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ON A THEOREM OF STEINITZ AND LEVY

BY

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ABSTRACT. Let $\sum_{n\in\omega}h(n)$ be a conditionally convergent series in a real Banach space B. Let $S(h)$ denote the set of sums of the convergent rearrangements of this series. A well-known theorem of Riemann states that $S(h) = B$ if $B = R$, the reals. A generalization of Riemann's Theorem, due independently to Levy [L] and Steinitz [S], states that if B is finite dimensional, then $S(h)$ is a linear manifold in B of dimension > 0. Another generalization of Riemann's Theorem [M] can be stated as an instance of the Levy-Steinitz Theorem in the Banach space of regulated real functions on the unit interval I. This instance generalizes to the Banach space of regulated B-valued functions on I, where B is finite dimensional, implying a generalization of the Levy-Steinitz Theorem.

1. Let $\omega = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, B a real Banach space, and let h be a function from ω into B. We say that the series $\sum_{n\in\omega}h(n)$ is conditionally convergent if there are two rearrangements of its terms, one resulting in a convergent series and the other a divergent one. It is unconditionally convergent if it is convergent for every rearrangement of its terms. If the series is conditionally (unconditionally) convergent we say that h is conditionally (unconditionally) summable.

Let $S(h)$ denote the set of those v in B that are sums (in the norm) of some convergent rearrangement of $\sum_{n\in\omega}h(n)$. It is well known, that if h is unconditionally summable then $S(h)$ has precisely one member.[1]

A hundred years ago Riemann showed:

THEOREM 1. *Let $B = R$. If h: $\omega \to B$ is conditionally convergent, then* $S(h) = R$.

Levy-Steinitz's Theorem (1905) generalizes as follows:

THEOREM 2. *Let B be finite dimensional.*[2] *If h: $\omega \to B$ is conditionally*

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[1] See [H], [O]. The converse is true if B is finite dimensional, but fails already in any infinite dimensional Hilbert space, as the example following Theorem 2 indicates.

[2] The proofs of Theorem 2 [L], [S], [B1] assume the Euclidean norm. Since all norms induce the same topology on R^n, $S(h)$ is independent of the particular norm chosen.

summable, then B has a subspace N of dimension > 0 and a member v_0 such that

$$S(h) = v_0 + N.$$

It is easy to construct an h from ω into any inifinite dimensional Hilbert space such that $S(h) = \{0\}$, but h is not unconditionally convergent (let e_n be an orthonormal sequence, and let $h(2n) = e_n/\sqrt{n+1} = -h(2n+1)$). Thus, Theorem 2 does not generalize to infinite dimensional Banach spaces. (See, however [**D1**].) It is a long standing conjecture that for arbitrary Banach space B and conditionally summable h from ω into B, $S(h)$ is a displacement of a closed additive subgroup of B.

Another generalization of Theorem 1 emerged from quite a different course of research ([**M**], see also [**G**]). It implies that if $h\colon \omega \to R$ is conditionally summable, then there is a chain of conditionally convergent subseries of $s = \sum_{n\in\omega} h(n)$, order isomorphic to the reals, such that any change in the list of their sums subject to some natural continuity restrictions is achieved by one rearrangement of the series s. A precise statement of this result is given in §2 as Theorem 3. We now describe it in a way that clarifies its relation with Theorem 2.

Let I denote the closed unit interval $[0, 1]$, and let $C(I, B)$ (respectively $\mathrm{Reg}(I, B)$) denote the Banach space of all continuous functions (respectively the functions having left and right limit everywhere) from I into B, endowed with the supremum norm. For $0 < x < 1$, $v \in B$ define $J_x^v \in \mathrm{Reg}(I, B)$ by

$$J_x^v(t) = 0, \quad t \leqslant x,$$
$$J_x^v(t) = v, \quad x < t.$$

Call a sequence $x = (x_n)_{n\in\omega}$ a dense sequence in I if x enumerates a dense subset of the open interval $(0, 1)$ with no repetitions. In the sequel, let x be a fixed dense sequence in I. With every $h\colon \omega \to B$ we associate $h^x\colon \omega \to \mathrm{Reg}(I, B)$ by setting $h^x(n) = J_{x_n}^{h(n)}$. The generalization of Theorem 1 to the present context depends on a proper notion of "conditionality" for h. We define it first in case $B = R$.

We call $h\colon \omega \to R$ x-conditional iff:

(1) for every $\varepsilon > 0$, $\{n\colon \|h(n)\| > \varepsilon\}$ is finite,

(2) for every $0 \leqslant a < b \leqslant 1$, we have

$$\sum \{h(n)\colon a < x_n < b, h(n) > 0\}$$
$$= -\sum \{h(n)\colon a < x_n < b, h(n) < 0\} = \infty.$$

Clearly if h is x-conditional, it is conditionally summable. It follows from [**G**, Theorem 3], that h^x is also conditionally summable.

THEOREM 3 [M]. *Let $B = R$, and let $h: \omega \to B$ be x-conditional. Then there is an $s_0 \in \mathrm{Reg}(I, B)$ such that*

$$S(h^x) = s_0 + C(I, B).$$

Stated this way, Theorem 3 is an instance of Theorem 2 in the infinite dimensional Banach space $\mathrm{Reg}(I, R)$. In §3 it is generalized as an instance of Theorem 2 in $\mathrm{Reg}(I, B)$, where B is finite dimensional (Theorem 4). In §4 we derive Theorem 5, which is another generalization of Theorem 1, from Theorem 4. Theorem 5 is then combined with Steinitz's work [S] to give the main result of this paper, Theorem 6, extending Theorem 2.

It is well known that the convergence of every subseries of a series in a Banach space is equivalent to its unconditional convergence (see e.g. [H]), and so every subseries of such a series is also unconditionally convergent, hence has one sum. Theorem 6 implies, by a way of contrast, that a conditionally convergent series in a finite dimensional Banach space B admits a nontrivial subspace $N \subseteq B$ and a chain of (conditionally convergent) subseries, order isomorphic to the reals, such that every continuous change in N of the sums of those subseries is accomplished by some rearrangement of the terms of the given series (Theorem 6′). We now turn to make these remarks precise.

2. We develop first some notation. Let $\prec$ be an ω-ordering of ω, i.e. a linear ordering of ω every intiial segment of which is finite. We denote by $n^{\prec}$ the nth member of ω under $\prec$, and for $C \subseteq \omega$ we set

$$\overset{\prec,n}{C} = C \cap \{0^{\prec}, 1^{\prec}, \ldots, (n-1)^{\prec}\}.$$

$[A]^{<\omega}$ denotes the family of all finite subsets of the set A. If $h: \omega \to B$, $F \in [\omega]^{<\omega}$ we set $h(F) = \sum_{n \in F} h(n)$.

We say that the ω-ordering $\prec$ sums h over $C \subseteq \omega$ if the sequence

$$v_n = h\left(\overset{\prec,n}{C}\right)$$

has a limit in B. If $\prec$ sums h over C we write

$$\sum_{n \in C}^{\prec} h(n) = \sum_{C}^{\prec} h = \lim_{n \to \infty} h\left(\overset{\prec,n}{C}\right).$$

Whenever $\prec$ is the natural ordering of ω we omit it from the notation.

Let **C** be a family of subsets of ω. We say that $\prec$ sums h over **C** iff for every $C \in$ **C**, $\prec$ sums h over C. We say that $\prec$ sums h uniformly over **C** iff for every $\varepsilon > 0$ there is an $n \in \omega$ such that for every $C \in$ **C**, $n < k, l$ we have:

$$\left\| h\left(\overset{\prec,k}{C}\right) - h\left(\overset{\prec,l}{C}\right) \right\| < \varepsilon.$$

Thus, h is unconditionally summable if and only if the natural order sums h over $P(\omega)$, the set of all subsets of ω (by the equivalence of subseries convergence and unconditional convergence). We leave to the interested reader the verification that if h is unconditionally summable, then the natural order sums h uniformly over $P(\omega)$, as does every other ω-ordering of ω, and that if any ω-ordering sums h over $P(\omega)$ then h is unconditionally summable.

We call $\mathsf{C} \subseteq P(\omega)$ a chain iff for any $C_1, C_2 \in \mathsf{C}$ we have $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. We consider a chain C as a linearly ordered set, with set-inclusion as the order. Every chain is order isomorphic to a subset of I (*Proof.* Let $g(n) = 2^{-(n+1)}$. Then $C \to \Sigma_C\, g$ is an order isomorphism of C into I.) Conversely, if T is a subset of I, then there is a chain C in $P(\omega)$ order isomorphic to T.

An I-chain is an indexed chain $\mathsf{C} = \{C_t: t \in I\}$ satisfying $C_0 = \varnothing$, $C_1 = \omega$ and $C_a \subsetneq C_b$ whenever $0 \leqslant a < b \leqslant 1$.

Assume now that $x = (x_n)_{n\in\omega}$ is a dense sequence in I. For $t \in I$ let $C_t = \{n \in \omega: x_n < t\}$. Then $\mathsf{C} = \{C_t: t \in I\}$ is an I-chain.

Let $h: \omega \to B$, $h^x: \omega \to \operatorname{Reg}(I, B)$ be as in §1. If for some ω-ordering $\prec$ of ω, $t \in I$, $\prec$ sums $h^x(n)(t) = J_{x_n}^{h(n)}(t)$ in B, then

$$\sum_{n\in\omega}^{\prec} h^x(n)(t) = \sum_{n\in C_t}^{\prec} h(n) = \sum_{C_t}^{\prec} h.$$

Thus, $\prec$ sums $h^x(n)(t)$ for every $t \in I$ iff $\prec$ sums h over C; that is, $\prec$ sums pointwise h^x iff $\prec$ sums h over C. Similarly, $\prec$ sums h^x in $\operatorname{Reg}(I, B)$ –i.e., uniformly on I–iff $\prec$ sums h uniformly over C.

We leave the easy proof of the following proposition to the reader. The nonbelievers are referred to Lemma 3 in §4, that extends it.

PROPOSITION 1. *Let h: $\omega \to R$ be conditionally summable. Then there is a dense sequence x in I such that h is x-conditional.*

If $\prec$ sums h over C, then a sum-function $\overset{\prec}{s}$ is defined over C by:

$$\overset{\prec}{s}(C) = \sum_{C}^{\prec} h \qquad (C \in \mathsf{C}).$$

We are now ready to eliminate h^x from the statement of Theorem 3:

THEOREM 3′. *Let $B = R$. Let h: $\omega \to B$ be conditionally summable. Then there is an I-chain $\mathsf{C} = \{C_t: t \in I\}$ such that:*

(i) *There is an ω-ordering $\prec_0$, that sums h uniformly over C. Set*

$$s_0(t) = \overset{\prec_0}{s}(C_t) \qquad (t \in I).$$

(ii) *For every continuous g: $I \to R$ with $g(0) = 0$ there is an ω-ordering $\prec$*

that sums h uniformly over C, *satisfying for every $t \in I$*:

$$\overset{\prec}{s}(C_t) = s_0(t) + g(t).$$

(iii) *For every ω-ordering $\prec$ of ω that sums h uniformly over* C *there is a continuous $g\colon I \to R$ satisfying $g(0) = 0$ such that for every $t \in I$*

$$\overset{\prec}{s}(C_t) = s_0(t) + g(t).$$

It is easy to derive Theorem 1 from (i) and (ii) as follows. Let $a \in R$. We have to show that for some ω-ordering $\prec$ of ω we have $\overset{\prec}{\sum}{}_{\omega} h = a$. Let $b = a - s_0(1)$. Let $g(t) = tb$. By (ii) pick an ω-ordering $\prec$ satisfying

$$\overset{\prec}{s}(C_t) = s_0(t) + g(t).$$

Then

$$\overset{\prec}{\sum_{\omega}} h = \overset{\prec}{s}(C_1) = s_0(1) + g(1) = a.$$

3. To generalize Theorem 3 we first extend the notion of being x-conditional to h from I into arbitrary Banach space B. Fix $h\colon \omega \to B$ and a dense sequence $x = (x_n)_{n\in\omega}$ in I. Let $F \in [\omega]^{<\omega}$, and assume $F = \{n_1, \ldots, n_r\}$, where $x_{n_1} < x_{n_2} < \cdots < x_{n_r}$. We shall say that F is proper for h with respect to x if:

$$\max_{1 < i < r} \left\| \sum_{j=1}^{i} h(n_j) \right\| < 2\|h(F)\|.$$

Geometrically, F is proper for h w.r.t. x if the polygonal line from 0 to $h(F)$ through $h(n_1)$, $h(n_1) + h(n_2)$, $h(n_1) + h(n_2) + h(n_3), \ldots$ never gets out or the ball centered at 0 and of radius $2\|h(F)\|$.[3]

We set $C_t = \{n\colon x_n < t\}$ for $t \in I$.

We say that $h\colon \omega \to B$ is x-conditional if h satisfies (1) and:

(3) For every $0 < a < b < 1$, $\{h(F)\colon F \in [C_b - C_a]^{<\omega}$, F is proper for h w.r.t. $x\}$ is dense in B.

(1) and (3) are equivalent to (1) and (2) if $B = R$. An x-conditional $h\colon \omega \to B$ exists if and only if B is a separable Banach space.

A straightforward generalization of the proof of Theorem 3 gives [M, Theorem 5]:

THEOREM 4. *Let B be arbitrary Banach space and assume that $h\colon \omega \to B$ is x-conditional. Then there is an $s_0 \in \mathrm{Reg}(I, B)$ such that*

$$S(h^x) = s_0 + C(I, B).$$

[3] The choice of 2 here is somewhat arbitrary for the sequel. Any constant > 1 could be chosen. See [M, §5].

4. Let B be arbitrary Banach space. Define a constant K_B as the smallest K such that for every finite sequence $(v_1, \ldots, v_n)$ in B satisfying $v_1 + \cdots + v_n = 0$, there is a rearrangement $(1', \ldots, n')$ of $(1, \ldots, n)$ such that

$$\max_{1 \leq i \leq n} \left\| \sum_{j=1}^{i} v_{j'} \right\| \leq K \max_{1 \leq i \leq n} \|v_i\|.$$

Clearly, $1 \leq K_B \leq \infty$. In **[B1]** Bergström bases his proof to Theorem 2 on the following Lemma (found also in **[S]**):

LEMMA 1. *Let B be finite dimensional. Then $K_B < \infty$.*

REMARK. 1. In **[B2]** Bergström shows that if E^d denotes the d-dimensional Euclidean space, then $K_{E^2} = \sqrt{5}/2$ and in general $K_{E^d} \sim \sqrt{d}$.

2. From Dvoretzky-Rogers' Theorem **[D2]** it follows that conversely, if $K_B < \infty$ then B is finite dimensional.

The following lemma is equivalent to Lemma 1.

LEMMA 2. *Let B be finite dimensional Banach space. Then there is a finite K such that whenever $v = v_1 + \cdots + v_n$, there is a rearrangement $(1', \ldots, n')$ of $(1, \ldots, n)$ such that*

$$\max_{1 \leq i \leq n} \min_{0 \leq t \leq 1} \left\| \sum_{j=1}^{i} v_{j'} - tv \right\| \leq K \max_{1 \leq i \leq n} \|v_i\|.$$

The smallest such K is K_B.

(*Hint*. Let $k \in \omega$ satisfy $\|v\| \leq k \cdot \max_{1 \leq i \leq n} \|v_i\|$.

Set $u = (1 - 1/k)v$ and apply Lemma 1 to the $(n + k)$-sequence $(v_1, \ldots, v_n, u, \ldots, u)$.)

COROLLARY. *If B is finite dimensional, $v = v_1 + \cdots + v_n$, $\|v_i\| \leq \|v\|/K_B$ then there is a rearrangement $(1', \ldots, n')$ of $(1, \ldots, n)$ such that*

$$\max_{1 \leq i \leq n} \|v_{1'} + \cdots + v_{i'}\| \leq 2\|v\|.$$

PROOF. Let $(1', \ldots, n')$ be given by Lemma 2. Let $1 \leq i \leq n$. Choose $0 \leq t \leq 1$ with

$$\|v_{1'} + \cdots + v_{i'} - tv\| \leq K_B \cdot \max_{1 \leq j \leq n} \|v_j\| \leq \|v\|.$$

Then $\|v_{1'} + \cdots + v_{i'}\| = \|(v_{1'} + \cdots + v_{i'} - tv) + tv\| \leq 2\|v\|$.

PROPOSITION 2. *Let B be arbitrary Banach space. Let $h\colon \omega \to B$ satisfy: $\{h(F)\colon F \in [\omega]^{<\omega}\}$ is dense in B. Let $A \subseteq \omega$ satisfy $\sum_{n \in A} \|h(n)\| < \infty$. Then $\{h(F)\colon F \in [\omega - A]^{<\omega}\}$ is dense in B.*

(We shall need the proposition only for finite A.)

PROOF. Let $v \in B$, $\varepsilon > 0$ be given. We shall find $F \in [\omega - A]^{<\omega}$ with $\|h(F) - v\| < \varepsilon$. Fix n such that

$$\sum_{n \leq m} \|h(m)\| < \frac{\varepsilon}{2}.$$

Let $A' = A \cap \{0, \ldots, n-1\}$. For $G \subseteq A'$ define:

$$V_G = \{h(F): F \in [\omega]^{<\omega}, F \cap A' = G\}.$$

Since $\{G: G \subseteq A'\}$ is finite, and $\{h(F): F \in [\omega]^{<\omega}\} = \bigcup_{G \subseteq A'} V_G$ is dense in B, there is a $G_0 \subseteq A'$ such that V_{G_0} is dense in B. Choose $F \in [\omega - A]^{<\omega}$, $H \in [A - A']^{<\omega}$ such that $v_1 = h(G_0 \cup F \cup H) = h(G_0) + h(F) + h(H)$ satisfies $\|v_1 - (v + h(G_0))\| < \varepsilon/2$. Then we have:

$$\begin{aligned} \|h(F) - v\| &= \|(h(F) + h(G_0) + h(H)) - (v + h(G_0)) - h(H)\| \\ &\leq \|v_1 - (v + h(G_0))\| + \|h(H)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

A corollary of Theorem 4 and the preceding remarks is

LEMMA 3. *Let B be finite dimensional. Let $h: \omega \to B$ satisfy* (1) *and*

(4) $$\{h(F): F \in [\omega]^{<\omega}\} \text{ is dense in } B.$$

Then there is a dense sequence $x = (x_n)_{n \in \omega}$ in I such that h is x-conditional.

PROOF. Let V be a countable dense set in B. Let $(v_n)_{n \in \omega}$ be an enumeration of V such that for each $v \in V$, $\{n: v_n = v\}$ is infinite. Let $B_n = \{u \in B: \|u - v_n\| < 1/(n+1)\}$.

We construct the sequence x in steps, defining in each step x_p for p in a fresh finite subset of ω. Assume that x_p is already defined for $p \in A$, where A is a finite subset of ω.

Step $2k$. Let $\delta = \max\{\|h(n)\|: n \in \omega - A\}$. Let $F = \{n \in \omega - A: \|h(n)\| = \delta\}$. Since h satisfies (1), F is finite. Pick $x_n \in (0, 1)$ for $n \in F$ so that $x_n \neq x_p$ for $p \in A$, and $x_n \neq x_m$ for $n \neq m$ $(n, m \in F)$.

Step $2k+1$. Let $I_1, \ldots, I_r$ be the components of $I - \{x_p: p \in A\}$. By Proposition 2, choose $F_q \in [\omega - A]^{<\omega}$ for $q = 1, \ldots, r$ so that $q \neq q'$ implies $F_q \cap F_{q'} = \varnothing$ and $h(F_q) \in B_k$.

Let $F_q = \{n_1, \ldots, n_k\}$ where the indices are chosen so that for $i = 1, \ldots, k$ we have:

$$\|h(n_1) + \cdots + h(n_i)\| \leq 2\|h(F_q)\|.$$

This is possible by the corollary to Lemma 2. Choose $x_{n_1} < x_{n_2} < \cdots < x_{n_k}$ in I_q, dividing I_q into intervals of equal length.

It is easy to check that $(x_n)_{n \in \omega}$ enumerates a dense set in I with no repetitions (Step $2k$ makes sure that x_n is defined for every $n \in \omega$).

Given $0 \leq a < b \leq 1$ there is a k_0 such that for every $k \geq k_0$ one of the intervals I_q considered in Step $2k + 1$ is included in (a, b), and so for some $F \in [C_b - C_a]^{<\omega}$, F is proper for h w.r.t. x and $h(F) \in B_k$. It follows that h is x-conditional.

REMARK. By an easy modification one shows that given any countable dense set X in I, the dense sequence $x = (x_n)_{n\in\omega}$ of Lemma 3 can be so chosen that $X = \{x_n : n \in \omega\}$.

Combining Lemma 3 and Theorem 4 we have:

THEOREM 5. *Let B be finite dimensional. Let h: $\omega \to B$ satisfy* (1) *and* (4). *Then there is a dense sequence x in I, and an $s_0 \in \mathrm{Reg}(I, B)$ such that*:

$$S(h^x) = s_0 + C(I, B).$$

5. We combine now Theorem 5 with Steinitz's ideas to generalize Theorem 2 as follows.

THEOREM 6. *Let B be finite dimensional, and let h: $\omega \to B$ be conditionally summable. Then there is a subspace N of B of dimension > 0, a dense sequence x in I, and an $s_0 \in \mathrm{Reg}(I, B)$ such that*:

$$S(h^x) = s_0 + C(I, N).$$

PROOF. It follows from Steinitz [S] that there are subspaces M, N of B such that $B = M \oplus N$, and if h_M: $\omega \to M$, h_N: $\omega \to N$ are determined by the equations $h(n) = h_M(n) + h_N(n)$, $h_M(n) \in M$, $h_N(n) \in N$, then

(i) $\sum_{n\in\omega}\|h_M(n)\| < \infty$,

(ii) $S(h_N) = N$.

(ii) implies that h_N satisfies (1) and that $\{h_N(F): F \in [\omega]^{<\omega}\}$ is dense in N. By Theorem 5 we pick a dense sequence $x = (x_n)_{n\in\omega}$ in I and $s_2 \in \mathrm{Reg}(I, N)$ such that:

$$S(h_N^x) = s_2 + C(I, N).$$

Since $\|h_M^x(n)\| = \|h_M(n)\|$, we have $\sum_{n\in\omega}\|h_M^x(n)\| < \infty$. Thus, there is an $s_1 \in \mathrm{Reg}(I, M)$ such that every ω-ordering $\prec$ of ω sums h_M^x to s_1 in $\mathrm{Reg}(I, M)$.

Since $h^x(n) = h_M^x(n) + h_N^x(n)$, we see that an ω-ordering $\prec$ sums h^x iff it sums h_N^x, and whenever $\prec$ sums h_N^x to f it sums h^x to $s_1 + f$.

Let $s_0 = s_1 + s_2$. Then $s_0 \in \mathrm{Reg}(I, B)$ and we have:

$$S(h) = s_1 + S(h_N) = s_1 + (s_2 + C(I, N)) = s_0 + C(I, N).$$

We restate Theorem 6, using the lexicon of §2:

THEOREM 6′. *Let B be finite dimensional Banach space. Let h: $\omega \to B$ be conditionally summable. Then B has a subspace N of dimension > 0, and there is an I-chain $\mathrm{C} = \{C_t: t \in I\}$ of subsets of ω so that*:

(i) *There is an ω-ordering $\prec_0$ that sums h uniformly over* C. *Let*

$$s_0(t) = \overset{\prec_0}{s}(C_t) \qquad (t \in I).$$

(ii) *For every continuous $g\colon I \to N$ satisfying $g(0) = 0$ there is an ω-ordering $\prec$ that sums h uniformly over* C, *satisfying for every $t \in I$*

$$\overset{\prec}{s}(C_t) = s_0(t) + g(t).$$

(iii) *For every ω-ordering $\prec$ that sums h uniformly over* C *there is a continuous $g\colon I \to N$ satisfying $g(0) = 0$, such that for every $t \in I$*

$$\overset{\prec}{s}(C_t) = s_0(t) + g(t).$$

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