INVARINANCE OF THE $L$-REGULARITY OF COMPACT SETS
IN $C^N$ UNDER HOLOMORPHIC MAPPINGS

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ABSTRACT. The property for a polynomially convex compact set $E$ in $C^N$
that the Siciak extremal function $\Phi_E$ be continuous or, equivalently, that $E$
satisfy some Bernstein type inequality, is proved to be invariant under a
large class of holomorphic mappings with values in $C^M (M < N)$ including
all open holomorphic mappings. Local specifications of this result are also
given.

0. Introduction. Let $E$ be a polynomially convex compact set in $C^N$ and let
$P_n (C^N)$ denote the space of all polynomials from $C^N$ to $C^1$ of degree at most
$n$. It is known [8] that if $f$ is a holomorphic function in a neighborhood of $E$
then

$$\lim_{n \to \infty} \left[ \text{dist}_E(f, P_n(C^N)) \right]^{1/n} < 1,$$

where the distance from $f$ to $P_n(C^N)$ is taken in the sense of the supremum
norm on $E$, denoted by $\| \|_E$.

Conversely, in order that each continuous function $f$ defined on $E$ and
satisfying there (*) be continuable to a holomorphic function in a neigh-
borhood of $E$, the compact set $E$ must satisfy some regularity conditions (see
[1]) of the type of Bernstein's (or Markov's) inequality for polynomials, and in
the case where the compact set $E$ is sufficiently big, they are equivalent to the
continuity in $C^N$ of Siciak's extremal function of $E$ (see [8] and [9]):

$$\Phi_E(z) = \sup\{|p(z)|^{1/n}: p \in P_n(C^N), \|p\|_E < 1, n > 1\}$$

for $z \in C^N$. In the sequel a compact set $E \subset C^N$, for which the function $\Phi_E$
is continuous in $C^N$, is said to be $L$-regular (compare [9]). By a result of
Zaharjuta [11] (see also [9]), in order that $E$ be $L$-regular it suffices that the
function $\Phi_E$ be continuous in $E$.

If $N = 1$, the function $\log \Phi_E$ is known to be equal to the Green function
of the unbounded component of the set $C^1 \setminus E$ with pole at $\infty$, and hence the
question about the $L$-regularity of compact sets in $C^1$ is well explored. Incomparably less is known about this problem in the case where $N > 1$. Some criteria of $L$-regularity can be found in [8], [9] and [1]. Additional information on the class of $L$-regular subsets of $C^N$ can be derived from the main result of this paper (Theorem 3.5) saying that the property for a polynomially convex, compact set $E$ in $C^N$ to be $L$-regular is invariant under a large class $\mathcal{F}$ of holomorphic mappings in a neighborhood of $E$, with values in $C^M$ ($M < N$), including, in particular, all open holomorphic mappings. Thus if $M = 1$, the class $\mathcal{F}$ consists of all nonconstant holomorphic functions in a neighborhood of $E$. This result has been probably unknown even in the case where $N = 1$.

Next we give local specifications of the main result and examine the invariance under holomorphic mappings of the property for $E$ to be $L$-regular at a point $a \in E$, which stands, by definition, for the continuity of the extremal function $\Phi_E$ at $a$. Here we distinguish two cases (Theorems 3.8 and 3.12) according as $E$ is a compact subset of $R^N$ or $C^N$ and leave open the problem of equivalence of both results (see Remark 3.10 and Question 3.11).

1. Properties (A) and (B).

1.1. Let $E$ be a compact set in $C^N$ and let $C(E)$ denote the Banach space of complex-valued continuous functions defined on $E$ with the supremum norm $\| \|_E$. Given an open set $U$ in $C^N$ let $\Theta_E(U)$ be the Banach space of all bounded continuous functions defined on $E \cup U$ holomorphic in $U$, with the supremum norm on $E \cup U$. We denote by $N_E(U)$ the kernel of the natural restriction $r_U: \Theta_E(U) \ni f \to f|_E \in C(E)$ and by $A_E(U)$ its range endowed with the quotient topology $\Theta_E(U)/N_E(U)$.

Given a subset $F$ of $E$ we define

$$\Theta_E(F) = \operatorname{ind lim}_{U \supset E, \text{open}} \Theta_E(U)$$

and

$$N_E(F) = \operatorname{ind lim}_{U \supset F, \text{open}} N_E(U).$$

The mappings $r_U$ define the restriction $r: \Theta_E(F) \to C(E)$ with its kernel $N_E(F)$ and its range

$$A_E(F) = \operatorname{ind lim}_{U \supset F, \text{open}} A_E(U).$$

Let us consider an increasing sequence $(H_n)$ of vector subspaces of the space $\Theta_E(F)$ and a nondecreasing sequence $(m_n)$ of positive numbers. Following Baouendi and Goulaouic [1] (see also [10]) we define properties (A) and (B) of the quadruplet $(E, F, (H_n), (m_n))$ as follows.
PROPERTY (A). For any function \( f \in C(E) \), if

\[
\limsup_{n \to \infty} \left[ \text{dist}_E(f, r(H_n)) \right]^{1/m_n} < 1,
\]

then \( f \in A_E(F) \).

PROPERTY (B). For any real number \( b > 1 \) there exist an open neighborhood \( U \) of \( F \) and a constant \( C > 0 \) such that for any \( h \in H_n \) \( (n = 1, 2, \ldots) \) there exists \( g \in N_E(F) \) for which \( h + g \in \partial_E(U) \) and

\[
\sup_{z \in U} |h(z) + g(z)| < Cb^{m_n}\|h\|_E.
\]

1.2. If the compact sets \( E \) and \( F \) are so big that \( N_E(F) = 0 \), then Property (B) of the quadruplet \( (E, F, (H^n), (m_n)) \) yields the following:

PROPERTY (B'). For any real number \( b > 1 \) there exist an open neighborhood \( U \) of \( F \) and a constant \( C > 0 \) such that each \( h \in H^n \) \( (n = 1, 2, \ldots) \) belongs to \( \partial_E(U) \) and

\[
\sup_{z \in U} |h(z)| < Cb^{m_n}\|h\|_E.
\]

In some special cases we shall also prove that (B') implies (B) and \( N_E(F) = 0 \) (see Proposition 3.2).

We note that Property (B') is an analogue of the well-known Bernstein-Walsh inequality for polynomials which can be written in terms of the extremal function \( \Phi_E \) as follows.

1.3. INEQUALITY. For any polynomial \( p \in P_n(C^N) \) \( (n = 1, 2, \ldots) \), we have

\[
|p(z)| < \|p\|_E[\Phi_E(z)]^n, \quad z \in C^N.
\]

An important role in our considerations will be played by the following theorem due to Baouendi and Goulaouic [1] (the case where \( E = F \subset R^N \)).

1.4. THEOREM. Assume that for each \( a \in (0, 1) \), \( \sum_{n=1}^{\infty} a^{m_n} < \infty \) and \( \limsup_{n \to \infty} m_{n+1}/m_n < \infty \). Then Properties (A) and (B) for the quadruplet \( (E, F, (H^n), (m_n)) \) are equivalent.

The proof of the implication (B) \( \Rightarrow \) (A) is easy while the converse implication can be proved in the same manner as in [10] (the case where \( H_n = P_n(C^N) \) and \( m_n = n, n = 1, 2, \ldots \)).

1.5. REMARK. Notice that we have not assumed the spaces \( H_n \) of Theorem 1.4 to be finite dimensional. However it can be proved in the case where \( E = F \) is polynomially convex that Property (B) holds only if \( \dim r(H_n) = O(m_n^N) \), as \( n \to \infty \) (see [7]).

2. Invariance of Property (B) under holomorphic mappings. We start with a lemma which is a version for families of holomorphic functions of a theorem due to Siciak [8, Theorem 10.2] being a generalization to \( C^N \) of a known Bernstein-Walsh Theorem. The lemma was first stated in [5]. Here we give its
elegant proof communicated to us by Siciak. Another proof can be derived from Theorem 2 in [11].

2.1. LEMMA. Let $A(U)$ denote the Banach space of all bounded holomorphic functions defined in an open set $U$ in $\mathbb{C}^N$, equipped with the supremum norm on $U$, $\| \|_U$. For each polynomially convex compact set $E \subset U$ there exist constants $C > 0$ and $a \in (0, 1)$, both $C$ and $a$ independent of $f \in A(U)$ and $n$, such that

$$\text{dist}_E(f, P_n(\mathbb{C}^N)) < C\|f\|_U a^n$$

for all $f$ in $A(U)$ and $n = 1, 2, \ldots$.

PROOF. Given a polynomially convex compact set $E \subset U$, we can find a polynomial polyhedron $P \subset U$ such that $E \subset \text{int } P$ (see e.g. [2, Lemma 2.7.4]). Then for sufficiently small $r > 0$, the set

$$E_r = \bigcup_{a \in E} B(a, r),$$

where $B(a, r)$ denotes the closed ball $\{z \in \mathbb{C}^N: |z - a| < r\}$, is contained in $P$ together with its polynomially convex hull

$$\hat{E}_r = \{z \in \mathbb{C}^N: |p(z)| < \|p\|_E \text{ for all } p \in P_n(\mathbb{C}^N) \text{ and } n > 1\}.$$

Since we have (see [8])

$$\Phi_{B(a, r)}(z) = \max\{1, |z - a|/r\}, \quad z \in \mathbb{C}^N, \quad 2(1)$$

and

$$\Phi_{E_1}(z) < \Phi_{E_2}(z), \quad z \in \mathbb{C}^N, \text{ whenever } E_1 \supset E_2, \quad 2(2)$$

which immediately follows from the definition of the extremal function, the set $E'$ is $L$-regular and so is the set $\hat{E}'$ because for every compact set $E \subset \mathbb{C}^N$ we have

$$\Phi_E(z) = \Phi_{\hat{E}}(z), \quad z \in \mathbb{C}^N.$$

Consequently the set $E$ of the lemma can be assumed to be $L$-regular. Then there exists an $R > 1$ such that

$$D_R = \{z \in \mathbb{C}^N: \Phi_E(z) < R\} \subset U.$$

Take any $b \in (1, R)$ and define

$$\mathcal{E}_b = \left\{f \in C(E): \sup_{n > 0} b^n \text{ dist}_E(f, P_n(\mathbb{C}^N)) < \infty \right\}.$$

$\mathcal{E}_b$ is a Banach space with the norm

$$|f|_b = \|f\|_E + \sup_{n > 0} b^n \text{ dist}_E(f, P_n(\mathbb{C}^N)).$$

By the above-mentioned result of Siciak [8, Theorem 10.2], there is a natural inclusion $\varphi_b: A(U) \to \mathcal{E}_b$. One can also easily check that the graph of the mapping $\varphi_b$ is closed. Hence $\varphi_b$ is continuous, which gives the result.
2.2. Given a compact set \( E \subset \mathbb{C}^n \) let
\[
\Phi_E^*(z) = \limsup_{w \to z} \Phi_E(w)
\]
and
\[
c(E) = \limsup_{|z| \to \infty} \left\lfloor \frac{|z|}{\Phi_E^*(z)} \right\rfloor.
\]
The number \( c(E) \) is called the \( \mathbb{C}^n \)-capacity of \( E \) (see [9], [11]; if \( N = 1 \), \( c(E) \) is equal to the logarithmic capacity of \( E \)). By 2(2), if \( E_1 \subset E_2 \), then \( c(E_1) \leq c(E_2) \). If \( E \) is \( L \)-regular at a point \( a \in \mathbb{C}^N \), then by Inequality 1.3 and 2(1), \( c(E) > 0 \).

The following lemma will play a crucial role in the proof of the main result of this section (Proposition 2.9).

2.3. **Lemma.** Suppose \( E \) is a polynomially convex compact set in \( \mathbb{C}^N \) and \( F \) is a subset of \( E \). Let \( h \) be a holomorphic mapping defined in an open neighborhood \( U \) of \( E \), with values in \( \mathbb{C}^M \) (\( M < N \)). Assume that \( c(h(E)) > 0 \). Then, if Property (B) holds for the quadruplet \( (E, F, (P_n(\mathbb{C}^N)), (n)) \), it also holds for the quadruplet \( (E, F, (H_n), (n)) \), where \( H_n = \{ p \circ h : p \in P_n(\mathbb{C}^M) \} \), \( n = 1, 2, \ldots \).

**Proof.** By Theorem 1.4 it suffices to show that Property (A) holds for \( (E, F, (H_n), (n)) \). To this aim take a function \( f \in C(E) \) such that
\[
\|f - p_n \circ h\|_E < C a^n, \quad n = 1, 2, \ldots, \tag{2.3}
\]
where \( p_n \in P_n(\mathbb{C}^M) \), \( C > 0 \) and \( a \in (0, 1) \), the constants \( C \) and \( a \) being independent of \( n \). We wish to show that \( f \in A_E(F) \). Observe that by 2(3), for each \( n \),
\[
\|p_n\|_{h(E)} = \|p_n \circ h\|_E < C + \|f\|_E.
\]
We may obviously suppose that \( h \) is bounded in \( U \). Since \( c(h(E)) > 0 \), we have
\[
B = \sup\{\Phi_{h(E)}(w) : w \in h(U)\} < \infty.
\]
Hence by Inequality 1.3,
\[
B_n = \sup_{z \in U} |p_n(h(z))| = \sup_{w \in h(U)} |p_n(w)| < \|p_n\|_{h(E)} B^n,
\]
whence
\[
B_n < (C + \|f\|_E) B^n < D^n
\]
for each \( n \), with an appropriate constant \( D > 0 \). By Lemma 2.1 we can find constants \( D_1 > 0 \) and \( d \in (0, 1) \) such that
\[
\text{dist}_E(p_n \circ h, P_k(\mathbb{C}^N)) < D_1 D^n d^k
\]
for \( k > 1, n > 1 \). Then by setting \( k = mn \), where the positive integer \( m \) is so chosen that \( D d^m < d \), we get
\[
\text{dist}_E(p_n \circ h, P_{mn}(\mathbb{C}^N)) < D_1 d^n, \quad n > 1,
\]
whence by 2(3),
\[ \text{dist}_E(f, P_{mn}(C^N)) < D_2d_1^n, \]
for \( n > 1 \), where \( D_2 > \max(C, D_1) \) and \( d_1 = \max(a, d) < 1 \). Therefore, since for \( mn < k < m(n + 1) \),
\[ \text{dist}_E(f, P_k(C^N)) < \text{dist}_E(f, P_{mn}(C^N)) < D_2(d_1^{n/k})^k < D_2d_2^k \]
with \( d_2 \in (0, 1) \) independent of \( n \), we get
\[ \limsup_{k \to \infty} \left[ \text{dist}_E(f, P_k(C^N)) \right]^{1/k} < 1, \]
and since the quadruplet \((E, F, (P_k(C^N)), (k))\) has Property (A), it follows that \( f \in A_E(f) \), as claimed.

2.4. Remark. It occurs that \( c(E) = 0 \) but \( c(h(E)) > 0 \), e.g. take \( E = [0, 1] \times \{0\} \subset \mathbb{R}^2 \) and set \( h(z_1, z_2) = z_1 \).

We shall need the following

2.5. Lemma. Let \( E \) be a compact subset of \( C^N \) with \( c(E) > 0 \). Suppose \( h \) is a holomorphic mapping in a connected open set \( U, E \subset U \), with values in \( C^M(M < N) \), such that
\[ \sup\{ c(7^*) : F \subset h(U), F \text{ compact} \} > 0. \]
Then \( c(h(E)) > 0 \).

Proof. Suppose \( c(h(E)) = 0 \). Then \( h(E) \) is globally \( C^M \)-polar (see [9, Corollary 3.9]), i.e. one could find a plurisubharmonic function \( p \) in \( C^M \) such that \( p(w) = -\infty \) for \( w \in h(E) \). Then the function \( q = p \circ h \) is plurisubharmonic in \( U \), and by the assumptions on \( h \), \( q \equiv -\infty \) in \( U \). Since \( q(z) = -\infty \) for \( z \in E \), the set \( E \) is locally \( C^N \)-polar, whence by a recent result of Josefson [3], it should be globally \( C^N \)-polar. Consequently by [9, Theorem 3.10], we would have \( c(E) = 0 \), a contradiction.

2.6. Remark. The assumption that \( f \) is nonconstant in \( U \) is not sufficient for the above lemma to hold. Take, e.g., \( E = \{(x_1, x_2, x_3) : 0 < x_i < 1, \ i = 1, 2, 3\} \subset \mathbb{R}^3 \) and \( h(z_1, z_2, z_3) = (z_1, z_1) \).

2.7. Given a compact set \( E \) in \( C^N \) let \( h \) be a holomorphic mapping in an open neighborhood \( U \) of \( E \), with values in \( C^M \) \((M < N)\). In the sequel we shall be interested in \( h \) such that for a given subset \( F \) of \( E \), the triplet \((h, E, F)\) satisfies the following hypothesis:

\( (H) \) For each \( a \in F \) and each bounded open set \( V \) such that \( F \subset V \subset \overline{V} \subset U \), the set \( h(V) \) is \( L \)-regular at \( h(a) \).

Notice that if the mapping \( h \) is open in an open neighborhood \( W \) of \( F \) then by 2(1) and 2(2) the triplet \((h, E, F)\) satisfies \((H)\). In particular, if \( M = 1 \) and \( h \) is nonconstant in any connected component \( W \) of \( U \) such that \( W \cap F \neq \emptyset \), then by the open mapping theorem for holomorphic functions the triplet \((h, E, F)\) satisfies \((H)\).
An example of a triplet \((h, E, F)\) satisfying (H) with a nonopen \(h\) is given by \(h(z_1, z_2) = (z_1, z_1z_2)\) for \((z_1, z_2) \in \mathbb{C}^2\) and \(E = F = [0, 1] \times [0, 1]\).

From Lemma 2.5 we derive

2.8. **Corollary.** If \(h: U \to \mathbb{C}^M (M < N)\) is holomorphic and \(U\) is connected, and the triplet \((h, E, E)\) satisfies (H), then \(c(E) > 0\) implies \(c(h(E)) > 0\).

More generally, if there exists a point \(a \in E\) such that \((h, E, \{a\})\) satisfies (H), and for the connected component \(V_a\) of the set \(U\) which contains \(a\), we have \(c(V_a \cap E) > 0\), then \(c(h(E)) > 0\).

Now we can prove

2.9. **Proposition.** Let \(E\) be a polynomially convex, compact set in \(\mathbb{C}^N\) and \(h\) a holomorphic mapping in an open set \(U \supset E\), with values in \(\mathbb{C}^M (M < N)\). Assume that \(c(h(E)) > 0\). Then, for any subset \(F\) of \(E\):

1°. If \((E, F, (P_n(\mathbb{C}^N)), (\gamma))\) has Property (B) and the triplet \((h, E, F)\) satisfies (H), and \(N_E(F) = 0\), then the quadruplet \((h(E), h(F), (P_n(\mathbb{C}^M)), (\gamma))\) has Property (B').

2°. If \(M = N\) and \(h\) is a biholomorphism, then if \((E, F, (P_n(\mathbb{C}^N)), (\gamma))\) has Property (B) then the quadruplet \((h(E), h(F), (P_n(\mathbb{C}^M)), (\gamma))\) also has this property.

**Proof.** In both 1° and 2°, if \((E, F, (P_n(\mathbb{C}^N)), (\gamma))\) satisfies (B), then by virtue of Lemma 2.3, so does the quadruplet \((E, F, (H_n), (\gamma)),\) where \(H_n = \{p \circ h: p \in P_n(\mathbb{C}^M)\}\). It follows that for each \(b > 1\) there exist a bounded open set \(V, F \subset V \subset \overline{V} \subset U\), and a constant \(c > 0\) such that for each \(n\) and each \(p \in P_n(\mathbb{C}^M)\) one can find \(g_p \in N_E(V)\) such that

\[
\sup_{z \in V} |p(h(z)) + g_p(z)| < Cb^{n/2}\|p \circ h\|_E. \tag{2(4)}
\]

Now with the assumptions of case 1°, it follows that \(g_p = 0\), whence

\[
\sup_{w \in h(V)} |p(w)| = \sup_{z \in V} |p(h(z))| < Cb^{n/2}\|p \circ h\|_E = Cb^{n/2}\|p\|_{n(E)}.
\]

The set \(h(V)\) need not be a neighborhood of \(h(F)\). However, by (H), there is an open neighborhood \(W\) of \(h(F)\) such that

\[
\Phi_{h(V)}(w) < b^{1/2}, \quad w \in W,
\]

and then by Inequality 1.3 we get

\[
\sup_{w \in W} |p(w)| < Cb^n\|p\|_{n(E)},
\]

which completes the proof of case 1°.

2°. Since \(h\) is a biholomorphism, the set \(W = h(V)\) is an open neighborhood of \(h(F)\) for each open set \(V\) such that \(F \subset V \subset U\), and by 2(4) we get

\[
\sup_{w \in W} |p(w) + g_p(h^{-1}(w))| < Cb^n\|p\|_{n(E)}.
\]
whence since \( g_p \circ h^{-1} \in N_{h(E)}(h(F)) \), we get the result.

3. Applications to the \( L \)-regularity. In this section our attention will be devoted to Properties (B) and (B') in the case where \( H_n = P_n(C^N) \) and \( m_n = n, n = 1, 2, \ldots \) Then, given a compact set \( E \subset C^N \) and a subset \( F \) of \( E \), we shall shortly write \((E, F) \in (B) \) (resp. \((E, F) \in (B') \)) if Property (B) (resp. (B')) holds for \((E, F, (P_n(C^N)), (n)) \).

3.1. We note that \((E, F) \in (B') \) if and only if \( E \) is \( L \)-regular at every point \( a \in F \).

3.2. Proposition. 1°. For any polynomially convex compact set \( E \) in \( C^N \), \((E, E) \in (B') \) if and only if \((E, E) \in (B) \) and \( N_E(E) = 0 \).

2°. If \( E \subset R^N \) then for any subset \( F \) of \( E \), \((E, F) \in (B') \) if and only if \((E, F) \in (B) \) and \( N_E(F) = 0 \).

Proof. By virtue of (1.2) in both 1° and 2° it suffices to prove that \((E, F) \in (B') \) implies \( N_E(F) = 0 \).

1°. Take a function \( f \in N_E(U) \), where \( U \) is an open neighborhood of \( E \). Since \((E, E) \in (B') \), the set \( E \) is \( L \)-regular, whence for each \( R > 1 \) the set

\[ E_R = \{ z \in C^N : \Phi_E(z) < R \} \]

is compact and \( E \subset \text{int } E_R \). Since \( E = E_R \), the polynomially convex hull of \( E \), we can find \( R > 1 \) such that \( E_R = E \subset U \). Then by Lemma 2.1 there exist constants \( C > 0 \) and \( a \in (0, 1) \), and a sequence of polynomials \( p_n \in P_n(C^N) \) \((n = 1, 2, \ldots) \), such that

\[ \| f - p_n \|_{E_R} < C a^n, \quad n = 1, 2, \ldots. \]

If \( 1 < R' < \min(R, 2/(1 + a)) \), then \( E_{R'} \subset E_R \) and since \( f = 0 \) on \( E \), by Inequality 1.3 we get

\[ \| f \|_{E_{R'}} \leq \| f - p_n \|_{E_{R'}} + \| p_n \|_{E_{R'}} < C a^n + \| p_n \|_{E(R')} < C [a^n + (2a/(1 + a))^n] \]

for \( n = 1, 2, \ldots, \) whence \( f = 0 \) on \( E_{R'} \), and, consequently, \( N_E(E) = 0 \).

2°. It suffices to show that if \( E \) is \( L \)-regular at \( b \in E \) then \( N_E(b) = 0 \). To do this take a function \( f \in N_E(U) \), where \( U \) is an open neighborhood of \( b \). We can find three bounded closed parallelepipeds \( K_1, K_2, \) and \( K_3 \), such that \( b \in \text{int } K_1, K_i \subset \text{int } K_{i+1}, i = 1, 2, K_2 \subset U \) and \( E \subset K_3 \). Then by Lemma 2.1 there exist polynomials \( p_n \in P_n(C^N) \) \((n = 1, 2, \ldots) \) such that

\[ \| f - p_n \|_{K_2} < C a^n \]

with \( C > 0 \) and \( a \in (0, 1) \), both \( C \) and \( a \) independent of \( n \). By [6, Lemma 12.3], there exist polynomials \( l_k \in P_k(C^N) \) \((k = 1, 2, \ldots) \) and constants \( D > 0 \) and \( d \in (0, 1) \) such that

\[ \| l_k - 1 \|_{K_2} < D d^k, \quad \| l_k \|_{K_2 \setminus \text{int } K_2} < D d^k \quad \text{and} \quad \| l_k \|_{K_2} < D k^N \]

for \( k = 1, 2, \ldots \). Write \( r_{k,n} = l_k p_n \) \((k > 1, n > 1) \). By Inequality 1.3 there is
a constant \(A > 0\) such that
\[
\|p_n\|_{K_j} < A^n, \quad n = 1, 2, \ldots
\]
Then for each \(k \geq 1\) and \(n > 1\), we have
\[
\|r_{k,n}\|_{E \cap (K_j \setminus \text{int} K_2)} < \|l_k\|_{K_j \setminus \text{int} K_2} \|p_n\|_{K_j} < DA^nd^k,
\]
\[
\|r_{k,n}\|_{E \cap (K_j \setminus \text{int} K_1)} < DkN \|p_n\|_{E \cap K_2} < DCk^Na^n
\]
and
\[
\|r_{k,n}\|_{E \cap K_1} < (D + 1)Ca^n.
\]
Now choose an integer \(m > 0\) such that \(Ad^m < e = \max(a, d) < 1\) and set \(r_n = r_{mn,n}\) for \(n = 1, 2, \ldots\). Then by the above inequalities there exists \(n_0 > 0\) such that
\[
\|r_n\| < D_1e^n \quad \text{for} \quad n > n_0,
\]
with an appropriate constant \(D_1 > 0\) and \(e = (e + 1)/2\). Hence by Inequality 1.3, we can find a compact neighborhood \(V\) of the point \(b\), \(V \subset K_1\), such that
\[
\|r_n\|_V \to 0, \quad \text{as} \quad n \to \infty.
\]
On the other hand, for each \(n\),
\[
\|r_n - p_n\|_{K_1} < \|l_{mn} - 1\|_{K_1} \|p_n\|_{K_1} < De^n,
\]
whence
\[
\|f\|_V < \|f - p_n\|_V + \|p_n - r_n\|_V + \|r_n\|_V \to 0
\]
as \(n \to \infty\), which yields \(f = 0\) on \(V\). This gives the result.

3.3. Remark. If \(E\) is not polynomially convex, then Proposition 3.2(1°) fails to hold (take, e.g., for \(N = 1\), \(E = \{|z| = 1\} \cup \{0\}\); then \(\Phi_E(z) = \max(1, |z|)\) but \(N_E(E) \neq 0\).

We also note that by the Stone-Weierstrass theorem, if \(E \subset \mathbb{R}^N\) then \(E\) is polynomially convex.

3.4. Question. For any compact set \(E = \hat{E} \subset \mathbb{C}^N\) does the \(L\)-regularity of \(E\) at a point \(a \in E\) imply \(N_E(a) = 0\)?

We note that this is the case when \(N = 1\) (see Remark 3.10 and the proof of Theorem 3.12).

Now we can prove the main result of this paper.

3.5. Theorem. Let \(E\) be a polynomially convex, \(L\)-regular compact set in \(\mathbb{C}^N\) and \(h\) a holomorphic mapping in an open neighborhood \(U\) of \(E\), with values in \(\mathbb{C}^M\) \((M < N)\), such that the triplet \((h, E, E)\) satisfies \((H)\). Then \(h(E)\) is \(L\)-regular.

Proof. Since \(E\) is \(L\)-regular, then by 2(1) and Inequality 1.3, \(c(E) > 0\). By Corollary 2.8 we then have \(c(h(E)) > 0\), and by Proposition 3.2(1°) we get
3.6. **COROLLARY.** If \( E = \hat{E} \subseteq \mathbb{C}^N \) is \( L \)-regular and \( h: E \subseteq U \to \mathbb{C}^1 \) is holomorphic and nonconstant in any connected component \( W \) of \( U \) such that \( W \cap E \neq \emptyset \), then the set \( h(E) \) is also \( L \)-regular.

3.7. **REMARK.** If \( E \neq \hat{E} \), then Theorem 3.5 fails to hold; take, e.g., \( E = \{ |z| = 1 \} \cup \{ \frac{1}{2} \} \subseteq \mathbb{C}^1 \) and \( h(z) = z^{-1} \); then \( h(E) \) is not \( L \)-regular at \( 2 \in h(E) \).

Now we wish to give local versions of Theorem 3.5. Owing to Proposition 3.2(2°), by a similar argument to that of the proof of Theorem 3.5 we get

3.8. **THEOREM.** If a compact set \( E \subseteq \mathbb{R}^N \) is \( L \)-regular at a point \( a \in E \) and \( h: E \subseteq U \to \mathbb{C}^M \) (\( M < N \)) is holomorphic in \( U \) and such that the triplet \( (h, E, \{ a \}) \) satisfies \( (H) \), then \( h(E) \) is \( L \)-regular at \( h(a) \).

We cannot prove Theorem 3.8 for any compact set \( E = \hat{E} \subseteq \mathbb{C}^N \) (see Question 3.4). Nevertheless we shall give a little weaker (or equivalent—see Remark 3.10 and Question 3.11) version of this result.

3.9. **DEFINITION** (compare [9]). A compact set \( E \subseteq \mathbb{C}^N \) is said to satisfy condition \( (L^1) \) at a point \( a \in E \), if for each \( r > 0 \) the set \( E \cap B(a, r) \) is \( L \)-regular at \( a \); \( B(a, r) \) being the closed ball with centre \( a \) and radius \( r \).

3.10. **REMARK.** It is obvious that if \( E \) satisfies \( (L^1) \) at \( a \in E \), then it is \( L \)-regular at \( a \). Conversely, if \( N = 1 \) and \( a \in E = \hat{E} \) (or \( a \in \partial(C^1 \setminus E) \), if \( E \neq \hat{E} \)) then the \( L \)-regularity of \( E \) at \( a \) implies that \( E \) satisfies \( (L^1) \) at \( a \) (see e.g. [4, (5.1.15')]).

3.11. **Question.** Suppose \( E = \hat{E} \subseteq \mathbb{C}^N \) is \( L \)-regular at \( a \in E \). Does \( E \) then have to satisfy \( (L^1) \) at \( a \)?

3.12. **THEOREM.** With the assumptions of Theorem 3.8 on \( h \), for any compact set \( E \in \mathbb{C}^N \), if \( E \) satisfies \( (L^1) \) at \( a \in E \), so does the set \( h(E) \) at \( h(a) \).

**Proof.** If \( E \) satisfies \( (L^1) \) at \( a \in E \), then for each \( r > 0 \), \( c(E \cap B(a, r)) > 0 \), and by Corollary 2.8 we have \( c(h(E)) > 0 \).

Moreover, it follows from Lemma 2.1 and Inequality 1.3 that for each \( r > 0 \), \( N_{E \cap B(a, r)}(a) > 0 \) (see the proof of Proposition 3.2). For each \( r > 0 \) there exists \( s > 0 \) such that

\[
h(E \cap B(a, s)) \subset h(E) \cap B(h(a), r).
\]

Hence, since by Proposition 2.9(1°) the set \( h(E \cap B(a, s)) \) is \( L \)-regular at \( h(a) \), by 2(2) so is the set \( h(E) \cap B(h(a), r) \), which means that \( h(E) \) satisfies \( (L^1) \) at \( h(a) \), as claimed.

**REFERENCES**


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