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DENSE SUBGROUPS OF LIE GROUPS. II

BY

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ABSTRACT. Let G be a dense analytic subgroup of an analytic group L. Then G contains a maximal (CA) closed normal analytic subgroup M and a closed abelian subgroup $A = Z(G) \times E$, where E is a closed vector subgroup of G, such that $G = M \cdot A$, $M \cap A = Z(G)$, $\overline{M} = M \cdot \overline{Z(G)}$, and $L = M \cdot \overline{A}$.

We also indicate the extent to which a (CA) analytic group is uniquely determined by its center and a dense analytic subgroup.

1. Introduction. By an analytic group and an analytic subgroup of a Lie group we mean a connected Lie group and connected Lie subgroup, respectively. If G and H are Lie groups and ϕ is one-to-one (continuous) homomorphism from G into H, ϕ will be called an immersion. ϕ will be called closed or dense, as $\phi(G)$ is closed or dense in H. G_0 and $Z(G)$ will denote the identity component group and center of G, respectively.

If G is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of G, topologized with the generalized compact-open topology. G will be called (CA) if $I(G)$, the Lie subgroup of $A(G)$ consisting of all inner automorphisms of G, is closed in $A(G)$. It is well known that G is (CA) if, and only if, its universal covering group is (CA).

A brief version of our main result (Theorem 2.1) is stated below. It represents a significant generalization of Zerling [7].

Let G be a dense analytic subgroup of an analytic group L. Then G contains a maximal (CA) closed normal analytic subgroup M and a closed abelian subgroup $A = Z(G) \times W \times Y$, where Y is a closed vector subgroup of G and W is a closed vector subgroup of L, such that $G = M \cdot A$, $M \cap A = Z(G)$, $\overline{M} = M \cdot \overline{Z(G)}$, and $L = M \cdot \overline{A}$.

If G is a normal analytic subgroup of an analytic group H, then each element h of H induces an automorphism of G, Namely, $g \to hgh^{-1}$. We will denote this homomorphism of H into $A(G)$ by ρ_{GH}. $I_H(h)$ will denote the inner automorphism of H determined by $h \in H$. More generally, if A is a

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subset of H, $I_H(A)$ will denote the set of all inner automorphisms of H determined by elements of A. $I_H(H)$ will be written as $I(H)$, and the mapping $h \to I_H(h)$ of H onto $I(H)$ will be denoted by I_H.

If N is an analytic group and H is an analytic subgroup of $A(N)$, then $N \circledS H$ will denote the semidirect product of N and H. On the other hand, if G is an analytic group containing a closed normal analytic subgroup N and a closed analytic subgroup H, such that $G = NH$, $N \cap H = \{e\}$, and such that the restriction of ρ_{NG} to H is one-to-one, we will frequently identify G with $N \circledS \rho_{NG}(H)$ and H with $\rho_{NG}(H)$, that is, we may write $G = N \circledS H$.

In Zerling [5] we proved the following theorem.

MAIN STRUCTURE THEOREM. *Let G be a non-(CA) analytic group. Then there exist a (CA) analytic group M, a toral group T in $A(M)$, and a dense vector subgroup V of T, such that:*

(i) *$H = M \circledS T$ is a (CA) analytic group.*

(ii) *G is isomorphic to the dense analytic subgroup $M \circledS V$ of H.*

(iii) *$Z(G)$ is contained in M.*

(iv) *$Z_0(G) = Z_0(H)$, and $\pi(Z(H))$ is finite, where π is the natural projection of H onto T. Moreover, if $G/Z(G)$ is homeomorphic to Euclidean space, then $Z(G) = Z(H)$.*

(v) *Each automorphism σ of G can be extended to an automorphism $\varepsilon(\sigma)$ of H, such that ε: $A(G) \to A(H)$ is a closed immersion.*

We will frequently use this theorem in §§2 and 3.

In §3 we relate the concept of (CA) completion of a Lie algebra as discussed in van Est [3], [4] and the Main Structure Theorem in order to improve upon some results in Zerling [6]. In particular we will indicate (Theorem 3.3) the extent to which a (CA) analytic group is uniquely determined by its center and a dense analytic subgroup.

However, in order to make our presentation more self-contained we will first state most of those results from our bibliography which are used in our proofs. The result of Goto [1] is modified to better suit our needs. Also, those bibliographical results concerning the (CA) completion of a Lie algebra will be stated in §3. In all cases the notation is consistent with our notation.

GOTO [1, THEOREM]. *Let G be a dense analytic subgroup of an analytic group L and suppose that G contains a maximal normal analytic subgroup N which contains the commutator subgroup of G and is also closed in L. Then for each maximal compact subgroup K of L there exists a closed vector subgroup V of G, such that $G = NV$, $N \cap V = \{e\}$, and $L = N\overline{V}$, where $N \cap \overline{V}$ is finite and $\overline{V}$ is a toral subgroup of L which is central in K. Moreover, L is diffeomorphic to the space $N \times \overline{V}$.*

VAN EST [2, THEOREM 2.2.1]. *If G is a dense (CA) analytic subgroup of an analytic group L, then $Z(L) = \overline{Z(G)}$, $L = G \cdot \overline{Z(G)}$, and L is also (CA).*

ZERLING [5, LEMMA 2.2]. *Let M be an analytic group and let K be a compact analytic subgroup of $A(M)$. Let F be a closed central subgroup of M, such that each element of K keeps F elementwise fixed. Let $m \in M$ and suppose that $\sigma(m) \cdot m^{-1}$ is in F for all σ in K. Then $\sigma(m) = m$ for all σ in K.*

ZERLING [6, LEMMA 2.1]. *Maintaining the notation of the Main Structure Theorem, we have that $Z(G)$ is of finite index in $Z(H)$.*

ZERLING [6, LEMMA 3.1]. *Let L be an analytic group. Let M and H be a closed normal analytic subgroup and a closed abelian analytic subgroup of L, respectively, such that $L = MH$, $M \cap H = \{e\}$. Let G be a dense analytic subgroup of L and let S be a subset of H. Then $\rho_{ML}(S)$ is closed in $A(M)$ if, and only if, $\rho_{GL}(S)$ is closed in $A(G)$.*

ZERLING [6, COROLLARY TO LEMMA 3.2]. *Let us maintain the notation of the Main Structure Theorem and let L be a (CA) analytic group containing G as a dense analytic subgroup. Then*

$$\dim L = \dim H + \dim Z(L) - \dim Z(G) \geqslant \dim H.$$

ZERLING [6, THEOREMS 3.1 AND 3.4]. *Let us maintain the notation of the Main Structure Theorem and let L be an analytic group with the following properties, which we know to be exhibited by H.*

(i) *L is (CA).*

(ii) *There is a dense immersion $f\colon G \to L$.*

(iii) *$Z(f(G))$ is of finite index in $Z(L)$.*

Then L is diffeomorphic to H, and $Z(f(G))$ is closed in L. If we replace (iii) *by*

(iii)′ *$Z(f(G))$ is of countably infinite index in $Z(L)$,*

then $\dim L = \dim H$ *and $Z(f(G))$ is still closed in L.*

2. Main results.

LEMMA 2.1. *Let $G = MV$, $M \cap V = \{e\}$ be the canonical decomposition of G given in the Main Structure Theorem. Then M is a maximal (CA) closed normal analytic subgroup of G.*

PROOF. Let M' be a closed normal analytic subgroup of G properly containing M. Let W denote the projection of M' onto V. Then $M' = M \cdot W$, $M \cap W = \{e\}$. We will prove that M' is non-(CA) by showing that $I_{M'}(W)$ is closed in $I(M')$, but not in $A(M')$.

Since the closure of $\rho_{MG}(V)$ in $A(M)$ is T, a toral group, clearly $\rho_{MM'}(W)$ is not closed in $A(M)$. Therefore, from Zerling [6, Lemma 3.1] $I_{M'}(W)$ is not closed in $A(M')$.

Now suppose that $\{I_{M'}(w_n)\}$ converges to $I_{M'}(mw)$ in $I(M')$. Then $\{I_G(w_n)\}$

converges to $I_G(mw)$ in $I(G)$. But $I_G(W)$ is closed in $I(G)$. This can be seen most directly by observing that $I(G) = I_G(M) \cdot I_G(V)$, $I_G(M) \cap I_G(V) = \{e\}$, since $Z(G)$ is contained in M. Therefore, $I_G(mw) = I_G(\bar{w})$ for some $\bar{w} \in W$. Hence, $mw\bar{w}^{-1} \in Z(G)$. So $m \in Z(G)$ and $I_{M'}(m) = e$. Thus, $\{I_{M'}(w_n)\}$ converges to $I_{M'}(w)$ in $I(M')$ and, consequently, $I_{M'}(W)$ is closed in $I(M')$. We have now proved that M' is non-(CA).

THEOREM 2.1. *Let f: $G \to L$ be a proper dense immersion of an analytic group G into an analytic group L. Then there exist closed vector subgroups W and Y of G and a maximal (CA) closed normal analytic subgroup M of G, which contains $Z(G)$, such that $G = MWY$, $MW \cap Y = M \cap W = \{e\}$, and WY is a closed vector subgroup of G, and such that*

$$L = f(M) \cdot f(W) \cdot \overline{f(Z(G)) \cdot f(Y)},$$

where $\overline{f(M)} = f(M) \cdot \overline{f(Z(G))}$ and $f(W)$ and $\overline{f(M)} \cdot f(W)$ are closed in L. Moreover, $f(W) \cap Z(L) = \{e\}$.

PROOF. If G is (CA), then $L = f(G) \cdot \overline{f(Z(G))}$ from van Est [**2**, Theorem 2.2.1]. We now assume that G is non-(CA) and will adopt the notation of the Main Structure Theorem.

Let K_L be an arbitrarily fixed maximal compact subgroup of L and let K be a maximal compact subgroup of $\overline{I(G)}$ containing $\rho_{GL}(K_L)$. Since G is non-(CA) we can appeal to Goto [**1**]: Let N be a maximal analytic subgroup of $I(G)$, which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$. Then there is a closed vector subgroup V' of $I(G)$, such that

$$I(G) = NV', \qquad N \cap V' = \{e\}, \tag{1}$$

and $\overline{I(G)} = N\overline{V'}$, where $T' = \overline{V'}$ is a central toral subgroup of K. Moreover, $N \cap T'$ is finite and the space of $\overline{I(G)}$ is diffeomorphic to the product space $N \times T'$. In the proof of the Main Structure Theorem $I_G(M) = N$ and $I_G(V) = V'$.

We now show that $\overline{f(M)} = f(M) \cdot \overline{f(Z(G))}$. To this end we construct the analytic subgroup $P = \overline{f(M)} \cdot f(V)$ of L. Since $Z(G) \subset M$ from the Main Structure Theorem, $\overline{f(Z(G))} \subset Z(P)$. Also, if $\bar{m}v \in Z(P)$, $\bar{m} \in \overline{f(M)}$, $v \in f(V)$, then $\rho_{GP}(\bar{m}) \cdot \rho_{GP}(v) = e$. Since $N \cap V' = \{e\}$ from (1), $\rho_{GP}(v) = e$ and so $v \in Z(G)$. Again, since $Z(G)$ is contained in M, $v = e$, and so $Z(P)$ is contained in $\overline{f(M)}$. Thus,

$$\overline{f(Z(G))} \subset Z(P) \subset \overline{f(M)}. \tag{2}$$

Let $d \in Z(P)$. Then from (2) there is a sequence $\{g_n\}$ in $f(M)$ converging to d in P. Therefore, $\{\rho_{GP}(g_n)\}$ converges to e in $A(G)$. Since N is closed in $A(G)$, $\{\rho_{GP}(g_n)\}$ converges to e in N and, therefore, in $I(G)$. Since $G/Z(G)$ is isomorphic to $I(G)$, there is a sequence of central elements of G such that $\{c_{n_r}^{-1} \cdot g_{n_r}\} = \{b_{n_r}\}$ converges to e in G, where $\{g_{n_r}\}$ is a subsequence of $\{g_n\}$.

So $\{b_{n_r}\}$ converges to e in P. Hence $\{c_{n_r}\} = \{g_{n_r} \cdot b_{n_r}^{-1}\}$ converges to d in P. Thus $d \in \overline{f(Z(G))}$. Therefore

$$Z(P) \subset \overline{f(Z(G))}$$

and so

$$\overline{f(Z(G))} = Z(P) \tag{3}$$

from (2).

Since $\rho_{GP}(\overline{f(M)}) = \overline{N = \rho_{GP}}(f(M))$ because N is closed in $A(G)$, $\overline{f(M)} \subset f(M) \cdot Z(P)$. So $Z(\overline{f(M)})$ is contained in $f(M) \cdot Z(P)$. Therefore,

$$\overline{f(M)} = f(M) \cdot Z\overline{(f(M))} \subset f(M) \cdot Z(P) \underset{(3)}{=} f(M) \cdot \overline{f(Z(G))} \underset{(2)}{\subset} \overline{f(M)},$$

where the first equality follows form van Est [2], since M is (CA). We now have

$$\overline{f(M)} = f(M) \cdot \overline{f(Z(G))}. \tag{4}$$

Let us return to $P = \overline{f(M)} \cdot f(V)$. If $\overline{m} = v$, then $\rho_{GP}(\overline{m}) \cdot \rho_{GP}(v^{-1}) = e$. So $v = e$ from (1). Thus $\overline{f(M)} \cap f(V) = \{e\}$, and so $f(V)$ is closed in P. Now let J denote a maximal analytic subgroup of P which contains $\overline{f(M)}$ and is closed in L. Then from Goto [1] there exists a closed vector subgroup U of P such that $P = J \cdot U$, $J \cap U = \{e\}$, and such that $L = J \cdot \overline{U}$, where $\overline{U}$ is a central toral subgroup of K_L and $J \cap \overline{U}$ is finite. Therefore, since $\rho_{GL}(\overline{U}) \subset \rho_{GL}(K_L) \subset K$, and since T' is central in K, we see that T' centralizes $\rho_{GL}(\overline{U})$.

Let π: $P \to f(V)$ be the natural projection and let $\pi(J) = f(W)$, where W is a vector subgroup of V. Then $f(W)$ is a closed vector subgroup of $f(V)$ and since J contains $\overline{f(M)}$ we see that

$$J = \overline{f(M)} \cdot f(W), \qquad \overline{f(M)} \cap f(W) = \{e\}. \tag{5}$$

Therefore,

$$P = \overline{f(M)} \cdot f(W) \cdot U \quad \text{and} \quad L = \overline{f(M)} \cdot f(W) \cdot \overline{U}. \tag{6}$$

Let $W' = \rho_{GP}(f(W))$. Then since $P = \overline{f(M)} \cdot f(V)$ and $\rho_{GP}(\overline{f(M)}) = N = \rho_{GP}(M)$, we have $I(G) = \rho_{GP}(P) = N \cdot W' \cdot \rho_{GP}(U)$ by (6). Since $Z(P)$ is contained in $\overline{f(M)}$ by (2), we see that $\rho_{GP}(U) \cap (N \cdot W') = \{e\}$. In particular, $\rho_{GP}(U)$ is a vector subgroup of $I(G)$, which is isomorphic to U.

Let $U' = \rho_{GP}(U)$, and let $U' = U'_q \cdot U'_{q-1} \cdots U'_1$ be a direct product decomposition of U' into one dimensional vector subgroups. For I_G: $G \to I(G)$ we see that MW is the complete inverse image of NW', and we let H_i, $1 \leqslant i \leqslant q$, denote the identity component group of the complete inverse image of U'_i. Each H_i is closed in G. Since the restriction of I_G to H_i is a homomorphism of H_i onto U'_i having kernel $Z(G) \cap H_i$, we see that $Z(G)$

$\cap H_i$ is connected and $H_i = (Z(G) \cap H_i) \cdot Y_i$, where Y_i is a closed one dimensional vector subgroup of H_i, such that

$$I_G(Y_i) = U_i'. \tag{7}$$

Therefore,

$$G = M \cdot W \cdot (Z(G) \cap H_q) \cdot Y_q \cdots (Z(G) \cap H_1) \cdot Y_1$$

$$= M \cdot W \cdot Y_q \cdot Y_{q-1} \cdots Y_1.$$

If $mwy_q \cdots y_1 = e$, then $I_G(m) \cdot I_G(w) \cdot I_G(y_q) \cdots I_G(y_1) = e$. Since $I(G) = N \cdot W' \cdot U'$, $NW' \cap U' = N \cap W' = \{e\}$, we see that $I_G(w) = I_G(y_i) = e$ for each i. Therefore, $m = w = y_i = e$. Hence, $W \cdot Y_q \cdot Y_{q-1} \cdots Y_1$ is closed in G. We now show that it is actually a closed vector subgroup of G.

Let $M_2 = M \cdot W \cdot Y_q \cdot Y_{q-1} \cdots Y_2$. M_2 is closed and normal in G and $G = M_2 \cdot Y_1$, $M_2 \cap Y_1 = \{e\}$. Let ψ: $Y_1 \to A(M_2)$ be given by $\psi(y_1)(m_2) = y_1 m_2 y_1^{-1}$. Since $Z(G)$ is contained in M_2, and since Y_1 is abelian, we see that ψ is an immersion. $\psi(Y_1)$ is not closed in $A(M_2)$, since $\underline{I_G(Y_1) = U_1'}$ is not closed in $A(G)$; Zerling [6, Lemma 3.1]. Consider $M_2 \circledS \overline{\psi(Y_1)}$, where $\overline{\psi(Y_1)}$ is the closure of $\psi(Y_1)$ in $A(M_2)$. $\overline{\psi(Y_1)}$ is a toral group.

Let $y_1 \in Y_1$ and let $x = wy_q \cdots y_2 \in WY_q \cdots Y_2$. Then $I_G(\psi(y_1)(x)) = I_G(y_1 \cdot x \cdot y_1^{-1}) = u_1' \cdot w' \cdot u_q' \cdots u_2' \cdot u_1'^{-1}$, where $u_i' = I_G(y_i) \in U_i' \subset K$ and $w' = I_G(w) \in W' \subset T'$. Since T' is central in K, and because U' is abelian, we see that $I_G(y_1 \cdot x \cdot y_1^{-1}) = I_G(x)$. Therefore, $\psi(y_1)(x) \cdot x^{-1}$ is in $Z(G)$. Thus, $\sigma(x) \cdot x^{-1}$ is in $Z(G)$ for each σ in $\overline{\psi(Y_1)}$ and each x in $W \cdot Y_q \cdots Y_2$.

$Z(G)$ is a closed central subgroup of M_2 and each element of $\overline{\psi(Y_1)}$ keeps $Z(G)$ elementwise fixed. Therefore, from Zerling [5, Lemma 2.2] we see that $\sigma(x) = x$ for each $\sigma \in \overline{\psi(Y_1)}$ and each $x \in W \cdot Y_q \cdot Y_{q-1} \cdots Y_2$. Hence, $W \cdot Y_q \cdots Y_1$ is a closed vector subgroup of G. Let $Y = Y_q \cdot Y_{q-1} \cdots Y_1$. Then $G = M \cdot W \cdot Y$, $(MW) \cap Y = M \cap W = \{e\}$. The maximality of M as a (CA) closed normal analytic subgroup of G follows from Lemma 2.1.

Since $\rho_{GP}(U) = \rho_{GP}(f(Y)) = U'$ from (7), we see that

$$U \subset Z(P) \cdot f(Y) = \overline{f(Z(G))} \cdot f(Y), \tag{8}$$

where the last equality follows from (3). Therefore we have proved that

$$L \underset{(6)}{=} \overline{f(M)} \cdot f(W) \cdot \bar{U} \underset{(4)}{=} f(M) \cdot f(W) \cdot \overline{f(Z(G))} \cdot \bar{U}$$

$$\underset{(8)}{\subset} f(M) \cdot f(W) \cdot \overline{f(Z(G)) f(Y)},$$

that is,

$$L = f(M)\cdot f(W)\cdot \overline{f(Z(G))\cdot f(Y)},$$

which we claimed in our theorem.

Finally we show that $f(W) \cap Z(L) = \{e\}$. Indeed, since $J \cap Z(L)$ is contained in $Z(P)$, it is contained in $\overline{f(M)}$ from (2). But $\overline{f(M)} \cap f(W) = \{e\}$ from (5). Hence, $f(W) \cap Z(L) = \{e\}$.

COROLLARY 1. *Maintaining the notation of Theorem* 2.1, *if* $\overline{f(Z(G))}$ *is compact, then*:

(i) *Y can be selected so that* $L = \overline{f(M)}\cdot f(W)\cdot \overline{f(Y)}$, *where* $\overline{f(Y)}$ *is a toral group.*

(ii) $\overline{f(Y)} \cap (\overline{f(M)}\cdot f(W))$ *is contained in* $\overline{f(Z(G))}\cdot F$, *where F is a finite subgroup of* $\overline{f(M)}$.

(iii) $f(W) \cap (\overline{f(M)}\cdot \overline{f(Y)}) = \{e\}$.

PROOF. (i) If $\overline{f(Z(G))}$ is compact, then from (8) of Theorem 2.1 we have $\bar{U} \subset \overline{f(Z(G))}\cdot \overline{f(Y)}$, and so from (6) of Theorem 2.1

$$L = \overline{f(M)}\, f(W)\cdot \bar{U} \subset \overline{f(M)}\cdot f(W)\cdot \overline{f(Y)},$$

since $\overline{f(Z(G))}$ is contained in $\overline{f(M)}$. That is, $L = \overline{f(M)}\cdot f(W)\cdot \overline{f(Y)}$ as we claimed. From (7) of Theorem 2.1, we see that $f(Y) \subset \bar{U}\cdot \overline{f(Z(G))}$. Therefore, since $\bar{U}$ is a toral group and since $\overline{f(Y)} \subset \bar{U}\cdot \overline{f(Z(G))}$, we see that $\overline{f(Y)}$ is a toral group.

(ii) Suppose that $x \in \overline{f(Y)} \cap (\overline{f(M)}\cdot f(W))$. Then $x = \bar{z}\cdot\bar{u}$, $\bar{z} \in \overline{f(Z(G))}$, $\bar{u} \in \bar{U}$, since $\overline{f(Y)} \subset \overline{f(Z(G))}\cdot \bar{U}$ as we just showed above in (i). Therefore $\bar{u} = \bar{z}^{-1}\cdot x$, and so $\bar{u} \in \overline{f(M)}\cdot f(W)$. So $\bar{u} \in J \cap \bar{U}$, which is a finite subgroup of P, and consequently in $\overline{f(M)}$. Call this finite group F. Hence $x \in \overline{f(Z(G))}\cdot F$. Thus, we have shown that $\overline{f(Y)} \cap (\overline{f(M)}\cdot f(W))$ is contained in $\overline{f(Z(G))}\cdot F$, as we claimed.

(iii) Suppose that $f(W) \cap (\overline{f(M)}\cdot \overline{f(Y)}) \neq \{e\}$. Then there exist $w \in f(W)$, $\bar{m} \in \overline{f(M)}$, and $\bar{y} \in \overline{f(Y)}$ so that $w = \bar{m}\bar{y}$. Therefore, $\bar{y} \in \overline{f(Y)} \cap J$. But $(\overline{f(Y)} \cap J) \subset \overline{f(Z(G))}\cdot F \subset \overline{f(M)}$ from (ii). Therefore, $\bar{y} \in \overline{f(M)}$ and so $w \in \overline{f(M)}$. Thus, $w = e$ from (5) of Theorem 2.1. Hence, $f(W) \cap (\overline{f(M)}\cdot \overline{f(Y)}) = \{e\}$, as we claimed.

COROLLARY 2. *Maintaining the notation of Theorem* 2.1, *if* $f(Z(G))$ *is closed in L, then Y can be selected so that* $L = f(MW)\cdot \overline{f(Y)}$, *where* $\overline{f(Y)}$ *is a toral group such that* $f(MW) \cap \overline{f(Y)}$ *is finite.*

PROOF. If $f(Z(G))$ is closed in L, then $\overline{f(M)} = f(M)$ from Theorem 2.1. Hence $G = P$ and we can take $Y_i = U_i$ in the proof of Theorem 2.1; so $Y = U$. Our claim then follows from (6) of Theorem 2.1.

REMARK. In the proof of Theorem 2.1 P was a device which we constructed in order to apply Goto's Theorem. Since U was in P, and not in G, we needed to construct Y in G in order to carry out our proof. In Corollary 2, however, the use of P as a device is not necessary because M is already closed in L.

COROLLARY 3. *Maintaining the notation of Theorem* 2.1, *if L is* (CA) *and $Z(L)$ is compact, then* $W = \{e\}$.

PROOF. Since $Z(L)$ is compact, $\overline{f(Z(G))}$ is compact. Since $f(W) \cap (\overline{f(M)} \cdot \overline{f(Y)}) = \{e\}$ from Corollary 1, and since $f(W) \cap Z(L) = \{e\}$ from Theorem 2.1, we see from Corollary 1 that $L = (\overline{f(M)} \cdot \overline{f(Y)}) \circledS f(W)$. Let $Q = \overline{f(M)} \cdot \overline{f(Y)}$. Then $\rho_{QL}(f(W))$ is not closed in $A(Q)$, since $\rho_{GL}(f(W)) = W' \subset V'$ is not closed in $A(G)$; Zerling [**6**, Lemma 3.1]. Hence L is properly dense in $(\overline{f(M)} \cdot \overline{f(Y)}) \circledS T_1$, where T_1 is the closure of $f(W)$ in $A(Q)$. This is a contradiction from van Est [**2**]. Hence $W = \{e\}$.

3. (CA) **Lie groups and Lie algebras.** Following van Est [**3**] we define a Lie subalgebra $\mathfrak{S}$ of a Lie algebra $\mathfrak{L}$ to be dense in $\mathfrak{L}$, if there exists an analytic group L with Lie algebra $\mathfrak{L}$ in which the analytic subgroup generated by $\mathfrak{S}$ is dense. We also say that ψ is a dense imbedding of a Lie algebra $\mathfrak{S}$ into a Lie algebra $\mathfrak{L}$ if ψ is a Lie algebra isomorphism of $\mathfrak{S}$ into $\mathfrak{L}$ such that $\psi(\mathfrak{S})$ is a dense subalgebra of $\mathfrak{L}$.

In [**3**, Theorem 4.1] van Est proved that for each Lie algebra $\mathfrak{S}$ there exists a unique (up to isomorphism) Lie algebra $\mathfrak{S}_{(CA)}$ such that:

(i) $\mathfrak{S}_{(CA)}$ is a (CA) Lie algebra, whose center coincides with the center of $\mathfrak{S}$.

(ii) There exists a dense imbedding ψ of $\mathfrak{S}$ into $\mathfrak{S}_{(CA)}$.

(iii) With any dense imbedding ψ' of $\mathfrak{S}$ into a (CA) Lie algebra $\mathfrak{L}$ there exists a dense imbedding η of $\mathfrak{S}_{(CA)}$ into $\mathfrak{L}$ so that $\psi' = \eta\psi$.

The Lie algebra $\mathfrak{S}_{(CA)}$ described above is called the (CA)-completion or (CA)-closure of $\mathfrak{S}$. We now relate this concept with the Main Structure Theorem in order to improve upon some results in Zerling [**6**]. First we state the following useful result from van Est [**3**, Lemma 5.4]: Let G be an analytic group with Lie algebra $\mathfrak{S}$. Suppose that $\mathfrak{S}$ is a dense ideal of a Lie algebra $\mathfrak{L}$, such that the center of $\mathfrak{L}$ is contained in $\mathfrak{S}$. Then there exists an analytic group L with Lie algebra $\mathfrak{L}$ that contains G as a dense analytic subgroup.

THEOREM 3.1. *Let us maintain the notation of the Main Structure Theorem. Let $\mathfrak{S}$ and $\mathfrak{H}$ be the Lie algebras of G and H, respectively. Then* $\mathfrak{H} \simeq \mathfrak{S}_{(CA)}$.

PROOF. Since $\mathfrak{S}$ is a dense subalgebra of $\mathfrak{S}_{(CA)}$, we have from van Est [**3**, Lemma 5.4] that there exists an analytic group $G_{(CA)}$ having $\mathfrak{S}_{(CA)}$ as its Lie algebra and containing G as a dense analytic subgroup. Therefore, from the corollary to Lemma 3.2 of Zerling [**6**] we see that $\dim H \leqslant \dim G_{(CA)}$.

On the other hand, since $\mathfrak{H}$ is a (CA) Lie algebra containing $\mathfrak{S}$ as a dense subalgebra, there is a dense imbedding from $\mathfrak{G}_{(CA)}$ into $\mathfrak{H}$. Since $\dim H \leqslant \dim G_{(CA)}$ we see that this imbedding is an isomorphism of $\mathfrak{G}_{(CA)}$ onto $\mathfrak{H}$. Hence, our theorem is proved.

THEOREM 3.2. *Let us maintain the notation of the Main Structure Theorem and let $f\colon G \to L$ be a dense immersion of G into a (CA) analytic group L.*

(i) *If $Z(f(G))$ is of finite index in $Z(L)$, then L is locally isomorphic and diffeomorphic to H, and $Z(f(G))$ is closed in L.*

(ii) *If $Z(f(G))$ is of countably infinite index in $Z(L)$, then L is locally isomorphic to H and $Z(f(G))$ is closed in L.*

PROOF. (i) In Lemma 2.1 of Zerling [6] we proved that $Z(G)$ is of finite index in $Z(H)$ and in Theorem 3.1 of that paper we showed that L is diffeomorphic to H and that $Z(f(G))$ is closed in L. Since G is a dense analytic subgroup of the (CA) analytic group L, there is a dense imbedding of $\mathfrak{G}_{(CA)}$ into $\mathfrak{L}$. Hence $\mathfrak{H} \simeq \mathfrak{L}$, since $\mathfrak{G}_{(CA)}$ is isomorphic to $\mathfrak{H}$ from Theorem 3.1 and $\dim H = \dim L$ from the diffeomorphism between H and L given above. Thus, H and L are locally isomorphic.

(ii) In Theorem 3.4 of Zerling [6] we proved that $\dim L = \dim H$ and that $Z(f(G))$ is closed in L. Now by repeating the argument in (i) above we see that H and L are locally isomorphic.

THEOREM 3.3. *Let L_1 and L_2 be (CA) analytic groups and let G_1 and G_2 be isomorphic proper dense analytic subgroups of L_1 and L_2, respectively.*

(i) *If $Z(L_1)$ and $Z(L_2)$ are both finite, the L_1 and L_2 are diffeomorphic and locally isomorphic.*

(ii) *If $Z(L_1)$ and $Z(L_2)$ are both discrete, then L_1 and L_2 are locally isomorphic.*

PROOF. (i) Since $Z(L_1)$ and $Z(L_2)$ are both finite, $Z(G_1)$ and $Z(G_2)$ are both finite. Thus, G_1 and G_2 are non-(CA). Let us now maintain the notation of the Main Structure Theorem with the obvious subscript modification for G_1 and G_2. We have $H_1 \simeq H_2$ and from (i) of Theorem 3.2 we have that H_1 and L_1 are diffeomorphic and locally isomorphic, as are H_2 and L_2. Hence L_1 and L_2 are diffeomorphic and locally isomorphic.

(ii) Replace "finite" by "discrete" in the proof of (i) above and then apply (ii) of Theorem 3.2.

EXAMPLE. Let $\mathbf{R}$, $\mathbf{C}$, and T denote the group of real numbers, the group of complex numbers, and the one dimensional toral group, respectively. For $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ and $r, s, t, u \in \mathbf{R}$ let $(\alpha, \beta, \gamma, \delta; r, s, t, u)$ denote the 5×5 matrix whose right-hand column consists from top to bottom of $\alpha, \beta, \gamma, \delta$, and 1, and whose main diagonal consists from top to bottom of $e^{2\pi ir}$, $e^{2\pi is}$, $e^{2\pi it}$, $e^{2\pi iu}$, and 1, and whose other entries are 0.

Let $L_1 = \{(\alpha, \beta, \gamma, \delta; r, s, t, u)\colon \alpha, \beta, \gamma, \delta \in \mathbf{C}; r, s, t, u \in \mathbf{R}\}$. Then $L_1 \simeq$

$\mathbf{C}^4 \circledS T^4$, $Z(L_1) = \{e\}$, and L_1 is (CA). Let μ and ν be fixed irrational numbers and let $L_2 = \{(\alpha, \beta, \gamma, \delta; r, s, t, \mu t)$: $\alpha, \beta, \gamma, \delta \in \mathbf{C}$; $r, s, t \in \mathbf{R}\}$. Then $L_2 \simeq \mathbf{C}^4 \circledS (T^2 \times \mathbf{R})$, $Z(L_2) = \{e\}$, and L_2 is non-(CA). Let $G = \{(\alpha, \beta, \gamma, \delta; r, \nu r, t, \mu t)$: $\alpha, \beta, \gamma, \delta \in \mathbf{C}$; $r, t \in \mathbf{R}\}$. Then $G \simeq \mathbf{C}^4 \circledS (\mathbf{R} \times \mathbf{R})$, $Z(G) = \{e\}$, and G is non-(CA).

This example shows that L_1 and L_2 both have trivial center and both possess G as a dense analytic subgroup; yet, L_1 and L_2 do not even have the same dimension. Thus, the condition in Theorem 3.3 that L_1 and L_2 both be (CA) cannot be removed. It is also true that the conditions on $Z(L_1)$ and $Z(L_2)$ in Theorem 3.3 cannot be removed, as we see by observing that T^2 and T^3 both possess a dense one dimensional vector subgroup.

BIBLIOGRAPHY

1. M. Goto, *Analytic subgroups of* GL $(n, \mathbf{R})$, Tôhoku Math. J. (2) **25** (1973), 197–199.

2. W. T. van Est, *Dense imbeddings of Lie groups*, Nederl. Akad. Wetensch. Proc. Ser. A **54** = Indag. Math. **13** (1951), 321–328.

3. ———, *Dense imbeddings of Lie groups*. II, Nederl. Akad. Wetensch. Proc. Ser. A **55** = Indag. Math. **14** (1952), 255–274.

4. ———, *Some theorems on (CA) Lie algebras*. I, II, Nederl. Akad. Wetensch. Proc. Ser. A **55** = Indag. Math. **14** (1952), 546–568.

5. D. Zerling, *Some theorems on (CA) analytic groups*, Trans. Amer. Math. Soc. **205** (1975), 181–192.

6. ———, *Some theorems on (CA) analytic groups*. II, Tôhoku Math. J. (2) **29** (1977), 325–333.

7. D. Zerling, *Dense subgroups of Lie groups*, Proc. Amer. Math. Soc. **62** (1977), 349–352.

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