TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 246, December 1978

INVARIANCE OF THE L-REGULARITY OF COMPACT SETS IN $\mathbf{C}^N$ UNDER HOLOMORPHIC MAPPINGS

BY

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ABSTRACT. The property for a polynomially convex compact set E in $\mathbf{C}^N$ that the Siciak extremal function Φ_E be continuous or, equivalently, that E satisfy some Bernstein type inequality, is proved to be invariant under a large class of holomorphic mappings with values in $\mathbf{C}^M$ ($M < N$) including all open holomorphic mappings. Local specifications of this result are also given.

0. Introduction. Let E be a polynomially convex compact set in $\mathbf{C}^N$ and let $P_n(\mathbf{C}^N)$ denote the space of all polynomials from $\mathbf{C}^N$ to $\mathbf{C}^1$ of degree at most n. It is known [8] that if f is a holomorphic function in a neighborhood of E then

$$\limsup_{n\to\infty}\left[\operatorname{dist}_E(f, P_n(\mathbf{C}^N))\right]^{1/n} < 1,$$

where the distance from f to $P_n(\mathbf{C}^N)$ is taken in the sense of the supremum norm on E, denoted by $\| \ \|_E$.

Conversely, in order that each continuous function f defined on E and satisfying there (*) be continuable to a holomorphic function in a neighborhood of E, the compact set E must satisfy some regularity conditions (see [1]) of the type of Bernstein's (or Markov's) inequality for polynomials, and in the case where the compact set E is sufficiently big, they are equivalent to the continuity in $\mathbf{C}^N$ of Siciak's extremal function of E (see [8] and [9]):

$$\Phi_E(z) = \sup\left\{|p(z)|^{1/n}: p \in P_n(\mathbf{C}^N), \|p\|_E < 1, n > 1\right\}$$

for $z \in \mathbf{C}^N$. In the sequel a compact set $E \subset \mathbf{C}^N$, for which the function Φ_E is continuous in $\mathbf{C}^N$, is said to be *L-regular* (compare [9]). By a result of Zaharjuta [11] (see also [9]), in order that E be L-regular it suffices that the function Φ_E be continuous in E.

If $N = 1$, the function $\log \Phi_E$ is known to be equal to the Green function of the unbounded component of the set $\mathbf{C}^1 \setminus E$ with pole at ∞, and hence the

Received by the editors June 18, 1977.

AMS (MOS) subject classifications (1970). Primary 41A10, 32A10; Secondary 41A25, 31C10, 31C15, 32E20.

Key words and phrases. Extremal function, Green function, approximation of analytic functions, polynomials, Bernstein inequality.

question about the L-regularity of compact sets in $\mathbf{C}^1$ is well explored. Incomparably less is known about this problem in the case where $N > 1$. Some criteria of L-regularity can be found in **[8]**, **[9]** and **[1]**. Additional information on the class of L-regular subsets of $\mathbf{C}^N$ can be derived from the main result of this paper (Theorem 3.5) saying that the property for a polynomially convex, compact set E in $\mathbf{C}^N$ to be L-regular is invariant under a large class $\mathfrak{F}$ of holomorphic mappings in a neighborhood of E, with values in $\mathbf{C}^M$ $(M \leqslant N)$, including, in particular, all open holomorphic mappings. Thus if $M = 1$, the class $\mathfrak{F}$ consists of all nonconstant holomorphic functions in a neighborhood of E. This result has been probably unknown even in the case where $N = 1$.

Next we give local specifications of the main result and examine the invariance under holomorphic mappings of the property for E to be L-regular at a point $a \in E$, which stands, by definition, for the continuity of the extremal function Φ_E at a. Here we distinguish two cases (Theorems 3.8 and 3.12) according as E is a compact subset of $\mathbf{R}^N$ or $\mathbf{C}^N$ and leave open the problem of equivalence of both results (see Remark 3.10 and Question 3.11).

1. Properties (A) and (B).

1.1. Let E be a compact set in $\mathbf{C}^N$ and let $C(E)$ denote the Banach space of complex-valued continuous functions defined on E with the supremum norm $\| \ \|_E$. Given an open set U in $\mathbf{C}^N$ let $\mathcal{O}_E(U)$ be the Banach space of all bounded continuous functions defined on $E \cup U$ holomorphic in U, with the supremum norm on $E \cup U$. We denote by $N_E(U)$ the kernel of the natural restriction r_U: $\mathcal{O}_E(U) \ni f \to f_{|E} \in C(E)$ and by $A_E(U)$ its range endowed with the quotient topology $\mathcal{O}_E(U)/N_E(U)$.

Given a subset F of E we define

$$\mathcal{O}_E(F) = \operatorname*{ind\,lim}_{U \supset E,\, U \text{ open}} \mathcal{O}_E(U)$$

and

$$N_E(F) = \operatorname*{ind\,lim}_{U \supset F,\, U \text{ open}} N_E(U).$$

The mappings r_U define the restriction r: $\mathcal{O}_E(F) \to C(E)$ with its kernel $N_E(F)$ and its range

$$A_E(F) = \operatorname*{ind\,lim}_{U \supset F,\, U \text{ open}} A_E(U).$$

Let us consider an increasing sequence (H_n) of vector subspaces of the space $\mathcal{O}_E(F)$ and a nondecreasing sequence (m_n) of positive numbers. Following Baouendi and Goulaouic **[1]** (see also **[10]**) we define properties (A) and (B) of the quadruplet $(E, F, (H_n), (m_n))$ as follows.

PROPERTY (A). For any function $f \in C(E)$, if

$$\limsup_{n\to\infty} \left[\operatorname{dist}_E(f, r(H_n))\right]^{1/m_n} < 1,$$

then $f \in A_E(F)$.

PROPERTY (B). For any real number $b > 1$ there exist an open neighborhood U of F and a constant $C > 0$ such that for any $h \in H_n$ ($n = 1, 2, \ldots$) there exists $g \in N_E(F)$ for which $h + g \in \mathcal{O}_E(U)$ and

$$\sup_{z\in U} |h(z) + g(z)| \leqslant Cb^{m_n}\|h\|_E.$$

1.2. If the compact sets E and F are so big that $N_E(F) = 0$, then Property (B) of the quadruplet $(E, F, (H_n), (m_n))$ yields the following:

PROPERTY (B′). For any real number $b > 1$ there exist an open neighborhood U of F and a constant $C > 0$ such that each $h \in H_n$ ($n = 1, 2, \ldots$) belongs to $\mathcal{O}_E(U)$ and

$$\sup_{z\in U} |h(z)| \leqslant Cb^{m_n}\|h\|_E.$$

In some special cases we shall also prove that (B′) implies (B) and $N_E(F) = 0$ (see Proposition 3.2).

We note that Property (B′) is an analogue of the well-known Bernstein-Walsh inequality for polynomials which can be written in terms of the extremal function Φ_E as follows.

1.3. INEQUALITY. For any polynomial $p \in P_n(\mathbf{C}^N)$ ($n = 1, 2, \ldots$), we have

$$|p(z)| \leqslant \|p\|_E[\Phi_E(z)]^n, \qquad z \in \mathbf{C}^N.$$

An important role in our considerations will be played by the following theorem due to Baouendi and Goulaouic **[1]** (the case where $E = F \subset \mathbf{R}^N$).

1.4. THEOREM. *Assume that for each* $a \in (0, 1)$, $\sum_{n=1}^{\infty} a^{m_n} < \infty$ *and* $\limsup_{n\to\infty} m_{n+1}/m_n < \infty$. *Then Properties* (A) *and* (B) *for the quadruplet* $(E, F, (H_n), (m_n))$ *are equivalent.*

The proof of the implication (B) $\Rightarrow$ (A) is easy while the converse implication can be proved in the same manner as in **[10]** (the case where $H_n = P_n(\mathbf{C}^N)$ and $m_n = n$, $n = 1, 2, \ldots$).

1.5. REMARK. Notice that we have not assumed the spaces H_n of Theorem 1.4 to be finite dimensional. However it can be proved in the case where $E = F$ is polynomially convex that Property (B) holds only if $\dim r(H_n) = O(m_n^N)$, as $n \to \infty$ (see **[7]**).

2. Invariance of Property (B) under holomorphic mappings. We start with a lemma which is a version for families of holomorphic functions of a theorem due to Siciak **[8**, Theorem 10.2] being a generalization to $\mathbf{C}^N$ of a known Bernstein-Walsh Theorem. The lemma was first stated in **[5]**. Here we give its elegant proof communicated to us by Siciak. Another proof can be derived from Theorem 2 in [11].

2.1. LEMMA. *Let $A(U)$ denote the Banach space of all bounded holomorphic functions defined in an open set U in $\mathbf{C}^N$, equipped with the supremum norm on U, $\| \|_U$. For each polynomially convex compact set $E \subset U$ there exist constants $C > 0$ and $a \in (0, 1)$, both C and a independent of $f \in A(U)$ and n, such that*

$$\operatorname{dist}_E(f, P_n(\mathbf{C}^N)) \leqslant C\|f\|_U a^n$$

for all f in $A(U)$ and $n = 1, 2, \ldots$.

PROOF. Given a polynomially convex compact set $E \subset U$, we can find a polynomial polyhedron $P \subset U$ such that $E \subset \operatorname{int} P$ (see e.g. [2, Lemma 2.7.4]). Then for sufficiently small $r > 0$, the set

$$E^r = \bigcup_{a \in E} B(a, r),$$

where $B(a, r)$ denotes the closed ball $\{z \in \mathbf{C}^N: |z - a| \leqslant r\}$, is contained in P together with its polynomially convex hull

$$\hat{E}^r = \left\{z \in \mathbf{C}^N: |p(z)| \leqslant \|p\|_E \text{ for all } p \in P_n(\mathbf{C}^N) \text{ and } n \geqslant 1\right\}.$$

Since we have (see [8])

$$\Phi_{B(a, r)}(z) = \max\{1, |z - a|/r\}, \qquad z \in \mathbf{C}^N, \tag{2(1)}$$

and

$$\Phi_{E_1}(z) \leqslant \Phi_{E_2}(z), \qquad z \in \mathbf{C}^N, \text{ whenever } E_1 \supset E_2, \tag{2(2)}$$

which immediately follows from the definition of the extremal function, the set E^r is L-regular and so is the set $\hat{E}^r$ because for every compact set $E \subset \mathbf{C}^N$ we have

$$\Phi_E(z) = \Phi_{\hat{E}}(z), \qquad z \in \mathbf{C}^N.$$

Consequently the set E of the lemma can be assumed to be L-regular. Then there exists an $R > 1$ such that

$$D_R = \left\{z \in \mathbf{C}^N: \Phi_E(z) < R\right\} \subset U.$$

Take any $b \in (1, R)$ and define

$$\mathcal{E}_b = \left\{f \in C(E): \sup_{n > 0} b^n \operatorname{dist}_E(f, P_n(\mathbf{C}^N)) < \infty\right\}.$$

$\mathcal{E}_b$ is a Banach space with the norm

$$|f|_b = \|f\|_E + \sup_{n > 0} b^n \operatorname{dist}_E(f, P_n(\mathbf{C}^N)).$$

By the above-mentioned result of Siciak [8, Theorem 10.2], there is a natural inclusion $\varphi_b: A(U) \to \mathcal{E}_b$. One can also easily check that the graph of the mapping φ_b is closed. Hence φ_b is continuous, which gives the result.

2.2. Given a compact set $E \subset \mathbf{C}^N$ let

$$\Phi_E^*(z) = \limsup_{w \to z} \Phi_E(w)$$

and

$$c(E) = \limsup_{|z| \to \infty} \left[|z| / \Phi_E^*(z) \right].$$

The number $c(E)$ is called the $\mathbf{C}^N$-*capacity* of E (see [9], [11]; if $N = 1$, $c(E)$ is equal to the logarithmic capacity of E). By 2(2), if $E_1 \subset E_2$, then $c(E_1) \leqslant c(E_2)$. If E is L-regular at a point $a \in C^N$, then by Inequality 1.3 and 2(1), $c(E) > 0$.

The following lemma will play a crucial role in the proof of the main result of this section (Proposition 2.9).

2.3. Lemma. *Suppose E is a polynomially convex compact set in $\mathbf{C}^N$ and F is a subset of E. Let h be a holomorphic mapping defined in an open neighborhood U of E, with values in $\mathbf{C}^M$ ($M \leqslant N$). Assume that $c(h(E)) > 0$. Then, if Property* (B) *holds for the quadruplet $(E, F, (P_n(\mathbf{C}^N)), (n))$, it also holds for the quadruplet $(E, F, (H_n), (n))$, where $H_n = \{p \circ h : p \in P_n(\mathbf{C}^M)\}$, $n = 1, 2, \ldots$.*

Proof. By Theorem 1.4 it suffices to show that Property (A) holds for $(E, F, (H_n), (n))$. To this aim take a function $f \in C(E)$ such that

$$\|f - p_n \circ h\|_E \leqslant Ca^n, \qquad n = 1, 2, \ldots, \tag{2(3)}$$

where $p_n \in P_n(\mathbf{C}^M)$, $C > 0$ and $a \in (0, 1)$, the constants C and a being independent of n. We wish to show that $f \in A_E(F)$. Observe that by 2(3), for each n,

$$\|p_n\|_{h(E)} = \|p_n \circ h\|_E \leqslant C + \|f\|_E.$$

We may obviously suppose that h is bounded in U. Since $c(h(E)) > 0$, we have

$$B = \sup\{\Phi_{h(E)}(w) : w \in h(U)\} < \infty.$$

Hence by Inequality 1.3,

$$B_n = \sup_{z \in U} |p_n(h(z))| = \sup_{w \in h(U)} |p_n(w)| \leqslant \|p_n\|_{h(E)} B^n,$$

whence

$$B_n \leqslant (C + \|f\|_E) B^n \leqslant D^n$$

for each n, with an appropriate constant $D > 0$. By Lemma 2.1 we can find constants $D_1 > 0$ and $d \in (0, 1)$ such that

$$\operatorname{dist}_E\left(p_n \circ h, P_k(\mathbf{C}^N)\right) \leqslant D_1 D^n d^k$$

for $k \geqslant 1$, $n \geqslant 1$. Then by setting $k = mn$, where the positive integer m is so chosen that $Dd^m \leqslant d$, we get

$$\operatorname{dist}_E\left(p_n \circ h, P_{mn}(\mathbf{C}^N)\right) \leqslant D_1 d^n, \qquad n \geqslant 1,$$

whence by 2(3),

$$\operatorname{dist}_E\big(f, P_{mn}(\mathbf{C}^N)\big) \leqslant D_2 d_1^n,$$

for $n \geqslant 1$, where $D_2 \geqslant \max(C, D_1)$ and $d_1 = \max(a, d) < 1$. Therefore, since for $mn \leqslant k < m(n + 1)$,

$$\operatorname{dist}_E\big(f, P_k(\mathbf{C}^N)\big) \leqslant \operatorname{dist}_E\big(f, P_{mn}(\mathbf{C}^N)\big) \leqslant D_2\big(d_1^{n/k}\big)^k \leqslant D_2 d_2^k$$

with $d_2 \in (0, 1)$ independent of n, we get

$$\limsup_{k\to\infty}\left[\operatorname{dist}_E\big(f, P_k(\mathbf{C}^N)\big)\right]^{1/k} < 1,$$

and since the quadruplet $(E, F, (P_k(\mathbf{C}^N)), (k))$ has Property (A), it follows that $f \in A_E(F)$, as claimed.

2.4. REMARK. It occurs that $c(E) = 0$ but $c(h(E)) > 0$, e.g. take $E = [0, 1] \times \{0\} \subset \mathbf{R}^2$ and set $h(z_1, z_2) = z_1$.

We shall need the following

2.5. LEMMA. *Let E be a compact subset of $\mathbf{C}^N$ with $c(E) > 0$. Suppose h is a holomorphic mapping in a connected open set U, $E \subset U$, with values in $\mathbf{C}^M (M \leqslant N)$, such that*

$$\sup\{c(F)\colon F \subset h(U), F \text{ compact}\} > 0.$$

Then $c(h(E)) > 0$.

PROOF. Suppose $c(h(E)) = 0$. Then $h(E)$ is globally C^M-polar (see [9, Corollary 3.9]), i.e. one could find a plurisubharmonic function p in $\mathbf{C}^M$ such that $p(w) = -\infty$ for $w \in h(E)$. Then the function $q = p \circ h$ is plurisubharmonic in U, and by the assumptions on h, $q \not\equiv -\infty$ in U. Since $q(z) = -\infty$ for $z \in E$, the set E is locally $\mathbf{C}^N$-polar, whence by a recent result of Josefson [3], it should be globally $\mathbf{C}^N$-polar. Consequently by [9, Theorem 3.10], we would have $c(E) = 0$, a contradiction.

2.6. REMARK. The assumption that f is nonconstant in U is not sufficient for the above lemma to hold. Take, e.g., $E = \{(x_1, x_2, x_3)\colon 0 \leqslant x_i \leqslant 1,\ i = 1, 2, 3\} \subset \mathbf{R}^3$ and $h(z_1, z_2, z_3) = (z_1, z_1)$.

2.7. Given a compact set E in $\mathbf{C}^N$ let h be a holomorphic mapping in an open neighborhood U of E, with values in $\mathbf{C}^M$ $(M \leqslant N)$. In the sequel we shall be interested in h such that for a given subset F of E, the triplet (h, E, F) satisfies the following hypothesis:

(H) For each $a \in F$ and each bounded open set V such that $F \subset V \subset \overline{V} \subset U$, the set $h(\overline{V})$ is L-regular at $h(a)$.

Notice that if the mapping h is open in an open neighborhood W of F then by 2(1) and 2(2) the triplet (h, E, F) satisfies (H). In particular, if $M = 1$ and h is nonconstant in any connected component W of U such that $W \cap F \neq \varnothing$, then by the open mapping theorem for holomorphic functions the triplet (h, E, F) satisfies (H).

An example of a triplet (h, E, F) satisfying (H) with a nonopen h is given by $h(z_1, z_2) = (z_1, z_1 z_2)$ for $(z_1, z_2) \in \mathbf{C}^2$ and $E = F = [0, 1] \times [0, 1]$.

From Lemma 2.5 we derive

2.8. COROLLARY. *If $h: U \to \mathbf{C}^M$ $(M \leqslant N)$ is holomorphic and U is connected, and the triplet (h, E, E) satisfies* (H), *then $c(E) > 0$ implies $c(h(E)) > 0$.*

More generally, if there exists a point $a \in E$ such that $(h, E, \{a\})$ satisfies (H), *and for the connected component V_a of the set U which contains a, we have $c(V_a \cap E) > 0$, then $c(h(E)) > 0$.*

Now we can prove

2.9. PROPOSITION. *Let E be a polynomially convex, compact set in $\mathbf{C}^N$ and h a holomorphic mapping in an open set $U \supset E$, with values in $\mathbf{C}^M$ $(M \leqslant N)$. Assume that $c(h(E)) > 0$. Then, for any subset F of E:*

1°. *If $(E, F, (P_n(\mathbf{C}^N)), (n))$ has Property* (B) *and the triplet (h, E, F) satisfies* (H), *and $N_E(F) = 0$, then the quadruplet $(h(E), h(F), (P_n(\mathbf{C}^M)), (n))$ has Property* (B′).

2°. *If $M = N$ and h is a biholomorphism, then if $(E, F, (P_n(\mathbf{C}^N)), (n))$ has Property* (B) *then the quadruplet $(h(E), h(F), (P_n(\mathbf{C}^M)), (n))$ also has this property.*

PROOF. In both 1° and 2°, if $(E, F, (P_n(\mathbf{C}^N)), (n))$ satisfies (B), then by virtue of Lemma 2.3, so does the quadruplet $(E, F, (H_n), (n))$, where $H_n = \{p \circ h: p \in P_n(\mathbf{C}^M)\}$. It follows that for each $b > 1$ there exist a bounded open set V, $F \subset V \subset \bar{V} \subset U$, and a constant $C > 0$ such that for each n and each $p \in P_n(\mathbf{C}^M)$ one can find $g_p \in N_E(V)$ such that

$$\sup_{z \in V} |p(h(z)) + g_p(z)| \leqslant Cb^{n/2}\|p \circ h\|_E. \tag{2(4)}$$

Now with the assumptions of case 1°, it follows that $g_p = 0$, whence

$$\sup_{w \in h(\bar{V})} |p(w)| = \sup_{z \in \bar{V}} |p(h(z))| \leqslant Cb^{n/2}\|p \circ h\|_E = Cb^{n/2}\|p\|_{h(E)}.$$

The set $h(\bar{V})$ need not be a neighborhood of $h(F)$. However, by (H), there is an open neighborhood W of $h(F)$ such that

$$\Phi_{h(\bar{V})}(w) < b^{1/2}, \qquad w \in W,$$

and then by Inequality 1.3 we get

$$\sup_{w \in W} |p(w)| \leqslant Cb^n\|p\|_{h(E)},$$

which completes the proof of case 1°.

2°. Since h is a biholomorphism, the set $W = h(V)$ is an open neighborhood of $h(F)$ for each open set V such that $F \subset V \subset U$, and by 2(4) we get

$$\sup_{w \in W} |p(w) + g_p(h^{-1}(w))| \leqslant Cb^n\|p\|_{h(E)},$$

whence since $g_p \circ h^{-1} \in N_{h(E)}(h(F))$, we get the result.

3. Applications to the L-regularity. In this section our attention will be devoted to Properties (B) and (B′) in the case where $H_n = P_n(\mathbf{C}^N)$ and $m_n = n$, $n = 1, 2, \ldots$. Then, given a compact set $E \subset \mathbf{C}^N$ and a subset F of E, we shall shortly write $(E, F) \in (\mathrm{B})$ (resp. $(E, F) \in (\mathrm{B}')$) if Property (B) (resp. (B′)) holds for $(E, F, (P_n(\mathbf{C}^N)), (n))$.

3.1. We note that $(E, F) \in (\mathrm{B}')$ if and only if E is L-regular at every point $a \in F$.

3.2. Proposition. 1°. *For any polynomially convex compact set E in $\mathbf{C}^N$, $(E, E) \in (\mathrm{B}')$ if and only if $(E, E) \in (\mathrm{B})$ and $N_E(E) = 0$.*

2°. *If $E \subset \mathbf{R}^N$ then for any subset F of E, $(E, F) \in (\mathrm{B}')$ if and only if $(E, F) \in (B)$ and $N_E(F) = 0$.*

Proof. By virtue of (1.2) in both 1° and 2° it suffices to prove that $(E, F) \in (\mathrm{B}')$ implies $N_E(F) = 0$.

1°. Take a function $f \in N_E(U)$, where U is an open neighborhood of E. Since $(E, E) \in (\mathrm{B}')$, the set E is L-regular, whence for each $R > 1$ the set

$$E_R = \{z \in \mathbf{C}^N \colon \Phi_E(z) < R\}$$

is compact and $E \subset \operatorname{int} E_R$. Since $E = \hat{E}$, the polynomially convex hull of E, we can find $R > 1$ such that $E_R = \hat{E}_R \subset U$. Then by Lemma 2.1 there exist constants $C > 0$ and $a \in (0, 1)$, and a sequence of polynomials $p_n \in P_n(\mathbf{C}^N)$ $(n = 1, 2, \ldots)$, such that

$$\|f - p_n\|_{E_R} \leqslant Ca^n, \qquad n = 1, 2, \ldots.$$

If $1 < R' \leqslant \min(R, 2/(1 + a))$, then $E_{R'} \subset E_R$ and since $f = 0$ on E, by Inequality 1.3 we get

$$\|f\|_{E_{R'}} \leqslant \|f - p_n\|_{E_{R'}} + \|p_n\|_{E_{R'}} \leqslant Ca^n + \|p_n\|_E (R')^n \leqslant C\big[a^n + (2a/(1 + a))^n\big]$$

for $n = 1, 2, \ldots$, whence $f = 0$ on $E_{R'}$, and, consequently, $N_E(E) = 0$.

2°. It suffices to show that if E is L-regular at $b \in E$ then $N_E(b) = 0$. To do this take a function $f \in N_E(U)$, where U is an open neighborhood of b. We can find three bounded closed parallelepipeds K_1, K_2, and K_3, such that $b \in \operatorname{int} K_1$, $K_i \subset \operatorname{int} K_{i+1}$, $i = 1, 2$, $K_2 \subset U$ and $E \subset K_3$. Then by Lemma 2.1 there exist polynomials $p_n \in P_n(\mathbf{C}^N)$ $(n = 1, 2, \ldots)$ such that

$$\|f - p_n\|_{K_2} \leqslant Ca^n$$

with $C > 0$ and $a \in (0, 1)$, both C and a independent of n. By [6, Lemma 12.3], there exist polynomials $l_k \in P_k(\mathbf{C}^N)$ $(k = 1, 2, \ldots)$ and constants $D > 0$ and $d \in (0, 1)$ such that

$$\|l_k - 1\|_{K_1} \leqslant Dd^k, \quad \|l_k\|_{K_3 \setminus \operatorname{int} K_2} \leqslant Dd^k \quad \text{and} \quad \|l_k\|_{K_2} \leqslant Dk^N$$

for $k = 1, 2, \ldots$. Write $r_{k,n} = l_k p_n$ $(k \geqslant 1, n \geqslant 1)$. By Inequality 1.3 there is a constant $A > 0$ such that

$$\|p_n\|_{K_3} \leq A^n, \quad n = 1, 2, \ldots.$$

Then for each $k \geq 1$ and $n \geq 1$, we have

$$\|r_{k,n}\|_{E \cap (K_3 \setminus \operatorname{int} K_2)} \leq \|l_k\|_{K_3 \setminus \operatorname{int} K_2} \|p_n\|_{K_3} \leq DA^n d^k,$$

$$\|r_{k,n}\|_{E \cap (K_2 \setminus \operatorname{int} K_1)} \leq Dk^N \|p_n\|_{E \cap K_2} \leq DCk^N a^n$$

and

$$\|r_{k,n}\|_{E \cap K_1} \leq (D + 1)Ca^n.$$

Now choose an integer $m > 0$ such that $Ad^m \leq e = \max(a, d) < 1$ and set $r_n = r_{mn,n}$ for $n = 1, 2, \ldots$. Then by the above inequalities there exists $n_0 > 0$ such that

$$\|r_n\|_E \leq D_1 e_1^n \quad \text{for } n \geq n_0,$$

with an appropriate constant $D_1 > 0$ and $e_1 = (e + 1)/2$. Hence by Inequality 1.3, we can find a compact neighborhood V of the point b, $V \subset K_1$, such that

$$\|r_n\|_V \to 0, \quad \text{as } n \to \infty.$$

On the other hand, for each n,

$$\|r_n - p_n\|_{K_1} \leq \|l_{mn} - 1\|_{K_1} \|p_n\|_{K_1} \leq De^n,$$

whence

$$\|f\|_V \leq \|f - p_n\|_V + \|p_n - r_n\|_V + \|r_n\|_V \to 0$$

as $n \to \infty$, which yields $f = 0$ on V. This gives the result.

3.3. Remark. If E is not polynomially convex, then Proposition 3.2(1°) fails to hold (take, e.g., for $N = 1$, $E = \{|z| = 1\} \cup \{0\}$; then $\Phi_E(z) = \max(1, |z|)$ but $N_E(E) \neq 0$).

We also note that by the Stone-Weierstrass theorem, if $E \subset \mathbf{R}^N$ then E is polynomially convex.

3.4. *Question*. For any compact set $E = \hat{E} \subset \mathbf{C}^N$ does the L-regularity of E at a point $a \in E$ imply $N_E(a) = 0$?

We note that this is the case when $N = 1$ (see Remark 3.10 and the proof of Theorem 3.12).

Now we can prove the main result of this paper.

3.5. Theorem. *Let E be a polynomially convex, L-regular compact set in $\mathbf{C}^N$ and h a holomorphic mapping in an open neighborhood U of E, with values in $\mathbf{C}^M$ ($M \leq N$), such that the triplet* (h, E, E) *satisfies* (H). *Then $h(E)$ is L-regular.*

Proof. Since E is L-regular, then by 2(1) and Inequality 1.3, $c(E) > 0$. By Corollary 2.8 we then have $c(h(E)) > 0$, and by Proposition 3.2(1°) we get

$N_E(E) = 0$. Therefore by Proposition 2.9(1°) and by 3.1, the set $h(E)$ is L-regular.

3.6. COROLLARY. *If $E = \hat{E} \subset \mathbf{C}^N$ is L-regular and h: $E \subset U \to \mathbf{C}^1$ is holomorphic and nonconstant in any connected component W of U such that $W \cap E \neq \varnothing$, then the set $h(E)$ is also L-regular.*

3.7. REMARK. If $E \neq \hat{E}$, then Theorem 3.5 fails to hold; take, e.g., $E = \{|z| = 1\} \cup \{\frac{1}{2}\} \subset \mathbf{C}^1$ and $h(z) = z^{-1}$; then $h(E)$ is not L-regular at $2 \in h(E)$.

Now we wish to give local versions of Theorem 3.5. Owing to Proposition 3.2(2°), by a similar argument to that of the proof of Theorem 3.5 we get

3.8. THEOREM. *If a compact set $E \subset \mathbf{R}^N$ is L-regular at a point $a \in E$ and h: $E \subset U \to \mathbf{C}^M$ $(M \leqslant N)$ is holomorphic in U and such that the triplet $(h, E, \{a\})$ satisfies* (H), *then $h(E)$ is L-regular at $h(a)$.*

We cannot prove Theorem 3.8 for any compact set $E = \hat{E} \subset \mathbf{C}^N$ (see Question 3.4). Nevertheless we shall give a little weaker (or equivalent–see Remark 3.10 and Question 3.11) version of this result.

3.9. DEFINITION (compare [9]). A compact set $E \subset \mathbf{C}^N$ is said to satisfy *condition* (L^1) at a point $a \in E$, if for each $r > 0$ the set $E \cap B(a, r)$ is L-regular at a; $B(a, r)$ being the closed ball with centre a and radius r.

3.10. REMARK. It is obvious that if E satisfies (L^1) at $a \in E$, then it is L-regular at a. Conversely, if $N = 1$ and $a \in E = \hat{E}$ (or $a \in \partial(\mathbf{C}^1 \setminus E)$, if $E \neq \hat{E}$) then the L-regularity of E at a implies that E satisfies (L^1) at a (see e.g. [4, (5.1.15′)]).

3.11. *Question*. Suppose $E = \hat{E} \subset \mathbf{C}^N$ is L-regular at $a \in E$. Does E then have to satisfy (L^1) at a?

3.12. THEOREM. *With the assumptions of Theorem* 3.8 *on h, for any compact set E in $\mathbf{C}^N$, if E satisfies* (L^1) *at $a \in E$, so does the set $h(E)$ at $h(a)$.*

PROOF. If E satisfies (L^1) at $a \in E$, then for each $r > 0$, $c(E \cap B(a, r)) > 0$, and by Corollary 2.8 we have $c(h(E)) > 0$.

Moreover, it follows from Lemma 2.1 and Inequality 1.3 that for each $r > 0$, $N_{E \cap B(a, r)}(a) = 0$ (see the proof of Proposition 3.2). For each $r > 0$ there exists $s > 0$ such that

$$h(E \cap B(a, s)) \subset h(E) \cap B(h(a), r).$$

Hence, since by Proposition 2.9(1°) the set $h(E \cap B(a, s))$ is L-regular at $h(a)$, by 2(2) so is the set $h(E) \cap B(h(a), r)$, which means that $h(E)$ satisfies (L^1) at $h(a)$, as claimed.

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