TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 246, December 1978

QUALITY CONTROL FOR MARKOV CHAINS AND FREE BOUNDARY PROBLEMS [1]

BY

ROBERT F. ANDERSON AND AVNER FRIEDMAN

ABSTRACT. A machine can manufacture any one of n Markov chains $P_x^{\lambda_j}$ $(1 < j < n)$; the $P_x^{\lambda_j}$ are defined on the space of all sequences $x = \{x(m)\}$ $(1 < m < \infty)$ and are absolutely continuous (in finite times) with respect to one another. It is assumed that chains $P_x^{\lambda_j}$ evolve in a random way, dictated by a Markov chain $\theta(m)$ with n states, so that when $\theta(m) = j$ the machine is producing $P_x^{\lambda_j}$. One observes the σ-fields of $x(m)$ in order to determine when to inspect $\theta(m)$. With each product $P_x^{\lambda_j}$ there is associated a cost c_j. One inspects θ at a sequence of times (each inspection entails a certain cost) and stops production when the state $\theta = n$ is reached. The problem is to find an optimal sequence of inspections. This problem is reduced, in this paper, to solving a certain free boundary problem. In case $n = 2$ the latter problem is solved.

0. **Introduction.** Let X be a fixed countable subset of the real line. Let $\theta(t)$ $(t = 0, 1, 2, \dots)$ be a Markov chain with n states $1, 2, \dots, n$, and with transition probability matrix $p_{i,j}$. With each state i we associate a Markov chain $P_x^{\lambda_i}$ defined on the space Ω_1 of sequences $(x_0, x_1, x_2, \dots)$ where each x_i varies in X. We assume that the $P_x^{\lambda_i}$ are distinct from each other and absolutely continuous (in finite time) with respect to one another. Denote by $E^{i,x}$ the expectation corresponding to the random evolution of the $P_x^{\lambda_j}$ in accordance with the chain $\theta(t)$ starting at $\theta = i$ and x.

Let $K_1, \dots, K_{n-1}$ be given positive numbers. Let $c_1, \dots, c_n$ be given nonnegative numbers and define a function $f(\theta)$ by $f(i) = c_i$ if $i = 1, 2, \dots, n$. Let $\tau = (\tau_1, \tau_2, \dots)$ be an increasing sequence of "inspection times" in the sense that τ_i assumes only nonnegative integer values and each set $(\tau_i \leqslant s)$ (s nonnegative integer) depends only on the coordinates $x_0, x_1, \dots, x_s$ and on the knowledge of $\theta(\tau_j)$ for all $1 \leqslant j \leqslant i - 1$.

Throughout this paper we shall use the notation

Received by the editors June 13, 1976.

AMS (MOS) subject classifications (1970). Primary 62N10, 90B25; Secondary 60J05, 60J75, 60K30, 90A15, 93E15, 93E99.

Key words and phrases. Markov chain, quality control, stopping time, inspection time, quasi variational inequality.

[1] This paper is partially supported by National Science Foundation Grant MPS 72-04959 A02.

$$\int_a^b g(s)\,ds = g(a) + g(a+1) + \cdots + g(b) \tag{0.1}$$

where a, b are integers and $0 \leqslant a < b$.

Consider the cost function

$$J_x^i(\tau) = E^{i,x}\left[K_i + \sum_{j=1}^{n-1} K_j\left[\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\right]\right]$$
$$+ E^{i,x}\left[\int_0^{\tau_1 - 1} f(\theta(s))\,ds + \sum_{j=1}^{n-1}\sum_{l=1}^{\infty} I_{\theta(\tau_l)=j}\int_{\tau_l - 1}^{\tau_{l+1}-1} f(\theta(s))\,ds\right]. \tag{0.2}$$

The problem considered in this paper is to find and characterize a sequence of inspection times $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \ldots)$ such that

$$J_x^i(\bar{\tau}) = \inf_{\tau} J_x^i(\tau). \tag{0.3}$$

This is called a *quality control problem*. The same problem in the case of continuous-parameter Markov processes was studied by the authors in [1], [2]. The problem was reduced to solving a certain elliptic quasi variational inequality (q.v.i.). We shall establish a similar reduction also in the present setting of Markov chains. Analogously to the q.v.i. of [1], [2] we shall obtain here a "discrete" q.v.i. In the special case where $n = 2$ we shall solve the q.v.i.

The development of this paper proceeds parallel to [2]. Some of the results follow similarly to [2], and these will be mentioned only briefly. There are, however, some novel features in the present Markov chain setting.

In §1 we introduce the random evolution process (x, θ). We choose a model as in [2, Appendix] which displays very clearly the structure of this evolution.

In §2 we introduce the p-process and prove results analogous to Theorems 2.1, 2.2 of [2]. The quality control problem is introduced in §3, where it is reduced to solving a certain "discrete" q.v.i.

In §4 we solve the q.v.i. in case $n = 2$ under some monotonicity assumption.

1. **The (x, θ) process.** It will be convenient to denote the discrete parameter of various Markov chains by t; thus the parameter t will take values $t = 0, 1, 2, \ldots$. We fix a countable set X of points on the real line and denote by Ω_1 the space of all sequences $\omega = (x_0, x_1, x_2, \ldots)$ with $x_i \in X$. Viewing ω as a function $x = x(t) = x(t, \omega)$ on the nonnegative integers with values in X, we write $x_t = x(t) = x(t, \omega)$, $t = 0, 1, 2, \ldots$.

Let $\theta(t)$ be a Markov chain with n states $1, 2, \ldots, n$ defined on a probability space Ω_0 of all sequences $\omega' = (\theta_0, \theta_1, \theta_2, \ldots)$ where each θ_i may take values $1, 2, \ldots, n$. Viewing ω' as a function $\theta = \theta(t) = \theta(t, \omega')$, we write $\theta_t = \theta(t) = \theta(t, \omega')$, $t = 0, 1, 2, \ldots$. Denote the transition probability matrix of $\theta(t)$ by $p_{i,j}$.

Let $P_x^{\lambda_i}$ $(i = 1, \ldots, n)$ be n distinct Markov chains defined on Ω_1 and absolutely continuous (in finite time) with respect to one another. Denoting the transition probability matrix of $P_x^{\lambda_i}$ by $p_{j,k}^{\lambda_i}$ we then have, for each pair (j, k),

$$\text{either } p_{j,k}^{\lambda_i} = 0 \text{ for all } 1 \leqslant i \leqslant n \text{ or } p_{j,k}^{\lambda_i} > 0 \text{ for all } 1 \leqslant i \leqslant n. \tag{1.1}$$

We are interested in an explicit construction of the random evolution of the $P_x^{\lambda_i}$ in accordance with the law of $\theta(t)$. First we write down what, intuitively speaking, the transition probabilities should be:

$$\begin{aligned} P_{i,x}(\theta(t) = j, x(t) \in B) = \sum_{\rho=0}^{t-1}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)} \\ \cdot \sum_{u_i + u_{\gamma_1} + \cdots + u_{\gamma_\rho} < t}{}' p_{i,i}^{u_i-1} p_{i,\gamma_1} p_{\gamma_1,\gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1,\gamma_2} p_{\gamma_2,\gamma_2}^{u_{\gamma_2}-1} \cdots p_{\gamma_\rho,j} p_{j,j}^{t-u_i-u_{\gamma_1}-\cdots-u_{\gamma_\rho}} \\ \cdot P_x^{\lambda_i} \otimes_{u_i}^{0} P_{x(u_i)}^{\lambda_{\gamma_1}} \otimes_{u_i+u_{\gamma_1}}^{u_i} P_{x(u_i+u_{\gamma_1})}^{\lambda_{\gamma_2}} \otimes \cdots \\ \otimes_{u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}}^{u_i+u_{\gamma_1}+\cdots+u_{\gamma_{\rho-1}}} P_{x(u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho})}^{\lambda_j} (x(t) \in B) \end{aligned} \tag{1.2}$$

for $i = 1, \ldots, n$; $x \in X$, where B is any subset of X. Here, the notation

$$\sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}{}' \qquad \text{for } \rho \geqslant 1$$

means that summation is extended over all integers $\gamma_1, \ldots, \gamma_\rho$ varying from 1 to n such that

$$i \neq \gamma_1 \neq \gamma_2 \neq \cdots \neq \gamma_{\rho-1} \neq j;$$

for $\rho = 0$ it means that $i \neq j$, i.e., the sum is empty if $i = j$, and consists of one term if $i \neq j$. The summation

$$\sum_{\rho=0}^{t-1}{}^{*}$$

means that ρ varies over $0, 1, 2, \ldots, t-1$ with one exception: if $i = j$ then there is no term with $\rho = 0$ and instead there appears the term

$$p_{i,i}^{t} P_x^{\lambda_i}(x(t) \in B);$$

we refer to this term as the term corresponding to $\rho = -1$. Finally, the notation

$$\sum_{u_i + u_{\gamma_1} + \cdots + u_{\gamma_\rho} < t}{}'$$

means that the summation is extended over all integers $u_i, u_{\gamma_1}, \ldots, u_{\gamma_\rho}$ such that

$$u_i \geqslant 1, \quad u_{\gamma_1} \geqslant 1, \ldots, u_{\gamma_\rho} \geqslant 1, \quad \text{and} \quad u_i + u_{\gamma_1} + \cdots + u_{\gamma_\rho} < t.$$

The concept of the tensor product

$$P_x^1 \otimes_{u_1}^{0} P_{x(u_1)}^2 \otimes \cdots \otimes_{u_1+\cdots+u_m}^{u_1+\cdots+u_{m-1}} P_{x(u_1+\cdots+u_m)}^{m+1}$$

used in (1.2) is the same as in [1], [2] (which is taken from [3]) with the obvious adaptation to the discrete parameter case.

Let $\Omega = \Omega_0 \otimes \Omega_1$ and denote by $\mathfrak{F}_t$ and $\mathfrak{M}_t$ the σ-fields generated by the first $t + 1$ coordinates of $(x_0, x_1, x_2, \dots)$ and of $\{(\theta_0, \theta_1, \theta_2, \dots), (x_0, x_1, x_2, \dots)\}$ respectively.

THEOREM 1.1. *The $P_{i,x}$ define a Markov process with respect to $\mathfrak{M}_t$ and Ω.*

PROOF. It suffices to verify the Chapman-Kolmogorov equation

$$P_{i,x}(\theta(t) = j, x(t) \in B)$$
$$= \sum_{l=1}^{n} \sum_{y} P_{i,x}(\theta(s) = l, x(s) = y) P_{l,y}(\theta(t-s) = j, x(t-s) \in B) \quad (1.3)$$

where s is any integer, $1 < s < t - 1$, and B is any subset of X. The right-hand side of (1.3) is equal to

$$\sum_{l=1}^{n} \sum_{y} \sum_{\mu=0}^{s-1}{}^{*} \sum_{(i,\alpha_1,\dots,\alpha_\mu,l)}{}' \sum_{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<s}{}' \sum_{\nu=0}^{t-s-1}{}^{*} \sum_{(l,\beta_1,\dots,\beta_\nu,j)}{}'$$
$$\cdot \sum_{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t-s}{}' p_{i,i}^{u_i-1} p_{i,\alpha_1} p_{\alpha_1,\alpha_1}^{u_{\alpha_1}-1} \cdots p_{\alpha_\mu,l} p_{l,l}^{s-u_i-\cdots-u_{\alpha_\mu}} p_{l,l}^{u_l-1} p_{l,\beta_1} p_{\beta_1,\beta_1}^{u_{\beta_1}-1}$$
$$\cdots p_{\beta_\nu,j} p_{j,j}^{t-s-u_l-\cdots-u_{\beta_\nu}} P_x^{\lambda_i} \otimes_{u_i}^{0} P_{x(u_i)}^{\lambda_{\alpha_1}} \otimes \cdots$$
$$\otimes_{u_i+\cdots+u_{\alpha_\mu}}^{u_i+\cdots+u_{\alpha_\mu}-1} P_{x(u_i+\cdots+u_{\alpha_\mu})}^{\lambda_l} (x(s) = y)$$
$$\cdot P_y^{\lambda_l} \otimes_{u_l}^{0} P_{x(u_l)}^{\lambda_{\beta_1}} \otimes \cdots \otimes_{u_1+\cdots+u_{\beta_\nu}}^{u_l+\cdots+u_{\beta_\nu}-1} P_{x(u_l+\cdots+u_{\beta_\nu})}^{\lambda_j} (x(t-s) \in B). \quad (1.4)$$

Summing over y and combining the two factors P^{λ_l} as in [2, following (A.7)] we deduce that the sum over y of the tensor products is equal to

$$P_x^{\lambda_i} \otimes_{u_i}^{0} P_{x(u_i)}^{\lambda_{\alpha_1}} \otimes \cdots \otimes_{u_i+\cdots+u_{\alpha_\mu}}^{u_i+\cdots+u_{\alpha_\mu}-1} P_{x(u_i+\cdots+u_{\alpha_\mu})}^{\lambda_l} \otimes_{s+u_l}^{u_i+\cdots+u_{\alpha_\mu}} P_{x(s+u_l)}^{\lambda_{\beta_1}}$$
$$\otimes \cdots \otimes_{s+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}}^{s+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}-1} P_{x(s+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu})}^{\lambda_j} (x(t) \in B).$$

Next we substitute $u_l + s \to u_l$. The sum

$$\sum_{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t-s}{}'$$

becomes a sum

$$\sum_{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t,\, u_l>s+1}{}'$$

where the prime " ′ " in the last summation indicates that $u_{\beta_1} \geq 1, \dots, u_{\beta_\nu} \geq 1$.

We next substitute $u_l \to u_i + u_{\alpha_1} + \cdots + u_{\alpha_\mu} + u_l$. The last sum becomes a summation over $u_l, u_{\beta_1}, \dots, u_{\beta_\nu}$ subject to

$$\sum_{\substack{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t \\ u_l>s+1-u_i-u_{\alpha_1}-\cdots-u_{\alpha_\mu}}}{}'$$

and the prime " ' " indicates that $u_{\beta_1} \geqslant 1, \ldots, u_{\beta_\nu} \geqslant 1$.

The effect of the two substitutions is to transform (1.4) into the sum (cf. [2])

$$\sum_{l=1}^{n} \sum_{\mu=0}^{s-1}{}^{*} \sum_{(i,\alpha_1,\ldots,\alpha_\mu,l)}' \sum_{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<s}' \sum_{\nu=0}^{t-s-1}{}^{*} \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}'$$

$$\cdot \sum_{\substack{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t \\ u_l>s+1-u_i-u_{\alpha_1}-\cdots-u_{\alpha_\mu}}} p_{i,i}^{u_i-1} p_{i,\alpha_1} \cdots p_{\alpha_\mu,l} p_{l,l}^{u_l-1} p_{l,\beta_1}^{u_{\beta_1}-1} p_{\beta_1,\beta_1} \cdots$$

$$p_{\beta_\nu,j} p_{j,j}^{t-u_i-u_{\alpha_1}-\cdots-u_{\alpha_\mu}-u_l-u_{\beta_1}-\cdots-u_{\beta_\nu}} P_x^{\lambda_i} \otimes_{u_i}^{0} P_{x(u_i)}^{\lambda_{\alpha_1}} \otimes \cdots \otimes_{u_i+\cdots+u_{\alpha_\mu}}^{u_i+\cdots+u_{\alpha_\mu-1}}$$

$$\cdot P_{x(u_i+\cdots+u_{\alpha_\mu})}^{\lambda_l} \otimes_{u_i+\cdots+u_{\alpha_\mu}+u_l}^{u_i+\cdots+u_{\alpha_\mu}} P_{x(u_i+\cdots+u_{\alpha_\mu}+u_l)}^{\lambda_{\beta_1}} \otimes_{u_i+\cdots+u_l+u_{\beta_1}}^{u_i+\cdots+u_{\alpha_\mu}+u_l}$$

$$\cdot P_{x(u_i+\cdots+u_l+u_{\beta_1})}^{\lambda_{\beta_2}} \otimes \cdots \otimes_{u_i+\cdots+u_{\beta_\nu}}^{u_i+\cdots+u_{\beta_{\nu-1}}} P_{x(u_i+\cdots+u_{\beta_\nu})}^{\lambda_j} (x(t) \in B). \tag{1.5}$$

The left-hand side of (1.3) is equal to

$$\sum_{\rho=0}^{t-1}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}' \sum_{u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}<t}' p_{i,i}^{u_i-1} p_{i,\gamma_1} p_{\gamma_1,\gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1,\gamma_2} \cdots$$

$$p_{\gamma_\rho,j} p_{j,j}^{t-u_i-u_{\gamma_1}-\cdots-u_{\gamma_\rho}} P_x^{\lambda_i} \otimes_{u_i}^{0} P_{x(u_i)}^{\lambda_{\gamma_1}} \otimes \cdots$$

$$\otimes_{u_i+\cdots+u_{\gamma_\rho}}^{u_i+\cdots+u_{\gamma_{\rho-1}}} P_{x(u_i+\cdots+u_{\gamma_\rho})}^{\lambda_j} (x(t) \in B). \tag{1.6}$$

We have to prove that the expressions in (1.5) and (1.6) are equal.

Denote the general term under the summation in (1.6) by

$$I(i, \gamma_1, \ldots, \gamma_\rho, j; u_i, u_{\gamma_1}, \ldots, u_{\gamma_\rho}).$$

Then the general term under the summation in (1.5) is precisely

$$I(i, \alpha_1, \ldots, \alpha_\mu, l, \beta_1, \ldots, \beta_\nu, j; u_i, u_{\alpha_1}, \ldots, u_{\alpha_\mu}, u_l, u_{\beta_1}, \ldots, u_{\beta_\nu}).$$

Thus it remains to prove the following combinatorial lemma.

LEMMA 1.2. *For any positive integers s, t with $s < t$,*

$$\sum_{l=1}^{n} \sum_{\mu=0}^{s-1}{}^{*} \sum_{(i,\alpha_1,\ldots,\alpha_\mu,l)}' \sum_{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<s}' \sum_{\nu=0}^{t-s-1}{}^{*} \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}'$$

$$\cdot \sum_{\substack{u_i+u_{\alpha_1}+\cdots+u_{\alpha_\mu}+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t \\ u_l>s+1-u_i-u_{\alpha_1}-\cdots-u_{\alpha_\mu}}}' I(i, \alpha_1, \ldots, \alpha_\mu, l, \beta_1, \ldots, \beta_\nu, j;$$

$$u_i, u_{\alpha_1}, \ldots, u_{\alpha_\mu}, u_l, u_{\beta_1}, \ldots, u_{\beta_\nu})$$

$$= \sum_{\rho=0}^{t-1}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}' \sum_{v_i+v_{\gamma_1}+\cdots+v_{\gamma_\rho}<t}' I(i, \gamma_1, \ldots, \gamma_\rho, j; v_i, v_{\gamma_1}, \ldots, v_{\gamma_\rho}). \tag{1.7}$$

This lemma is entirely different from the corresponding combinatorial lemma used in [2].

PROOF OF LEMMA 1.2. Each term on the left-hand side of (1.7) corresponding to $\mu \geqslant 0$, $\nu \geqslant 0$ appears also on the right-hand side of (1.7) with

$$\gamma_k = \alpha_k \quad (1 \leqslant k \leqslant \mu), \quad \gamma_{\mu+1} = l, \quad \gamma_{\mu+m+1} = \beta_m \quad (1 \leqslant m \leqslant \nu),$$
$$v_i = u_i, \quad v_{\gamma_h} = u_{\gamma_h}. \tag{1.8}$$

The terms corresponding to $\mu = -1$, $\nu \geqslant 0$ arise when $l = i$, and then there are no α's and

$$u_i + u_{\beta_1} + \cdots + u_{\beta_\nu} \leqslant t, \qquad u_i \geqslant s + 1.$$

These terms also appear on the right-hand side of (1.7) (they are given by (1.8) with no α's). Similarly, the terms with $\nu = -1$, $\mu \geqslant 0$ which appear on the left-hand side of (1.7) appear also on the right-hand side. Finally, the term corresponding to $\mu = -1$, $\nu = -1$ occurs only if $i = j$ and in that case it is precisely the term on the right-hand side of (1.7) corresponding to $\rho = -1$.

It remains to show that each term which appears on the right-hand side of (1.7) with $\rho \geqslant 0$ appears also on the left-hand side and that this correspondence is given by (the one-to-one mapping) (1.8).

Consider the case $\rho > 0$. Let

$$\sigma_0 = \inf\{\sigma; v_i + v_{\gamma_1} + \cdots + v_{\gamma_\sigma} \geqslant s\}.$$

Suppose first that

$$v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}} = s. \tag{1.9}$$

If $\sigma_0 < \rho$ then define α's, β's and u's by (1.8) with

$$l = \gamma_{\sigma_0+1}, \qquad \mu = \sigma_0, \qquad \nu = \rho - \sigma_0 - 1.$$

Since $v_i + v_{\gamma_1} + \cdots + v_{\gamma_\mu} = s$ and $v_i \geqslant 1$, $v_{\gamma_1} \geqslant 1, \ldots, v_{\gamma_\mu} \geqslant 1$, we have $\mu \leqslant s - 1$. Similarly, since

$$u_{\gamma_{\sigma_0+1}} + \cdots + u_{\gamma_\rho} \leqslant t - (u_i + u_{\gamma_1} + \cdots + u_{\gamma_{\sigma_0}}) = t - s$$

and $\gamma_m \geqslant 1$, we must have $\nu \leqslant t - s - 1$. Therefore in order for the term $I(i, \gamma_1, \ldots, \gamma_\rho, j; v_i, v_{\gamma_1}, \ldots, v_{\gamma_\rho})$ to appear on the left-hand side of (1.7) we must show that the restriction

$$u_l \geqslant s + 1 - u_i - u_{\alpha_1} - \cdots - u_{\alpha_\mu}$$

is satisfied. But this follows immediately from (1.9) and the fact that $u_l \geqslant 1$.

If $\sigma_0 = \rho$ then the given term appears on the left-hand side of (1.7) with $l = j$, $\nu = -1$.

So far we have assumed that (1.9) holds. We now assume that (1.9) does not hold, i.e.,

$$v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}} > s. \tag{1.10}$$

If $\sigma_0 > 0$ then we take $l = \gamma_{\sigma_0}$, $\mu = \sigma_0 - 1$, $\nu = \rho - \sigma_0$ in the definition (1.8). Since

$$u_i + u_{\gamma_1} + \cdots + u_{\gamma_{\sigma_0-1}} < s, \qquad u_i \geqslant 1, u_{\gamma_h} \geqslant 1,$$

we have $\mu \leqslant s - 1$. Also

$$v_{\gamma_{\sigma_0+1}} + \cdots + v_{\gamma_\rho} \leqslant t - (v_i + v_{\gamma_1} + \cdots + v_{\gamma_{\sigma_0}}) < t - s$$

so that $\rho - \sigma_0 < t - s$, i.e., $\nu \leqslant t - s - 1$. Thus it remains to show that

$$u_l \geqslant s + 1 - u_i - u_{\alpha_1} - \cdots - u_{\alpha_\mu}.$$

But this follows immediately from (1.10).

If $\sigma_0 = 0$ we take $l = i$ and proceed as in the last case. This completes the proof of the lemma.

Having proved Theorem 1.1, we denote by $P^{i,x}$ and $E^{i,x}$ the probabilities and expectations corresponding to the transition probabilities $P_{i,x}$. Recall that the probability space is Ω and that the σ-fields are the $\mathfrak{M}_t$.

We shall now extend formula (1.2).

LEMMA 1.3. *Let* $A \in \mathfrak{F}_t$, $t = 0, 1, 2, \ldots$. *Then*

$$\int_A I_{\theta(t)=j}\, dP^{i,x} = \sum_{\rho=0}^{t-1}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}{}' \sum_{u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}<t}{}'$$

$$p_{i,i}^{u_i-1} p_{i,\gamma_1} p_{\gamma_1,\gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1,\gamma_2} p_{\gamma_2,\gamma_2}^{u_{\gamma_2}-1} \cdots p_{\gamma_\rho,j} p_{j,j}^{t-u_i-u_{\gamma_1}-\cdots-u_{\gamma_\rho}} P_x^{\lambda_i} \otimes P_{x(u_i)}^{\lambda_{\gamma_1}}$$

$$\otimes_{u_i+u_{\gamma_1}}^{u_i} P_{x(u_i+u_{\gamma_1})}^{\lambda_{\gamma_2}} \otimes \cdots \otimes_{u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}}^{u_i+u_{\gamma_1}+\cdots+u_{\gamma_{\rho-1}}} P_{x(u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho})}^{\lambda_j}(A). \tag{1.11}$$

PROOF. The proof is similar to the proof of Lemma A.3 in [2]. It suffices to prove (1.11) for a cylindrical set

$$A = (x(t_1) \in B_1, \ldots, x(t_m) \in B_m), \qquad t_1 < t_2 < \cdots < t_m.$$

Let

$$A_i = (x(t_1) \in B_1, \ldots, x(t_i) \in B_i), \qquad 1 \leqslant i \leqslant m,$$

$$C_i = (x(t_{i+1} - t_i) \in B_{i+1}, \ldots, x(t_m - t_i) \in B_m),$$

so that $A = A_m$. By the Markov property of the (x, θ) chain and by (1.2) we obtain, after substituting $u_l + t_{m-1} \to u_l$,

$$\int_A I_{\theta(t)=j}\, dP^{i,x} = \sum_{l=1}^{n} p_{l,l}^{-t_{m-1}} \int_{A_{m-1}} I_{\theta(t_{m-1})=l} \sum_{\nu=0}^{t-1}{}^{*} \sum_{(i,\beta_1,\ldots,\beta_\nu,l)}{}'$$

$$\cdot \sum_{\substack{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t \\ u_l > t_{m-1}+1}}{}' p_{l,l}^{u_l-1} p_{l,\beta_1} p_{\beta_1,\beta_1}^{u_{\beta_1}-1} \cdots p_{\beta_\nu,j} p_{j,j}^{t-u_l-\cdots-u_{\beta_\nu}} P_{x(t_{m-1})}^{\lambda_l}$$

$$\otimes_{u_l-t_{m-1}}^{0} P_{x(u_l-t_{m-1})}^{\lambda_{\beta_1}} \otimes \cdots \otimes_{u_l+\cdots+u_{\beta_\nu}-t_{m-1}}^{u_l+\cdots+u_{\beta_{\nu-1}}-t_{m-1}}$$

$$\cdot\, P_{x(u_l+\cdots+u_{\beta_\nu}-t_{m-1})}^{\lambda_j}(C_{m-1})\, dP^{i,x}. \tag{1.12}$$

Using the Markov property we can write the right-hand side in the form

$$\sum_{k=1}^{n} \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{l=1}^{n} p_{l,l}^{-t_{m-1}} \sum_{\mu=0}^{t_{m-1}-t_{m-2}-1}{}^{*} \sum_{(k,\alpha_1,\ldots,\alpha_\mu,l)}{}' \sum_{\nu=0}^{t-t_{m-1}-1}{}^{*}$$

$$\cdot \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}{}' \sum_{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<t_{m-1}-t_{m-2}}{}' \sum_{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t}{}' p_{k,k}^{u_k-1} p_{k,\alpha_1} p_{\alpha_1,\alpha_1}^{u_{\alpha_1}-1}$$

$$\cdots p_{l,l}^{t_{m-1}-t_{m-2}-u_k-\cdots-u_{\alpha_\mu}} p_{l,l}^{u_l-t_{m-1}-1} p_{l,\beta_1} \cdots p_{j,j}^{t_m-u_l-\cdots-u_{\beta_\nu}}$$

$$\cdot P_{x(t_{m-2})}^{\lambda_k} \otimes_{u_k}^{0} P_{x(u_k)}^{\lambda_{\alpha_1}} \otimes \cdots \otimes_{u_k+\cdots+u_{\alpha_\mu}}^{u_k+\cdots+u_{\alpha_\mu-1}}$$

$$\cdot P_{x(u_k+\cdots+u_{\alpha_\mu})}^{\lambda_l}\Big[\big(x(t_{m-1}-t_{m-2})\in B_{m-1}\big)$$

$$\cdot P_{x(t_{m-1}-t_{m-2})}^{\lambda_l} \otimes_{u_l-t_{m-1}}^{0} P_{x(u_l-t_{m-1})}^{\lambda_{\alpha_1}} \otimes \cdots$$

$$\otimes_{u_l+\cdots+u_{\beta_\nu}-t_{m-1}}^{u_l+\cdots+u_{\beta_{\nu-1}}-t_{m-1}} P_{x(u_l+\cdots+u_{\beta_\nu}-t_{m-1})}^{\lambda_j}(C_{m-1})\Big]\, dP^{i,x}.$$

We now make the substitution $u_k + t_{m-2} \to u_k$ which transforms

$$\sum_{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<t_{m-1}-t_{m-2}}{}' \quad \text{into} \quad \sum_{\substack{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<t_{m-1}\\ u_k>t_{m-2}+1}}{}'$$

and then make the substitution $u_l - u_k - u_{\alpha_1} - \cdots - u_{\alpha_\mu} \to u_l$ which transforms

$$\sum_{u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t}{}' \quad \text{into} \quad \sum_{\substack{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t\\ u_l>t_{m-1}+1-u_k-u_{\alpha_1}-\cdots-u_{\alpha_\mu}}}{}' .$$

Using the rules (A.16), (A.17) of [2] we finally obtain

$$\int_A I_{\theta(t)=j}\, dP^{i,x} = \sum_{k=1}^{n} p_{k,k}^{-t_{m-2}} \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{l=1}^{n} \sum_{\mu=0}^{t_{m-1}-t_{m-2}-1}{}^{*}$$

$$\cdot \sum_{(k,\alpha_1,\ldots,\alpha_\mu,l)}{}' \sum_{\nu=0}^{t-t_{m-1}-1}{}^{*} \sum_{(l,\beta_1,\ldots,\beta_\nu,j)}{}' \sum_{\substack{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}<t_{m-1}\\ u_k>t_{m-2}+1}}{}'$$

$$\cdot \sum_{\substack{u_k+u_{\alpha_1}+\cdots+u_{\alpha_\mu}+u_l+u_{\beta_1}+\cdots+u_{\beta_\nu}<t\\ u_l>t_{m-1}+1-u_k-u_{\alpha_1}-\cdots-u_{\alpha_\mu}}}{}'$$

$$\cdot\Big[p_{k,k}^{u_k-1} p_{k,\alpha_1} \cdots p_{l,l}^{u_l-1} p_{l,\beta_1} \cdots p_{j,j}^{t_m-u_l-u_{\alpha_1}-\cdots-u_{\beta_\nu}}$$

$$\cdot P_{x(t_{m-2})}^{\lambda_k} \otimes_{u_k-t_{m-2}}^{0} P_{x(u_k-t_{m-2})}^{\lambda_{\alpha_1}} \otimes \cdots$$

$$\otimes_{u_k+\cdots+u_{\alpha_\mu}-t_{m-2}}^{u_k+\cdots+u_{\alpha_{\mu-1}}-t_{m-2}} P_{x(u_k+\cdots+u_{\alpha_\mu}-t_{m-2})}^{\lambda_l} \otimes \cdots$$

$$\otimes_{u_k+u_{\alpha_1}+\cdots+u_{\beta_\nu}-t_{m-2}}^{u_k+u_{\alpha_1}+\cdots+u_{\beta_{\nu-1}}-t_{m-2}} P_{x(u_k+u_{\alpha_1}+\cdots+u_{\beta_\nu}-t_{m-2})}^{\lambda_j}(C_{m-2})\Big] dP^{i,x}. \quad (1.13)$$

We now apply a slightly different version of Lemma 1.2 whereby instead of $t_{m-2} = 0$ we have $t_{m-2} \geqslant 0$. We conclude that

$$\int_A I_{\theta(t)=j}\, dP^{i,x} = \sum_{k=1}^{n} p_{k,k}^{-t_{m-2}} \int_{A_{m-2}} I_{\theta(t_{m-2})=k} \sum_{\rho=0}^{t-t_{m-2}-1}{}^{*} \sum_{(k,\gamma_1,\ldots,\gamma_\rho,j)}{}' \cdot \sum_{\substack{u_k+\cdots+u_{\gamma_\rho}<t \\ u_k > t_{m-2}+1}}{}' [\cdots] dP^{i,x} \tag{1.14}$$

where the expression in $[\cdots]$ is the same as on the right-hand side of (1.13). Formula (1.14) is analogous to (1.12), except that $m-1$ has been replaced by $m-2$. Proceeding in this way step by step and setting $t_0 = 0$, $B_0 = X$, $A_0(x(t_0) \in B_0)$, $C_0 = A$, we arrive at (1.11) with $m-1$ and t_{m-1} replaced by 0 and t_0 respectively. But this relation is precisely the assertion of the lemma.

2. **The p-process.** In view of the assumption (1.1) we have

$$\frac{dP_x^{\lambda_j}}{dP_x^{\lambda_i}} \Big| \mathfrak{F}_t = \frac{p^{\lambda_j}_{x,x(1)} p^{\lambda_j}_{x(1),x(2)} \cdots p^{\lambda_j}_{x(t-1),x(t)}}{p^{\lambda_i}_{x,x(1)} p^{\lambda_i}_{x(1),x(2)} \cdots p^{\lambda_i}_{x(t-1),x(t)}} \tag{2.1}$$

on all paths for which both the numerator and the denominator do not vanish, and $P_x^{\lambda_j} = 0$, $P_x^{\lambda_i} = 0$ on all the remaining paths. Let

$$z_{i,j}(s,t) = \frac{dP_x^{\lambda_i} \otimes_s^0 P^{\lambda_j}_{x(s)}}{dP_x^{\lambda_i}} \Big| \mathfrak{F}_t \qquad (s < t). \tag{2.2}$$

Then we have

$$z_{i,j}(s,t) = \frac{p^{\lambda_j}_{x(s),x(s+1)} p^{\lambda_j}_{x(s+1),x(s+2)} \cdots p^{\lambda_j}_{x(t-1),x(t)}}{p^{\lambda_i}_{x(s),x(s+1)} p^{\lambda_i}_{x(s+1),x(s+2)} \cdots p^{\lambda_i}_{x(t-1),x(t)}} \qquad (s < t) \tag{2.3}$$

on the paths for which the numerator and denominator do not vanish. Clearly $z_{i,j}(t,t) = 1$.

As in [2] we define

$$\bar{p}_{i,j}(t) = P^{i,x}[\theta(t) = j | \mathfrak{F}_t] \frac{dP^{i,x}}{dP_x^{\lambda_i}} \Big| \mathfrak{F}_t. \tag{2.4}$$

We then have (cf. [1], [2])

$$\bar{p}_{i,j}(t) = \sum_{\rho=0}^{t-1}{}^{*} \sum_{(i,\gamma_1,\ldots,\gamma_\rho,j)}{}' \sum_{u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}<t}{}' p_{i,i}^{u_i-1} p_{i,\gamma_1} p_{\gamma_1,\gamma_1}^{u_{\gamma_1}-1} p_{\gamma_1,\gamma_2} \cdots p_{\gamma_\rho,j} p_{j,j}^{t-u_i-u_{\gamma_1}-\cdots-u_{\gamma_\rho}} z_{i,\gamma_1}(u_i, u_i+u_{\gamma_1}) z_{i,\gamma_2}(u_i+u_{\gamma_1}, u_i+u_{\gamma_1}+u_{\gamma_2}) \cdots z_{i,j}(u_i+u_{\gamma_1}+\cdots+u_{\gamma_\rho}, t). \tag{2.5}$$

We now introduce the probabilities

$$\bar{P}^{p,x} = \sum_{i=1}^{n} p_i P^{i,x} \qquad \left(p = (p_1,\ldots,p_n), p_i \geqslant 0, \sum_{i=1}^{n} p_i = 1\right) \tag{2.6}$$

and the process

$$X(t) = \big(x_x(t), p_1(p, t), \ldots, p_n(p, t)\big) \tag{2.7}$$

where $t = 0, 1, 2, \ldots$; $x_x(t)$ is $x(t)$ with $x(0) = x$ and

$$p_j(p, t) = \bar{E}^{p,x}\big[\theta(t) = j \mid \mathfrak{F}_t\big]. \tag{2.8}$$

Here $\bar{E}^{p,x}$ is the expectation corresponding to the probability $\bar{P}^{p,x}$. As in [2] we have

$$p_j(p, t) = \sum_{i=1}^{n} p_i \bar{p}_{i,j}(t) \bigg/ \left[\sum_{l=1}^{n} p_l z_{i,l}(0, t) \sum_{k=1}^{n} \bar{p}_{l,k}(t)\right]. \tag{2.9}$$

THEOREM 2.1. *The process $X(t)$ is a Markov process, with respect to the σ-fields $\mathfrak{F}_t$ and the measures $\bar{P}^{p,x}$.*

The proof is similar to the proof of the corresponding result in the Appendix of [2] except that now we use Lemma 1.3 instead of Lemma A.2 of [2].

3. **The quality control problem.** Using the notation (0.1), we introduce the cost function (0.2) and, more generally, the cost

$$J_x^p(\tau) = \bar{E}^{p,x}\left[K(p) + \sum_{l=1}^{\infty} K(\theta(\tau_l))\, I_{\theta(\tau_l)\neq n}\right]$$

$$+ \bar{E}^{p,x}\left[\int_0^{\tau_1 - 1} f(\theta(s))\, ds + \sum_{l=1}^{\infty} I_{\theta(\tau_l)\neq n} \int_{\tau_l - 1}^{\tau_{l+1}-1} f(\theta(s))\, ds\right] \tag{3.1}$$

where $K(p) = K_i$ if $p = (p_1, \ldots, p_n)$, $p_1 = \cdots = p_{i-1} = 0$, $p_i \neq 0$; if the process $\theta(t)$ is such that $p_{i,j}(t) = 0$ whenever $j < i$ then no restrictions are made on the K_i, but if the process $\theta(t)$ can go in both directions then we require that $K_1 = K_2 = \cdots = K_{n-1}$. Here $\tau = (\tau_1, \tau_2, \ldots)$ is a *sequence of inspection times*, i.e.,

$$\tau_1 = \sigma_1, \quad \tau_{m+1} = \tau_m + \sum_{l=1}^{n-1} I_{\theta(\tau_m)=l}\, \sigma_{m,l}(\phi_{\tau_m}) \qquad (m \geq 1) \tag{3.2}$$

where σ_1, $\sigma_{m,l}$ are stopping times with respect to $\mathfrak{F}_t$ with nonnegative integer values, and ϕ is the shift operator: $\phi_s x(t) = x(t + s)$. It is understood that $\tau_{m+i} = \infty$ $(i \geq 1)$ on the set $\tau_m = \infty$. Also, in (3.1), $K(\theta(\tau_l))I_{\theta(\tau_l)\neq n}$ and $\int_{\tau_l - 1}^{\tau_{l+1}-1} f(\theta(s))\, ds$ do not appear whenever $\tau_l = \infty$. We shall denote by $\mathfrak{A}$ the class of all sequences of inspection times. We are interested in the problem of characterizing $\bar{\tau}_p \in \mathfrak{A}$ such that

$$J_x^p(\bar{\tau}_p) = \inf_{\tau \in \mathfrak{A}} J_x^p(\tau). \tag{3.3}$$

Denote by $A_{x,p}$ the generator of the Markov process occurring in Theorem

2.1. Thus, $A_{x,p}$ is defined by

$$A_{x,p}u(x,p) = \bar{E}^{p,x}\big[u(x(1), p(p,1)) - u(x,p)\big] \tag{3.4}$$

where $p(p,t) = (p_1(p,t), \dots, p_n(p,t))$.

Using the Markov property one can establish, by induction on t ($t = 1, 2, \dots$), Dynkin's formula

$$\bar{E}^{p,x}\big[u(x_x(t), p(p,t))\big] - u(x,p)$$
$$= \bar{E}^{p,x}\left[\int_0^{t-1} A_{x_x(s),p(p,s)}u(x_x(s), p(p,s))\,ds\right]. \tag{3.5}$$

We can now proceed as in [2] to reduce the problem of characterizing an optimal $\bar{\tau}_p$ as in (3.3) to the problem of solving the following quasi variational inequality (q.v.i.) for a function $V(x,p)$:

$$V(x,p) \leqslant K(p) + \sum_{j=1}^{n-1} p_j V(x, e_j) \tag{3.6}$$

where e_j is the jth unit vector $(0, 0, \dots, 0, 1, 0, \dots, 0)$,

$$A_{x,p}V(x,p) + \sum_{j=1}^{n} c_j p_j \geqslant 0, \tag{3.7}$$

$$\left[A_{x,p}V(x,p) + \sum_{j=1}^{n} c_j p_j\right]\left[K(p) + \sum_{j=1}^{n-1} p_j V(x,e_j) - V(x,p)\right] = 0 \tag{3.8}$$

where the p_j vary in the set $p_j \geqslant 0$, $\sum_{j=1}^{n} p_j = 1$ and x varies in X.

Let

$$S = \left\{(x,p);\ x \in X, p = (p_1, \dots, p_n), p_j \geqslant 0,\ \sum_{j=1}^{n} p_j = 1,\right.$$
$$\left. V(x,p) = K(p) + \sum_{j=1}^{n-1} p_j V(x,e_j)\right\}. \tag{3.9}$$

Define the $\mathcal{F}_t$ stopping times:

$$\sigma_*^p = \text{hitting time of the set } S \text{ by } X(t) = (x(t), p(p,t)),$$

$$\sigma_*^l = \sigma_*^p \text{ when } p = e_l,$$

$$\bar{\tau}_1^p = \sigma_*^p,\ \bar{\tau}_{m+1}^p = \bar{\tau}_m^p + \sum_{l=1}^{n-1} I_{\theta(\bar{\tau}_m^p) = l}\,\sigma_*^l(\phi_{\bar{\tau}_m^p}),$$

$$\bar{\tau}^p = (\bar{\tau}_1^p, \bar{\tau}_2^p, \bar{\tau}_3^p, \dots). \tag{3.10}$$

THEOREM 3.1. *Let $V(x,p)$ be a solution of the q.v.i.* (3.6) – (3.8). *If the* σ_*^p

are finite valued then

$$V(x,p) = \inf_{\tau \in \mathcal{Q}} J_x^p(\tau) = J_x^p(\bar{\tau}^p). \tag{3.11}$$

The proof is similar to the proof of the corresponding result in [2] and will therefore be omitted.

In the special case where

$$p_{i,j} = 0 \quad \text{if } 1 \leqslant j < i \leqslant n \tag{3.12}$$

the q.v.i. reduces to a sequence of simpler q.v.i. analogous to (4.35)–(4.37) in [2].

Another type of simplification of (3.6)–(3.8) occurs when

$$p^{\lambda_i}_{x(s),x(t)} = p^{\lambda_i}_{x(0),x(t-s)} \equiv p^{\lambda_i}_{t-s} \quad \text{if } 0 \leqslant s < t. \tag{3.13}$$

In this case the numbers

$$\bar{P}^{p,x}\big[\, p_j(p,t) \in B;\ 1 \leqslant j \leqslant n \big]$$

do not depend on x and, consequently, the process

$$p_j(p,t) \qquad (1 \leqslant j \leqslant n) \text{ with measures } \bar{P}^{p,0} \tag{3.14}$$

is a Markov process. We shall denote its generator by A_p.

Denote by R_i^* the (countable) range of the process $(p_j(e_i, t);\ 1 \leqslant j \leqslant n)$ and let $R^* = \bigcup_{i=1}^{n-1} R_i^*$.

One is interested in the quality control problem mainly for the initial values $p = e_i$. In case (3.13) holds it then suffices to solve the q.v.i. in the set R^* only. Thus we have to solve a "discrete" q.v.i.

In the next section we shall solve the discrete q.v.i. in a case when $n = 2$.

4. **Solution of the discrete q.v.i. in case** $n = 2$. We assume that (3.12), (3.13) hold and that $n = 2$. Thus $p_{2,1} = 0$, $p_{2,2} = 1$. To rule out a trivial case, we assume that $p_{1,1} > 0$, $p_{1,2} > 0$.

Let

$$p_j^{\lambda_1} = p^{\lambda_1}_{i,i+j}, \qquad p_j^{\lambda_2} = p^{\lambda_2}_{i,i+j} \tag{4.1}$$

where $p^{\lambda_k}_{k,l}$ is the transition probability matrix of $P_x^{\lambda_k}$. Denote by N the set of j's for which $p_j^{\lambda_i} \neq 0$, and let

$$\mu_j = p_j^{\lambda_2}/p_j^{\lambda_1} \qquad (j \in N). \tag{4.2}$$

Since

$$\sum_{j \in N} p_j^{\lambda_i} = 1 \qquad (i = 1, 2), \tag{4.3}$$

we must have

$$\sum_{j \in N} p_j^{\lambda_1}\mu_j = 1. \tag{4.4}$$

Since $p_{2,1} = 0$, $p_{2,2} = 1$, we have

$$\bar{p}_{1,1}(1) = p_{1,1}, \quad \bar{p}_{1,2}(1) = p_{1,2}, \qquad \bar{p}_{2,1}(t) = 0, \bar{p}_{2,2}(t) = 1,$$

Hence, (2.9) for $t = 1$ simplifies to

$$p_1(p, 1) = \frac{p_1 p_{1,1}}{p_1 + p_2 \mu_{x(1)-x(0)}}, \quad p_2(p, 1) = \frac{p_1 p_{1,2} + p_2 \mu_{x(1)-x(0)}}{p_1 + p_2 \mu_{x(1)-x(0)}}.$$

Defining

$$T_j^1(p_1, p_2) = \frac{p_1 p_{1,1}}{p_1 + p_2 \mu_j}, \quad T_j^2(p_1, p_2) = \frac{p_1 p_{1,2} + p_2 \mu_j}{p_1 + p_2 \mu_j},$$

we can write the generator A_p ($p = p_1, p_2$) in the form

$$\begin{aligned} A_p g(p) &= \bar{E}^p[g(p(p, 1)) - g(p)] \\ &= p_1 \bar{E}^{e_1}[g(p(p, 1)) - g(p)] + p_2 \bar{E}^{e_2}[g(p(p, 1)) - g(p)] \\ &= \bar{E}^{e_1}[(p_1 + p_2 z_{1,2}(0, 1))(g(p(p, 1)) - g(p))] \\ &= \sum_{j \in N} p_j^{\lambda_1}(p_1 + p_2 \mu_j)[g(T_j^1(p_1, p_2), T_j^2(p_1, p_2)) - g(p_1, p_2)] \end{aligned} \tag{4.5}$$

where we have used the facts

$$\bar{E}^{e_1}[g] = \bar{E}^{e_2}[g z_{1,2}(0, 1)] \quad (g = g(p(p, 1))), \quad z_{1,2}(0, 1) = \mu_{x(1)-x(0)}$$

and the notation $e_1 = (0, 1)$, $e_2 = (1, 0)$.

Define

$$y = p_1/p_2, \qquad g(y) = g(p_1, p_2).$$

Then, as easily verified,

$$A_p g(p_1, p_2) = L_y g(y) \tag{4.6}$$

where

$$L_y g(y) = \frac{1}{1 + y} \sum_{j \in N} p_j^{\lambda_1}(1 + y\mu_j)[g(T_j(y)) - g(y)] \tag{4.7}$$

and

$$T_j(y) = \frac{p_{1,2}}{p_{1,1}} + \frac{\mu_j}{p_{1,1}} y \qquad (j \in N). \tag{4.8}$$

We shall impose the monotonicity condition

$$T_j(y) > y \qquad (j \in N), \tag{4.9}$$

that is

$$\mu_j \geqslant p_{1,1} \qquad (j \in N). \tag{4.10}$$

This condition implies that

$$\frac{p_1(p, t)}{p_2(p, t)} > \frac{p_1(p, s)}{p_2(p, s)} \quad \text{if } t > s.$$

It is easily verified that

$$A_p p_2 = -p_{1,2} p_2. \tag{4.11}$$

We shall need Dynkin's formula

$$\bar{E}^{e_1}[g(p(\tau))] - g(p) = \bar{E}^{e_1}\left[\int_0^{\tau - 1} A_p g(p(s))\, ds\right] \quad (p(0) = e_1) \tag{4.12}$$

where g is any function defined on the discrete set R^* and τ is any bounded $\mathcal{F}_t$ stopping time.

Let b be any positive number and let τ_b be the first time such that $(p_1(t)/p_2(t)) \geq b$. (Notice $p_1(\tau_b)/p_2(\tau_b)$ is not necessarily equal to b.) Applying (4.12) with $g(p_1, p_2) = p_2$ and $\tau = \tau_b \wedge m$ $(m > 0)$ and using (4.11), we conclude that

$$\bar{E}^{e_1}[\tau_b \wedge m] \leq C$$

where C is a constant independent of M. Taking $m \to \infty$ we conclude that

$$\bar{E}^{e_1}[\tau_b] < \infty. \tag{4.13}$$

Notice that the proof of (4.13) does not exploit the monotonicity assumption (4.9).

We now assume for simplicity that $c_1 = 0$ (but $c_2 > 0$), and set $c = c_2$, $K = K_1$.

Recalling (4.6), (4.7), the q.v.i. (3.6)–(3.8) can be written in terms of the function $V(y) = V(p_1, p_2)$ $(y = p_1/p_2)$ in the form

$$L_y V(y) + \frac{cy}{1+y} \geq 0 \quad \text{in } \hat{R}, \tag{4.14}$$

$$V(y) \leq K + \frac{V(0)}{1+y} \quad \text{in } \hat{R}, \tag{4.15}$$

$$\left[L_y V(y) + \frac{cy}{1+y}\right]\left[K + \frac{V(0)}{1+y} - V(y)\right] = 0 \quad \text{in } \hat{R} \tag{4.16}$$

where $\hat{R}$ is the (discrete) range of $p_1(t)/p_2(t)$ when $p_1(0) = 0, p_2(0) = 1$.

THEOREM 4.1. *Let* (4.1), (4.9) *hold. Then there exist a unique* $b \in \hat{R}, b > 0$ *and a unique function* $V(y)$ *defined on* $\hat{R}$ *such that*

$$L_y V(y) + \frac{cy}{1+y} = 0 \quad \textit{if } y \in \hat{R}, 0 \leq y < b, \tag{4.17}$$

$$L_y V(y) + \frac{cy}{1+y} > 0 \quad \textit{if } y \in \hat{R}, y > b, \tag{4.18}$$

$$V(y) = K + \frac{V(0)}{1+y} \quad \textit{if } y \in \hat{R}, y \geq b, \tag{4.19}$$

$$V(y) < K + \frac{V(0)}{1+y} \quad \textit{if } y \in \hat{R}, 0 \leq y < b. \tag{4.20}$$

Notice that $V(y)$ is then a solution of the q.v.i. (4.14)–(4.16). In view of (4.13), $\tau_b < \infty$ and therefore Theorem 3.1 can be applied to conclude that

$$V(0) = \inf_{\tau \in \mathcal{Q}} J_x^1(\tau) = J_x^1(\bar{\tau}^1). \tag{4.21}$$

The optimal inspection is then to inspect at time τ_b (given that $p(0) = (0, 1)$); let $p(t)$ start again at $p(0) = (0, 1)$ and again inspect at time τ_b, etc.

PROOF OF THEOREM 4.1. Suppose b is such that (4.17), (4.19) hold. By Dynkin's formula we then get (with $E = \bar{E}^{e_1}$, $P = \bar{P}^{e_1}$, $y(t) = p_1(t)/p_2(t)$)

$$E\left[K + \frac{V(0)}{1 + y(\tau_b)}\right] - V(0) = E\left[-\int_0^{\tau_b - 1} \frac{cy(s)}{1 + y(s)}\, ds\right],$$

or

$$V(0)E\left[\frac{y(\tau_b)}{1 + y(\tau_b)}\right] = K + cE\left[\int_0^{\tau_b - 1} \frac{y(s)}{1 + y(s)}\, ds\right]. \tag{4.22}$$

Setting

$$H(b, dz) = P[y(\tau_b) \in z], \tag{4.23}$$

$$L(b, dz) = \sum_{s < \tau_b - 1} P[y(s) \in z], \tag{4.24}$$

we then obtain from (4.22) an expression for $V(0)$:

$$V(0) = \frac{K + c\int \frac{z}{1+z} L(b, dz)}{\int \frac{z}{1+z} H(b, dz)} \equiv Q(b). \tag{4.25}$$

Notation. For any $b \in \hat{R}$ we denote by $\hat{b}$ the number in $\hat{R}$ immediately to the right of b.

LEMMA 4.2. *For any* $b \in \hat{R}$,

$$L(\hat{b}, db) = P(y(\tau_b) = b). \tag{4.26}$$

PROOF. The left-hand side is equal to

$$\sum_{s < \tau_b - 1} P(y(s) = b) = \sum_{s < \tau_b - 1} P(y(s) = b)$$

$$+ \sum_{\tau_b - 1 < s < \tau_{\hat{b}} - 1} P(y(s) = b). \tag{4.27}$$

The first sum on the right-hand side is equal to zero (by the definition of τ_b). Since $\tau_{\hat{b}} - \tau_b \leq 1$ by the monotonicity assumption, and $\tau_{\hat{b}} - \tau_b = 1$ if and only if $y(\tau_b) = b$, the second sum on the right-hand side of (4.27) is equal to

$$\sum_{s = \tau_b} P(y(s) = b) = P(y(\tau_b) = b),$$

and (4.26) follows.

We also have the relation

$$L(\hat{b}, db) = H(b, db); \tag{4.28}$$

however this relation will not be needed.

LEMMA 4.3. *The following formula holds.*

$$\int \frac{z}{1+z} H(b, dz) = p_{1,2} \int \frac{1}{1+z} L(b, dz). \tag{4.29}$$

PROOF. Since $L_y(1) = 0$,

$$L_y\left(\frac{y}{1+y}\right) = -L_y\left(\frac{1}{1+y}\right) = \frac{p_{1,2}}{1+y}$$

by (4.11). Hence, by Dynkin's formula,

$$E\left[\frac{y(\tau_b)}{1+y(\tau_b)}\right] = E\left[\int_0^{\tau_b - 1} \frac{p_{1,2}}{1+y(s)}\, ds\right].$$

Recalling (4.23) and (4.24), (4.29) follows.

Using Lemma 4.3, we can rewrite the expression $Q(b)$ introduced in (4.25) in the form

$$Q(b) = \frac{K + c\int \frac{z}{1+z} L(b, dz)}{p_{1,2}\int \frac{1}{1+z} L(b, dz)}. \tag{4.30}$$

Our plan now is to show that there is a unique b which minimizes $Q(b)$ and then show that the function V defined by (4.17), (4.19) also satisfies (4.18), (4.20). The uniqueness of the minimal b implies the uniqueness assertion of Theorem 4.1.

For any $b \in \hat{R}$, we compute from (4.30)

$$\begin{aligned} Q(\hat{b}) - Q(b) = &\left[c\int \frac{z}{1+z} L(\hat{b}, dz) - c\int \frac{z}{1+z} L(b, dz)\right] \\ &\Big/\left[p_{1,2}\int \frac{1}{1+z} L(\hat{b}, dz)\right] \\ &+\left[K + c\int \frac{z}{1+z} L(b, dz)\right]\frac{1}{p_{1,2}}\left[\int \frac{1}{1+z} L(b, dz) - \int \frac{1}{1+z} L(\hat{b}, dz)\right] \\ &\Big/\left[\left(\int \frac{1}{1+z} L(\hat{b}, dz)\right)\cdot\left(\int \frac{1}{1+z} L(b, dz)\right)\right]. \end{aligned} \tag{4.31}$$

By the strict monotonicity of the y-process

$$\int_{z<y} h(z) L(\bar{\bar{y}}, dz) = \int_{z<y} h(z) L(\bar{y}, dz) \quad \text{if } y < \bar{y} < \bar{\bar{y}}. \tag{4.32}$$

Hence

$$\int \frac{1}{1+z} L(\hat{b}, dz) - \int \frac{1}{1+z} L(b, dz) = \frac{1}{1+b} L(\hat{b}, db).$$

Since $b \in \hat{R}$ and the y-process is monotone, $P(y(\tau_b) = b) > 0$ so that, by Lemma 4.2, also $L(\hat{b}, db) > 0$. It follows that

$$\begin{aligned}\operatorname{sgn}[Q(\hat{b}) - Q(b)] &\\ &= \operatorname{sgn}\left[cb \int \frac{1}{1+z} L(b, dz) - c \int \frac{z}{1+z} L(b, dz) - K\right] \\ &= \operatorname{sgn}\left[c \int \frac{b-z}{1+z} L(b, dz) - K\right]. \end{aligned} \tag{4.33}$$

Let

$$S(b) = \int \frac{b-z}{1+z} L(b, dz). \tag{4.34}$$

Then, using (4.32),

$$\begin{aligned} S(\hat{b}) - S(b) &= \int \frac{\hat{b}-z}{1+z} L(\hat{b}, dz) - \int \frac{b-z}{1+z} L(b, dz) \\ &= \int \frac{\hat{b}-b}{1+z} L(b, dz) + \frac{1}{1+b} L(\hat{b}, db). \end{aligned} \tag{4.35}$$

The right-hand side is larger than $(\hat{b} - b)L(b, d0) = \hat{b} - b$. Hence $S(b)$ is strictly increasing on $\hat{R}$ and $S(b) \to \infty$ if $b \to \infty$.

From (4.33), (4.34) we then conclude that there is a unique b in $\hat{R}$ such that

$$\begin{aligned} Q(\hat{y}) - Q(y) &< 0 \quad \text{if } y < b, y \in \hat{R}, \\ Q(\hat{y}) - Q(y) &> 0 \quad \text{if } y \geqslant b, y \in \hat{R}. \end{aligned} \tag{4.36}$$

The point b is then the unique minimum of $Q(y), y \in \hat{R}$.

We next define $V(y)$ for $y \geqslant b, y \in \hat{R}$ by (4.19) and for $0 \leqslant y < b, y \in \hat{R}$ by (4.17) (using iteration and the strict monotonicity of the y-process). It remains to show that (4.18) and (4.20) hold.

To prove (4.18) we compute

$$L_y\left(K + \frac{V(0)}{1+y}\right) = \frac{cy}{1+y} - \frac{V(0)p_{1,2}}{1+y}.$$

Thus, (4.18) would follow from

$$cb - V(0)p_{1,2} > 0. \tag{4.37}$$

If we prove that

$$L_y\left(V(y) - K - \frac{V(0)}{1+y}\right) < 0 \quad \text{if } y \in \hat{R}, y < b \tag{4.38}$$

then by the maximum principle (which holds, since the y-process is monotone) and the fact that

$$V(y) - K - \frac{V(0)}{1+y} \begin{cases} = 0 & \text{if } y = b, \\ < 0 & \text{if } y = 0, \end{cases}$$

it follows that (4.20) holds. Since the left-hand side of (4.38) is equal to $cy - V(0)p_{1,2}$, (4.20) is thus a consequence of

$$cy - p_{1,2}V(0) < 0 \quad \text{if } y \in \hat{R}, y < b. \tag{4.39}$$

Thus, to complete the proof of the theorem it remains to prove (4.37), (4.39). Inserting $V(0)$ from (4.25) (or rather (4.30)) into (4.37), (4.39) we find that these two inequalities reduce to

$$cb \int \frac{1}{1+z} L(b, dz) - \left(K + c \int \frac{z}{1+z} L(b, dz)\right) > 0,$$

$$cy \int \frac{1}{1+z} L(b, dz) - \left(K + c \int \frac{z}{1+z} L(b, dz)\right) < 0 \qquad (y < b);$$

but these inequalities clearly follow from (4.33) and (4.36).

References

1. R. F. Anderson and A. Friedman, *A quality control problem and quasi variational inequalities*, Arch. Rational Mech. Anal. **63** (1977), 205–252.

2. ———, *Multi-dimensional quality control problems and quasi variational inequalities*, Trans. Amer. Math. Soc. **246** (1978), 31–76.

3. D. W. Stroock and S. R. S. Varadhan, *Diffusion processes with continuous coefficients*. I, Comm. Pure Appl. Math. **22** (1969), 345–400. MR **40** #6641.

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

Department of Mathematics, Northwestern University, Evanston, Illinois 60201