different kinds of tangencies at the various boundaries $B_i$; we will therefore need to modify the gradient flow at certain points of $M - M_0$, and we will need to be careful not to create nonwandering points in the process.

Our starting point, then, is the flow described in

LEMMA 6. *Given $M$ and $M_0$ as in Lemma 5, there exists a completely unstable flow $\phi$ on $M$ satisfying the following nondegeneracy condition at $\partial M_0$:*
*If $\dot{\phi}$ is tangent to $\partial M_0$ at $m \in B_i$, then $\dot{\phi}|\partial M_0$ is transverse at $m$ to $TB_i$.*

PROOF. As before, there exists a function on $M$ with no critical points. Its gradient (relative to any metric) gives a completely unstable flow which is nonexplosive by **[9]**. The nondegeneracy condition above is the simplest version of "condition $L$" in **[10]**, which is generic. But a perturbation of the given gradient flow is still completely unstable; this perturbation is our desired $\phi$.

To keep control of our modifications of $\phi$ in $M - M_0$, we will restrict ourselves to certain invariant "bands" of orbits which leave $M_0$ and never return. To find these bands, we formulate the idea of "escape sets".

DEFINITION. Suppose $m \in \partial M_0$ and $\dot{\phi}(m)$ is not tangent to $\partial M_0$. We say that $m$ is a *positive* (resp. *negative*) *escape point* if for all $t > 0$ (resp. $t < 0$)

$$\phi(t, m) \notin M_0.$$

We note (by complete instability) that every point of $M_0$ must eventually escape $M_0$ in both directions. If a point crosses $B_i$ into $C_i$ and later (or earlier) returns to $M_0$, it must do so via $B_i$; if its orbit is not tangent to $B_i$ at either point, then nearby points of $B_i$ return to $B_i$ at nearby times. Thus, the (forward) first-return map from $\partial M_0$ to itself is defined continuously on the complement of the (positive) escape points and the points flowing to tangencies. The domain of this map is open.

It will be useful to assume that each $B_i$ contains escape points of one or the other type. To this end, we show

LEMMA 7. *If $B_i$ contains neither positive nor negative escape points, then $C_i$ is compact.*

PROOF. There are finitely many points of tangency on $B_i$, say $y_1, \ldots, y_k$. Strictly speaking, one of these might "escape" in both directions, but in that case, we can shrink $B_i$ into $M_0$ slightly and replace the "escaping" tangency with a (unique) nearby tangency that reenters $B_i$ in both directions. We can assume, therefore, that for each tangency $y_j$ we can take a maximal interval $\gamma_j$ of the orbit of $y_j$ intersecting $C_i$, that the endpoints of this interval are transverse to $B_i$, and that $\gamma_j$ is compact. The intersections of $\gamma_1, \ldots, \gamma_k$ with $B_i$ form a finite set of points; its complement is a finite set of intervals on which $\dot{\phi}$ is transverse to $B_i$. Since there are no escape points, the first-return map takes those intervals at which $\dot{\phi}$ points out of $M_0$ into the intervals at which it points in. The orbit segments joining these intervals form a finite number of bands in $C_i$ whose boundary is the union of the $\gamma_i$. Since the ends of $\gamma_i$ are transverse to $B_i$, the $\gamma_i$ are interior to the closure of the "bands" in $C_i$. This closure is therefore a subset of $C_i$ which is compact, but open in $C_i$; since $C_i$ is connected, the lemma follows.

By adjoining all the compact $C_i$ to $M_0$, we can assume that each boundary component contains escape points. We will now complete the construction, distinguishing two subcases:

*Case* A. There are at least two noncompact $C_i$.

*Case* B. Only $C_0$ is compact.

We will concentrate on Case A, and return to Case B at the end.

*Case* A. Pick an escape point $p_i$ on each boundary component $B_i$. Since $M_0$ contains both positive and negative escape points, we can make sure that our choices include at least one positive and one negative escape point. By renumbering, we assume that $p_0$ is a negative escape point, and $p_1$ is a positive escape point. The others (if any) can be of either type, although we note which are positive and which are negative, for future reference.

Now, using a "thickening" technique as in Lemma 3, we modify $\phi$ so as to replace the (full) orbit of each $p_i$ with a band of parallel orbits running through $M$. The band from $p_i$ intersects $B_i$ in an interval of escape points, $E_i$; let $F_i$ denote the semiorbit of $E_i$ which escapes from $M_0$. The subset $M_1 = M_0 \cup F_0 \cup F_1 \cup \cdots \cup F_n$ is closed in $M$, and its boundary consists of part of $M_0$ together with $2n + 2$ semiorbits of endpoints under $\phi$. We will modify $\phi$ only inside $M_1$, and also will insure that any orbit which enters and then leaves $M_1$ does so via the same points as it did under the unmodified flow. This will help insure complete instability.

We now turn to the model of $M_0$ as $X - K$; we label $x_1$, as well as all other $x_i$ whose associated escape set $E_i$ is positive, as "sinks", and label as "sources" all $x_i$ with $E_i$ negative. Using Lemma 4, we construct a flow on $X$ with $\Omega = K$, sinks and sources as specified, $x_0$ in $\alpha(x) \cup \omega(x)$ for all $x \in X$, and with Auslander recurrence off $K$. This gives a flow $\psi$ on $M_0$ with $\Omega(\psi) \subset \partial M_0$ and an explosive situation in $M_0$.

To define our final flow, we will begin by defining the foliation by integral curves, and then checking that these can be oriented consistently with a flow. Basically, we would like $\psi$ inside $M_0$ and $\phi$ outside, but there are two difficulties: (i) at $B_i$, $i = 1, \ldots, n$, $\psi$ is everywhere transverse to $B_i$, but $\phi$ has, in general, some tangencies, while (ii) at $B_0$, both $\psi$ and $\phi$ may have tangencies, but they can be different.

To remedy this situation, we start by picking orbits $\gamma_i$ for $\psi$ with one end at $B_0$ and the other at $B_i$, $i = 1, \ldots, n$. Again using the technique of Lemma 3, we thicken each $\gamma_i$ into a "band" of orbits, $\Gamma_i$, joining each $B_i$ to $B_0$.

Now, each $\Gamma_i$ hits $\partial M_0$ in two closed intervals, which we shall call $G_i \subset B_0$ and $H_i \subset B_i$, $i = 1, \ldots, n$. We would like to have every tangency of $\phi$ with $B_i$ occur inside $H_i$, and for its effect to be pushed across $\Gamma_i$ to escape $M_0$ via $G_i$ into $F_0$; we would also like every tangency of $\psi$ with $B_0$ to escape from $M_0$ into $F_0$. We need therefore to modify $\phi$ (or $\psi$) so that $H_i$ and $E_i$ cover $B_i$.

This is, in fact, easy to do. For $i = 1, \ldots, n$, let $U_i$ be a collar neighborhood of $B_i$ in $M_0$–that is, $U_i$ is homeomorphic to the product of a closed interval and a circle, $U_i = [0, \varepsilon] \times S^1$, with $B_i = \{0\} \times S^1$. Define on each $U_i$ a diffeomorphism $\Delta_i$: $U_i \to U_i$ which preserves the levels $\{t\} \times S^1$ of $U_i$, is the identity near $\{\varepsilon\} \times S^1$, but near $B_i = \{0\} \times S^1$ is a homeomorphism of $S^1$ taking $H_i$ into an interval whose interior contains $\operatorname{clos}[S^1 - E_i]$. The image of the integral curves of $\psi$ under $\Delta_i$ is a new flow, with $B_i \subset \operatorname{int} H_i \cup \operatorname{int} E_i$.

To control the behavior at $B_0$, we do a similar thing; but first we single out a closed interval $G_0 \subset \operatorname{int}[H_1 \cap E_1]$, and follow it back by $\psi$ to an interval $H_0 \subset B_0$. Now, we define $\Delta_0$ on a collar neighborhood $U_0$ of $B_0$ so that $\operatorname{clos}[S^1 - E_0] \subset \operatorname{int} \Delta_0(H_0)$. We will call this new flow on $M_0$ $\psi$ as well, and use $H_0$, $H_1$, etc. in place of $\Delta_0(H_0)$, $\Delta_1(H_1)$, etc.

We now wish to "patch in" the tangencies at the various boundary components. First, we worry about tangencies of $\phi$ with $B_i$, $i = 1, \ldots, n$. Let $V_i = [0, \varepsilon] \times S^1$ be a collar neighborhood of $B_i$ in $C_i$ ($B_i = \{0\} \times S^1$). We can think of integral curves of $\phi$ in $V_i$ as graphs of functions $S^1 \to [0, \varepsilon]$; a tangency "external" to $M_0$ occurs where a graph has a local minimum value 0, while an "internal" tangency occurs at a zero maximum. In either case, our nondegeneracy condition on tangencies insures that nearby integral curves have similar extrema, and by picking $\varepsilon$ small, we can assume that each level of $V_i$ contains exactly the same kinds of tangencies. For clarity, we separate our treatment into two cases: internal and external tangency.

Let $q \in B_i$ be a tangency of $\phi$ "internal" to $M_0$. The nearby tangencies on other levels of $V_i$ form a curve, $Q(\tau)$, extending from $q = Q(0) \in B_i = \{0\} \times S^1$ to $Q(1) \in \{1\} \times S^1$. (See Figure 5.) For $0 < \tau \leqslant 1$, there is an orbit segment of $\phi$ inside $[0, \tau] \times S^1$, tangent to $\{\tau\} \times S^1$ at $Q(\tau)$, and intersecting $B_i$ in a pair of points. By picking $V_i$ sufficiently small, we can assume that all of these segments hit $B_i$ well inside $H_i$. Let the orbit segment through $Q(\frac{1}{2})$ intersect $B_i$ at the pair of points $q^-$, $q^+$. Denote by $[q^-, q^+]$ the interval in $B_i$ containing $q$, and note that for $0 < \tau < \frac{1}{2}$, the orbit segment hitting $Q(\tau)$ intersects $[q^-, q^+]$ at a pair of points, which we can call $q^-(\tau)$, $q^+(\tau)$. With a little smoothing inside $M_0$, near $B_i$, we can link up the (unmodified) integral curves of $\phi$ crossing $B_i$ just outside $[q^-, q^+]$ with (slightly modified) integral curves of $\psi$ in $\Gamma_i$, which cross over to $B_0$ and escape into $F_0$. Let $\alpha^-$ and $\alpha^+$ denote the two semiorbits of $\psi$, hitting $B_i$ at $q^-$ and $q^+$, crossing $M_0$ via $\Gamma_i$ to

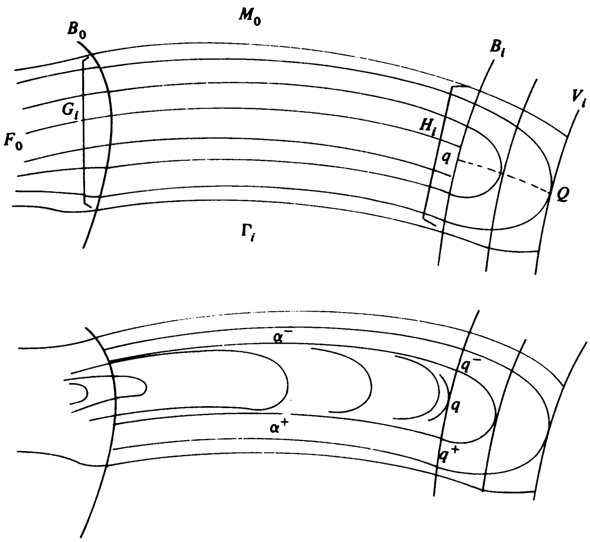


FIGURE 5
Surgery at an Internal Tangency

$E_0$, and thence following $\phi$ to escape into $F_0$. The region $A \subset F_0 \cup \Gamma_i \cup V_i$ bounded by $\alpha^-$, $\alpha^+$ and the orbit-segment of $\phi$ from $q^-$ to $q^+$ (including the boundary curve in $A$) is a closed region in $M$, diffeomorphic to the plane region

$$\{(x,y) \in R^2 | x^2 + y^2 \leq 1 \text{ and } y > 0\};$$

similarly the region $A^* \subset V_i$ bounded by the orbit segment from $q^-$ to $q^+$ and the interval $[q^-, q^+]$ is diffeomorphic to the closed upper half-disc, with $[q^-, q^+]$ mapping to the $x$-axis. Thus $A^* - B_i$ is diffeomorphic to $A$, and the foliation of $A^* - B_i$ by orbit segments of $\phi$ then induces a foliation of $A$ with the boundary leaf pieced together from $\alpha^-$, $\alpha^+$, and the orbit segment from $q^-$ to $q^+$. This extends the foliation by integral curves of $\phi$ slightly outside $A^*$ and by integral curves of $\psi$ in $\Gamma_i - A$ to a foliation of part of $M$ including all of $A$. We have, in effect, taken the "band", obtained by letting $[q^-, q^+]$ flow under $\psi$ into $F_0$, and glued into it a kind of "Reeb component" which matches up with the tangencies near $q$. We note for future reference that in this extension we have insured that the "first-return" map of this new foliation on points of $B_i$ slightly outside $[q^-, q^+]$ is identical with the "first-return" map of $\phi$; we could, with slightly more care, insure that the new foliation on $A$ was an extension of the original foliation on $A^*$, so that the "first-return" map is unaltered by this surgery.

Now, if $q \in B_i$ is a tangency of $\phi$ "external" to $M_0$, we follow an analogous procedure. Again, $q$ belongs to a curve $Q(\tau)$ of tangencies, with $Q(0) = q$ and $Q(1) \in \{1\} \times S^1$. This time, however, the orbit segment through $Q(\tau)$,

$0 < \tau < 1$, crosses $\{1\} \times S^1$ instead of $B_i = \{0\} \times S^1$. Let $r^-, r^+ \in \{1\} \times S^1$ be the intersection of the orbit segment through $Q(\frac{1}{2})$ with $\{1\} \times S^1$ and $s^-, s^+$ the intersection of the orbit segment through $q = Q(0)$ with $\{1\} \times S^1$.

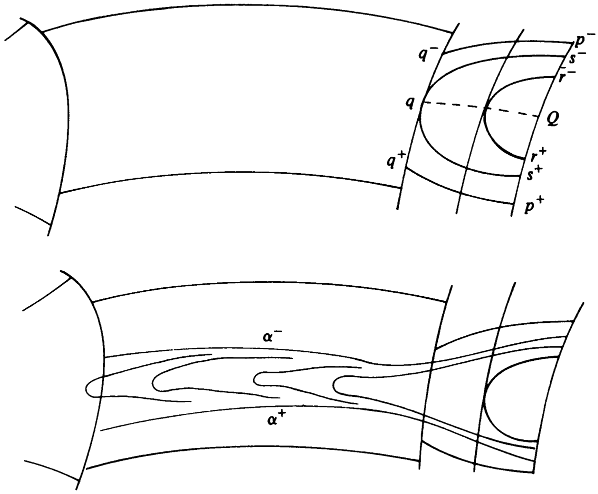


FIGURE 6

Surgery at an External Tangency

The intervals $[r^-, r^+] \subset [s^-, s^+]$ in $\{1\} \times S^1$ will be those containing $Q(1)$. We now pick a small interval $[q^-, q^+]$ in $B_i$ containing $q$, and let $p^-$, $p^+ \in \{1\} \times S^1$ be the points where the orbits of $q^-, q^+$ cross $\{1\} \times S^1$. Of course, $[s^-, s^+] \subset [p^-, p^+]$. Again, let $\alpha^-$, $\alpha^+$ denote semiorbits of $q^-$, $q^+$ under $\psi$ crossing $M_0$ via $\Gamma_i$ and escaping via $\phi$ into $F_0$. Let $A \subset F_0 \cup \Gamma_i \cup V_i$ be the region in $M$ bounded by $\alpha^-$, $\alpha^+$, the $\phi$-orbit segments $[q^-, p^-]$ and $[q^+, p^+]$, the intervals $[p^-, r^-]$ and $[r^+, p^+]$ in $\{1\} \times S^1$, and the $\phi$-orbit segment through $Q(\frac{1}{2})$. Let $A^* \subset V_i$ be the region bounded by the orbit-segment through $q$, the intervals $[s^-, r^-]$ and $[r^+, s^+]$ in $\{1\} \times S^1$, and the orbit-segment through $Q(\frac{1}{2})$. Now, the only point of $A^*$ in $B_i$ is $q$, so $A$ and $A^* - B_i = A^* - \{q\}$ are diffeomorphic. We can assume the diffeomorphism is the identity on the orbit-segment through $Q(\frac{1}{2})$, and that it respects the first-return map in the sense that if a point $x$ of $[p^-, r^-]$ lies in $[s^-, r^-]$, and if $y$ is the other end (in $[r^+, s^+]$) of the $\phi$-orbit segment through $x$, then the diffeomorphic image of $x$ lies on the same $\phi$-orbit segment as the diffeomorphic image of $y$.

Again, the diffeomorphism $A^* - \{q\} \to A$ carries the foliation by orbit segments of $\phi$ to a foliation of $A$ whose boundary leaves are the orbit segment through $Q(\frac{1}{2})$ and the two half-lines $\alpha^- \cup [q^-, p^-]$, $\alpha^+ \cup [q^+, p^+]$. Thus, we have again "patched" $\phi$ to $\psi$ near $q$, in such a way that the new flow has the same first-return map on $(s^-, s^+)$ as $\phi$, while points of $[p^-, s^-]$ and $[s^+, p^+]$ do not return to $B_i$.

These two processes tell us how to "patch" $\phi$ to $\psi$ near tangencies to $B_i$; elsewhere on $B_i$, both flows are transverse to $B_i$, so with a little smoothing we can join up integral curves of $\psi$ in $M_0$ to integral curves of $\phi$ in $M - M_0$. On the complement of $H_i$ in $B_i$, we can orient this "patched" foliation by integral curves consistently, since $x_i$ is a "sink" precisely if $\phi$ points out of $M_0$, and so on. In $H_i$, the tangencies of $\phi$ force some reversal of direction all along the part of $A$ trailing into $F_0$. However, we note that there must be an even number of such reversals, and the "strips" of the various "$A$" in $F_0 \cup \Gamma_i$ are all parallel, so that the reversals of orientation can be filled in between the various strips and still end up consistent on the boundary of $F_0 \cup \Gamma_i$.

We need now to take care of the tangencies at $B_0$. The situation here is a little more complicated, in that both $\phi$ and $\psi$ can have tangencies at $B_0$. However, we have managed things in such a way that the tangencies of $\phi$ all occur inside the interval $G_0$; thus, we can mimic the whole process just described to extend the flow across the interior of $\Gamma_1$ into $H_0$, so that the tangencies of $\phi$ with $B_0$ "escape" into $F_1$. On the other hand, the tangencies of $\psi$ with $B_0$ all occur inside $E_0$, the "escape set" of $\phi$ into $F_0$. Thus, we can push all $\psi$-tangencies directly into strips that escape into $F_0$ immediately, without crossing $M_0$ at all. The details of labelling are slightly different, but it is clear that again the process we have employed near $B_i$ can be adapted to this situation. Finally, the complement in $B_0$ of these two kinds of situations consists again of a set where the flows are both transverse to $B_0$, and we can "patch". The orientation of the foliation, again, presents no problem, for we only need to reverse the orientation on a series of "strips" between the various "$A$"'s, all contained in the strip $F_0$. The total number of reversals must, again, be even, so that we have the right orientation on the boundary of $F_0$ in $M - M_0$.

To complete the consideration of Case A, we need to show that the flow constructed in this manner is completely unstable. To this end, we note first that our constructions have all been maneuvered in such a way that the first-return map of the new flow to $\partial M_0$ agrees with a restriction of the first-return map of $\phi$ to $\partial M_0$.

We note that an open subset of $C_i - F_i$ $(i \geqslant 1)$ will trace out, under the new flow, an open set in $M$ whose intersection with $C_i$ is a subset of its orbit under the original flow $\phi$, and so points of $C_i - F_i$ $(i \geqslant 1)$ continue to wander under the new flow. On the other hand, points of $C_i \cap F_i$ cross $B_i$ at most once, and if they do so, they then eventually escape $M_0$ via $E_0$. Points of $C_0 - F_0$ which enter $M_0$ either escape into the interior of $F_0$ or cross via $\psi$ to some $E_i$, $i \geqslant 1$, and vanish into $C_i$. Points of $F_0$ my reenter $F_0$, but they do so, if at all, via a finite sequence of distinct and disjoint "strips" coming from the various "$A$"'s of our construction. Finally, the boundary of each $F_i$ is part of a band of "parallel" orbits in $C_i$, and so these again wander. Since every orbit in $M_0$ leaves $M_0$ (under $\psi$) in finite time, the above considerations sketch a proof that the new flow is completely unstable. On the other hand, our construction of $\psi$ insures that the Auslander recurrent set intersects $M_0$ in a nonempty set, and so the flow is explosive.

*Case* B. If $M - M_0$ is connected, our construction needs to be modified a little. In this case, we find a positive escape point $p^+$ and a negative escape point $p^-$ in $B_0$, and again "thicken" the two orbits to obtain a positive escape interval $E^+$ and a negative interval $E^-$, both in $B_0$. These are of course disjoint closed intervals. Their semiorbits escaping into $M - M_0$ are two "strips", labelled $F^+$ and $F^-$, respectively. We now use Lemma 4 to build a flow $\psi$ on $M_0$; we note that by assumption $X$ is a polygon with at least two pairs of identified edges. We put in a prolongation cycle of the flow across one pair, and "thicken" the nonbounding curve corresponding to another pair of edges into a band $\Gamma$ of parallel orbits. The band $\Gamma$ crosses $B_0$ at two disjoint intervals: $G$, where orbits enter $M_0$, and $H$, where they leave. We then use a diffeomorphism on a collar of $B_0$ in $M_0$ to make sure that $E^-$ and $G$ cover $B_0$, and that $H \subset$ interior $E^-$. The situation is sketched in Figure 7: if we ignore orientation for a moment, we see that the two boundary-orbits of $\Gamma$ in $M_0$ can be extended backward into $E^-$ and forward into $E^-$, and that these extended lines in $M$ separate $C_0 = M - M_0$ into four sets. One of these is a strip inside $F^-$ bounded at $B_0$ by $H$; we call this strip $\Gamma^-$. Two other strips, flanking $\Gamma^-$ on either side, touch $B_0$ at the two intervals of $B_0 - \{H \cup G\}$; we label these $F_1^-$ and $F_2^-$. Finally, the complement of $F_1^- \cup \Gamma^- \cup F_2^-$ in $C_0 = M - M_0$ is a more complicated manifold, abutting $M_0$ on $G$; call this $C^+$. Now, using our technique as before, we can extend the flow on $M_0 - \Gamma$ into $F_1^- \cup F_2^-$ by extending each tangency of $\psi$ with $B_0 - \{H \cup G\}$ into a "strip" $A$ in $F_1^-$ or $F_2^-$; $\{M_0 - \Gamma\} \cup F_1^- \cup F_2^-$ is then an invariant set, homeomorphic to $M_0 - \Gamma$, on which the flow is wandering. We note that each interval of $B_0 - \{H \cup G\}$ will contain an odd number of tangencies of $\psi$ with $B_0$, so a consistent flow orientation can be defined on $F_1 \cup \Gamma^- \cup F_2^-$ and the new orientation on $\Gamma^-$ is the negative of the original one on $E^-$. Now, we use similar techniques to push tangencies of $\phi$ (which are all inside $G$) across $\Gamma$ and into $\Gamma^-$. There will be an even number of tangencies of $\phi$ with $B_0$ in $G$, so again the re-orientation (which will of course extend into $\Gamma^-$) will present no problem. Finally, if we make sure that each of the two orbits bounding these four sets is "thickened", it is easy to see that there are no new nonwandering points: the flow on $C^+ \cup \Gamma \cup \Gamma^-$ is homeomorphic to the original flow $\phi$ on $C - F^-$, and the flow on $\{M_0 - \Gamma\} \cup F_1^- \cup F_2^-$ is homeomorphic to the flow $\psi$ on $X - \{x_0\}$.

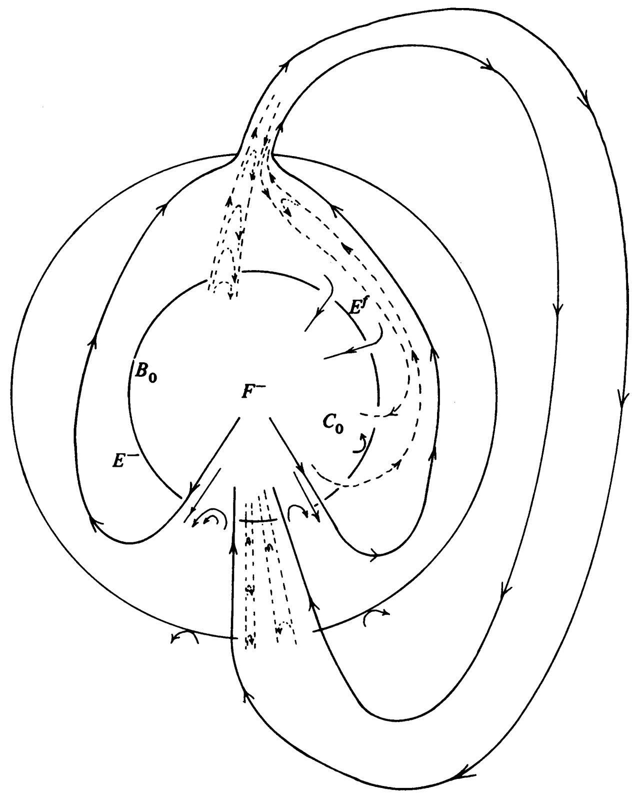


FIGURE 7
Case B

But then the two cases A and B have both been taken care of, and we have shown:

PROPOSITION 3. *If $M$ is an open 2-manifold not embeddable in any closed 2-manifold, then $M$ supports an explosive flow.*

This proposition is the last case of the theorem stated at the outset: $R^1$ and $R^2$ are the only open manifolds without explosive flows.

## REFERENCES


1. J. Auslander, *Generalized recurrence in dynamical systems*, Contributions to Differential Equations 3 (1964), 65–74.

2. M. Hirsch, *Differential topology*, Springer-Verlag, Berlin and New York, 1976.

3. ______, *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. (2) 73 (1961), 566–571.

4. B. von Kerekjarto, *Vorlesungen über Topologie*. I. *Flächentopologie*, Grundlehren der math. Wissenschaften, Band 8, Springer-Verlag, Berlin, 1923.

5. M. Krych, *Note on structural stability of open manifolds*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. XXII (1974), 1033–1038.

6. S. Lefschetz, *Introduction to topology*, Princeton Univ. Press, Princeton, N. J., 1942.

7. P. Mendes, *On stability of dynamical systems on open manifolds*, J. Differential Equations **16** (1974), 144–167.

8. Z. Nitecki, *Explosions in completely unstable dynamical systems*, in *Dynamical Systems*, Proc. Univ. of Florida Internat. Sympos., Edited by A. R. Bednarek and L. Cesari, Academic Press, New York, 1977, pp. 473–475.

9. ______, *Explosions in completely unstable flows*. I. *Preventing explosions*, Trans. Amer. Math. Soc. **245** (1978), 43–61.

10. P. Percell, *Structural stability on manifolds with boundary*, Topology **12** (1973), 123–144.

11. I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269.

12. F. Takens and W. White, *Vector fields with no nonwandering points*, Amer. J. Math. **98** (1976), 415–425.

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