IN Variance of the $L$-regularity of compact sets in $C^N$ under holomorphic mappings

By

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Abstract. The property for a polynomially convex compact set $E$ in $C^N$ that the Siciak extremal function $\Phi_E$ be continuous or, equivalently, that $E$ satisfy some Bernstein type inequality, is proved to be invariant under a large class of holomorphic mappings with values in $C^M (M < N)$ including all open holomorphic mappings. Local specifications of this result are also given.

0. Introduction. Let $E$ be a polynomially convex compact set in $C^N$ and let $P_n (C^N)$ denote the space of all polynomials from $C^N$ to $C^l$ of degree at most $n$. It is known [8] that if $f$ is a holomorphic function in a neighborhood of $E$ then

$$\lim sup_{n \to \infty} \left[ \text{dist}_E (f, P_n (C^N)) \right]^{1/n} < 1,$$

where the distance from $f$ to $P_n (C^N)$ is taken in the sense of the supremum norm on $E$, denoted by $\| \|_E$.

Conversely, in order that each continuous function $f$ defined on $E$ and satisfying there ($\ast$) be continuable to a holomorphic function in a neighborhood of $E$, the compact set $E$ must satisfy some regularity conditions (see [1]) of the type of Bernstein's (or Markov's) inequality for polynomials, and in the case where the compact set $E$ is sufficiently big, they are equivalent to the continuity in $C^N$ of Siciak's extremal function of $E$ (see [8] and [9]):

$$\Phi_E(z) = \sup \{|p(z)|^{1/n} : p \in P_n (C^N), \|p\|_E < 1, n > 1 \}$$

for $z \in C^N$. In the sequel a compact set $E \subset C^N$, for which the function $\Phi_E$ is continuous in $C^N$, is said to be L-regular (compare [9]). By a result of Zaharjuta [11] (see also [9]), in order that $E$ be L-regular it suffices that the function $\Phi_E$ be continuous in $E$.

If $N = 1$, the function $\log \Phi_E$ is known to be equal to the Green function of the unbounded component of the set $C^l \setminus E$ with pole at $\infty$, and hence the
question about the $L$-regularity of compact sets in $C^1$ is well explored. Incomparably less is known about this problem in the case where $N > 1$. Some criteria of $L$-regularity can be found in [8], [9] and [1]. Additional information on the class of $L$-regular subsets of $C^N$ can be derived from the main result of this paper (Theorem 3.5) saying that the property for a polynomially convex, compact set $E$ in $C^N$ to be $L$-regular is invariant under a large class $\mathcal{F}$ of holomorphic mappings in a neighborhood of $E$, with values in $C^M$ ($M < N$), including, in particular, all open holomorphic mappings. Thus if $M = 1$, the class $\mathcal{F}$ consists of all nonconstant holomorphic functions in a neighborhood of $E$. This result has been probably unknown even in the case where $N = 1$.

Next we give local specifications of the main result and examine the invariance under holomorphic mappings of the property for $E$ to be $L$-regular at a point $a \in E$, which stands, by definition, for the continuity of the extremal function $\Phi_E$ at $a$. Here we distinguish two cases (Theorems 3.8 and 3.12) according as $E$ is a compact subset of $R^N$ or $C^N$ and leave open the problem of equivalence of both results (see Remark 3.10 and Question 3.11).

1. Properties (A) and (B).

1.1. Let $E$ be a compact set in $C^N$ and let $C(E)$ denote the Banach space of complex-valued continuous functions defined on $E$ with the supremum norm $\|f\|_E$. Given an open set $U$ in $C^N$ let $\Theta_E(U)$ be the Banach space of all bounded continuous functions defined on $E \cup U$ holomorphic in $U$, with the supremum norm on $E \cup U$. We denote by $N_E(U)$ the kernel of the natural restriction $r_U: \Theta_E(U) \ni f \mapsto f|_E \in C(E)$ and by $A_E(U)$ its range endowed with the quotient topology $\Theta_E(U)/N_E(U)$.

Given a subset $F$ of $E$ we define

$$\Theta_E(F) = \lim\inf_{U \supset F, \; U \text{ open}} \Theta_E(U)$$

and

$$N_E(F) = \lim\inf_{U \supset F, \; U \text{ open}} N_E(U).$$

The mappings $r_U$ define the restriction $r: \Theta_E(F) \to C(E)$ with its kernel $N_E(F)$ and its range

$$A_E(F) = \lim\inf_{U \supset F, \; U \text{ open}} A_E(U).$$

Let us consider an increasing sequence $(H_n)$ of vector subspaces of the space $\Theta_E(F)$ and a nondecreasing sequence $(m_n)$ of positive numbers. Following Baouendi and Goulaouic [1] (see also [10]) we define properties (A) and (B) of the quadruplet $(E, F, (H_n), (m_n))$ as follows.
Property (A). For any function \( f \in C(E) \), if
\[
\lim_{n \to \infty} \sup \left[ \text{dist}_E(f, r(H_n)) \right]^{1/m_n} < 1,
\]
then \( f \in A_E(F) \).

Property (B). For any real number \( b > 1 \) there exist an open neighborhood \( U \) of \( F \) and a constant \( C > 0 \) such that for any \( h \in H_n \) \( (n = 1, 2, \ldots) \) there exists \( g \in N_E(F) \) for which \( h + g \in \partial_E(U) \) and
\[
\sup_{z \in U} |h(z) + g(z)| \leq Cb^{m_n} \|h\|_E.
\]

1.2. If the compact sets \( E \) and \( F \) are so big that \( N_E(F) = 0 \), then Property (B) of the quadruplet \( (E, F, (H^n), (m_n)) \) yields the following:

Property (B'). For any real number \( b > 1 \) there exist an open neighborhood \( U \) of \( F \) and a constant \( C > 0 \) such that each \( h \in H^n \) \( (n = 1, 2, \ldots) \) belongs to \( \partial_E(U) \) and
\[
\sup_{z \in U} |h(z)| \leq Cb^{m_n} \|h\|_E.
\]

In some special cases we shall also prove that (B') implies (B) and \( N_E(F) = 0 \) (see Proposition 3.2).

We note that Property (B') is an analogue of the well-known Bernstein-Walsh inequality for polynomials which can be written in terms of the extremal function \( \Phi_E \) as follows.

1.3. INEQUALITY. For any polynomial \( p \in P_n(C^N) \) \( (n = 1, 2, \ldots) \), we have
\[
|p(z)| < \|p\|_E [\Phi_E(z)]^n, \quad z \in C^N.
\]

An important role in our considerations will be played by the following theorem due to Baouendi and Goulaouic [1] (the case where \( E = F \subset \mathbb{R}^N \)).

1.4. Theorem. Assume that for each \( a \in (0, 1), \Sigma_{n=1}^{\infty} a^m_n < \infty \) and \( \lim \sup_{n \to \infty} m_{n+1}/m_n < \infty \). Then Properties (A) and (B) for the quadruplet \( (E, F, (H^n), (m_n)) \) are equivalent.

The proof of the implication (B) \( \Rightarrow \) (A) is easy while the converse implication can be proved in the same manner as in [10] (the case where \( H_n = P_n(C^N) \) and \( m_n = n, n = 1, 2, \ldots \)).

1.5. Remark. Notice that we have not assumed the spaces \( H_n \) of Theorem 1.4 to be finite dimensional. However it can be proved in the case where \( E = F \) is polynomially convex that Property (B) holds only if \( \dim r(H_n) = O(m_n^N) \), as \( n \to \infty \) (see [7]).

2. Invariance of Property (B) under holomorphic mappings. We start with a lemma which is a version for families of holomorphic functions of a theorem due to Siciak [8, Theorem 10.2] being a generalization to \( C^N \) of a known Bernstein-Walsh Theorem. The lemma was first stated in [5]. Here we give its
elegant proof communicated to us by Siciak. Another proof can be derived from Theorem 2 in [11].

2.1. **Lemma.** Let $A(U)$ denote the Banach space of all bounded holomorphic functions defined in an open set $U$ in $\mathbb{C}^N$, equipped with the supremum norm on $U, \| \cdot \|_U$. For each polynomially convex compact set $E \subset U$ there exist constants $C > 0$ and $a \in (0, 1)$, both $C$ and $a$ independent of $f \in A(U)$ and $n$, such that

$$\text{dist}_E(f, P_n(\mathbb{C}^N)) < C\|f\|_E a^n$$

for all $f$ in $A(U)$ and $n = 1, 2, \ldots$.

**Proof.** Given a polynomially convex compact set $E \subset U$, we can find a polynomial polyhedron $P \subset U$ such that $E \subset \text{int } P$ (see e.g. [2, Lemma 2.7.4]). Then for sufficiently small $r > 0$, the set

$$E' = \bigcup_{a \in E} B(a, r),$$

where $B(a, r)$ denotes the closed ball $\{z \in \mathbb{C}^N: |z - a| < r\}$, is contained in $P$ together with its polynomially convex hull

$$\hat{E}' = \{z \in \mathbb{C}^N: |p(z)| < \|p\|_E \text{ for all } p \in P_n(\mathbb{C}^N) \text{ and } n > 1\}.$$

Since we have (see [8])

$$\Phi_{B(a, r)}(z) = \max\{1, |z - a|/r\}, \quad z \in \mathbb{C}^N,$$

and

$$\Phi_{E_1}(z) < \Phi_{E_2}(z), \quad z \in \mathbb{C}^N, \text{ whenever } E_1 \supset E_2,$$

which immediately follows from the definition of the extremal function, the set $E'$ is $L$-regular and so is the set $\hat{E}'$ because for every compact set $E \subset \mathbb{C}^N$ we have

$$\Phi_E(z) = \Phi_{E'}(z), \quad z \in \mathbb{C}^N.$$

Consequently the set $E$ of the lemma can be assumed to be $L$-regular. Then there exists an $R > 1$ such that

$$D_R = \{z \in \mathbb{C}^N: \Phi_E(z) < R\} \subset U.$$

Take any $b \in (1, R)$ and define

$$\mathcal{E}_b = \left\{f \in C(E): \sup_{n > 0} b^n \text{dist}_E(f, P_n(\mathbb{C}^N)) < \infty \right\}.$$

$\mathcal{E}_b$ is a Banach space with the norm

$$|f|_b = \|f\|_E + \sup_{n > 0} b^n \text{dist}_E(f, P_n(\mathbb{C}^N)).$$

By the above-mentioned result of Siciak [8, Theorem 10.2], there is a natural inclusion $\varphi_b: A(U) \rightarrow \mathcal{E}_b$. One can also easily check that the graph of the mapping $\varphi_b$ is closed. Hence $\varphi_b$ is continuous, which gives the result.
2.2. Given a compact set $E \subset \mathbb{C}^N$ let

$$
\Phi^*_E(z) = \limsup_{w \to z} \Phi_E(w)
$$

and

$$
c(E) = \limsup_{|z| \to \infty} \left\lfloor \frac{|z|}{\Phi^*_E(z)} \right\rfloor.
$$

The number $c(E)$ is called the $\mathbb{C}^N$-capacity of $E$ (see [9], [11]; if $N = 1$, $c(E)$ is equal to the logarithmic capacity of $E$). By 2(2), if $E_1 \subset E_2$, then $c(E_1) < c(E_2)$. If $E$ is $L$-regular at a point $a \in \mathbb{C}^N$, then by Inequality 1.3 and 2(1), $c(E) > 0$.

The following lemma will play a crucial role in the proof of the main result of this section (Proposition 2.9).

2.3. Lemma. Suppose $E$ is a polynomially convex compact set in $\mathbb{C}^N$ and $F$ is a subset of $E$. Let $h$ be a holomorphic mapping defined in an open neighborhood $U$ of $E$, with values in $\mathbb{C}^M$ ($M < N$). Assume that $c(h(E)) > 0$. Then, if Property (B) holds for the quadruplet $(E, F, (P_n(\mathbb{C}^N)), (n))$, it also holds for the quadruplet $(E, F, (H_n), (n))$, where $H_n = \{ p \circ h : p \in P_n(\mathbb{C}^M) \}$, $n = 1, 2, \ldots$.

Proof. By Theorem 1.4 it suffices to show that Property (A) holds for $(E, F, (H^n), (n))$. To this aim take a function $f \in C(E)$ such that

$$
\|f - p_n \circ h\|_E < Ca^n, \quad n = 1, 2, \ldots, 2(3)
$$

where $p_n \in P_n(\mathbb{C}^M)$, $C > 0$ and $a \in (0, 1)$, the constants $C$ and $a$ being independent of $n$. We wish to show that $f \in A_E(F)$. Observe that by (2.3), for each $n$,

$$
\|p_n\|_{h(E)} = \|p_n \circ h\|_E < C + \|f\|_E.
$$

We may obviously suppose that $h$ is bounded in $U$. Since $c(h(E)) > 0$, we have

$$
B = \sup \{ \Phi_{h(E)}(w) : w \in h(U) \} < \infty.
$$

Hence by Inequality 1.3,

$$
B_n = \sup_{z \in U} |p_n(h(z))| = \sup_{w \in h(U)} |p_n(w)| < \|p_n\|_{h(E)} B^n,
$$

whence

$$
B_n < (C + \|f\|_E)B^n < D^n
$$

for each $n$, with an appropriate constant $D > 0$. By Lemma 2.1 we can find constants $D_1 > 0$ and $d \in (0, 1)$ such that

$$
\text{dist}_E\left( p_n \circ h, P_k(\mathbb{C}^N) \right) < D_1 D^nd^k
$$

for $k > 1, n > 1$. Then by setting $k = mn$, where the positive integer $m$ is so chosen that $Dd^{mn} < d$, we get

$$
\text{dist}_E\left( p_n \circ h, P_{mn}(\mathbb{C}^N) \right) < D_1 d^n, \quad n > 1,
$$
whence by \(2(3),\)
\[
\text{dist}_E(f, P_{mn}(C^N)) < D_2 d_1^n,
\]
for \(n > 1,\) where \(D_2 > \max(C, D_1)\) and \(d_1 = \max(a, d) < 1.\) Therefore, since for \(mn < k < m(n + 1),\)
\[
\text{dist}_E(f, P_k(C^N)) < \text{dist}_E(f, P_{mn}(C^N)) < D_2 (d_1^{n/k})^k < D_2 d_2^k
\]
with \(d_2 \in (0, 1)\) independent of \(n,\) we get
\[
\limsup_{k \to \infty} \left[ \frac{\text{dist}_E(f, P_k(C^N))}{1/k} \right] < 1,
\]
and since the quadruplet \((E, F, (P_k(C^N)), (k))\) has Property (A), it follows that \(f \in A_F(E),\) as claimed.

2.4. **Remark.** It occurs that \(c(E) = 0\) but \(c(h(E)) > 0,\) e.g. take \(E = [0, 1] \times \{0\} \subset \mathbb{R}^2\) and set \(h(z_1, z_2) = z_1.\)

We shall need the following

2.5. **Lemma.** Let \(E\) be a compact subset of \(C^N\) with \(c(E) > 0.\) Suppose \(h\) is a holomorphic mapping in a connected open set \(U,\) \(E \subset U,\) with values in \(C^M (M < N)\), such that
\[
\sup\{c(F) : F \subset h(U), F \text{ compact}\} > 0.
\]
Then \(c(h(E)) > 0.\)

**Proof.** Suppose \(c(h(E)) = 0.\) Then \(h(E)\) is globally \(C^M\)-polar (see [9, Corollary 3.9]), i.e. one could find a plurisubharmonic function \(p\) in \(C^M\) such that \(p(w) = -\infty\) for \(w \in h(E).\) Then the function \(q = p \circ h\) is plurisubharmonic in \(U,\) and by the assumptions on \(h,\) \(q \equiv -\infty\) in \(U.\) Since \(q(z) = -\infty\) for \(z \in E,\) the set \(E\) is locally \(C^N\)-polar, whence by a recent result of Josefson [3], it should be globally \(C^N\)-polar. Consequently by [9, Theorem 3.10], we would have \(c(E) = 0,\) a contradiction.

2.6. **Remark.** The assumption that \(f\) is nonconstant in \(U\) is not sufficient for the above lemma to hold. Take, e.g., \(E = \{(x_1, x_2, x_3) : 0 < x_i < 1, i = 1, 2, 3\} \subset \mathbb{R}^3\) and \(h(z_1, z_2, z_3) = (z_1, z_1).\)

2.7. Given a compact set \(E\) in \(C^N\) let \(h\) be a holomorphic mapping in an open neighborhood \(U\) of \(E,\) with values in \(C^M (M < N).\) In the sequel we shall be interested in \(h\) such that for a given subset \(F\) of \(E,\) the triplet \((h, E, F)\) satisfies the following hypothesis:
\[
\text{(H)} \text{ For each } a \in F \text{ and each bounded open set } V \text{ such that } F \subset V \subset V^c \subset U, \text{ the set } h(V) \text{ is } L\text{-regular at } h(a).
\]
Notice that if the mapping \(h\) is open in an open neighborhood \(W\) of \(F\) then by \(2(1)\) and \(2(2)\) the triplet \((h, E, F)\) satisfies \((H)\).

In particular, if \(M = 1\) and \(h\) is nonconstant in any connected component \(W\) of \(U\) such that \(W \cap F \neq \emptyset,\) then by the open mapping theorem for holomorphic functions the triplet \((h, E, F)\) satisfies \((H).\)
An example of a triplet \((h, E, F)\) satisfying \((H)\) with a nonopen \(h\) is given by \(h(z_1, z_2) = (z_1, z_1 z_2)\) for \((z_1, z_2) \in \mathbb{C}^2\) and \(E = F = [0, 1] \times [0, 1]\).

From Lemma 2.5 we derive

2.8. **Corollary.** If \(h: U \to \mathbb{C}^M (M < N)\) is holomorphic and \(U\) is connected, and the triplet \((h, E, E)\) satisfies \((H)\), then \(c(E) > 0\) implies \(c(h(E)) > 0\).

More generally, if there exists a point \(a \in E\) such that \((h, E, \{a\})\) satisfies \((H)\), and for the connected component \(V_a\) of the set \(U\) which contains \(a\), we have \(c(V_a \cap E) > 0\), then \(c(h(E)) > 0\).

Now we can prove

2.9. **Proposition.** Let \(E\) be a polynomially convex, compact set in \(\mathbb{C}^N\) and \(h\) a holomorphic mapping in an open set \(U \supset E\), with values in \(\mathbb{C}^M (M < N)\). Assume that \(c(h(E)) > 0\). Then, for any subset \(F\) of \(E\):

1°. If \((E, F, (P_n(\mathbb{C}^N)), (n))\) has Property \((B)\) and the triplet \((h, E, E)\) satisfies \((H)\), and \(N_E(F) = 0\), then the quadruplet \((h(E), h(F), (P_n(\mathbb{C}^M)), (n))\) has Property \((B')\).

2°. If \(M = N\) and \(h\) is a biholomorphism, then if \((E, F, (P_n(\mathbb{C}^N)), (n))\) has Property \((B)\) then the quadruplet \((h(E), h(F), (P_n(\mathbb{C}^M)), (n))\) also has this property.

**Proof.** In both 1° and 2°, if \((E, F, (P_n(\mathbb{C}^N)), (n))\) satisfies \((B)\), then by virtue of Lemma 2.3, so does the quadruplet \((E, F, (H_n), (n))\), where \(H_n = \{p \circ h : p \in P_n(\mathbb{C}^M)\}\). It follows that for each \(b > 1\) there exist a bounded open set \(V, F \subset V \subset \overline{V} \subset U\), and a constant \(C > 0\) such that for each \(n\) and each \(p \in P_n(\mathbb{C}^M)\) one can find \(p_\alpha \in N_E(V)\) such that

\[
\sup_{z \in V} |p(h(z)) + p_\alpha(z)| < Cb^{n/2}\|p \circ h\|_E. \tag{2(4)}
\]

Now with the assumptions of case 1°, it follows that \(g_\alpha = 0\), whence

\[
\sup_{w \in h(V)} |p(w)| = \sup_{z \in V} |p(h(z))| < Cb^{n/2}\|p \circ h\|_E = Cb^{n/2}\|p\|_{h(E)}.
\]

The set \(h(V)\) need not be a neighborhood of \(h(F)\). However, by \((H)\), there is an open neighborhood \(W\) of \(h(F)\) such that

\[
\Phi_{h(V)}(w) < b^{1/2}, \quad w \in W,
\]

and then by Inequality 1.3 we get

\[
\sup_{w \in W} |p(w)| \leq C b^{n}\|p\|_{h(E)}.
\]

which completes the proof of case 1°.

2°. Since \(h\) is a biholomorphism, the set \(W = h(V)\) is an open neighborhood of \(h(F)\) for each open set \(V\) such that \(F \subset V \subset U\), and by 2(4) we get

\[
\sup_{w \in W} |p(w) + g_\alpha(h^{-1}(w))| < C b^{n}\|p\|_{h(E)}.
\]
3. Applications to the $L$-regularity. In this section our attention will be
devoted to Properties (B) and (B') in the case where $H_n = P_n(C^N)$ and
$m_n = n, n = 1, 2, \ldots$ Then, given a compact set $E \subset C^N$ and a subset $F$ of
$E$, we shall shortly write $(E, F) \in (B)$ (resp. $(E, F) \in (B')$) if Property (B)
(resp. (B')) holds for $(E, F, (P_n(C^N)), (n))$.

3.1. We note that $(E, F) \in (B')$ if and only if $E$ is $L$-regular at every point
$a \in F$.

3.2. Proposition. 1°. For any polynomially convex compact set $E$ in $C^N$,
$(E, E) \in (B')$ if and only if $(E, E) \in (B)$ and $N_E(E) = 0$.
2°. If $E \subset R^N$ then for any subset $F$ of $E$, $(E, F) \in (B')$ if and only if
$(E, F) \in (B)$ and $N_E(F) = 0$.

Proof. By virtue of (1.2) in both 1° and 2° it suffices to prove that
$(E, E) \in (B')$ implies $N_E(F) = 0$.

1°. Take a function $f \in N_E(U)$, where $U$ is an open neighborhood of $E$.
Since $(E, E) \in (B')$, the set $E$ is $L$-regular, whence for each $R > 1$ the set
$E_R = \{ z \in C^N : \Phi_{E}(z) \leq R \}$
is compact and $E \subset \text{int } E_R$. Since $E = \widehat{E}$, the polynomially convex hull of $E$,
we can find $R > 1$ such that $E_R = \widehat{E}_R \subset U$. Then by Lemma 2.1 there exist
constants $C > 0$ and $a \in (0, 1)$, and a sequence of polynomials $p_n \in P_n(C^N)$
$(n = 1, 2, \ldots)$, such that
$$\| f - p_n \|_{E_R} \leq C a^n, \quad n = 1, 2, \ldots.$$  
If $1 < R' < \min(R, 2/(1 + a))$, then $E_{R'} \subset E_R$ and since $f = 0$ on $E$, by
Inequality 1.3 we get
$$\| f \|_{E_R} \leq \| f - p_n \|_{E_R} + \| p_n \|_{E_R} \leq C a^n + \| p_n \|_{E}(R')^n \leq C[a^n + (2a/(1 + a))^n]$$
for $n = 1, 2, \ldots$, whence $f = 0$ on $E_{R'}$, and, consequently, $N_E(E) = 0$.

2°. It suffices to show that if $E$ is $L$-regular at $b \in E$ then $N_E(b) = 0$. To
do this take a function $f \in N_E(U)$, where $U$ is an open neighborhood of $b$.
We can find three bounded closed parallelepipeds $K_1, K_2$, and $K_3$, such that
$b \in \text{int } K_1, K_i \subset \text{int } K_{i+1}, i = 1, 2, K_2 \subset U$ and $E \subset K_3$. Then by Lemma 2.1
there exist polynomials $p_n \in P_n(C^N)$ $(n = 1, 2, \ldots)$ such that
$$\| f - p_n \|_{K_2} \leq C a^n$$
with $C > 0$ and $a \in (0, 1)$, both $C$ and $a$ independent of $n$. By [6, Lemma
12.3], there exist polynomials $l_k \in P_k(C^N)$ $(k = 1, 2, \ldots)$ and constants
$D > 0$ and $d \in (0, 1)$ such that
$$\| l_k - 1 \|_{K_3} \leq D d^k, \quad \| l_k \|_{K_3 \setminus \text{int } K_2} \leq D d^k \quad \text{and} \quad \| l_k \|_{K_2} \leq D k^N$$
for $k = 1, 2, \ldots$. Write $r_{k,n} = l_k p_n (k > 1, n > 1)$. By Inequality 1.3 there is
a constant $A > 0$ such that
\[ \|p_n\|_{K_j} < A^n, \quad n = 1, 2, \ldots. \]
Then for each $k > 1$ and $n > 1$, we have
\[ \|r_{k,n}\|_{E \cap (K_j \setminus \text{int} K_j)} \leq \|p_n\|_{K_j} \leq DA^{-d}d^k, \]
\[ \|r_{k,n}\|_{E \cap (K_j \setminus \text{int} K_j)} \leq Dk^N\|p_n\|_{E \cap K_j} \leq DCk^Na^n \]
and
\[ \|r_{k,n}\|_{E \cap K_j} \leq (D + 1)Ca^n. \]

Now choose an integer $m > 0$ such that $Ad^m < e = \max(a, d) < 1$ and set $r_n = r_{mn,n}$ for $n = 1, 2, \ldots$. Then by the above inequalities there exists $n_0 > 0$ such that
\[ \|r_n\|_E \to D_1e_1^n \quad \text{for} \quad n > n_0, \]
with an appropriate constant $D_1 > 0$ and $e_1 = (e + 1)/2$. Hence by Inequality 1.3, we can find a compact neighborhood $V$ of the point $b$, $V \subset K_1$, such that
\[ \|r_n\|_V \to 0, \quad \text{as} \quad n \to \infty. \]

On the other hand, for each $n$,
\[ \|r_n - p_n\|_{K_1} \leq \|m_{mn} - 1\|_{K_1}\|p_n\|_{K_1} \leq De^n, \]
whence
\[ \|f\|_V \leq \|f - p_n\|_V + \|p_n - r_n\|_V + \|r_n\|_V \to 0 \]
as $n \to \infty$, which yields $f = 0$ on $V$. This gives the result.

3.3. REMARK. If $E$ is not polynomially convex, then Proposition 3.2(1°) fails to hold (take, e.g., for $N = 1$, $E = \{|z| = 1\} \cup \{0\}$; then $\Phi_E(z) = \max(1, |z|)$ but $\mathcal{N}_E(E) \neq 0$).

We also note that by the Stone-Weierstrass theorem, if $E \subset \mathbb{R}^N$ then $E$ is polynomially convex.

3.4. QUESTION. For any compact set $E = \hat{E} \subset \mathbb{C}^N$ does the $L$-regularity of $E$ at a point $a \in E$ imply $\mathcal{N}_E(a) = 0$?

We note that this is the case when $N = 1$ (see Remark 3.10 and the proof of Theorem 3.12).

Now we can prove the main result of this paper.

3.5. THEOREM. Let $E$ be a polynomially convex, $L$-regular compact set in $\mathbb{C}^N$ and $h$ a holomorphic mapping in an open neighborhood $U$ of $E$, with values in $\mathbb{C}^M$ ($M < N$), such that the triplet $(h, E, E)$ satisfies (H). Then $h(E)$ is $L$-regular.

PROOF. Since $E$ is $L$-regular, then by 2(1) and Inequality 1.3, $c(E) > 0$. By Corollary 2.8 we then have $c(h(E)) > 0$, and by Proposition 3.2(1°) we get...
\( N_E(E) = 0 \). Therefore by Proposition 2.9(1°) and by 3.1, the set \( h(E) \) is \( L \)-regular.

3.6. Corollary. If \( E = \hat{E} \subset \mathbb{C}^N \) is \( L \)-regular and \( h: E \subset U \rightarrow \mathbb{C}^1 \) is holomorphic and nonconstant in any connected component \( W \) of \( U \) such that \( W \cap E \neq \emptyset \), then the set \( h(E) \) is also \( L \)-regular.

3.7. Remark. If \( E \neq \hat{E} \), then Theorem 3.5 fails to hold; take, e.g., \( E = \{|z| = 1\} \cup \{\frac{1}{2}\} \subset \mathbb{C}^1 \) and \( h(z) = z^{-1} \); then \( h(E) \) is not \( L \)-regular at \( 2 \in h(E) \).

Now we wish to give local versions of Theorem 3.5. Owing to Proposition 3.2(2°), by a similar argument to that of the proof of Theorem 3.5 we get

3.8. Theorem. If a compact set \( E \subset \mathbb{R}^N \) is \( L \)-regular at a point \( a \in E \) and \( h: E \subset U \rightarrow \mathbb{C}^M \) \((M < N)\) is holomorphic in \( U \) and such that the triplet \((h, E, \{a\})\) satisfies (H), then \( h(E) \) is \( L \)-regular at \( h(a) \).

We cannot prove Theorem 3.8 for any compact set \( E = \hat{E} \subset \mathbb{C}^N \) (see Question 3.4). Nevertheless we shall give a little weaker (or equivalent—see Remark 3.10 and Question 3.11) version of this result.

3.9. Definition (compare [9]). A compact set \( E \subset \mathbb{C}^N \) is said to satisfy condition \((L^1)\) at a point \( a \in E \), if for each \( r > 0 \) the set \( E \cap B(a, r) \) is \( L \)-regular at \( a \); \( B(a, r) \) being the closed ball with centre \( a \) and radius \( r \).

3.10. Remark. It is obvious that if \( E \) satisfies \((L^1)\) at \( a \in E \), then it is \( L \)-regular at \( a \). Conversely, if \( N = 1 \) and \( a \in E = \hat{E} \) (or \( a \in \partial(\mathbb{C}^1 \setminus E) \), if \( E \neq \hat{E} \)) then the \( L \)-regularity of \( E \) at \( a \) implies that \( E \) satisfies \((L^1)\) at \( a \) (see e.g. [4, (5.1.15')]).

3.11. Question. Suppose \( E = \hat{E} \subset \mathbb{C}^N \) is \( L \)-regular at \( a \in E \). Does \( E \) then have to satisfy \((L^1)\) at \( a \)?

3.12. Theorem. With the assumptions of Theorem 3.8 on \( h \), for any compact set \( E \subset \mathbb{C}^N \), if \( E \) satisfies \((L^1)\) at \( a \in E \), so does the set \( h(E) \) at \( h(a) \).

Proof. If \( E \) satisfies \((L^1)\) at \( a \in E \), then for each \( r > 0 \), \( c(E \cap B(a, r)) > 0 \), and by Corollary 2.8 we have \( c(h(E)) > 0 \).

Moreover, it follows from Lemma 2.1 and Inequality 1.3 that for each \( r > 0 \), \( N_{E \cap B(a, r)}(a) = 0 \) (see the proof of Proposition 3.2). For each \( r > 0 \) there exists \( s > 0 \) such that

\[ h(E \cap B(a, s)) \subset h(E) \cap B(h(a), r). \]

Hence, since by Proposition 2.9(1°) the set \( h(E \cap B(a, s)) \) is \( L \)-regular at \( h(a) \), by 2(2) so is the set \( h(E) \cap B(h(a), r) \), which means that \( h(E) \) satisfies \((L^1)\) at \( h(a) \), as claimed.

References


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