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PARABOLIC FUNCTION SPACES WITH MIXED NORM

BY

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ABSTRACT. The spaces $\mathcal{H}_\alpha^p$ of parabolic Bessel potentials were introduced by B. F. Jones and R. J. Bagby. We prove a Sobolev-type imbedding theorem for $\mathcal{H}_\alpha^{p_1,p_2}$ (multinormed versions of $\mathcal{H}_\alpha^p$) when α is a positive integer k, $1 < p_1, p_2 < \infty$. In particular this theorem holds for $W_{2l,l}^p$, since $\mathcal{H}_{2l}^p \equiv W_{2l,l}^p$. We use the concepts of parabolic Riesz transforms and half-time derivatives introduced by us elsewhere.

Introduction. Sobolev spaces $W_k^p(R^n)$ are usually defined as

$$W_k^p(R^n) = \left\{ f: f \in L^p(R^n),\ \sum_{|\beta|=0}^{k} \|D^\beta f\|_p < \infty \right\},$$

where $1 < p < \infty$, $D^\beta = (\partial/\partial x_1)^{\beta_1} \dots (\partial/\partial x_n)^{\beta_n}$, $|\beta| = \sum \beta_i$ and $\partial f/\partial x_j$ denotes the distribution derivative of f for all $j = 1, \dots, n$. Alternatively they may be defined as follows. For $\alpha > 0$, let G_α be defined on R^n by

$$G_\alpha(x) = (4\pi)^{-n/2}\Gamma(\alpha/2)^{-1} \int_0^\infty e^{-\pi x^2/\delta} e^{-\delta/4\pi} \delta^{(-n+\alpha)/2}\, d\delta/\delta.$$

Let $\mathcal{L}_\alpha^p(R^n) = \{g: g = G_\alpha * f, f \in L^p(R^n)\}$. It is known that (see for example **[12]**) $\mathcal{L}_\alpha^p(R^n)$ is a Banach space with norm $\|G_\alpha * f\|_{p,\alpha} = \|f\|_p$. The functions $G_\alpha * f$ are called Bessel potentials and are related to the negative fractional powers of a certain elliptic operator. It is also well known that for integer values k of α and $1 < p < \infty$ $\mathcal{L}_k^p(R^n) = W_k^p(R^n)$ both algebraically and topologically. Therefore we may take this as an alternate definition of the Sobolev spaces.

This also raises a question. Starting from the heat operator $\partial/\partial t - \Delta$ (and $\partial/\partial t + I - \Delta$) is it possible to define spaces of parabolic Bessel potentials? If the answer to this question is affirmative than is it possible to identify these new spaces with "Sobolev spaces" for integer values of α? As it turns out the answer to the first question is affirmative as was shown in **[1]**, **[6]**, and **[10]**. However, the answer to the second question is inconclusive. This was shown in **[10]**. This inconclusive answer is due to the fact that spaces $\mathcal{H}_\alpha^p$ of parabolic Bessel potentials are indeed different from the usual Sobolev spaces

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W_k^p for integer values k of α. This difference is made clear in [**10**]. For integer values k of α we denote the spaces of parabolic Bessel potentials by $T_k^p(R^{n+1})$. These are analogues to Sobolev spaces and *the aim of this article is to characterize these spaces and obtain an imbedding theorem*. In view of the fact that functions with different orders of integrability in different variables play an important role in partial differential equations [**8**], *we prove our results for the more general spaces $T_k^{p_1,p_2}(R^n \times R)$, that is, spaces with mixed norm.*

The spaces $T_k^p(R^{n+1})$ are closely related to the spaces $W_{2l,l}^p(R^{n+1})$ used in studying parabolic partial differential equations. $W_{2l,l}^p(R^{n+1})$ is defined to be the Banach space consisting of those elements of $L^p(R^{n+1})$ which have generalized derivatives of the form $D_t^r D_x^s f$ with $2r + s \leqslant 2l$ (cf. [**8**, *p.* 5]). *The norm is described there in detail and it is easy to show that $T_{2l}^p(R^{n+1}) = W_{2l,l}^p(R^{n+1})$ both algebraically and topologically. Our norm employs half-derivatives and is different from the norm on $W_{2l,l}^p$, but equivalent to it. This may be shown by employing the closed graph theorem.*

All the function spaces mentioned above, the parabolic Riesz and Bessel potentials, parabolic Riesz transforms and the half-derivative are defined in §2. Also included in §2 are known and needed results. §3 deals with parabolic Riesz potentials in multinormed spaces while §4 deals with boundedness of parabolic Bessel potentials in multinormed spaces. §5 is reserved for the imbedding theorem.

2. Preliminaries. For $(x, t) \in R^n \times R$ and $\alpha > 0$ the functions h_α and H_α are defined by

$$h_\alpha(x, t) = \begin{cases} C_\alpha t^{(\alpha-n-2)/2} e^{-x^2/4t}, & t > 0, \\ 0, & t \leqslant 0, \end{cases} \tag{2.1}$$

$$H_\alpha(x, t) = e^{-t} h_\alpha(x, t), \tag{2.2}$$

where $x^2 = \Sigma x_i^2$ and $C_\alpha = ((4\pi)^{n/2}\Gamma(\alpha/2))^{-1}$. The functions h_2 and H_2 are fundamental solutions of the equations $(\partial/\partial t - \Delta)u = 0$ and $(\partial/\partial t + I - \Delta)u = 0$ respectively, where Δ denotes the n-dimensional Laplacian. Corresponding to H_α there is a linear operator J_α on $L^p(R^{n+1})$ defined by $J_\alpha(f) = H_\alpha ** f$ where $**$ denotes convolution in x and t respectively. $J_\alpha(f)$ *is called the parabolic Bessel potential* and the space of all such functions, i.e., $\{J_\alpha(f), f \in L^p(R^{n+1})\}$ is denoted by $\mathcal{H}_\alpha^p(R^{n+1})$. Since $\|H_\alpha\|_1 = 1$, $H_\alpha ** f \in L^p(R^{n+1})$. $\mathcal{H}_\alpha^p(R^{n+1})$ is a Banach space with norm $\|J_\alpha(f)\|_{p;\alpha} = \|f\|_p$.

For $0 < \alpha < n + 2$ and $f \in L^p(R^{n+1})$, $h_\alpha ** f$ is called the parabolic Riesz potential. Let $\mathcal{S}$ denote the space of rapidly decreasing functions defined on R^{n+1} and $\mathfrak{D}$, the space of C_0^∞-functions. If $\phi \in \mathcal{S}$ the Fourier transform of ϕ is defined by

$$\hat{\phi}(x, t) = (2\pi)^{-(n+1)/2} \iint e^{-ix\zeta} e^{-it\tau} \phi(\zeta, \tau)\, d\zeta\, d\tau.$$

It is known that, [6], [11], if $\beta \equiv (2\pi)^{-(n+1)/2}$, then

$$\hat{H}_\alpha(x, t) = \beta(1 + x^2 + it)^{-\alpha/2}, \qquad \alpha > 0, \tag{2.3}$$

$$\hat{h}_\alpha(x, t) = \beta(x^2 + it)^{-\alpha/2}, \qquad 0 < \alpha < n + 2. \tag{2.4}$$

Because of the equations (2.3) and (2.4) it is possible to define negative fractional powers of $(\partial/\partial t + I - \Delta)$ and $(\partial/\partial t - \Delta)$ by employing H_α and h_α.

The following result exposes the relationship between H_α and h_α through their Fourier transforms.

LEMMA 2.1 (SAMPSON). *If $\alpha > 0$, there exist bounded measures μ, μ_1, μ_2 such that*

(i) $(x^2 + it)^{\alpha/2} = (1 + x^2 + it)^{\alpha/2}\hat{\mu}$, *and*

(ii) $(1 + x^2 + it)^{\alpha/2} = \hat{\mu}_1 + (x^2 + it)^{\alpha/2}\hat{\mu}_2$.

REMARK 2.2. The expression for μ, i.e., $\mu = \delta + \sum_{j=1}^{\infty} A_{j\alpha} H_{2j}$, $\sum |A_{j\alpha}| < \infty$ was obtained by Sampson in [11]. The expression for μ_1 (and μ_2) is $\delta + \Phi(x, t)$ where $\Phi \in L^1(R^{n+1})$ and δ denotes the Dirac distribution. This fact may be proved exactly as in [12, p. 134].

The following theorem, which is a special case of a more general result [12], concerning Riesz potentials will be useful in later sections.

THEOREM 2.3. *Let $0 < \gamma < 1$, $1 < p < q < \infty$ and $q^{-1} = p^{-1} - \gamma$. Then, for all $f \in L^p(R^1)$*

$$\left\| \int_{-\infty}^{\infty} |s|^{-1+\gamma} f(t - s)\, ds \right\|_q \leqslant C\|f\|_p,$$

C being independent of f.

DEFINITION 2.4. Let $1 < p < \infty$ and $f \in L^p(R^{n+1})$. Then for $j = 1, \ldots, n$ define the n parabolic Riesz transforms $P_1, \ldots, P_n$ associated with the n coordinates of the space variable by

$$P_j(f)(x, t) = \lim_{\varepsilon \to 0} \int_{|(y,s)| > \varepsilon} \frac{\partial h_1}{\partial y_j}(y, s) f(x - y, t - s)\, d(y, s). \tag{2.5}$$

These are generalizations of Riesz transforms to the parabolic case [10]. By using the Fourier Multiplier theorem it can be shown that these are bounded operators on $L^p(R^{n+1})$, $1 < p < \infty$. The motivation for this definition is explained in [10]. There, it is also proved that $f \in \mathcal{H}_\alpha^p(R^{n+1})$ if and only if f, $\partial(h_1 ** f)/\partial t$ and $\partial f/\partial x_j$ for $j = 1, \ldots, n$ belong to $\mathcal{H}_{\alpha-1}^p(R^{n+1})$. This result suggests the following definition.

DEFINITION 2.5. Let $f \in \mathcal{H}_\alpha^p(R^{n+1})$, $1 < p < \infty$, and $\alpha \geqslant 1$. The operator D_0: $\mathcal{H}_\alpha^p \to \mathcal{H}_{\alpha-1}^p$ associated with the multiplier $it/(x^2 + it)^{1/2}$ is said to define a "half-derivative".

It must be noted that $D_0 f$ is not a half-derivative in the strictest sense of the word. For, if $f \in \mathcal{S}$, then $(D_0^2 f)^\Lambda = (\partial f/\partial t)^\Lambda + (\partial(h_2 ** \Delta f)/\partial t)^\Lambda$. The same result which motivated the definition of D_0 also motivates the definition of certain function spaces resembling the Sobolev spaces $W_k^p(R^n)$.

DEFINITION 2.6. Let $\beta = (\beta_0, \ldots, \beta_n)$, where β_i are nonnegative integers and k be a nonnegative integer. If $1 < p < \infty$, then the class of functions

$$T_k^p(R^{n+1}) = \left\{ f: f \in L^p(R^{n+1}), \|f\|_{p;k} = \sum_{|\beta|=0}^{k} \|D^\beta f\|_p < \infty \right\} \quad (2.6)$$

is a Banach space, where $D^\beta = D_0^{\beta_0} \cdots D_n^{\beta_n}$ and for $i = 1, \ldots, n$, $D_i^{\beta_i} = \partial/\partial x_i$.

From the main result of **[10]**, it is now easy to conclude that for $\alpha = k$, $\mathcal{H}_\alpha^p = T_k^p$ both algebraically and topologically. From the alternate definition of Sobolev spaces discussed in the introduction and the similar definition of T_k^p introduced above we are forced to conclude that these spaces must be useful in dealing with parabolic differential equations. *A simple application is included at the end of this article.*

In this article we concentrate on the multinormed versions of these spaces. In other words we deal with the spaces $T_k^{p_1,p_2}$ as subspaces of L^{p_1,p_2}. The precise definition is as follows.

DEFINITION 2.7. Let β and k be as in Definition 2.6. If $1 < p_1, p_2 < \infty$, the class of functions

$$T_k^{p_1,p_2}(R^{n+1}) = \left\{ f: f \in L^{p_1 p_2}(R^{n+1}), \|f\|_{p_1 p_2;k} = \sum_{|\beta|}^{k} \|D^\beta f\|_{p_1 p_2} < \infty \right\} \quad (2.7)$$

is a Banach space. By $L^{p_1,p_2}(R^{n+1})$ we mean those functions f for which

$$\|f\|_{p_1,p_2} = \left(\int \left(\int |f(x,t)|^{p_1} \, dx \right)^{p_2/p_1} dt \right)^{1/p_2} < \infty.$$

A theory of real variables for functions of this type may be found in [3].

3. Parabolic Riesz potentials in multinormed spaces. We present two results in this section. First, if $q_1 > p_1$ and $q_2 > p_2$ we will exhibit a necessary and sufficient condition for $h_\alpha ** f$ to be in L^{q_1,q_2} when f is in L^{p_1,p_2}. Second, if $1 \leqslant p_1 < \infty$ and $1 < p_2 < \infty$ then the operators P_j: $L^{p_1,p_2} \to L^{p_1,p_2}$ are bounded. Recall that

$$(h_\alpha ** f)(x,t) = \int_0^\infty \int_{R^n} h_\alpha(\zeta, \tau) f(x - \zeta, t - \tau) \, d\zeta \, d\tau, \quad (3.1)$$

where h_α is as given by equation (2.1).

THEOREM 3.1. *Let $f \in L^{p_1,p_2}(R^{n+1})$ and $h_\alpha ** f$ be as above. Let $0 < \alpha < n + 2$, $1 \leqslant p_i < q_i < \infty$, $i = 1, 2$ and $\alpha/2 = (n/2)(p_1^{-1} - q_1^{-1}) + (p_2^{-1} -$*

q_2^{-1}). *Then*

$$\|h_\alpha ** f\|_{q_1,q_2} \leqslant C\|f\|_{p_1p_2}, \tag{3.2}$$

where the constant C is independent of f.

Professor Richard Bagby has informed us that this result has been established in **[2]** and hence no proof will be presented here.

LEMMA 3.2. *Let* $1 < p_1 < \infty$, $1 < p_2 < \infty$ *and* $f \in L^{p_1p_2}(R^{n+1})$. *Then, for* $j = 1, \ldots, n$, *the parabolic Riesz transforms* P_j *defined by equation* (2.5) *satisfy*

$$\|P_jf\|_{p_1,p_2} \leqslant C\|f\|_{p_1,p_2}, \tag{3.3}$$

for some constant C independent of f.

To prove this lemma one has only to consider the multipliers associated with the operators P_j and apply Corollary 1, p. 234 of Lizorkin **[9]**. I thank Professor Richard Bagby for pointing out this work by Lizorkin.

4. Parabolic Bessel potentials. In this section we characterize the space $\mathcal{H}_\alpha^{p_1,p_2}$ for $\alpha \geqslant 1$ and establish the boundedness of the operators $f \to \mu ** f$, $\mu_1 ** f$, $\mu_2 ** f$ and $D_0 f$. With the exception of D_0 all these operators are defined on L^{p_1,p_2} whereas D_0 is defined on $\mathcal{H}_\alpha^{p_1,p_2}$ for any $\alpha \geqslant 1$.

THEOREM 4.1. *Let* $\alpha \geqslant 1$, $1 < p_1 < \infty$ *and* $1 < p_2 < \infty$. *Then* $f \in \mathcal{H}_\alpha^{p_1,p_2}(R^{n+1})$ *if and only if* f, $D_0 f$ *and, for* $j = 1, \ldots, n$, $\partial f/\partial x_j$ *are all in* $\mathcal{H}_{\alpha-1}^{p_1,p_2}(R^{n+1})$. *Moreover, the two norms*

$$\begin{cases} \|f\|_{p_1,p_2;\alpha}, \\ \|f\|_{p_1,p_2;\alpha-1} + \sum_1^n \left\| \dfrac{\partial f}{\partial x_j} \right\|_{p_1,p_2;\alpha-1} + \|D_0 f\|_{p_1,p_2;\alpha-1} \end{cases} \tag{4.1}$$

are equivalent.

A detailed proof of this theorem when $p_1 = p_2 = p$ is given in **[10]**. Below we shall prove Theorem 4.1 in a very brief manner. The following lemma is needed.

LEMMA 4.2. *Let* $f \in L^{p_1,p_2}(R^{n+1})$, $1 < p_1, p_2 < \infty$. *Then the operators* $f \to \mu ** f$, $\mu_1 ** f$ *and* $\mu_2 ** f$ *are bounded on* $L^{p_1,p_2}(R^{n+1})$.

PROOF. Recall that $d\mu = \delta + \sum A_{j\alpha} H_{2j}(x, t)\, dx\, dt$ from Remark 2.2. Hence

$$(\mu ** f)(x, t) = f(x, t) + \left(\left(\sum A_{j\alpha} H_{2j}\right) ** f\right)(x, t)$$

and

$$\begin{aligned}\|\mu ** f\|_{p_1,p_2} &\leq \|f\|_{p_1,p_2} + \left\|\left(\sum A_{j\alpha}H_{2j}\right) ** f\right\|_{p_1,p_2}\\ &\leq \|f\|_{p_1,p_2} + \left\|\sum A_{j\alpha}H_{2j}\right\|_1 \|f\|_{p_1,p_2}\\ &\leq \|f\|_{p_1,p_2}\left(1 + \sum |A_{j\alpha}|\, \|H_{2j}\|_1\right)\\ &= \|f\|_{p_1,p_2}\left(1 + \sum |A_{j\alpha}|\right).\end{aligned}$$

This completes the proof of one part of the lemma. As for μ_1 and μ_2 we recall once again from Remark 2.2 that $d\mu_1 = d\mu_2 = \delta + \Phi(x, t)\, dx\, dt$ where $\Phi \in L^1(R^n \times R)$. Thus for $i = 1, 2$

$$\begin{aligned}\|\mu_i ** f\|_{p_1,p_2} &\leq \|f\|_{p_1,p_2} + \|\Phi ** f\|_{p_1,p_2}\\ &\leq \|f\|_{p_1,p_2} + \|\Phi\|_1 \|f\|_{p_1,p_2}\\ &= (1 + \|\Phi\|_1)\|f\|_{p_1,p_2}.\end{aligned}$$

PROOF OF THEOREM 4.1. Let $f \in \mathcal{H}_\alpha^{p_1,p_2}$. Then $f = J_\alpha(g)$ for some $g \in L^{p_1,p_2}(R^{n+1})$. Let $\{g_m\}$ be a sequence in $\mathcal{S}$ converging to g in $L^{p_1p_2}$-norm and $f_m = J_\alpha(g_m)$. Then as in [10], [12]

$$\left(\frac{\partial f_m}{\partial x_j}\right)^\Lambda = \left(J_{\alpha-1}\left(g_m^j\right)\right)^\Lambda, \qquad g_m^j = P_j(\mu ** g_m).$$

By Lemmas 3.2 and 4.2 it now follows that

$$\left\|\frac{\partial f_m}{\partial x_j}\right\|_{p_1,p_2;\alpha-1} \leq C\|f_m\|_{p_1,p_2;\alpha},$$

and since $\mathcal{S}$ is dense in L^{p_1,p_2} we immediately have

$$\left\|\frac{\partial f}{\partial x_j}\right\|_{p_1,p_2;\alpha-1} \leq C\|f\|_{p_1,p_2;\alpha}. \tag{4.2}$$

Because $\mathcal{H}_\alpha^{p_1,p_2} \subset \mathcal{H}_{\alpha-1}^{p_1,p_2}$, we also have

$$\|f\|_{p_1,p_2;\alpha-1} \leq C\|f\|_{p_1,p_2;\alpha}. \tag{4.3}$$

With regard to D_0, as in [10], we have

$$\left(\frac{\partial}{\partial t}(h_1 ** f_m)\right)^\Lambda = \left(J_{\alpha-1}\left(g_m^0\right)\right)^\Lambda, \tag{4.4}$$

where $g_m^0 = \mu ** g_m - \mu ** \sum P_j^2 g_m$. Once again because of Lemmas 3.2 and 4.2, we conclude from (4.4) that

$$\left\|\frac{\partial}{\partial t}(h_1 ** f_m)\right\|_{p_1,p_2;\alpha-1} = \|g_m^0\|_{p_1,p_2} \leq C\|g_m\|_{p_1,p_2} = C\|f_m\|_{p_1,p_2;\alpha} \tag{4.5}$$

and extend it by continuity to $f \in \mathcal{H}_\alpha^{p_1,p_2}$. By combining inequalities (4.2), (4.3) and (4.5) we obtain

$$\|f\|_{p_1,p_2;\alpha-1} + \sum \left\| \frac{\partial f}{\partial x_j} \right\|_{p_1,p_2;\alpha-1} + \|D_0 f\|_{p_1,p_2;\alpha-1} \leqslant C\|f\|_{p_1,p_2;\alpha} \quad (4.6)$$

thus completing one half of our theorem.

To prove the converse, as in **[10]**, we first note that if f, $\partial f/\partial x_j$ for $j = 1, \ldots, n$ and $D_0 f$ are in $\mathcal{H}_{\alpha-1}^{p_1,p_2}$ then for some $g \in L^{p_1,p_2}$ and $\partial g/\partial x_j \in L^{p_1,p_2}$, $f = J_{\alpha-1}(g)$, $\partial f/\partial x_j = J_{\alpha-1}(\partial g/\partial x_j)$ for $j = 1, \ldots, n$ and $D_0 f = J_{\alpha-1}(D_0 g)$.

Since g and $\partial g/\partial x_j \in L^{p_1,p_2}(R^{n+1})$, there exists a sequence $\{g_m\} \subset \mathfrak{D}$ so that $g_m \to g$ and $\partial g_m/\partial x_j \to \partial g/\partial x_j$ in L^{p_1,p_2}-norm. Since J_1 is an isomorphism from $\mathbb{S} \to \mathbb{S}$, $g_m = J_1(U_m)$ for some $U_m \in \mathbb{S}$, $\forall m$. Therefore $\hat{g}_m = (1 + x^2 + it)^{-1/2}\hat{U}_m$ and, as in **[10]**, with the aid of Lemma 2.1 we establish that

$$\hat{U}_m = (\mu_1 ** g_m)^{\Lambda} + \left[\mu_2 ** \left\{ \sum_1^n P_j \frac{\partial g_m}{\partial x_j} + \frac{\partial}{\partial t}(h_1 ** g_m) \right\} \right]^{\Lambda}. \quad (4.7)$$

From Lemmas 3.2 and 4.2 it now follows that

$$\|U_m\|_{p_1,p_2} \leqslant C\left\{ \|g_m\|_{p_1,p_2} + \sum \left\| \frac{\partial g_m}{\partial x_j} \right\|_{p_1,p_2} + \|D_0 g_m\|_{p_1,p_2} \right\}. \quad (4.8)$$

Since $f_m = J_{\alpha-1}(g_m) = J_{\alpha-1}(J_1(U_m)) = J_\alpha(U_m)$, we have $\|f_m\|_{p_1,p_2;\alpha} = \|U_m\|_{p_1,p_2}$. Combining this fact with the inequality (4.8) and extending it to the full space $\mathcal{H}_\alpha^{p_1,p_2}$ we obtain

$$\|f\|_{p_1,p_2;\alpha} \leqslant C\left\{ \|f\|_{p_1,p_2;\alpha-1} + \sum \left\| \frac{\partial f}{\partial x_j} \right\|_{p_1,p_2;\alpha-1} + \|D_0 f\|_{p_1,p_2;\alpha-1} \right\}.$$

This completes the proof of our theorem.

We now single out a part of the above proof and state it as a lemma for future purposes.

LEMMA 4.3. *The operator* D_0: $\mathcal{H}_\alpha^{p_1,p_2}(R^{n+1}) \to \mathcal{H}_{\alpha-1}^{p_1,p_2}(R^{n+1})$ *is bounded, provided* $\alpha \geqslant 1$ *and* $1 < p_1, p_2 < \infty$.

5. An imbedding theorem. The imbedding theorem we have in mind is similar to Sobolev's theorem **[12]**. Specifically it is an imbedding theorem for the parabolic analogues $T_k^{p_1,p_2}(R^{n+1})$ (in particular for $W_{2l,l}^{p}(R^{n+1})$) of the Sobolev spaces W_k^p. *Our approach depends on the representation of* $f \in T_k^{p_1,p_2}$ *in terms of* h_1, P_j *and* D_0. This approach is different from that which was employed in the proof of Sobolev's theorem **[12]**. There the function f was given a representation which is a generalization of the one dimensional case where one represents a function as the integral of its derivative.

THEOREM 5.1. *For any integer $k > 0$ let $k/2 = (n/2)(p_1^{-1} - q_1^{-1}) + (p_2^{-1} - q_2^{-1})$.*

(i) *If $q_1, q_2 < \infty$, then $T_k^{p_1,p_2}(R^{n+1}) \subseteq L^{q_1,q_2}(R^{n+1})$ and the inclusion map is continuous.*

(ii) *If $q_1 = q_2 = \infty$, then the restriction of an $f \in T_k^{p_1,p_2+\varepsilon}(R^{n+1})$ to any set $R^n \times F$ is continuous, $F \subset R$ being compact.*

(iii) *If $q_1 = q_2 = \infty$, then the restriction of an $f \in T_k^{p_1,p_2+\varepsilon}(R^{n+1})$ to any compact set $E \subset R^{n+1}$ is in L^{r_1,r_2} for all r_1, r_2 such that $1 \leqslant r_1, r_2 < \infty$.*

(iv) *If $(n/2p_1) + (1/p_2) < k/2$ then the restriction of an $f \in T_k^{p_1,p_2}(R^{n+1})$ to any compact set $E \subset R^{n+1}$ is continuous.*

PROOF. Assume that $k = 1$. Our proof depends on the identity

$$f = H_1 ** \left\{ \sum P_j\left(\frac{\partial f}{\partial x_j}\right) + D_0 f \right\} \tag{5.1}$$

which is easily verified by applying Fourier transform if $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in both $T_1^{p_1,p_2}$ and $T_1^{q_1,q_2}$, equation (5.1) extends to $T_1^{q_1,q_2}$ by continuity. From this identity it now follows that

$$\begin{aligned} \|f\|_{q_1,q_2} &\leqslant \left\| \sum P_j\left(\frac{\partial f}{\partial x_j}\right) + D_0 f \right\|_{p_1,p_2} \\ &\leqslant C\left(\sum \left\| \frac{\partial f}{\partial x_j} \right\|_{p_1,p_2} + \|D_0 f\|_{p_1,p_2} \right) \\ &\leqslant C\|f\|_{p_1,p_2;1}. \end{aligned}$$

This proves part (i)

Part (ii) also follows from the same identity. In fact, if $f \in \mathcal{H}_1^{p_1,p_2}$ and $g = \sum P_j(\partial f/\partial x_j) + D_0 f$, then

$$\begin{aligned} \|h_1 ** g\|_\infty &\leqslant \int_0^\infty \|h_1 * g\|_\infty(\tau, t - \tau)\, d\tau \\ &\leqslant \int_0^\infty \|h_1\|_{p_1'}(\tau)\|g\|_{p_1}(t - \tau)\, d\tau \\ &\leqslant C\int_0^\infty \tau^{-1+p_2^{-1}}\|g\|_{p_1}(t - \tau)\, d\tau, \end{aligned}$$

where $\|h_1\|_{p_1'}$ is computed as in the proof of Theorem 3.1. Since f has compact support (as a function of t) in a symmetric interval $[-a, a]$, g also has compact support in the same interval as a function of t. This may be verified by computing the Fourier transforms of f and g and applying the Paley-Wiener theorem [5]. Thus, the supremum of f over $R^n \times [-a, a]$ is given by

$$\begin{aligned}\|h_1 ** g\|_{\infty,\infty} &\leqslant C\left\|\int_0^\infty \tau^{(1/p_2)-1}\|g\|_{p_1}(t-\tau)\,d\tau\right\|_\infty \\ &\leqslant C\left(\int_{-2a}^{2a} \tau^{1/p_2'}\right)^{1/(p_2'-\beta)}\|g\|_{p_1,p_2+\varepsilon} \\ &\leqslant C\|g\|_{p_1,p_2+\varepsilon} \leqslant C\|g\|_{p_1,p_2},\end{aligned}$$

where $\varepsilon > 0$ and $p_2' - \beta$ is the conjugate of $p_2 + \varepsilon$. The last inequality is a consequence of the fact that g has compact support with respect to the variable t.

For part (iii) once again we consider the representation $h_1 ** g$. Let s_1 and s_2 be such that $(1/p_i) + (1/s_i) - 1 = (1/r_i)$ for $i = 1, 2$, where $r_1, r_2 < \infty$ and meet all the requirements of Young's inequality. It is easy to see that the lower bounds p_1 and p_2 are imposed on r_1 and r_2 respectively by Young's inequality. Now

$$\begin{aligned}\|h_1 * g\|_{r_1}(\tau, t-\tau) &\leqslant \|h_1\|_{s_1}(\tau)\|g\|_{p_1}(t-\tau) \\ &= C\tau^{(1/p_2)+(n/2r_1)-1}\|g\|_{p_1}(t-\tau).\end{aligned}$$

Hence

$$\begin{aligned}\|h_1 ** g\|_{r_1,r_2} &\leqslant \left\|\int_0^\infty \|h_1\|_{s_1}(\tau)\|g\|_{p_1}(t-\tau)\,d\tau\right\|_{r_2} \\ &\leqslant \|h_1\|_{s_1,s_2}\|g\|_{p_1,p_2},\end{aligned}$$

where

$$\|h_1\|_{s_1,s_2} = C\left(\int_{-2a}^{2a} t^{((1/p_2)-(n/2r_1)-1)s_2}\,dt\right)^{(1/s_2)}.$$

However, since

$$s_2\left(\frac{1}{p_2} + \frac{n}{2r_1} - 1\right) = s_2\left(\frac{1}{r_2} - \frac{1}{s_2} + \frac{n}{2r_1}\right) = \frac{s_2 n}{2r_1} + \frac{s_2}{r_2} - 1$$

is greater than -1, $\|h_1\|_{s_1,s_2} < \infty$. On the other hand if $1 \leqslant r_i \leqslant p_i$ then $f \in L^{r_1,r_2}(R^{n+1})$ since $L^{p_1,p_2}(E) \subset L^{r_1,r_2}(E)$.

Finally, for part (iv) under the assumption $(n/2p_1) + (1/p_2) < 1/2$, we have

$$\|h_1\|_{p_1'}(t) = Ct^{(1/2)-(n/2p_1)-1}.$$

But $p_2'((1/2) - (n/2p_1) - 1) > p_2'(1/p_2 - 1) = -1$ and hence on any compact set $[k_1, k_2]$

$$\left(\int_{k_1}^{k_2} \left(\|h_1\|_{p_1'}(t)\right)^{p_2'}\right)^{(1/p_2')} < \infty. \tag{5.2}$$

If $\{f_m\}$ is a sequence in $\mathcal{S}$ converging to f in $T_1^{p_1,p_2}$, then in view of (5.2)

$$\|h_1 ** (g_n - g_m)\|_{\infty,\infty} \leqslant C\|f_n - f_m\|_{p_1,p_2;1}.$$

Thus on compact sets $\{f_m\}$ converges uniformly. Since each f_m is a continuous function the limit function f may be taken to be continuous.

To conclude the proof of the theorem we argue by induction and show that the case of $k \geq 2$ may be reduced to the case $k \geq 1$. In part (i) $f \in T_k^{p_1,p_2}(R^{n+1})$ implies that, for $j = 1, \ldots, n$, $\partial f/\partial x_j$ and $D_0 f$ belong to $T_{k-1}^{p_1,p_2}(R^{n+1})$. Hence the induction hypothesis implies that $\partial f/\partial x_j$ for $j = 1, \ldots, n$ and $D_0 f \in L^{\lambda_1,\lambda_2}$ where

$$\frac{k-1}{2} = \frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{\lambda_1}\right) + \left(\frac{1}{p_2} - \frac{1}{\lambda_2}\right),$$

that is $f \in T_1^{\lambda_1,\lambda_2}$. The case $k = 1$ now implies that $f \in L^{q_1 q_2}$, where

$$\begin{aligned}\frac{1}{2} &= \frac{n}{2}\left(\frac{1}{\lambda_1} - \frac{1}{q_1}\right) + \left(\frac{1}{\lambda_2} - \frac{1}{q_2}\right)\\ &= \frac{n}{2p_1} + \frac{1}{p_2} - \frac{k-1}{2} - \frac{n}{2q_1} - \frac{1}{q_2}\\ &= \frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{q_1}\right) + \left(\frac{1}{p_2} - \frac{1}{q_2}\right) - \frac{k-1}{2},\end{aligned}$$

which is a simple restatement of the condition imposed in the hypothesis. The other parts can be proved in a similar manner.

REMARK 5.1 (APPLICATION). Suppose $u \in T_2^2(R^{n+1})$. Consider the operator $A - \partial/\partial t = \Sigma_1^n \, a_{ij}(x, t)\partial^2/\partial x_i \partial x_j - \partial/\partial t$ with bounded measurable coefficients defined in R^{n+1}. Then it is easy to show that in $L^2(R^{n+1})$

$$Au - \frac{\partial u}{\partial t} = \left[\sum_1^n (a_{ij} - \delta_{ij})P_i P_j - I\right]\left(\frac{\partial}{\partial t} - \Delta\right)u$$

by utilizing the theory of Fourier transforms, with δ_{ij} denoting the Kronecker delta. Since P_i are bounded on $L^2(R^{n+1})$, it follows immediately that $\|(A - \partial/\partial t)u\|_2 \leq C\|(\partial/\partial t - \Delta)u\|_2$, where the constant C depends on the coefficients a_{ij}. Let $\tilde{T} = A - \partial/\partial t$ and $T = \Delta - \partial/\partial t$. Then,

$$\begin{aligned}\tilde{T}u - Tu &= \sum_1^n (a_{ij} - \delta_{ij})P_i P_j\left(\frac{\partial}{\partial t} - \Delta\right)u\\ &= -STu, \quad \text{say.}\end{aligned}$$

If $\|a_{ij} - \delta_{ij}\|_\infty$ *is sufficiently small*, then

$$\|\tilde{T} - T\| \leq C\|T\| \quad < \alpha/\|T^{-1}\|, \qquad 0 < \alpha < 1.$$

This is possible since the assumption on the size of $\|a_{ij} - \delta_{ij}\|_\infty$ allows us the use of the inequality $C\|T\| \, \|T^{-1}\| < \alpha$. By a Neumann series type argument it is now easy to deduce that $\tilde{T}$, i.e. $A - \partial/\partial t$: $T_2^2 \to L^2$ has an inverse. This argument can easily be extended to operators $A - \partial/\partial t$: $T_k^{p_1,p_2} \to T_{k-2}^{p_1,p_2}$, $k \geq 2$.

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