ON THE FREE BOUNDARY OF
A QUASI VARIATIONAL INEQUALITY ARISING
IN A PROBLEM OF QUALITY CONTROL

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Abstract. In some recent work in stochastic optimization with partial
observation occurring in quality control problems, Anderson and Friedman
[1], [2] have shown that the optimal cost can be determined as a solution of the
quasi variational inequality
\[ Mw(p) + f(p) > 0, \quad w(p) < \psi(p; w), \]
\[ (Mw(p) + f(p))(w(p) - \psi(p; w)) = 0 \]
in the simplex \( p_i > 0, \sum_{i=1}^{n} p_i = 1 \). Here \( f, \psi \) are given functions of \( p \), \( \psi \) is a
functional of \( w \), and \( M \) is a given elliptic operator degenerating on the
boundary. This system has a unique solution when \( M \) does not degenerate
in the interior of the simplex. The aim of this paper is to study the free
boundary, that is, the boundary of the set where \( w(p) < \psi(p; w) \).

1. Introduction. In the model considered by Anderson and Friedman [1], [2]
one is interested in finding an optimal sequence of increasing inspection times
\( \tau_i \) which minimize the cost function
\[
J^p_x(\tau) \equiv E^x_p \left[ Ke^{-\alpha \tau_1} + \int_0^{\tau_1} f(\theta(s))e^{-\alpha s} \, ds \right. \\
+ \sum_{i=1}^{\infty} I_{\theta(\tau_i) \neq n} \left[ Ke^{-\alpha \tau_{i+1}} + \int_{\tau_i}^{\tau_{i+1}} f(\theta(s))e^{-\alpha s} \, ds \right] \right]; 
\]
where \( \theta(s) \) is a Markov process with \( n \) states 1, 2, \ldots, \( n \) and \( Q \)-matrix \( (q_{ij}) \);
\( f(i) = c_i > 0, K > 0, \alpha > 0 \), and the \( \tau_i \) depend only on the information
given by \( \theta(\tau_1), \ldots, \theta(\tau_{i-1}) \) and the \( \sigma \)-fields \( \mathcal{F}_i \) of the process \( x(t) \) which is defined
as follows: Let \( w(t) + \lambda_i t \) be a \( n \)-dimensional Brownian motion with drift \( \lambda_i \)
\( (1 \leq i \leq n) \); then \( x(t) \) is the random evolution of these \( n \) diffusion processes
in accordance with \( \theta(t) \). Finally, \( p = (p_1, \ldots, p_n) \) is the initial distribution
of \( \theta(t) \), and \( x = x(0) \).

The problem of finding
\[
w(x, p) = \inf J^p_x(\tau) \]

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and characterizing an optimal sequence of inspections $\tau = \tau^* = (\tau_1^*, \tau_2^*, \ldots)$ is called a quality control problem. The motivation for this problem is explained in detail in [1], [2].

It is shown in [2] that $w(x, p)$ is independent of $x$. Further, the problem of finding $w = w(p)$ and $\tau^*$ is reduced to the problem of solving a quasi variational inequality (q.v.i.) of the form

$$Mw + \sum_{j=1}^{n} c_j p_j \geq 0, \quad w(p) \leq K + \sum_{j=1}^{n-1} w(e_j) p_j,$$

$$(Mw + \sum_{j=1}^{n} c_j p_j)(w(p) - K - \sum_{j=1}^{n-1} w(e_j) p_j) = 0 \quad (1.3)$$

in the set $A = \{ p_i > 0, \sum_{i=1}^{n} p_i = 1 \}$. Here $e_j = (\delta_{j,1}, \ldots, \delta_{j,n})$ and $M$ is an elliptic operator degenerating on $\partial A$. The q.v.i. is solved in [2] under the assumption that $M$ is nondegenerate in (the interior of) $A$. In §2 we recall this fact and also state some other results from [2] in a form which will be useful for the subsequent sections.

The aim of the present paper is to study the set

$$C^A = \left\{ p; w(p) < K + \sum_{j=1}^{n-1} w(e_j) p_j \right\} \quad (1.4)$$

and the free boundary $\Gamma^A = \partial C^A \cap A$. For this purpose it is convenient to make a change of coordinates $y_j = p_j/p_1$ and to transform the q.v.i. into a q.v.i. in the space

$$R_{n-1}^+ = \{(y_2, \ldots, y_n); y_i > 0 \text{ for } 2 < i < n\}.$$

Then $C^A$ and $\Gamma^A$ are transformed into sets which we designate by $C$ and $\Gamma$ respectively.

In §3 we find a sharp condition for the set $C$ to be bounded. In §4 we prove that, when $C$ is bounded, $\Gamma$ is a graph, monotone in each variable, i.e., a point $(y_2, \ldots, y_n)$ belongs to $C$ if and only if,

$$y_j < \Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$$

where $\Psi_j$ is a finite valued function. In §5 we prove that $\Gamma$ is given by $y_j = \Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$, the $\Psi_j$ are analytic, and $\partial \Psi_j/\partial y_i < 0$. Some concluding remarks are given in §6.

For a variational inequality (v.i.) for a function $u$ and an obstacle $\psi$, the support of the solution is, by definition, the closure of the set $\{ u < \psi \}$. The question of compact support of solutions of v.i. was first studied by Brezis [6]. Recent results on the support of solutions of some q.v.i. have been obtained in [3] and [4].
2. The q.v.i. Let

\[ A = \left\{ (p_1, \ldots, p_n); p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\} \]

and let \( \lambda_1, \ldots, \lambda_n \) be distinct \( \nu \)-dimensional vectors such that

\[ \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots, \lambda_n - \lambda_1 \]

are linearly independent; this condition implies, of course, that \( \nu > n - 1 \).

Let \( q_{i,j} \) \((1 < i, j < n)\) be real numbers satisfying:

\[ q_{i,j} > 0 \quad \text{if} \quad i \neq j, \quad \sum_{j=1}^{n} q_{i,j} = 0. \]

Finally, let \( K \) and \( \alpha \) be positive numbers and let \( c_1, \ldots, c_n \) be nonnegative numbers. Introduce the elliptic operator in \( A \):

\[ Mw(p) = \frac{1}{2} \sum_{i,j=1}^{n} q_{i,j} p_i p_j \left( \lambda_i - \sum_{l=1}^{n} \lambda_l p_l \right) \cdot \left( \lambda_j - \sum_{l=1}^{n} \lambda_l p_l \right) \frac{\partial^2 w(p)}{\partial p_i \partial p_j} + \sum_{i,j=1}^{n} q_{i,j} p_i \frac{\partial w(p)}{\partial p_j} - \alpha w(p). \]

Note that any \( n - 1 \) of the \( p_i \)'s can be taken as independent variables; the remaining \( p_i \), say \( p_{i_0} \), is then given by \( 1 - \sum_{i \neq i_0} p_i \).

We shall be interested in the q.v.i. (1.3) in the set \( A \), where \( e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the \( j \)-th component. As easily seen (see [2]) \( M \) is nondegenerate in (the interior of) \( A \) if and only if condition (2.1) holds. \( M \) is degenerate on all of \( \partial A \).

**Theorem 1.1** [2]. There exists a unique solution \( w \) of (1.3) such that

\[ w \in C(\overline{A}) \cap W^{2,r}_{\text{loc}}(A) \quad \text{for all} \quad 1 < r < \infty. \]  

We recall that \( w(p) > 0 \) if \( p \in \overline{A}, p \neq e_n \).

From (2.4) it follows that \( w(p) \) is continuously differentiable in \( A \). The set \( C^A \), defined by (1.4), is an open subset of \( A \); it is called the domain of continuation. The set \( \Gamma^A = \partial C^A \cap A \) (\( \partial C^A \) = boundary of \( C^A \)) is called the free boundary, and the set

\[ S^A = \left\{ p \in A; w(p) = K + \sum_{j=1}^{n-1} w(e_j) p_j \right\} \]

is called the stopping set. As shown in [2], the optimal inspections are performed when a certain process \( p(t) \), given explicitly in terms of the process \( x(t) \), exits the set \( C^A \); this explains the terminology of \( C^A, S^A \).

It will be convenient to use Cartesian coordinates \( y_i = p_i/p_1 (2 < i < n); \)
here the role of \( p_1 \) is incidental; \( p_1 \) may be replaced by any other fixed variable \( p_k \). Since \( Y = 1 + y_2 + \cdots + y_n = 1 + (p_2 + \cdots + p_n)/p_1 = 1/p_1 \), we have \( p_i = y_i/Y \) (2 \( \leq i \leq n \)).

Define \( R_{n-1}^+ \) by (1.5) and set \( u(y) = w(p), y_1 \equiv 1 \). Then (see [2]) \( Mw(p) = Lu(y) \) where

\[
Lu(y) = \frac{1}{2} \sum_{i,j=2}^{n} \mu_{ij} y_i y_j \frac{\partial^2 u(y)}{\partial y_i \partial y_j} + \sum_{j=2}^{n} b_j(y) \frac{\partial u(y)}{\partial y_j} - au(y) \tag{2.5}
\]

where

\[
\mu_{ij} = (\lambda_i - \lambda_1) \cdot (\lambda_j - \lambda_1),
\]

\[
b_j(y) = -(\lambda_j - \lambda_1) \cdot \lambda_1 y_j + (\lambda_j - \lambda_1) y_j \cdot \frac{\sum_{i=1}^{n} \lambda_i y_i}{Y} + \sum_{i=1}^{n} (q_{i,j} - q_{i,1}) y_i \tag{2.7}
\]

The q.v.i. (1.3) transforms into

\[
Lu(y) + \frac{1}{Y} \sum_{j=1}^{n} c_j y_j \geq 0, \quad u(y) \leq K + \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j,
\]

\[
\left( Lu(y) + \frac{1}{Y} \sum_{j=1}^{n} c_j y_j \right) \left( u(y) - K - \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j \right) = 0 \tag{2.8}
\]
in \( R_{n-1}^+ \), where

\[
u_j = w(e_j) \quad (1 \leq j \leq n - 1). \tag{2.9}
\]

Let \( \Omega_\delta \) be any family of bounded domains with smooth boundary \( \partial \Omega_\delta \) such that \( (\Omega_\delta \cup \partial \Omega_\delta) \subset A, \Omega_\delta \uparrow A \) as \( \delta \downarrow 0 \). Set

\[
\psi(p) = K + \sum_{j=1}^{n-1} w(e_j) p_j. \tag{2.10}
\]

For any \( \epsilon > 0 \) consider the elliptic problem

\[
Mw_{\epsilon, \delta} - \frac{1}{\epsilon} (w_{\epsilon, \delta} - \psi)^+ + \sum_{j=1}^{n} c_j p_j = 0 \quad \text{in} \ \tilde{\Omega}_\delta,
\]

\[
w_{\epsilon, \delta} = 0 \quad \text{on} \ \partial \tilde{\Omega}_\delta. \tag{2.11}
\]

Since \( M \) is nondegenerate in the closure of \( \tilde{\Omega}_\delta \), this problem has a unique solution. As shown in [2] (see also [5])

\[
w_{\epsilon, \delta} \to w_\epsilon \quad \text{as} \ \delta \to \infty, \quad w_\epsilon \to w \quad \text{as} \ \epsilon \to 0 \tag{2.12}
\]
uniformly in compact subsets of \( A \). The proof exploits the probabilistic interpretation of \( w_{\epsilon, \delta} \) as given in [5]. One can also prove that

\[
w_{\epsilon, \delta} \to w_{\delta}^* \quad \text{as} \ \epsilon \to 0, \quad w_{\delta}^* \to w \quad \text{as} \ \delta \to 0 \tag{2.13}
\]
uniformly in compact subsets of $A$. In fact, the proof (which is similar to the proof of (2.12) in [2]) exploits the standard representation of $w^*_\delta$ (as a solution of a v.i. in $\hat{\Omega}_\delta$ with zero Dirichlet data) and the fact that

if $\tau^*_\delta = \text{exit time of the process } p(t) \text{ from } \Omega_\delta$, then $\tau^*_\delta \to \infty$ as $\delta \to 0$.

The above result (2.13) is valid (with obvious changes in the proof) if we replace the boundary conditions $w_{\epsilon, \delta} = w^*_\delta = 0$ on $\partial \hat{\Omega}_\delta$ by the boundary conditions $w_{\epsilon, \delta} = w^*_\delta = g$ where $g$ is any bounded continuous function such that $g(p) < K + \sum_{j=1}^{n-1} \lambda_j p_j$. Taking, in particular, $g(p) = K + \sum_{j=1}^{n-1} \lambda_j p_j$ and going into the $y$-coordinates, we conclude:

**Theorem 2.2.** Let $\Omega_\delta$ be a family of bounded domains with smooth boundary $\partial \Omega_\delta$ such that

$$(\Omega_\delta \cup \partial \Omega_\delta) \subset R^n_+, \quad \Omega_\delta \uparrow R^n_{-1} \quad \text{as } \delta \downarrow 0.$$ Let $u_\delta$ be the solution of the v.i. (2.8) in $\Omega_\delta$ with

$$u_\delta = K + \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j \quad \text{on } \partial \Omega_\delta$$

(where the $u_j$ are given by (2.9)). Then $u_\delta(y) \to u(y)$ as $\delta \to 0$, uniformly in compact subsets of $R^n_-$.

Notice that $u_\delta \in W^{2,r}(\Omega_\delta)$ for any $1 < r < \infty$. Consequently, $u_\delta$ is continuously differentiable in $\overline{\Omega}_\delta$.

Later on we shall use the notation

$$\psi(y) = K + \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j, \quad y_1 \equiv 1, \quad (2.14)$$

$$C_\delta = \{ y \in \Omega_\delta; u_\delta(y) < \psi(y) \}. \quad (2.15)$$

3. **Boundedness of the domain of continuation.** In the $y$-space, the domain of continuation $C$ is given by

$$C = \{ y \in R^n_+; u(y) < \psi(y) \}. \quad (3.1)$$

In this section we shall prove, under some sharp conditions, that $C$ is a bounded set. That means that

$$\overline{C^A} \text{ does not intersect the set } p_1 = 0. \quad (3.2)$$

Notice that since $u(0) < K + u(0) = K + u_1 = \psi(0)$, $C$ contains an $R^n_{-1}$-neighborhood of the origin.

We introduce the numbers
\[ B_i = c_i + \sum_{j=1}^{n-1} q_{i,j} u_j - \alpha u_i - \alpha K \quad (1 \leq i < n - 1), \]

\[ B_n = c_n + \sum_{j=1}^{n-1} q_{n,j} u_j - \alpha K. \quad (3.3) \]

**Theorem 3.1.** The set \( C \) is bounded if

\[ B_i > 0 \quad \text{for} \quad 2 < i < n. \quad (3.4) \]

**Proof.** From (2.3) we get \( M_p_i = \sum_{i=1}^{n} q_{i} \alpha_p_i - \alpha p_i \). In terms of the \( y \)-coordinates we then have

\[ L\left( \frac{y_i}{Y} \right) = \frac{1}{Y} \left( \sum_{i=1}^{n} q_{i} y_i - \alpha y_i \right) \]

with the usual convention that \( y_1 = 1 \).

It follows that

\[ L\psi = -\alpha K + \frac{1}{Y} \sum_{j=1}^{n-1} u_j \left( \sum_{i=1}^{n} q_{i} y_i - \alpha y_i \right) = \frac{1}{Y} \sum_{i=1}^{n} \beta y_i \quad (3.5) \]

where

\[ \beta_i = \sum_{j=1}^{n-1} q_{i,j} u_j - \alpha u_i - \alpha K \quad (1 < i < n - 1), \]

\[ \beta_n = \sum_{j=1}^{n-1} q_{n,j} u_j - \alpha K. \quad (3.6) \]

Hence

\[ \frac{1}{Y} \sum_{i=1}^{n} c_i y_i + L\psi = \frac{1}{Y} \sum_{i=1}^{n} (c_i + \beta_i) y_i = \frac{1}{Y} \sum_{i=1}^{n} B_i y_i, \quad (3.7) \]

by Definition (3.3).

Set

\[ \nu = u_0 - \psi. \quad (3.8) \]

Then \( \nu \) is a solution, in \( \Omega \), of the v.i.

\[ -Lv < \frac{1}{Y} \sum_{i=1}^{n} B_i y_i, \quad \nu < 0, \]

\[ \left( -Lv - \frac{1}{Y} \sum_{i=1}^{n} B_i y_i \right) \nu = 0, \quad \nu = 0 \text{ on } \partial \Omega. \quad (3.9) \]

The assumption (3.4) implies that there exist positive constants \( R^*, \gamma \) such that
We shall compare \( v \) with the function

\[
z(y) = \begin{cases} 
\frac{N}{1 - \theta} \left[ \left( \frac{\log |y|}{\log R} \right)^{\theta} - \theta \frac{\log |y|}{\log R} \right] - N & \text{if } R_0 < |y| < R, \\
0 & \text{if } |y| > R
\end{cases}
\]

in the open set \( \Omega_{\delta, R_0} = \Omega_{\delta} \cap \{|y| > R_0\} \); here \( \theta \) is any number in the interval \((0, 1)\), and the positive constants \( N, R_0, R \) are to be determined below, and \( R_0 > R^* \).

We shall show that \( z \) satisfies in \( \Omega_{\delta, R_0} \) the v.i.

\[
-Lz < g, \quad z < 0, \quad (-Lz - g)z = 0
\]

and that

\[
g < \gamma, \quad \gamma \text{ as in (3.10)},
\]

\[
z < -E \equiv \inf_{|y|=R_0} v \quad \text{on } |y| = R_0,
\]

\[
z < 0 = v \quad \text{on } \partial \Omega_{\delta, R_0} \cap \{|y| > R_0\}.
\]

We begin by noting that

\[
|(L - \alpha)\log|y|| < \text{const.}, \quad \left| (L - \alpha)(\log|y||)^{\theta} \right| < \frac{\text{const.}}{(\log|y|)^{1-\theta}}.
\]

Consequently, if we set \( g = -Lz \) in \( \Omega_{\delta} \cap \{|y| > R_0\} \) then

\[
g < \frac{cN}{\log R_0} + \alpha z \quad \text{in } \Omega_{\delta} \cap \{|y| < R\},
\]

where \( c \) is a constant independent of \( R_0, \delta, N \).

Next, the function \( u_{\delta} \) is bounded in \( \Omega_{\delta} \) by a constant independent of \( \delta \).

Hence \( E < N_0 \) where \( N_0 \) is a positive constant independent of \( \delta, R_0 \). We now take \( N = N_0 + 1 \), so that (3.14) is reduced to

\[
\frac{N}{1 - \theta} \left[ \left( \frac{\log R_0}{\log R} \right)^{\theta} - \theta \frac{\log R_0}{\log R} \right] < 1.
\]

Since

\[
\frac{\partial z}{\partial |y|} = \frac{N \theta}{(1 - \theta)|y|} \left( \frac{1}{(\log R)^{\theta}(\log|y|)^{1-\theta}} - \frac{1}{\log R} \right),
\]

we have \( \partial z/\partial |y| > 0 \) if \( |y| < R \), \( \partial z/\partial |y| = 0 \) if \( |y| = R \). Also \( z(y) = 0 \) if \( |y| = R \). It follows that \( z < 0 \) if \( R_0 < |y| < R \), and \( z \) (extended by zero to
$|y| > R$ is continuously differentiable in $\{|y| > R_0\}$. Thus $z$ is a $W^{1,2}$ solution of (3.12) in $\Omega_{R_0}$ provided we define

$$z = 0 \quad \text{if } |y| > R. \quad (3.18)$$

From (3.16), (3.18) we see that (3.13) is satisfied if

$$\frac{cN}{\log R_0} + \alpha z \leq \gamma. \quad (3.19)$$

The assertion (3.15) is obvious, and thus it remains to verify (3.17), (3.19). Since $z < 0$, (3.19) would follow from

$$\frac{cN}{\log R_0} \leq \gamma. \quad (3.20)$$

We now choose first $R_0$ sufficiently large so that $R_0 > R^*$ and (3.20) holds. Then we choose $R$ sufficiently large so that (3.17) is satisfied.

Having completed the construction of $z$ satisfying (3.12)--(3.15), and recalling (3.9), (3.10), we can now employ the standard comparison theorem for v.i. and conclude that $z < v$ in $\Omega_{\delta, R_0}$. Hence $u_\delta - \psi = v = 0$ in $\Omega_{\delta, R}$. Noting that $R$ was independent of $\delta$, and taking $\delta \to 0$, we obtain, after using Theorem 2.1, $u - \psi = 0$ if $|y| > R$, i.e., the set $C$ is contained in the set where $|y| < R$.

We shall next show that condition (3.4) is sharp.

**Theorem 3.2.** If $B_j < 0$ for some $j$, $2 < j < n$, then $C$ is unbounded; in fact, there exists a cone

$$K_\eta = \{ y \in R_{n-1}^+; y_i < \eta \delta_j \text{ for } 2 < i < n, i \neq j \}, \quad \eta > 0, \quad (3.21)$$

and $R > 0$ such that $C$ contains the region

$$K_\eta \cap (|y| > R). \quad (3.22)$$

**Proof.** Since $B_j < 0$, we have

$$\frac{1}{Y} \sum_{i=1}^n B_i y_i < 0 \quad \text{in some set } K_\eta \cap (|y| > R). \quad (3.23)$$

From the v.i. for $v = u - \psi$ we have

$$-Lv < \frac{1}{Y} \sum_{i=1}^n B_i y_i < 0 \quad \text{a.e. in } K_\eta \cap (|y| > R).$$

Since also $v < 0$ in this domain, the strong maximum principle gives $v < 0$ in this domain.

**4. The shape of the free boundary.** We shall need the assumptions:

$$B_i > 0 \quad \text{for } 2 < i < n, \quad (4.1)$$

$$q_{i,1} = 0 \quad \text{for } 2 < j < n. \quad (4.2)$$
Theorem 4.1. If (4.1), (4.2) hold then
\[ \frac{\partial((u - \psi)/Y)}{\partial y_j} > 0 \quad \text{for } 2 < j < n. \] (4.3)

Corollary 4.2. If (4.1), (4.2) hold then, for any \( j, 2 < j < n \), there exists a function \( \Psi_j(y_2, \ldots , y_{j-1}, y_{j+1}, \ldots , y_n) \) such that the following is true: A point \( y = (y_2, \ldots , y_n) \) belongs to \( C \) if and only if
\[ y_j < \Psi_j(y_2, \ldots , y_{j-1}, y_{j+1}, \ldots , y_n). \] (4.4)

Indeed, this assertion means that, for any \( y = (y_2, \ldots , y_n) \in C \), the point \( y' = (y_2, \ldots , y_{j-1}, y_j', y_{j+1}, \ldots , y_n) \) belongs to \( C \) if \( y'_j < y_j \). Now, at the point \( y \) we have \( u - \psi < 0 \) and therefore also \( (u - \psi)/Y < 0 \). Because of (4.3) we then also have \( (u - \psi)/Y < 0 \) at \( y' \), i.e., \( u - \psi < 0 \) at \( y' \), which implies that \( y' \in C \).

Remark. The functions \( \Psi_j \) need not be finite valued. If, however, (3.4) is satisfied then \( C \) is a bounded set and, consequently, the \( \Psi_j \) are finite valued functions.

Proof of Theorem 4.1. Set \( v = u_s - \psi \) and introduce the function \( z \) by \( v = e^h z \) where \( h = -\log Y \). The function \( z \) is continuously differentiable in \( C_s \) and twice continuously differentiable in \( C_s \). We have
\[
\frac{\partial v}{\partial y_i} = e^h \left( \frac{\partial z}{\partial y_i} - \frac{z}{Y} \right), \quad \frac{\partial^2 v}{\partial y_i \partial y_j} = e^h \left( \frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y} \frac{\partial z}{\partial y_i} - \frac{1}{Y} \frac{\partial z}{\partial y_j} + \frac{2}{Y^2} z \right).
\]

Hence, in \( C_s \),
\[
\frac{1}{2} \sum_{i,j=2}^n \mu_{ij} y_i y_j \left( \frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y} \frac{\partial z}{\partial y_i} - \frac{1}{Y} \frac{\partial z}{\partial y_j} + \frac{2}{Y^2} z \right) + \sum_{j=2}^n b_j \frac{\partial z}{\partial y_j} - \frac{z}{Y} \sum_{j=2}^n b_j - az = - \frac{1}{Y} \left( \sum_{i=1}^n B_i y_i \right) e^{-h} = - \sum_{i=1}^n B_i y_i.
\]
Applying \( \partial / \partial y_i \) and setting \( w_i = \frac{\partial z}{\partial y_i} \), we get
\[
\frac{1}{2} \sum \mu_{ij} y_i y_j \left( \frac{\partial^2 w_i}{\partial y_i \partial y_j} - \frac{2}{Y} \frac{\partial w_i}{\partial y_i} + \frac{2}{Y^2} w_i + \frac{2}{Y^2} w_i - \frac{4}{Y^3} z \right) + \sum \mu_{ij} y_i \left( \frac{\partial w_i}{\partial y_i} - \frac{1}{Y} w_i - \frac{1}{Y} w_i + \frac{2}{Y^2} z \right) - aw_i + \sum b_j \frac{\partial w_i}{\partial y_j} + \sum b_j \frac{\partial w_i}{\partial y_i} - \frac{z}{Y} \sum b_j - \frac{w_i}{Y} \sum b_j + \frac{z}{Y^2} \sum b_j = B_i. \] (4.5)

Here and in the following calculations the summation index always varies from 2 to \( n \), unless otherwise specified.
We can rewrite the system (4.5) for \(2 < l < n\) in the more compact form
\[
\frac{1}{2} \sum \mu_{ij} y_i y_j \frac{\partial^2 w_i}{\partial y_i \partial y_j} + g_i \cdot \nabla w_i - \alpha w_i + \sum Q_{l,j} w_j = -B_l - Q_l z \tag{4.6}
\]
with suitable \(g_i, Q_{l,j}, Q_l\). We shall now compute the \(Q_{l,j}, Q_l\) without imposing, as yet, the restrictions (4.1), (4.2). We shall prove that
\[
Q_l = -q_{l,1}, \tag{4.7}
\]
\[
Q_{l,j} = q_{l,j} - q_{l,1} y_j. \tag{4.8}
\]

We begin with
\[
Q_l = -\frac{2}{Y^3} \sum \mu_{ij} y_i y_j + \frac{2}{Y^2} \sum \mu_{ij} y_i + \frac{1}{Y^2} \sum b_j - \frac{1}{Y} \sum \frac{\partial b_j}{\partial y_l}. \tag{4.9}
\]

Noticing that since
\[
\sum_{j=2}^{n} \left( \sum_{k=1}^{n} q_{k,j} y_k - q_{k,1} y_k y \right) = -\sum_{k=1}^{n} q_{k,1} y_k \left( 1 + \sum_{j=2}^{n} y_j \right) = -Y \sum_{k=1}^{n} q_{k,1} y_k,
\]
we have
\[
\sum b_j = -\sum (\lambda_j - \lambda_1) \cdot \lambda_1 y_j + \sum (\lambda_j - \lambda_1) y_j \cdot \frac{\lambda_1 + \sum \lambda_j y_j}{Y} - Y \sum_{j=1}^{n} q_{j,1} y_j \tag{4.10}
\]
and
\[
\sum \frac{\partial b_j}{\partial y_l} = -(\lambda_l - \lambda_1) \cdot \lambda_1 + (\lambda_l - \lambda_1) \cdot \frac{\lambda_1 + \sum \lambda_j y_j}{Y} - \frac{1}{Y} \sum (\lambda_j - \lambda_1) y_j \cdot \left( \lambda_1 + \sum \lambda_j y_j \right) - \sum_{j=1}^{n} q_{j,1} y_j - Y q_{l,1}. \tag{4.11}
\]

Substituting from (4.10), (4.11) into (4.9), we obtain
\[
Q_l = -\frac{2}{Y^3} \sum \mu_{ij} y_i y_j + \frac{2}{Y^2} \sum (\lambda_j - \lambda_1) \cdot (\lambda_l - \lambda_1) - \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) \cdot \lambda_1 y_j
\]
\[
+ \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) \cdot (\lambda_1 + \sum \lambda_j y_j) - \frac{1}{Y} \sum_{j=1}^{n} q_{j,1} y_j + \frac{1}{Y} (\lambda_l - \lambda_1) \cdot \lambda_1
\]
\[
- \frac{1}{Y^2} (\lambda_l - \lambda_1) \cdot (\lambda_1 + \sum \lambda_j y_j) - \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \lambda_1
\]
\[
+ \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) y_j \cdot (\lambda_1 + \sum \lambda_j y_j) + \frac{1}{Y} \sum q_{j,1} y_j - q_{l,1} = \sum_{i=1}^{11} J_i.
\]

Clearly \(J_5 + J_{10} = 0, J_4 = J_9\). Substituting
\[
\lambda_1 + \sum \lambda_i y_i = \sum (\lambda_i - \lambda_1) y_i + \lambda_1 Y \tag{4.12}
\]

into \( J_4 + J_9 \) we obtain

\[
J_1 + J_4 + J_9 = J_1 + 2J_4 = \frac{2}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \lambda_1.
\]

Adding this to \( J_2 + J_3 + J_8 \) we end up with

\[
\frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot (\lambda_j - \lambda_1).
\]

Adding this to \( J_6 + J_7 \) and substituting (4.12) into \( J_7 \), we obtain the sum zero.

Hence \( Q_i = J_{11} = - q_{i,1} \).

Next, if \( i \neq j \),

\[
Q_{i,j} = \frac{1}{Y^2} \sum_i \mu_i y_i y_j - \frac{1}{Y} \mu_j y_j + \frac{\partial b_j}{\partial y_i}
\]

\[
= \frac{1}{Y^2} \sum_i \mu_i y_i y_j - \frac{1}{Y} \mu_j y_j + (\lambda_j - \lambda_1) y_j \cdot \left( \frac{\lambda_j}{Y} - \frac{1}{Y^2} \left( \lambda_1 + \sum \lambda_i y_i \right) \right)
\]

\[
+ \frac{\partial}{\partial y_i} \sum_{k=1}^n (q_{k,j} - q_{k,1} y_j) y_k
\]

\[
= \frac{1}{Y^2} \sum_i \mu_i y_i y_j - \frac{1}{Y^2} (\lambda_j - \lambda_1) \cdot (\lambda_1 + \sum \lambda_i y_i) - \frac{1}{Y} \mu_j y_j
\]

\[
+ \frac{1}{Y} (\lambda_j - \lambda_1) \cdot y_j y_i + (q_{i,j} - q_{i,1} y_i) = \sum_{i=1}^5 J_i.
\]

Substituting (4.12) into \( J_2 \) we get

\[
J_1 + J_2 = - \frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot \lambda_1
\]

\[
= \frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot (\lambda_j - \lambda_1) - \frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot \lambda_j = -(J_3 + J_4).
\]

Hence \( Q_{i,j} = J_5 = q_{i,j} - q_{i,1} y_j \).

Finally,

\[
Q_{i,1} = \frac{1}{Y^2} \sum \mu_i y_i y_j + \frac{1}{Y} \sum \mu_i y_i - \frac{1}{Y} \sum \mu_i y_i - \frac{1}{Y} \sum b_i + \frac{\partial b_i}{\partial y_i}.
\]

Using (4.10) we find that
\[
Q_{l,l} = \frac{1}{Y^2} \sum \mu_j \varphi_j \varphi_j + \frac{1}{Y^2} \sum \mu_i \varphi_i - \frac{1}{Y} \sum \mu_i \varphi_i \\
- \frac{1}{Y} \mu_i \varphi_i + \frac{1}{Y} (\lambda_j - \lambda_1) \cdot \lambda_1 \varphi_j \\
- \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) \varphi_j \cdot (\lambda_1 + \sum \lambda_i \varphi_i) + \sum q_{l,1} \varphi_j - (\lambda_j - \lambda_1) \cdot \lambda_1 \\
+ \frac{1}{Y} (\lambda_j - \lambda_1) \cdot (\lambda_1 + \sum \lambda_i \varphi_i) + \frac{1}{Y} (\lambda_j - \lambda_1) \varphi_i \cdot \lambda_i \\
- \frac{1}{Y^2} (\lambda_j - \lambda_1) \varphi_i \cdot (\lambda_1 + \sum \lambda_i \varphi_i) \\
+ \frac{\partial}{\partial \varphi_i} \left( q_{l,1} \varphi_i - \sum_{k=1}^n q_{k,1} \varphi_k \right) = \sum_{k=1}^{12} J_i.
\]

Using (4.12) in \( J_6 \) we get

\[
J_1 + J_6 = - \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) \varphi_j \cdot \lambda_1 = - J_5.
\]

Using (4.12) in \( J_{11} \) we obtain \( J_2 + J_{11} = - \frac{1}{Y} (\lambda_j - \lambda_1) \varphi_i \cdot \lambda_1 \), which together with \( J_4 + J_{10} \) add up to zero. Substituting (4.12) in \( J_7 \) we also find that \( J_7 + J_8 + J_9 = 0 \). Hence \( Q_{l,l} = J_3 + J_{12} = q_{l,1} - q_{l,1} \varphi_i \).

We now make use of the conditions (4.1), (4.2) and deduce that

\[
Q_{l,j} = q_{l,j} > 0 \quad \text{if} \; l \neq j \\
Q_{l,l} = q_{l,l} < 0, \quad \text{(4.13)}
\]

\[
\sum_{j=2}^{n} Q_{l,j} = \sum_{j=2}^{n} q_{l,j} = \sum_{j=1}^{n} q_{l,j} = 0 \quad \text{(4.14)}
\]

and the right-hand side of (4.6) is

\[
- B_l - Q_{l,z} = - B_l < 0. \quad \text{(4.15)}
\]

Since conditions (4.13) – (4.15) hold, a fairly standard maximum principle for coupled elliptic systems can be applied \([4, \text{Theorem 2.1}]\) to conclude that

\[
w_l > 0 \quad \text{in} \; C_\delta; \quad \text{(4.16)}
\]

we use here the fact that the \( w_l \) are continuous in \( \overline{C_\delta} \) and vanish on \( \partial C_\delta \), and this is true because \( u - \psi \) and its first derivatives are continuous in \( \overline{C_\delta} \) and vanish on \( \partial C_\delta \). (We should point out that Theorem 2.1 in \([4]\) deals with the case where the leading part in (4.6) is \( \Delta w_l \), but the proof of the theorem extends to any nondegenerate principal elliptic operator.)

Taking \( \delta \to 0 \) in (4.16), assertion (4.3) follows.

\textbf{Remark 1.} If \( C \) is bounded then, by Theorem 3.2, condition (4.1) must hold. Hence, if \( q_{j,1} = 0 \) for \( 2 \leq j \leq n \) and if \( C \) is bounded then the assertion
of Corollary 4.2 regarding the shape of $C$ is valid.

**Remark 2.** If for some $j$, $2 < j < n$, $B_j < 0$ then assertion (4.4) of Corollary 4.2 is false for this $j$. Indeed, this follows immediately from Theorem 3.2.

**Remark 3.** Corollary 4.2 can also be stated in terms of the shape of $C^A$. We again stipulate that the role of $p_1$ can be given to any other variable $p_i$; if $q_{j,i} = 0$ for all $j \neq i$ then an assertion similar to Corollary 4.2 is valid.

### 5. Regularity of the free boundary.

**Theorem 5.1.** If (4.1), (4.2) hold then the free boundary $\Gamma$ is analytic.

That means that one can represent $\Gamma$ locally by analytic functions $y_j = \Phi_j(\tau_1, \ldots, \tau_{n-2})$.

**Proof.** We write the v.i. for $v = u - \psi$ in the form

\[-Lvv < f, \quad v < 0, \quad (-Lv - f)v < 0 (5.1)\]

where $f(y) = \sum_{i=1}^{n-1} B_i y_i$. Without loss of generality we may assume that $f(y) = 0$ implies $\nabla f(y) \neq 0$. We shall now use an argument of Caffarelli and Rivièrè [8] to show that

\[\text{if } y^0 \in \Gamma \text{ then } f(y^0) > 0. \quad (5.2)\]

Suppose (5.2) is false for some $y^0$. Denote by $\pi$ the hyperplane passing through $y^0$ and perpendicular to $\nabla f(y^0)$, and denote by $H$ the half space bounded by $\pi$ such that $f < 0$ in $H$. Then $H \cap R_{n-1}^+$ is contained in $C$ and therefore $Lv = f < 0$, $v < 0$ on $H \cap R_{n-1}^+$. Since, however, $v(y^0) = 0$, the strong maximum principle gives $\nabla v(y^0) \neq 0$, which is impossible, because $y^0 \in \Gamma$.

The assertion (5.2) shows that $f(y) > 0$ on $\Gamma$. Therefore the regularity theorem of Caffarelli [7] for the free boundary of a v.i. can be applied to (5.1). Since the set $C$ has the shape given by Corollary 4.2, we deduce that each point of $\Gamma$ is a point of positive density with respect to the stopping set $S = R_{n-1}^+ \setminus C$. Appealing to [7] we then conclude that $\Gamma$ is analytic.

**Lemma 5.2.** If (4.1), (4.2) hold then $\Gamma$ does not contain any line segment parallel to one of the $y_j$ axes.

**Proof.** By Theorem 5.1, $u$ is a $C^\infty$ function in $C \cup \Gamma$. Suppose $\Gamma$ contains a line segment $l$ parallel to the $y_2$ coordinate axis. Then the functions $w_j = (\partial/\partial y_j)(v/Y) (j \neq 2)$ vanish along $l$. Hence

\[\frac{\partial}{\partial y_j} w_2 = \frac{\partial}{\partial y_2} w_j = 0 \quad \text{along } l, j \neq 2, \quad (5.3)\]

so that also
\[
\frac{\partial}{\partial \nu} w_2 = 0 \quad \text{on } l, \nu = \text{normal to } \Gamma \text{ at } l. \quad (5.4)
\]

Now, \( w_2 \) satisfies in \( C \) equation \((4.6)\) for \( l = 2 \), and each \( w_j \) is \( > 0 \). By the strong maximum principle, \( w_2 > 0 \) in \( C \) and, since \( w_2 = 0 \) on \( \Gamma \), \( \partial w_2 / \partial \nu \neq 0 \) on \( \Gamma \). This contradicts \((5.4)\).

If we use the fact that the free boundary is analytic, then we can extend the proof of Lemma 5.2 to the case where \( l \) does not actually lie on \( \Gamma \) but is just tangent to \( \Gamma \) at some point \( y^0 \). (The relations \((5.3)\), \((5.4)\) are then valid at \( y^0 \).

We can therefore assert:

**Theorem 5.3.** Let \((4.1)\), \((4.2)\) hold. Then, for any \( j, 2 \leq j \leq n \), \( \Gamma \) can be represented in the form

\[
y_j = \Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \quad (5.5)
\]

for \((y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)\) in some bounded domain \( A_j \), and

\[
\frac{\partial \Psi_j}{\partial y_i} < 0 \quad \text{for each } i; \quad i = 2, \ldots, j-1, j+1, \ldots, n. \quad (5.6)
\]


**Remark 1.** In the special case where

\[
q_{i,j} = 0 \quad \text{whenever } j < i, \quad (6.1)
\]
a more general quality control problem was studied in [2] in which \( K \) was replaced by \( K_1, \ldots, K_n \). The corresponding q.v.i. is then replaced by \( n-1 \) q.v.i. for functions \( w_{n-i}(p_{n-i}, p_{n-i+1}, \ldots, p_n) (p_j > 0, \sum_{j=n-i}^{n} p_j = 1)\):

\[
M_{n-i} w_{n-i} = \frac{1}{2} \sum_{j,k=n-i}^{n} p_j p_k \left( \lambda_j - \sum_{l=n-i}^{n} \lambda_l p_l \right) \cdot \left( \lambda_k - \sum_{l=n-i}^{n} \lambda_l p_l \right) \frac{\partial^2 w_{n-i}}{\partial p_j \partial p_k} + \sum_{j,k=n-i}^{n} q_{j,k} p_j p_k \frac{\partial w_{n-i}}{\partial p_k} > - \sum_{j=n-i}^{n} c_j p_j, \]

\[
w_{n-i} < K_{n-i} + \sum_{j=n-i}^{n-1} p_j w_j(e_j),
\]

\[
\left( M_{n-i} w_{n-i} + \sum_{j=n-i}^{n} c_j p_j \right) \left( w_{n-i} - K_{n-i} - \sum_{j=n-i}^{n-1} p_j w_j(e_j) \right) = 0 \quad (6.2)
\]

where \( e_j = (p_j, \ldots, p_n) = (1, 0, \ldots, 0) \). It is natural to assume in this quality control problem that

\[
K_1 > K_2 > \cdots > K_n. \quad (6.3)
\]

We now define the \( B_j \) as in \((3.3)\), but with \( K = K_1 \), \( u_j = w_j(e_j) \), so that, in view of \((6.1)\),
\[ B_i = c_i + \sum_{j=i}^{n-1} q_{i,j} u_j - \alpha u_i - \alpha K_1 \quad (1 \leq i < n-1), \]

\[ B_n = c_n - \alpha K_1. \]  

The results of §§3–5 extend immediately to the q.v.i. (6.2). Taking note of condition (6.3) we conclude that under exactly the same conditions on the \( B_j \) as in §§3–5 we have precisely the same assertions for the continuation regions \( C = C_{n-i} \) and for the free boundaries \( \Gamma = \Gamma_{n-i} \) of the q.v.i. (6.2), \( 1 \leq i < n-1 \).

**Remark 2.** In case \( n = 2 \) the system (4.6) consists of just one equation. If \( q_{2,1} \neq 0 \) then \( q_{2,1} > 0 \) so that \( Q_{1,1} = q_{2,2} - q_{2,1} y_2 < 0 \) and \(- B_1 - Q_1 z = - B_2 + q_{2,1} z < 0 \) since \( B_2 > 0, z < 0 \). Thus the maximum principle gives \( w_i = w_2 > 0 \). We conclude that, if \( n = 2 \), the results of §§4, 5 remain valid without imposing the restriction \( q_{2,1} = 0 \).

**Remark 3.** Denote by \( w_\alpha(p), J^p_\alpha(\tau; \alpha) \) and \( u_{j, \alpha} \) the functions \( w(p), J^p_\alpha(\tau), u_j \) as functions of the parameter \( \alpha, \alpha > 0 \), and set

\[ B_i^* = c_i + \sum_{j=1}^{n-1} q_{i,j} u_{j,0}. \]  

It is clear that \( J^p_\alpha(\tau, \alpha) \uparrow J^p_\alpha(\tau, 0) \) as \( \alpha \downarrow 0 \) and that

\[ w_\alpha(p) \uparrow w_0(p), \quad u_{j, \alpha} \uparrow u_{j,0} \]  

as \( \alpha \downarrow 0 \). (6.6)

Suppose

\[ u_{j,0} < \infty \quad \text{for} \quad 1 \leq j \leq n-1. \]  

(6.7)

Then clearly,

\[ B_i^* > 0 \implies B_i > 0 \quad \text{if} \quad \alpha \text{ is sufficiently small}, \]  

(6.8)

so that the results of §§3–5 can be applied by imposing the simpler conditions

\[ B_i^* > 0 \quad (2 \leq i < n) \]  

(6.9)

provided \( \alpha \) is sufficiently small.

We claim that (6.7) is true if either (6.1) holds or

\[ q_{n,n} = 0, \quad q_{i,n} > 0 \quad \text{for} \quad 1 \leq i < n-1. \]  

(6.10)

Indeed, as shown in [2], any one of these conditions implies \( P[\theta(t) \neq n] \to 0 \) as \( t \to \infty \). Hence, by the Markov property,

\[ P[\theta(t) \neq n] \leq e^{-\gamma t} \quad \text{for some} \quad \gamma > 0. \]

This implies that \( J^p_\alpha(\tilde{\tau}, 0) \leq B < \infty \) where \( \tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2, \ldots), \tilde{\tau}_j = j, \) and \( B \) is a constant independent of \( p, \alpha, \) and (6.7) follows.

**Remark 4.** In case (6.1) holds, the system (4.6) for the unknown functions, say \( \tilde{w}_n \), is not coupled and we can get additional results by applying the maximum principle first to \( \tilde{w}_n \), then to \( \tilde{w}_{n-1} \), etc. For instance, if \( B_n > 0 \) then
\( \bar{w}_n > 0; \) if also \( B_{n-1} > 0 \) then also \( \bar{w}_{n-1} > 0. \)

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