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ON THE FREE BOUNDARY OF A QUASI VARIATIONAL INEQUALITY ARISING IN A PROBLEM OF QUALITY CONTROL[1]

BY

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ABSTRACT. In some recent work in stochastic optimization with partial observation occurring in quality control problems, Anderson and Friedman [1], [2] have shown that the optimal cost can be determined as a solution of the quasi variational inequality

$$Mw(p) + f(p) \geq 0, \quad w(p) \leq \psi(p; w),$$
$$(Mw(p) + f(p))(w(p) - \psi(p; w)) = 0$$

in the simplex $p_i > 0$, $\Sigma_{i=1}^n p_i = 1$. Here f, ψ are given functions of p, ψ is a functional of w, and M is a given elliptic operator degenerating on the boundary. This system has a unique solution when M does not degenerate in the interior of the simplex. The aim of this paper is to study the free boundary, that is, the boundary of the set where $w(p) < \psi(p; w)$.

1. **Introduction.** In the model considered by Anderson and Friedman [1], [2] one is interested in finding an optimal sequence of increasing inspection times τ_l which minimize the cost function

$$J_x^p(\tau) \equiv E_x^p\left[Ke^{-\alpha\tau_1} + \int_0^{\tau_1} f(\theta(s))e^{-\alpha s}\,ds + \sum_{l=1}^{\infty} I_{\theta(\tau_l)\neq n}\left[Ke^{-\alpha\tau_{l+1}} + \int_{\tau_l}^{\tau_{l+1}} f(\theta(s))e^{-\alpha s}\,ds\right]\right]; \quad (1.1)$$

here $\theta(s)$ is a Markov process with n states $1,2,\ldots,n$ and Q-matrix $(q_{i,j})$; $f(i) = c_i \geq 0$, $K > 0$, $\alpha > 0$, and the τ_l depend only on the information given by $\theta(\tau_1), \ldots, \theta(\tau_{l-1})$ and the σ-fields $\mathcal{F}_t$ of the process $x(t)$ which is defined as follows: Let $w(t) + \lambda_i t$ be a ν-dimensional Brownian motion with drift λ_i $(1 \leq i \leq n)$; then $x(t)$ is the random evolution of these n diffusion processes in accordance with $\theta(t)$. Finally, $p = (p_1, \ldots, p_n)$ is the initial distribution of $\theta(t)$, and $x = x(0)$.

The problem of finding

$$w(x,p) = \inf J_x^p(\tau) \quad (1.2)$$

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and characterizing an optimal sequence of inspections $\tau = \tau^* = (\tau_1^*, \tau_2^*, \dots)$ is called a *quality control problem*. The motivation for this problem is explained in detail in [1], [2].

It is shown in [2] that $w(x, p)$ is independent of x. Further, the problem of finding $w = w(p)$ and τ^* is reduced to the problem of solving a quasi variational inequality (q.v.i.) of the form

$$Mw + \sum_{j=1}^{n} c_j p_j \geqslant 0, \qquad w(p) \leqslant K + \sum_{j=1}^{n-1} w(e_j) p_j,$$

$$\left(Mw + \sum_{j=1}^{n} c_j p_j\right)\left(w(p) - K - \sum_{j=1}^{n-1} w(e_j) p_j\right) = 0 \tag{1.3}$$

in the set $A = \{p_i > 0, \sum_{i=1}^{n} p_i = 1\}$. Here $e_j = (\delta_{j,1}, \dots, \delta_{j,n})$ and M is an elliptic operator degenerating on ∂A. The q.v.i. is solved in [2] under the assumption that M is nondegenerate in (the interior of) A. In §2 we recall this fact and also state some other results from [2] in a form which will be useful for the subsequent sections.

The aim of the present paper is to study the set

$$C^A = \left\{ p;\, w(p) < K + \sum_{j=1}^{n-1} w(e_j) p_j \right\} \tag{1.4}$$

and the free boundary $\Gamma^A = \partial C^A \cap A$. For this purpose it is convenient to make a change of coordinates $y_j = p_j / p_1$ and to transform the q.v.i. into a q.v.i. in the space

$$R_{n-1}^{+} = \{(y_2, \dots, y_n);\, y_i > 0 \text{ for } 2 \leqslant i \leqslant n\}.$$

Then C^A and Γ^A are transformed into sets which we designate by C and Γ respectively.

In §3 we find a sharp condition for the set C to be bounded. In §4 we prove that, when C is bounded, Γ is a graph, monotone in each variable, i.e., a point $(y_2, \dots, y_n)$ belongs to C if and only if

$$y_j < \Psi_j(y_2, \dots, y_{j-1}, y_{j+1}, \dots, y_n)$$

where Ψ_j is a finite valued function. In §5 we prove that Γ is given by $y_j = \Psi_j(y_2, \dots, y_{j-1}, y_{j+1}, \dots, y_n)$, the Ψ_j are analytic, and $\partial \Psi_j / \partial y_i < 0$. Some concluding remarks are given in §6.

For a variational inequality (v.i.) for a function u and an obstacle ψ, the support of the solution is, by definition, the closure of the set $\{u < \psi\}$. The question of compact support of solutions of v.i. was first studied by Brezis [6]. Recent results on the support of solutions of some q.v.i. have been obtained in [3] and [4].

2. The q.v.i. Let

$$A = \left\{ (p_1, \ldots, p_n); p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}$$

and let $\lambda_1, \ldots, \lambda_n$ be distinct ν-dimensional vectors such that

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots, \lambda_n - \lambda_1 \tag{2.1}$$

are linearly independent; this condition implies, of course, that $\nu \geqslant n - 1$. Let $q_{i,j}$ $(1 \leqslant i, j \leqslant n)$ be real numbers satisfying:

$$q_{i,j} \geqslant 0 \quad \text{if } i \neq j, \qquad \sum_{j=1}^{n} q_{i,j} = 0. \tag{2.2}$$

Finally, let K and α be positive numbers and let $c_1, \ldots, c_n$ be nonnegative numbers. Introduce the elliptic operator in A:

$$Mw(p) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left(\lambda_i - \sum_{l=1}^{n} \lambda_l p_l \right) \cdot \left(\lambda_j - \sum_{l=1}^{n} \lambda_l p_l \right) \frac{\partial^2 w(p)}{\partial p_i \partial p_j}$$
$$+ \sum_{i,j=1}^{n} q_{i,j} p_i \frac{\partial w(p)}{\partial p_j} - \alpha w(p). \tag{2.3}$$

Note that any $n - 1$ of the p_i's can be taken as independent variables; the remaining p_i, say p_{i_0}, is then given by $1 - \sum_{i \neq i_0} p_i$.

We shall be interested in the q.v.i. (1.3) in the set A, where $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the jth component. As easily seen (see [2]) M is nondegenerate in (the interior of) A if and only if condition (2.1) holds. M is degenerate on all of ∂A.

THEOREM 1.1 [2]. *There exists a unique solution w of* (1.3) *such that*

$$w \in C(\bar{A}) \cap W^{2,r}_{\text{loc}}(A) \quad \textit{for all } 1 < r < \infty. \tag{2.4}$$

We recall that $w(p) > 0$ if $p \in \bar{A}, p \neq e_n$.

From (2.4) it follows that $w(p)$ is continuously differentiable in A. The set C^A, defined by (1.4), is an open subset of A; it is called the *domain of continuation*. The set $\Gamma^A = \partial C^A \cap A$ (∂C^A = boundary of C^A) is called the *free boundary*, and the set

$$S^A = \left\{ p \in A;\ w(p) = K + \sum_{j=1}^{n-1} w(e_j) p_j \right\}$$

is called the *stopping set*. As shown in [2], the optimal inspections are performed when a certain process $p(t)$, given explicitly in terms of the process $x(t)$, exits the set C^A; this explains the terminology of C^A, S^A.

It will be convenient to use Cartesian coordinates $y_i = p_i / p_1$ $(2 \leqslant i \leqslant n)$;

here the role of p_1 is incidental; p_1 may be replaced by any other fixed variable p_k. Since $Y \equiv 1 + y_2 + \cdots + y_n = 1 + (p_2 + \cdots + p_n)/p_1 = 1/p_1$, we have $p_i = y_i/Y$ $(2 \leqslant i \leqslant n)$.

Define R^+_{n-1} by (1.5) and set $u(y) = w(p), y_1 \equiv 1$. Then (see [2]) $Mw(p) = Lu(y)$ where

$$Lu(y) = \frac{1}{2} \sum_{i,j=2}^{n} \mu_{ij} y_i y_j \frac{\partial^2 u(y)}{\partial y_i \partial y_j} + \sum_{j=2}^{n} b_j(y) \frac{\partial u(y)}{\partial y_j} - \alpha u(y) \tag{2.5}$$

where

$$\mu_{ij} = (\lambda_i - \lambda_1) \cdot (\lambda_j - \lambda_1), \tag{2.6}$$

$$b_j(y) = -(\lambda_j - \lambda_1) \cdot \lambda_1 y_j + (\lambda_j - \lambda_1) y_j \cdot \frac{\sum_{i=1}^{n} \lambda_i y_i}{Y} + \sum_{i=1}^{n} (q_{i,j} - q_{i,1}) y_i. \tag{2.7}$$

The q.v.i. (1.3) transforms into

$$Lu(y) + \frac{1}{Y} \sum_{j=1}^{n} c_j y_j \geqslant 0, \qquad u(y) \leqslant K + \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j,$$

$$\left(Lu(y) + \frac{1}{Y} \sum_{j=1}^{n} c_j y_j \right) \left(u(y) - K - \frac{1}{Y} \sum_{j=1}^{n-1} u_j y_j \right) = 0 \tag{2.8}$$

in R^+_{n-1}, where

$$u_j = w(e_j) \qquad (1 \leqslant j \leqslant n-1). \tag{2.9}$$

Let $\tilde{\Omega}_\delta$ be any family of bounded domains with smooth boundary $\partial \tilde{\Omega}_\delta$ such that $(\tilde{\Omega}_\delta \cup \partial \tilde{\Omega}_\delta) \subset A$, $\tilde{\Omega}_\delta \uparrow A$ as $\delta \downarrow 0$. Set

$$\tilde{\psi}(p) = K + \sum_{j=1}^{n-1} w(e_j) p_j. \tag{2.10}$$

For any $\varepsilon > 0$ consider the elliptic problem

$$Mw_{\varepsilon,\delta} - \frac{1}{\varepsilon} \left(w_{\varepsilon,\delta} - \tilde{\psi} \right)^+ + \sum_{j=1}^{n} c_j p_j = 0 \quad \text{in } \tilde{\Omega}_\delta,$$

$$w_{\varepsilon,\delta} = 0 \quad \text{on } \partial \tilde{\Omega}_\delta. \tag{2.11}$$

Since M is nondegenerate in the closure of $\tilde{\Omega}_\delta$, this problem has a unique solution. As shown in [2] (see also [5])

$$w_{\varepsilon,\delta} \to w_\varepsilon \quad \text{as } \delta \to \infty, \qquad w_\varepsilon \to w \quad \text{as } \varepsilon \to 0 \tag{2.12}$$

uniformly in compact subsets of A. The proof exploits the probabilistic interpretation of $w_{\varepsilon,\delta}$ as given in [5]. One can also prove that

$$w_{\varepsilon,\delta} \to w^*_\delta \quad \text{as } \varepsilon \to 0, \qquad w^*_\delta \to w \quad \text{as } \delta \to 0 \tag{2.13}$$

uniformly in compact subsets of A. In fact, the proof (which is similar to the proof of (2.12) in [2]) exploits the standard representation of w_δ^* (as a solution of a v.i. in $\tilde{\Omega}_\delta$ with zero Dirichlet data) and the fact that

if τ_δ^* = exit time of the process $p(t)$ from Ω_δ, then $\tau_\delta^* \to \infty$ as $\delta \to 0$.

The above result (2.13) is valid (with obvious changes in the proof) if we replace the boundary conditions $w_{\varepsilon,\delta} = w_\delta^* = 0$ on $\partial\tilde{\Omega}_\delta$ by the boundary conditions $w_{\varepsilon,\delta} = w_\delta^* = g$ where g is any bounded continuous function such that $g(p) \leqslant K + \Sigma_{j=1}^{n-1} u_j p_j$. Taking, in particular, $g(p) = K + \Sigma_{j=1}^{n-1} u_j p_j$ and going into the y-coordinates, we conclude:

THEOREM 2.2. *Let Ω_δ be a family of bounded domains with smooth boundary $\partial\Omega_\delta$ such that*

$$(\Omega_\delta \cup \partial\Omega_\delta) \subset R_{n-1}^+, \quad \Omega_\delta \uparrow R_{n-1}^+ \quad \text{as } \delta \downarrow 0.$$

Let u_δ be the solution of the v.i. (2.8) *in Ω_δ with*

$$u_\delta = K + \frac{1}{Y}\sum_{j=1}^{n-1} u_j y_j \quad \text{on } \partial\Omega_\delta$$

(*where the u_j are given by* (2.9)). *Then $u_\delta(y) \to u(y)$ as $\delta \to 0$, uniformly in compact subsets of R_{n-1}^+.*

Notice that $u_\delta \in W^{2,r}(\Omega_\delta)$ for any $1 < r < \infty$. Consequently, u_δ is continuously differentiable in $\overline{\Omega}_\delta$.

Later on we shall use the notation

$$\psi(y) = K + \frac{1}{Y}\sum_{j=1}^{n-1} u_j y_j, \qquad y_1 \equiv 1, \tag{2.14}$$

$$C_\delta = \{y \in \Omega_\delta;\ u_\delta(y) < \psi(y)\}. \tag{2.15}$$

3. **Boundedness of the domain of continuation.** In the y-space, the domain of continuation C is given by

$$C = \{y \in R_{n-1}^+;\ u(y) < \psi(y)\}. \tag{3.1}$$

In this section we shall prove, under some sharp conditions, that C is a bounded set. That means that

$$\overline{C^A} \text{ does not intersect the set } p_1 = 0. \tag{3.2}$$

Notice that since $u(0) < K + u(0) = K + u_1 = \psi(0)$, C contains an R_{n-1}^+-neighborhood of the origin.

We introduce the numbers

$$B_i = c_i + \sum_{j=1}^{n-1} q_{i,j}u_j - \alpha u_i - \alpha K \qquad (1 \leq i \leq n-1),$$

$$B_n = c_n + \sum_{j=1}^{n-1} q_{n,j}u_j - \alpha K. \tag{3.3}$$

THEOREM 3.1. *The set C is bounded if*

$$B_i > 0 \quad \text{for } 2 \leq i \leq n. \tag{3.4}$$

PROOF. From (2.3) we get $Mp_l = \sum_{i=1}^{n} q_{i,l}p_i - \alpha p_l$. In terms of the y-coordinates we then have

$$L\left(\frac{y_l}{Y}\right) = \frac{1}{Y}\left(\sum_{i=1}^{n} q_{i,l}y_i - \alpha y_l\right)$$

with the usual convention that $y_1 = 1$.

It follows that

$$L\psi = -\alpha K + \frac{1}{Y}\sum_{j=1}^{n-1} u_j\left(\sum_{i=1}^{n} q_{i,j}y_i - \alpha y_j\right) = \frac{1}{Y}\sum_{i=1}^{n} \beta_i y_i \tag{3.5}$$

where

$$\beta_i = \sum_{j=1}^{n-1} q_{i,j}u_j - \alpha u_i - \alpha K \qquad (1 \leq i \leq n-1),$$

$$\beta_n = \sum_{j=1}^{n-1} q_{n,j}u_j - \alpha K. \tag{3.6}$$

Hence

$$\frac{1}{Y}\sum_{i=1}^{n} c_i y_i + L\psi = \frac{1}{Y}\sum_{i=1}^{n}(c_i + \beta_i)y_i = \frac{1}{Y}\sum_{i=1}^{n} B_i y_i, \tag{3.7}$$

by Definition (3.3).

Set

$$v = u_\delta - \psi. \tag{3.8}$$

Then v is a solution, in Ω_δ, of the v.i.

$$-Lv \leq \frac{1}{Y}\sum_{i=1}^{n} B_i y_i, \qquad v \leq 0,$$

$$\left(-Lv - \frac{1}{Y}\sum_{i=1}^{n} B_i y_i\right)v = 0, \qquad v = 0 \text{ on } \partial\Omega_\delta. \tag{3.9}$$

The assumption (3.4) implies that there exist positive constants R^*, γ such that

$$\frac{1}{Y}\sum_{i=1}^{n} B_i y_i \geq \gamma \quad \text{if } |y| \geq R^*. \tag{3.10}$$

We shall compare v with the function

$$z(y) = \begin{cases} \dfrac{N}{1-\theta}\left[\left(\dfrac{\log|y|}{\log R}\right)^{\theta} - \theta\dfrac{\log|y|}{\log R}\right] - N & \text{if } R_0 < |y| < R, \\ 0 \quad \text{if } |y| > R \end{cases} \tag{3.11}$$

in the open set $\Omega_{\delta, R_0} = \Omega_\delta \cap \{|y| > R_0\}$; here θ is any number in the interval (0, 1), and the positive constants N, R_0, R are to be determined below, and $R_0 > R^*$.

We shall show that z satisfies in Ω_{δ, R_0} the v.i.

$$-Lz \leq g, \qquad z \leq 0, \qquad (-Lz - g)z = 0 \tag{3.12}$$

and that

$$g < \gamma, \qquad \gamma \text{ as in (3.10)}, \tag{3.13}$$

$$z \leq -E \equiv \inf_{|y| = R_0} v \quad \text{on } |y| = R_0, \tag{3.14}$$

$$z \leq 0 = v \quad \text{on } \partial\Omega_{\delta, R_0} \cap \{|y| > R_0\}. \tag{3.15}$$

We begin by noting that

$$|(L-\alpha)\log|y|| \leq \text{const.}, \qquad \left|(L-\alpha)(\log|y|)^{\theta}\right| \leq \frac{\text{const.}}{(\log|y|)^{1-\theta}}.$$

Consequently, if we set $g = -Lz$ in $\Omega_\delta \cap \{R_0 < |y| < R\}$ then

$$g \leq \frac{cN}{\log R_0} + \alpha z \quad \text{in } \Omega_\delta \cap \{R_0 < |y| < R\}, \tag{3.16}$$

where c is a constant independent of R_0, δ, N.

Next, the function u_δ is bounded in Ω_δ by a constant independent of δ. Hence $E \leq N_0$ where N_0 is a positive constant independent of δ, R_0. We now take $N = N_0 + 1$, so that (3.14) is reduced to

$$\frac{N}{1-\theta}\left[\left(\frac{\log R_0}{\log R}\right)^{\theta} - \theta\frac{\log R_0}{\log R}\right] \leq 1. \tag{3.17}$$

Since

$$\frac{\partial z}{\partial|y|} = \frac{N\theta}{(1-\theta)|y|}\left(\frac{1}{(\log R)^{\theta}(\log|y|)^{1-\theta}} - \frac{1}{\log R}\right),$$

we have $\partial z/\partial|y| > 0$ if $|y| < R$, $\partial z/\partial|y| = 0$ if $|y| = R$. Also $z(y) = 0$ if $|y| = R$. It follows that $z < 0$ if $R_0 < |y| < R$, and z (extended by zero to

$|y| > R$) is continuously differentiable in $\{|y| > R_0\}$. Thus z is a $W^{1,2}$ solution of (3.12) in Ω_{δ, R_0} provided we define

$$z = 0 \quad \text{if } |y| > R. \tag{3.18}$$

From (3.16), (3.18) we see that (3.13) is satisfied if

$$\frac{cN}{\log R_0} + \alpha z \leqslant \gamma. \tag{3.19}$$

The assertion (3.15) is obvious, and thus it remains to verify (3.17), (3.19). Since $z \leqslant 0$, (3.19) would follow from

$$\frac{cN}{\log R_0} \leqslant \gamma. \tag{3.20}$$

We now choose first R_0 sufficiently large so that $R_0 > R^*$ and (3.20) holds. Then we choose R sufficiently large so that (3.17) is satisfied.

Having completed the construction of z satisfying (3.12)–(3.15), and recalling (3.9), (3.10), we can now employ the standard comparison theorem for v.i. and conclude that $z \leqslant v$ in Ω_{δ, R_0}. Hence $u_\delta - \psi = v = 0$ in $\Omega_{\delta, R}$. Noting that R was independent of δ, and taking $\delta \to 0$, we obtain, after using Theorem 2.1, $u - \psi = 0$ if $|y| > R$, i.e., the set C is contained in the set where $|y| < R$.

We shall next show that condition (3.4) is sharp.

THEOREM 3.2. *If $B_j < 0$ for some j, $2 \leqslant j \leqslant n$, then C is unbounded; in fact, there exists a cone*

$$K_\eta = \{y \in R^+_{n-1}; y_i < \eta y_j \text{ for } 2 \leqslant i \leqslant n, i \neq j\}, \quad \eta > 0, \tag{3.21}$$

and $R > 0$ such that C contains the region

$$K_\eta \cap (|y| > R). \tag{3.22}$$

PROOF. Since $B_j < 0$, we have

$$\frac{1}{Y} \sum_{i=1}^{n} B_i y_i < 0 \quad \text{in some set } K_\eta \cap (|y| > R). \tag{3.23}$$

From the v.i. for $v = u - \psi$ we have

$$-Lv \leqslant \frac{1}{Y} \sum_{i=1}^{n} B_i y_i < 0 \quad \text{a.e. in } K_\eta \cap (|y| > R).$$

Since also $v \leqslant 0$ in this domain, the strong maximum principle gives $v < 0$ in this domain.

4. **The shape of the free boundary.** We shall need the assumptions:

$$B_i \geqslant 0 \quad \text{for } 2 \leqslant i \leqslant n, \tag{4.1}$$

$$q_{j,1} = 0 \quad \text{for } 2 \leqslant j \leqslant n. \tag{4.2}$$

THEOREM 4.1. *If* (4.1), (4.2) *hold then*

$$\partial((u-\psi)/Y)/\partial y_j \geqslant 0 \quad \text{for } 2 \leqslant j \leqslant n. \tag{4.3}$$

COROLLARY 4.2. *If* (4.1), (4.2) *hold then, for any* j, $2 \leqslant j \leqslant n$, *there exists a function* $\Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$ *such that the following is true: A point* $y = (y_2, \ldots, y_n)$ *belongs to* C *if and only if*

$$y_j < \Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n). \tag{4.4}$$

Indeed, this assertion means that, for any $y = (y_2, \ldots, y_n) \in C$, the point $y' = (y_2, \ldots, y_{j-1}, y_j', y_{j+1}, \ldots, y_n)$ belongs to C if $y_j' < y_j$. Now, at the point y we have $u - \psi < 0$ and therefore also $(u-\psi)/Y < 0$. Because of (4.3) we then also have $(u-\psi)/Y < 0$ at y', i.e., $u - \psi < 0$ at y', which implies that $y' \in C$.

REMARK. The functions Ψ_j need not be finite valued. If, however, (3.4) is satisfied then C is a bounded set and, consequently, the Ψ_j are finite valued functions.

PROOF OF THEOREM 4.1. Set $v = u_\delta - \psi$ and introduce the function z by $v = e^h z$ where $h = -\log Y$. The function z is continuously differentiable in $\overline{C_\delta}$ and twice continuously differentiable in C_δ. We have

$$\frac{\partial v}{\partial y_i} = e^h\left(\frac{\partial z}{\partial y_i} - \frac{z}{Y}\right), \quad \frac{\partial^2 v}{\partial y_i \partial y_j} = e^h\left(\frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y}\frac{\partial z}{\partial y_i} - \frac{1}{Y}\frac{\partial z}{\partial y_j} + \frac{2}{Y^2} z\right).$$

Hence, in C_δ,

$$\frac{1}{2}\sum_{i,j=2}^{n} \mu_{ij} y_i y_j \left(\frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y}\frac{\partial z}{\partial y_i} - \frac{1}{Y}\frac{\partial z}{\partial y_j} + \frac{2}{Y^2} z\right)$$
$$+ \sum_{j=2}^{n} b_j \frac{\partial z}{\partial y_j} - \frac{z}{Y}\sum_{j=2}^{n} b_j - \alpha z = -\frac{1}{Y}\left(\sum_{i=1}^{n} B_i y_i\right)e^{-h} = -\sum_{i=1}^{n} B_i y_i.$$

Applying $\partial/\partial y_l$ and setting $w_l = \dfrac{\partial z}{\partial y_l}$, we get

$$\frac{1}{2}\sum \mu_{ij} y_i y_j \left(\frac{\partial^2 w_l}{\partial y_i \partial y_j} - \frac{2}{Y}\frac{\partial w_l}{\partial y_i} + \frac{2}{Y^2} w_l + \frac{2}{Y^2} w_i - \frac{4}{Y^3} z\right)$$
$$+ \sum \mu_{il} y_i \left(\frac{\partial w_l}{\partial y_i} - \frac{1}{Y} w_i - \frac{1}{Y} w_l + \frac{2}{Y^2} z\right) - \alpha w_l$$
$$+ \sum b_j \frac{\partial w_l}{\partial y_j} + \sum \frac{\partial b_j}{\partial y_l} w_j - \frac{z}{Y}\sum \frac{\partial b_j}{\partial y_l} - \frac{w_l}{Y}\sum b_j + \frac{z}{Y^2}\sum b_j = B_l. \tag{4.5}$$

Here and in the following calculations the summation index always varies from 2 to n, unless otherwise specified.

We can rewrite the system (4.5) for $2 \leqslant l \leqslant n$ in the more compact form

$$\frac{1}{2} \sum \mu_{ij} y_i y_j \frac{\partial^2 w_l}{\partial y_i \partial y_j} + g_l \cdot \nabla w_l - \alpha w_l + \sum Q_{l,j} w_j = -B_l - Q_l z \tag{4.6}$$

with suitable g_l, $Q_{l,j}$, Q_l. We shall now compute the $Q_{l,j}$, Q_l without imposing, as yet, the restrictions (4.1), (4.2). We shall prove that

$$Q_l = -q_{l,1}, \tag{4.7}$$

$$Q_{l,j} = q_{l,j} - q_{l,1} y_j. \tag{4.8}$$

We begin with

$$Q_l = -\frac{2}{Y^3} \sum \mu_{ij} y_i y_j + \frac{2}{Y^2} \sum \mu_{il} y_i + \frac{1}{Y^2} \sum b_j - \frac{1}{Y} \sum \frac{\partial b_j}{\partial y_l}. \tag{4.9}$$

Noticing that since

$$\sum_{j=2}^{n} \left(\sum_{k=1}^{n} q_{k,j} y_k - q_{k,1} y_j y_k \right) = -\sum_{k=1}^{n} q_{k,1} y_k \left(1 + \sum_{j=2}^{n} y_j \right) = -Y \sum_{k=1}^{n} q_{k,1} y_k,$$

we have

$$\sum b_j = -\sum (\lambda_j - \lambda_1) \cdot \lambda_1 y_j + \sum (\lambda_j - \lambda_1) y_j \cdot \frac{\lambda_1 + \sum \lambda_i y_i}{Y} - Y \sum_{j=1}^{n} q_{j,1} y_j \tag{4.10}$$

and

$$\sum \frac{\partial b_j}{\partial y_l} = -(\lambda_l - \lambda_1) \cdot \lambda_1 + (\lambda_l - \lambda_1) \cdot \frac{\lambda_1 + \sum \lambda_i y_i}{Y} - \frac{\sum (\lambda_j - \lambda_1) y_j \cdot \lambda_l}{Y}$$
$$- \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \left(\lambda_1 + \sum \lambda_i y_i \right) - \sum_{j=1}^{n} q_{j,1} y_j - Y q_{l,1}. \tag{4.11}$$

Substituting from (4.10), (4.11) into (4.9), we obtain

$$Q_l = -\frac{2}{Y^3} \sum \mu_{ij} y_i y_j + \frac{2}{Y^2} \sum (\lambda_j - \lambda_1) \cdot (\lambda_l - \lambda_1) - \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) \cdot \lambda_1 y_j$$
$$+ \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) \cdot \left(\lambda_1 + \sum \lambda_i y_i \right) - \frac{1}{Y} \sum_{j=1}^{n} q_{j,1} y_j + \frac{1}{Y} (\lambda_l - \lambda_1) \cdot \lambda_1$$
$$- \frac{1}{Y^2} (\lambda_l - \lambda_1) \cdot \left(\lambda_1 + \sum \lambda_i y_i \right) - \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \lambda_1$$
$$+ \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) y_j \cdot \left(\lambda_1 + \sum \lambda_i y_i \right) + \frac{1}{Y} \sum q_{j,1} y_j - q_{l,1} = \sum_{i=1}^{11} J_i.$$

Clearly $J_5 + J_{10} = 0$, $J_4 = J_9$. Substituting

$$\lambda_1 + \sum \lambda_i y_i = \sum (\lambda_i - \lambda_1) y_i + \lambda_1 Y \tag{4.12}$$

into $J_4 + J_9$ we obtain

$$J_1 + J_4 + J_9 = J_1 + 2J_4 = \frac{2}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \lambda_1.$$

Adding this to $J_2 + J_3 + J_8$ we end up with

$$\frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot (\lambda_l - \lambda_1).$$

Adding this to $J_6 + J_7$ and substituting (4.12) into J_7, we obtain the sum zero. Hence $Q_l = J_{11} = -q_{l,1}$.

Next, if $l \neq j$,

$$\begin{aligned} Q_{l,j} &= \frac{1}{Y^2} \sum_i \mu_{ij} y_i y_j - \frac{1}{Y} \mu_{lj} y_j + \frac{\partial b_j}{\partial y_l} \\ &= \frac{1}{Y^2} \sum_i \mu_{ij} y_i y_j - \frac{1}{Y} \mu_{lj} y_j + (\lambda_j - \lambda_1) y_j \cdot \left(\frac{\lambda_l}{Y} - \frac{1}{Y^2} \left(\lambda_1 + \sum \lambda_i y_i\right)\right) \\ &\quad + \frac{\partial}{\partial y_l} \sum_{k=1}^{n} (q_{k,j} - q_{k,1} y_j) y_k \\ &= \frac{1}{Y^2} \sum_i \mu_{ij} y_i y_j - \frac{1}{Y^2} (\lambda_j - \lambda_1) \cdot \left(\lambda_1 + \sum \lambda_i y_i\right) - \frac{1}{Y} \mu_{lj} y_j \\ &\quad + \frac{1}{Y} (\lambda_j - \lambda_1) \cdot y_j \lambda_l + (q_{l,j} - q_{l,1} y_i) = \sum_{i=1}^{5} J_i. \end{aligned}$$

Substituting (4.12) into J_2 we get

$$\begin{aligned} J_1 + J_2 &= -\frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot \lambda_1 \\ &= \frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot (\lambda_l - \lambda_1) - \frac{1}{Y} (\lambda_j - \lambda_1) y_j \cdot \lambda_l = -(J_3 + J_4). \end{aligned}$$

Hence $Q_{l,j} = J_5 = q_{l,j} - q_{l,1} y_j$.

Finally,

$$Q_{l,l} = \frac{1}{Y^2} \sum \mu_{ij} y_i y_j + \frac{1}{Y} \sum \mu_{il} y_l - \frac{1}{Y} \sum \mu_{il} y_i - \frac{1}{Y} \mu_{ll} y_l - \frac{1}{Y} \sum b_i + \frac{\partial b_l}{\partial y_l}.$$

Using (4.10) we find that

$$Q_{l,l} = \frac{1}{Y^2} \sum \mu_{ij} y_i y_j + \frac{1}{Y^2} \sum \mu_{il} y_l - \frac{1}{Y} \sum \mu_{il} y_i$$

$$- \frac{1}{Y} \mu_{ll} y_l + \frac{1}{Y} (\lambda_j - \lambda_1) \cdot \lambda_1 y_j$$

$$- \frac{1}{Y^2} \sum (\lambda_j - \lambda_1) y_j \cdot \left(\lambda_1 + \sum \lambda_i y_i\right) + \sum_{j=1}^{n} q_{j,1} y_j - (\lambda_l - \lambda_1) \cdot \lambda_1$$

$$+ \frac{1}{Y} (\lambda_l - \lambda_1) \cdot \left(\lambda_1 + \sum \lambda_i y_i\right) + \frac{1}{Y} (\lambda_l - \lambda_1) y_l \cdot \lambda_l$$

$$- \frac{1}{Y^2} (\lambda_l - \lambda_1) y_l \cdot \left(\lambda_1 + \sum \lambda_i y_i\right)$$

$$+ \frac{\partial}{\partial y_l} \left(q_{l,1} y_l - \sum_{k=1}^{n} q_{k,1} y_l y_k \right) = \sum_{k=1}^{12} J_i.$$

Using (4.12) in J_6 we get

$$J_1 + J_6 = - \frac{1}{Y^3} \sum (\lambda_j - \lambda_1) y_j \cdot \lambda_1 = -J_5.$$

Using (4.12) in J_{11} we obtain $J_2 + J_{11} = - \frac{1}{Y} (\lambda_l - \lambda_1) y_l \cdot \lambda_1$, which together with $J_4 + J_{10}$ add up to zero. Substituting (4.12) in J_9 we also find that $J_7 + J_8 + J_9 = 0$. Hence $Q_{l,l} = J_3 + J_{12} = q_{l,l} - q_{l,1} y_l$.

We now make use of the conditions (4.1), (4.2) and deduce that

$$Q_{l,j} = q_{l,j} \geqslant 0 \quad \text{if } l \neq j$$

$$Q_{l,l} = q_{l,l} \leqslant 0, \tag{4.13}$$

$$\sum_{j=2}^{n} Q_{l,j} = \sum_{j=2}^{n} q_{l,j} = \sum_{j=1}^{n} q_{l,j} = 0 \tag{4.14}$$

and the right-hand side of (4.6) is

$$- B_l - Q_l z = - B_l \leqslant 0. \tag{4.15}$$

Since conditions (4.13) – (4.15) hold, a fairly standard maximum principle for coupled elliptic systems can be applied [4, Theorem 2.1] to conclude that

$$w_l \geqslant 0 \quad \text{in } C_\delta; \tag{4.16}$$

we use here the fact that the w_l are continuous in $\overline{C_\delta}$ and vanish on ∂C_δ, and this is true because $u - \psi$ and its first derivatives are continuous in $\overline{C_\delta}$ and vanish on ∂C_δ. (We should point out that Theorem 2.1 in [4] deals with the case where the leading part in (4.6) is Δw_l, but the proof of the theorem extends to any nondegenerate principal elliptic operator.)

Taking $\delta \to 0$ in (4.16), assertion (4.3) follows.

REMARK 1. If C is bounded then, by Theorem 3.2, condition (4.1) must hold. Hence, if $q_{j,1} = 0$ for $2 \leqslant j \leqslant n$ and if C is bounded then the assertion of Corollary 4.2 regarding the shape of C is valid.

REMARK 2. If for some j, $2 \leqslant j \leqslant n$, $B_j < 0$ then assertion (4.4) of Corollary 4.2 is false for this j. Indeed, this follows immediately from Theorem 3.2.

REMARK 3. Corollary 4.2 can also be stated in terms of the shape of C^A. We again stipulate that the role of p_1 can be given to any other variable p_i; if $q_{j,i} = 0$ for all $j \neq i$ then an assertion similar to Corollary 4.2 is valid.

5. Regularity of the free boundary.

THEOREM 5.1. *If* (4.1), (4.2) *hold then the free boundary* Γ *is analytic.*

That means that one can represent Γ locally by analytic functions $y_j = \Phi_j(\tau_1, \ldots, \tau_{n-2})$.

PROOF. We write the v.i. for $v = u - \psi$ in the form

$$-Lv \leqslant f, \qquad v \leqslant 0, \qquad (-Lv - f)v \leqslant 0 \tag{5.1}$$

where $f(y) = \sum_{i=1}^{n} B_i y_i$. Without loss of generality we may assume that $f(y) = 0$ implies $\nabla f(y) \neq 0$. We shall now use an argument of Caffarelli and Rivière [8] to show that

$$\text{if } y^0 \in \Gamma \text{ then } f(y^0) > 0. \tag{5.2}$$

Suppose (5.2) is false for some y^0. Denote by π the hyperplane passing through y^0 and perpendicular to $\nabla f(y^0)$, and denote by H the half space bounded by π such that $f < 0$ in H. Then $H \cap R_{n-1}^+$ is contained in C and therefore $Lv = f < 0$, $v < 0$ on $H \cap R_{n-1}^+$. Since, however, $v(y^0) = 0$, the strong maximum principle gives $\nabla v(y^0) \neq 0$, which is impossible, because $y^0 \in \Gamma$.

The assertion (5.2) shows that $f(y) > 0$ on Γ. Therefore the regularity theorem of Caffarelli [7] for the free boundary of a v.i. can be applied to (5.1). Since the set C has the shape given by Corollary 4.2, we deduce that each point of Γ is a point of positive density with respect to the stopping set $S = R_{n-1}^+ \setminus C$. Appealing to [7] we then conclude that Γ is analytic.

LEMMA 5.2. *If* (4.1), (4.2) *hold then* Γ *does not contain any line segment parallel to one of the* y_j *axes.*

PROOF. By Theorem 5.1, u is a C^∞ function in $C \cup \Gamma$. Suppose Γ contains a line segment l parallel to the y_2 coordinate axis. Then the functions $w_j = (\partial/\partial y_j)(v/Y)$ $(j \neq 2)$ vanish along l. Hence

$$\frac{\partial}{\partial y_j} w_2 = \frac{\partial}{\partial y_2} w_j = 0 \quad \text{along } l, j \neq 2, \tag{5.3}$$

so that also

$$\frac{\partial}{\partial \nu} w_2 = 0 \quad \text{on } l, \nu = \text{normal to } \Gamma \text{ at } l. \tag{5.4}$$

Now, w_2 satisfies in C equation (4.6) for $l = 2$, and each w_j is $\geqslant 0$. By the strong maximum principle, $w_2 > 0$ in C and, since $w_2 = 0$ on Γ, $\partial w_2/\partial \nu \neq 0$ on Γ. This contradicts (5.4).

If we use the fact that the free boundary is analytic, then we can extend the proof of Lemma 5.2 to the case where l does not actually lie on Γ but is just tangent to Γ at some point y^0. (The relations (5.3), (5.4) are then valid at y^0.)

We can therefore assert:

THEOREM 5.3. *Let* (4.1), (4.2) *hold. Then, for any* j, $2 \leqslant j \leqslant n$, Γ *can be represented in the form*

$$y_j = \Psi_j(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \tag{5.5}$$

for $(y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$ *in some bounded domain* A_j, *and*

$$\frac{\partial \Psi_j}{\partial y_i} < 0 \quad \textit{for each } i; \qquad i = 2, \ldots, j-1, j+1, \ldots, n. \tag{5.6}$$

6. **Concluding remarks.**

REMARK 1. In the special case where

$$q_{i,j} = 0 \quad \text{whenever } j < i, \tag{6.1}$$

a more general quality control problem was studied in [2] in which K was replaced by $K_1, \ldots, K_n$. The corresponding q.v.i. is then replaced by $n-1$ q.v.i. for functions $w_{n-i}(p_{n-i}, p_{n-i+1}, \ldots, p_n)$ ($p_j > 0$, $\sum_{j=n-i}^n p_j = 1$):

$$M_{n-i} w_{n-i} \equiv \frac{1}{2} \sum_{j,k=n-i}^n p_j p_k \Big(\lambda_j - \sum_{l=n-i}^n \lambda_l p_l\Big) \cdot \Big(\lambda_k - \sum_{l=n-i}^n \lambda_l p_l\Big) \frac{\partial^2 w_{n-i}}{\partial p_j \partial p_k}$$

$$+ \sum_{j,k=n-i}^n q_{j,k} p_j \frac{\partial w_{n-i}}{\partial p_k} \geqslant - \sum_{j=n-i}^n c_j p_j,$$

$$w_{n-i} \leqslant K_{n-i} + \sum_{j=n-i}^{n-1} p_j w_j(e_j),$$

$$\Big(M_{n-i} w_{n-i} + \sum_{j=n-i}^n c_j p_j\Big)\Big(w_{n-i} - K_{n-i} - \sum_{j=n-i}^{n-1} p_j w_j(e_j)\Big) = 0 \tag{6.2}$$

where $e_j = (p_j, \ldots, p_n) = (1, 0, \ldots, 0)$. It is natural to assume in this quality control problem that

$$K_1 \geqslant K_2 \geqslant \cdots \geqslant K_n. \tag{6.3}$$

We now define the B_j as in (3.3), but with $K = K_1$, $u_j = w_j(e_j)$, so that, in view of (6.1),

$$B_i = c_i + \sum_{j=i}^{n-1} q_{i,j}u_j - \alpha u_i - \alpha K_1 \qquad (1 \leqslant i \leqslant n-1),$$

$$B_n = c_n - \alpha K_1. \tag{6.4}$$

The results of §§3–5 extend immediately to the q.v.i. (6.2). Taking note of condition (6.3) we conclude that under exactly the same conditions on the B_j as in §§3–5 we have precisely the same assertions for the continuation regions $C = C_{n-i}$ and for the free boundaries $\Gamma = \Gamma_{n-i}$ of the q.v.i. (6.2), $1 \leqslant i \leqslant n - 1$.

REMARK 2. In case $n = 2$ the system (4.6) consists of just one equation. If $q_{2,1} \neq 0$ then $q_{2,1} > 0$ so that $Q_{l,l} = q_{2,2} - q_{2,1}y_2 < 0$ and $-B_l - Q_l z = -B_2 + q_{2,1}z \leqslant 0$ since $B_2 \geqslant 0$, $z \leqslant 0$. Thus the maximum principle gives $w_l = w_2 \geqslant 0$. We conclude that, if $n = 2$, the results of §§4, 5 remain valid without imposing the restriction $q_{2,1} = 0$.

REMARK 3. Denote by $w_\alpha(p)$, $J_x^p(\tau; \alpha)$ and $u_{j,\alpha}$ the functions $w(p)$, $J_x^p(\tau)$, u_j as functions of the parameter α, $\alpha \geqslant 0$, and set

$$B_1^* = c_i + \sum_{j=1}^{n-1} q_{i,j}u_{j,0}. \tag{6.5}$$

It is clear that $J_x^p(\tau, \alpha)\uparrow J_x^p(\tau, 0)$ as $\alpha\downarrow 0$ and that

$$w_\alpha(p)\uparrow w_0(p), \qquad u_{j,\alpha}\uparrow u_{j,0} \quad \text{as } \alpha\downarrow 0. \tag{6.6}$$

Suppose

$$u_{j,0} < \infty \quad \text{for } 1 \leqslant j \leqslant n-1. \tag{6.7}$$

Then clearly,

$$B_1^* > 0 \text{ implies } B_i > 0 \quad \text{if } \alpha \text{ is sufficiently small,} \tag{6.8}$$

so that the results of §§3–5 can be applied by imposing the simpler conditions

$$B_i^* > 0 \qquad (2 \leqslant i \leqslant n) \tag{6.9}$$

provided α is sufficiently small.

We claim that (6.7) is true if either (6.1) holds or

$$q_{n,n} = 0, \qquad q_{i,n} > 0 \quad \text{for } 1 \leqslant i \leqslant n-1. \tag{6.10}$$

Indeed, as shown in [2], any one of these conditions implies $P[\theta(t) \neq n] \to 0$ as $t \to \infty$. Hence, by the Markov property,

$$P\big[\theta(t) \neq n\big] \leqslant e^{-\gamma t} \quad \text{for some } \gamma > 0.$$

This implies that $J_x^p(\tilde{\tau}, 0) \leqslant B < \infty$ where $\tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2, \dots)$, $\tilde{\tau}_j = j$, and B is a constant independent of p, x, and (6.7) follows.

REMARK 4. In case (6.1) holds, the system (4.6) for the unknown functions, say $\tilde{w}_l$, is not coupled and we can get additional results by applying the maximum principle first to $\tilde{w}_n$, then to $\tilde{w}_{n-1}$, etc. For instance, if $B_n \geqslant 0$ then

$\tilde{w}_n \geqslant 0$; if also $B_{n-1} \geqslant 0$ then also $\tilde{w}_{n-1} \geqslant 0$.

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