

DISTRIBUTION OF EIGENVALUES OF A TWO-PARAMETER SYSTEM OF DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. In this paper two simultaneous Sturm-Liouville systems are considered, the first defined for the interval $0 < x_1 < 1$, the second for the interval $0 < x_2 < 1$, and each containing the parameters λ and μ . Denoting the eigenvalues and eigenfunctions of the simultaneous systems by $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$, respectively, $j, k = 0, 1, \dots$, asymptotic methods are employed to derive asymptotic formulae for these expressions, as $j + k \rightarrow \infty$, when (j, k) is restricted to lie in a certain sector of the (x, y) -plane. These results constitute a further stage in the development of the theory related to the behaviour of the eigenvalues and eigenfunctions of multiparameter Sturm-Liouville systems and answer an open question concerning the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$.

1. Introduction. The importance of multiparameter Sturm-Liouville problems in mathematical physics has led in recent years to a revival of interest in this area of investigation after a period of relative neglect. However, most investigations to date have been concerned with problems related to oscillation and expansion theory (cf. [1], [3], [5], and [9]), and it has been pointed out by Atkinson [2, §4] that such fundamental problems as the determination of the behaviour of the eigenvalues and of the eigenfunctions of multiparameter Sturm-Liouville systems have not yet been resolved.

Stimulated by Atkinson's remark, the author was led in an earlier paper [6] to investigate the behaviour of the eigenvalues and eigenfunctions of the simultaneous two-parameter systems

$$y_1'' + (\lambda A_1(x_1) - \mu B_1(x_1) + q_1(x_1))y_1 = 0, \quad 0 \leq x_1 \leq 1, \quad ' = d/dx_1, \quad (1.1)$$

$$y_1(0)\cos \alpha_1 - y_1'(0)\sin \alpha_1 = 0, \quad 0 \leq \alpha_1 < \pi,$$

$$y_1(1)\cos \beta_1 - y_1'(1)\sin \beta_1 = 0, \quad 0 < \beta_1 < \pi, \quad (1.2)$$

and

$$y_2'' + (-\lambda A_2(x_2) + \mu B_2(x_2) + q_2(x_2))y_2 = 0, \quad 0 \leq x_2 \leq 1, \quad ' = d/dx_2, \quad (1.3)$$

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$$\begin{aligned} y_2(0)\cos \alpha_2 - y_2'(0)\sin \alpha_2 &= 0, & 0 \leq \alpha_2 < \pi, \\ y_2(1)\cos \beta_2 - y_2'(1)\sin \beta_2 &= 0, & 0 < \beta_2 \leq \pi. \end{aligned} \quad (1.4)$$

To explain the work of that paper (and also our present problem) in greater detail, let us recall that by an eigenvalue of the system (1.1)–(1.4) we mean a pair of numbers, (λ^*, μ^*) , such that when $\lambda = \lambda^*$ and $\mu = \mu^*$, (1.2i – 1) has, for $i = 1, 2$, a nontrivial solution, say $y_i(x_i, \lambda^*, \mu^*)$, which satisfies (1.2i). Furthermore, we also know that the eigenvalues and normalized eigenfunctions of the system (1.1)–(1.4) may be represented in the form $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}(x_1, x_2)$, respectively, $j, k = 0, 1, \dots$, where $(\lambda_{j,k}, \mu_{j,k})$ denotes that eigenvalue of (1.1)–(1.4) for which $y_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely j zeros in $0 < x_1 < 1$ and $y_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely k zeros in $0 < x_2 < 1$, while

$$\psi_{j,k}(x_1, x_2) = C_{j,k} \prod_{i=1}^2 y_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

and $C_{j,k}$ denotes a normalization constant. In [6] the positive quadrant of the (x, y) -plane was divided into three disjoint sectors, Ω , Ω_1 , and Ω_2 (each with vertex at the origin), and asymptotic formulae were derived for the $(\lambda_{j,k}, \mu_{j,k})$ and $\psi_{j,k}$, as $j + k \rightarrow \infty$, when (j, k) was restricted to lie in Ω and in Ω_1 , respectively (since the results for Ω_2 are similar to those for Ω_1). The choice of the central sector, Ω , was motivated by the fact that for (j, k) in that sector, the above formulae could be derived by means of standard techniques, that is, without having to appeal to transition point theory. However, in order to apply the same techniques to the case $(j, k) \in \Omega_1$, we imposed the further condition that $A_2'B_2 - A_2B_2' = 0$ for $0 < x_2 < 1$, and thus the results of [6] for $(j, k) \in \Omega_1$ are valid only under this hypothesis. Hence in this paper we propose to derive asymptotic formulae for the above expressions for $(j, k) \in \Omega_1$ under the hypothesis that $A_2'B_2 - A_2B_2' \neq 0$ for $0 < x_2 < 1$. The importance of the case considered here lies in the fact that the results obtained will enable us to answer an open question concerning the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$.

In what follows it will be supposed that A_1, B_1 , and q_1 are real and continuous in $0 < x_1 < 1$, with both A_1 and B_1 having absolutely continuous first derivatives in this interval, that q_2 is real and continuous in $0 < x_2 < 1$, that A_2 and B_2 are real and of class C^3 in some interval containing the interval $0 < x_2 < 1$ in its interior, that $A_2'B_2 - A_2B_2' \neq 0$ for $0 < x_2 < 1$, and lastly, that $\Delta = A_1B_2 - A_2B_1 \neq 0$ in I^2 (the product of the intervals $0 < x_1 < 1, 0 < x_2 < 1$). Furthermore, there is no loss of generality in assuming henceforth that the A_j, B_j , and Δ are positive for all values of x_1 and x_2 in I^2 ; for we know (see [6, Appendix A]) that this can always be achieved, if necessary, by introducing a nonsingular transformation in the parameters λ and μ .

After introducing notation and assumptions in §2, we concern ourselves in §3 with the asymptotic integration of (1.3). Here we encounter problems involving a simple transition point whose position varies with μ/λ ; and techniques from transition point theory are used to obtain estimates for the solutions of (1.3) for the various positions of the transition point. In §4 we fix our attention upon certain disjoint subsets Ω_i^* , $i = 1, \dots, 5$, of the sector Ω_1 introduced above, and use the results of §3 to obtain estimates for $\eta_{j,k}$ ($= \mu_{j,k}/\lambda_{j,k}$) for (j, k) in each of the Ω_i^* (see Theorem 4.1). These results enable us to determine the position of the transition point hitherto mentioned corresponding to $\eta_{j,k}$ and hence to deduce from the results of §3 estimates for the solutions of (1.3) when $\lambda = \lambda_{j,k}$ and $\mu = \mu_{j,k}$. We remark that the Ω_i^* have been introduced since there are essentially five different forms of solutions of (1.3) to consider, depending upon the position of the transition points, and hence upon the $\eta_{j,k}$. Thus by restricting (j, k) to a particular Ω_i^* , we are able to restrict the $\eta_{j,k}$, and hence the corresponding transition points, to intervals for which a common formula may be derived to describe the solutions of (1.3) when $\lambda = \lambda_{j,k}$, $\mu = \mu_{j,k}$, and $(j, k) \in \Omega_i^*$. In §5 we employ the results of §§3 and 4 together with estimates for the solutions of (1.1) (which are obtained by standard arguments) to derive asymptotic formulae for $\lambda_{j,k}$, $\mu_{j,k}$, and $\psi_{j,k}$ for (j, k) in each of the Ω_i^* (see Theorems 5.1–5.5). These results are then utilized in §6 to deduce some facts concerning the uniform boundedness of the $\psi_{j,k}$ and the dependence of the $\lambda_{j,k}$ and $\mu_{j,k}$ on the q_i ; extensions of the foregoing results are also discussed here.

A novel feature of our work is that we are able to demonstrate that smoothness of the coefficients of the system (1.1)–(1.4) does not ensure the uniform boundedness of the $\psi_{j,k}(x_1, x_2)$. Indeed, although we were able to show in [6] that the $\psi_{j,k}$ are uniformly bounded when (j, k) lies in the sector Ω (see above), we shall see in the sequel that this is certainly not the case for (j, k) in Ω_1 . In fact, for this latter sector, we shall show instead that the $j^{-1/3}\psi_{j,k}$ are uniformly bounded, and that this result is false if $1/3$ is replaced by any smaller number.

2. Notation and assumptions. In the sequel we shall put

$$P_1(x_1, \lambda, \mu) = \lambda A_1(x_1) - \mu B_1(x_1), \quad P_2(x_2, \lambda, \mu) = -\lambda A_2(x_2) + \mu B_2(x_2),$$

$$r(x_2) = A_2(x_2)/B_2(x_2),$$

and denote by b_1 and b_2 the infimum and supremum, respectively, of $A_1(x_1)/B_1(x_1)$ in $0 \leq x_1 \leq 1$ and by a_1 and a_2 the infimum and supremum, respectively, of $r(x_2)$ in $0 \leq x_2 \leq 1$. Moreover, we shall extend the definition of $q_2(x_2)$ to all real values of x_2 by putting $q_2(x_2) = q_2(0)$ for $x_2 < 0$ and $q_2(x_2) = q_2(1)$ for $x_2 > 1$. Observe that there exists the interval $[c, d]$, where $-1 < c < 0$ and $1 < d < 2$, such that in this interval $B_2(x_2)$, $r(x_2)$ are

positive and of class C^3 , $q_2(x_2)$ is continuous, $r'(x_2) \neq 0$ ($' = d/dx_2$), and $99a_1/100 < r(x_2) < a_2 + (b_1 - a_2)/100$.

To simplify matters, we shall assume for the remainder of this paper that $r'(x_2) < 0$ for $c < x_2 < d$ (see §6) and put $\delta = (r(c) - a_2)/2$. Then for $r(d) \leq t < r(c)$, we denote by $x(t)$ the point of $[c, d]$ at which $r(x_2) = t$ (note that in this interval $x(t)$ is of class C^3 and $x'(t) < 0$, where $' = d/dt$), put $c^* = x(a_2 + \delta)$, and let $\delta_1 = \min\{(c^* - c), (d - 1)\}$. For $c < x_2 < d$, $r(d) \leq t < r(c)$, we shall also put

$$\Psi(x_2, t) = \frac{3}{2} \int_{u=0}^1 u^{1/2} \left[B_2((1-u)x(t) + ux_2) \right. \\ \left. \times \int_{s=0}^1 -r'((1-su)x(t) + sux_2) ds \right]^{1/2} du,$$

$$\Psi_1(x_2, t) = \int_0^1 -r'((1-s)x(t) + sx_2) ds,$$

$$\Psi_2(x_2, t) = B_2(x_2)\Psi_1(x_2, t),$$

$$\Phi(x_2, t) = 2\Psi(x_2, t)/3\Psi_1^{3/2}(x_2, t),$$

$$\Phi_1(x_2, t) = \Psi_2^{1/2}(x_2, t)\Psi^{-1/3}(x_2, t),$$

$$\Phi_2(x_2, t) = (3\Phi(x_2, t)/2)^{2/3},$$

where all fractional powers have their positive values (observe that these functions are all positive and of class C^2 in the given rectangle), and denote by N_1 and N_2 the infimum and supremum, respectively, of $\Phi(x_2, t)$ in this rectangle.

Finally, we shall hereinafter consider the differential equation (1.3) as being defined in the interval $[c, d]$.

3. Asymptotic integration of (1.3).

3.0. *Introduction.* Throughout this section we suppose that $\lambda > 0$, put $t = \mu/\lambda$, and assume that t is held fixed at a value satisfying $a_1 < t < a_2 + \delta$. Then in order to obtain asymptotic formulae for the eigenvalues and eigenfunctions of the system (1.1)–(1.4), it is essential that we derive estimates for the solutions of (1.3) in $[c, d]$ for large values of λ under the given restriction on t . Accordingly, it is to this topic that this section is devoted and the method which we shall employ in achieving our ends will be to approximate (1.3) by a certain related equation which may be solved explicitly with the use of Bessel functions (see subsections 3.2–3.3). However, before passing to these approximations, we shall firstly concern ourselves in subsection 3.1 with introducing certain auxiliary variables which will play an important role

in the analysis to follow. In subsection 3.4 we derive some results concerning the zeros of solutions of (1.3).

Notation. (1) Throughout this section we shall write x^* for $x(t)$ (see §2). Observe that $c^* < x^* < 1$.

(2) For a any point of $[c, d]$, let $u(x_2, a)$ and $v(x_2, a)$ denote the solutions of (1.3) in $[c, d]$ satisfying $u(a, a) = 1, u'(a, a) = 0$ and $v(a, a) = 0, v'(a, a) = 1$, where $' = d/dx_2$.

(3) For a any complex number, let $U(z, a)$ and $V(z, a)$ denote the solutions of the equation

$$y'' + zy = 0, \quad ' = d/dz, \tag{3.1}$$

satisfying $U(a, a) = 1, U'(a, a) = 0$ and $V(a, a) = 0, V'(a, a) = 1$.

(4) Let $T(z), S(z)$ denote the solutions of (3.1) defined by

$$T(z) = [\kappa^* U(z, 0) + V(z, 0)], \quad S(z) = 3^{1/2}[\kappa^* U(z, 0) - V(z, 0)],$$

where $\kappa^* = \Gamma(1/3)/3^{1/3}\Gamma(2/3)$. Observe that $W(T, S)(z) = -2 \times 3^{1/2}\kappa^*$, where W denotes the Wronskian.

(5) For z real let

$$\rho(z) = (T^2(z) + S^2(z))^{1/2}, \quad \rho^*(z) = ((T'(z))^2 + (S'(z))^2)^{1/2},$$

where $' = d/dz$ and all fractional powers have their positive values.

(6) Let $\gamma(z)$ and $\gamma^*(z)$ be defined for all real values of z by the conditions that

$$\cos \gamma(z) = T(z)/\rho(z), \quad \sin \gamma(z) = S(z)/\rho(z)$$

and

$$\cos \gamma^*(z) = T'(z)/\rho^*(z), \quad \sin \gamma^*(z) = S'(z)/\rho^*(z),$$

with the indeterminate multiple of 2π in these definitions being fixed by choosing $\gamma(0) = \pi/3, \gamma^*(0) = -\pi/3$ and requiring that both $\gamma(z)$ and $\gamma^*(z)$ be continuous functions of z .

(7) Let $\kappa = (1/2 \times 3^{1/2}\kappa^*)^{1/2}$.

Finally, for the remainder of this section we shall drop subscripts and write x for x_2, B for B_2, q for q_2 , and P for P_2 in $c < x < d$.

3.1. *Auxiliary variables.* Let $r_1(x) = (r(x) - r(x^*))/(x - x^*)$ if $x \neq x^*$, $r_1(x^*) = r'(x^*)$, where $' = d/dx$ (see §2). It is a simple matter to verify that $r_1(x)$ is of class C^2 in $[c, d]$ and that in this interval $|r_1(x)|, |1/r_1(x)|, |r_1'(x)|$, and $|r_1''(x)|$ all remain less than some bound independent of x, t , and λ . Writing $P(x)$ for $P(x, \lambda, \mu)$ in $c < x < d$, it now follows that

$$P(x) = \lambda \chi^2(x)(x - x^*), \tag{3.2}$$

where $\chi^2(x) = B(x)|r_1(x)|$. Then in order to deal with fractional powers of P and of other functions that will appear below, let us henceforth agree to the

following convention: we shall take the argument of a positive quantity to be zero, take $\arg e^{i\nu}$ (ν real) to be ν , take the argument of a complex number which is a product of factors whose arguments have been specified to be the sum of the arguments of its factors, and lastly, interpret the expression a^ν ($a \neq 0$), where $\arg a$ has been specified and ν is real, in accordance with the rule $a^\nu = |a|^\nu \exp\{i\nu \arg a\}$. This convention enables us to determine $\arg \lambda$, $\arg \chi^2(x)$, and $\arg(x - x^*)$ for $x > x^*$, and if for $x < x^*$ we agree to take $\arg(x - x^*)$ to be π , then it is clear that the expressions

$$P^{1/2}(x) = \lambda^{1/2} \chi(x)(x - x^*)^{1/2}$$

and

$$w(x) = \int_{x^*}^x P^{1/2}(s) ds \quad (3.3)$$

are unambiguously defined in $[c, d]$. Furthermore, it is not difficult to verify that $w(x)$ may be expressed in the form

$$w(x) = (2/3)\lambda^{1/2} \chi_1(x)(x - x^*)^{3/2}, \quad (3.4)$$

where $\chi_1(x)$ is real, positive, and of class C^2 in $[c, d]$, and that in this interval $\chi_1(x)$, $1/\chi_1(x)$, $|\chi_1'(x)|$, and $|\chi_1''(x)|$, $' = d/dx$, all remain less than some bound, say M , independent of x , t , and λ .

Let us next introduce the function z defined by

$$z(x) = \lambda^{1/3} \chi_1^{2/3}(x)(x - x^*) \quad (3.5)$$

for $c \leq x \leq d$. Then it follows from (3.4) that $w(x) = (2/3)z^{3/2}(x)$, and hence we conclude from (3.3) that

$$z^{1/2}(x)z'(x) = P^{1/2}(x) \quad (3.6)$$

for $c \leq x \leq d$, where $' = d/dx$. From (3.2) and (3.5)–(3.6) we now have

$$z'(x) = \lambda^{1/3} B^{1/2}(x) |r_1(x)|^{1/2} \chi_1^{-1/3}(x),$$

and hence $z'(x) > 0$ in $[c, d]$. Moreover, it is clear that in this interval $\lambda^{-1/3}z'(x)$, $1/\lambda^{-1/3}z'(x)$, $|z''(x)/z'(x)|$, and $|\{z, x\}|$ all remain less than some bound independent of x , t , and λ , where $\{z, x\}$ denotes the Schwarzian derivative of z with respect to x .

Observe that as x runs from c to d , $w(x)$ traces out a curve in the w -plane such that $\arg w(x) = 3\pi/2$ in $c \leq x < x^*$, and as x runs from c to x^* , $|w(x)|$ strictly decreases from a value exceeding $2\lambda^{1/2}M^\# / 3$ (where $M^\# = \delta_1^{3/2} / M < 1$ and δ_1 is defined in §2) when $x = c$ to zero when $x = x^*$, while in the interval $x^* < x \leq d$, $\arg w(x) = 0$, and as x runs from x^* to d , $|w(x)|$ strictly increases from zero to a value exceeding $2\lambda^{1/2}M^\# / 3$ when $x = d$. Analogous results also hold for the path traced out by $z(x)$ as x runs from c to d .

3.2. *Estimates.* We are now going to use the results of subsection 3.1 to derive estimates for the solutions of (1.3) in $[c, d]$. Accordingly, let x_1 be any point of $[c, d]$. Then recalling the definition of $v(x, x_1)$ (as well as the other expressions) given in subsection 3.0, we may argue in a manner similar to that in [4, pp. 4-5] to verify that $v(x, x_1)$ satisfies the integral equation

$$(z')^{1/2}y(x) = (z_1')^{-1/2}V(z, z_1) + \int_{x_1}^x V(z, \zeta)(\zeta')^{-1/2}Q(s)y(s) ds, \quad (3.7)$$

$c \leq x \leq d$, where here $z = z(x)$, $z_1 = z(x_1)$, $\zeta = z(s)$, $z' = z'(x)$, $z_1' = z'(x_1)$, $\zeta' = z'(s)$, $Q(s) = (-q(s) + \{z, s\}/2)$, and $\{z, s\}$ denotes the Schwarzian derivative of z with respect to s . Assuming henceforth that x_2 is any point of $[c, d]$ satisfying $x_2 \geq x_1$, and that $\lambda > (2 \times 10^3/M^*)$ (hence it follows from subsection 3.1 that both $|w(c)|$ and $w(d)$ exceed 10^3), it is our intention to utilize (3.7) to arrive at an estimate for $v(x_2, x_1)$. In order to achieve this end, it is essential that we obtain approximations for $V(z_2, z_1)$ ($z_2 = z(x_2)$) for the various positions of z_1, z_2 , and accordingly, we shall now pass to such approximations.

Let $w_j = w(x_j)$, $\rho_j = \rho(z_j)$, and $\gamma_j = \gamma(z_j)$ for $j = 1, 2$, $\nu = 5/72$, and $\nu^\dagger = 7/72$. Then from the formulae

$$\begin{aligned} V(z_2, z_1) &= U(z_1, 0)V(z_2, 0) - V(z_1, 0)U(z_2, 0), \\ V'(z_2, z_1) &= U(z_1, 0)V'(z_2, 0) - V(z_1, 0)U'(z_2, 0), \end{aligned}$$

from the formulae given in [4, p. 82] expressing $U(z, 0)$, $V(z, 0)$ and their derivatives in terms of Bessel functions, and from the results given in [10, pp. 199, 202], it is easy to see that:

(1) if $|w(x)| \geq 1$ in $x_1 \leq x \leq x_2$, then

$$\begin{aligned} V(z_2, z_1) &= z_1^{-1/4}z_2^{-1/4} \left[\sin W_{12} \{1 + O(|w|^{-2})\} \right. \\ &\quad \left. + \cos W_{12} \{ \nu w_1^{-1} - \nu w_2^{-1} + O(|w|^{-3}) \} \right], \\ V'(z_2, z_1) &= z_1^{-1/4}z_2^{1/4} \left[\cos W_{12} \{1 + O(|w|^{-2})\} \right. \\ &\quad \left. - \sin W_{12} \{ \nu w_1^{-1} + \nu^\dagger w_2^{-1} + O(|w|^{-3}) \} \right], \end{aligned} \quad (3.8)$$

where $W_{12} = w_2 - w_1$ and $|w| = \min\{|w_1|, |w_2|\}$;

(2) if $w_2 \geq 1$, then

$$\begin{aligned} V(z_2, z_1) &= \kappa \rho_1 z_2^{-1/4} \left[\sin W_{12}^* \{1 + O(w_2^{-2})\} \right. \\ &\quad \left. - \cos W_{12}^* \{ \nu w_2^{-1} + O(w_2^{-3}) \} \right], \\ V'(z_2, z_1) &= \kappa \rho_1 z_2^{1/4} \left[\cos W_{12}^* \{1 + O(w_2^{-2})\} \right. \\ &\quad \left. - \sin W_{12}^* \{ \nu^\dagger w_2^{-1} + O(w_2^{-3}) \} \right], \end{aligned} \quad (3.9)$$

where $W_{12}^* = w_2 + \gamma_1 - \pi/4$;

(3) if $|w_1| \geq 1$ and $\arg w_1 = 3\pi/2$, then

$$\begin{aligned} V(z_2, z_1) &= C(x_1) \left[T(z_2) \{ 1 - iw_1^{-1} + O(|w_1|^{-2}) \} \right. \\ &\quad \left. + S(z_2) \{ O(|w_1|^{-3}) \} \right], \\ V'(z_2, z_1) &= C(x_1) \left[T'(z_2) \{ 1 - iw_1^{-1} + O(|w_1|^{-2}) \} \right. \\ &\quad \left. + S'(z_2) \{ O(|w_1|^{-3}) \} \right], \end{aligned} \quad (3.10)$$

where $C(x_1) = \kappa z_1^{-1/4} \exp\{i(w_1 + \pi/4)\}$ (note that in these equations the coefficients of $S(z_2)$ and $S'(z_2)$ are also $O(\exp\{-2|w_1|\})$).

Moreover, if $w_2 \geq 1$, then in (3.10) we may put

$$\begin{aligned} \kappa T(z_2) &= z_2^{-1/4} \left[\sin W_2 \{ 1 + \alpha w_2^{-2} + O(w_2^{-4}) \} \right. \\ &\quad \left. - \cos W_2 \{ \nu w_2^{-1} + \beta w_2^{-3} + O(w_2^{-5}) \} \right], \\ \kappa S(z_2) &= z_2^{-1/4} \left[\cos W_2 \{ 1 + O(w_2^{-2}) \} + \sin W_2 \{ \nu w_2^{-1} + O(w_2^{-3}) \} \right], \\ \kappa T'(z_2) &= z_2^{1/4} \left[\cos W_2 \{ 1 + \alpha^\dagger w_2^{-2} + O(w_2^{-4}) \} \right. \\ &\quad \left. - \sin W_2 \{ \nu^\dagger w_2^{-1} + \beta^\dagger w_2^{-3} + O(w_2^{-5}) \} \right], \\ \kappa S'(z_2) &= -z_2^{1/4} \left[\sin W_2 \{ 1 + O(w_2^{-2}) \} + \cos W_2 \{ \nu^\dagger w_2^{-1} + O(w_2^{-3}) \} \right], \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} W_2 &= w_2 + \pi/4, \quad \alpha = p_1(1/3), \quad \beta = -p_2(1/3), \\ \alpha^\dagger &= p_1(2/3), \quad \beta^\dagger = p_2(2/3), \end{aligned}$$

$$p_1(s) = -(4s^2 - 1)(4s^2 - 9)/128, \quad p_2(s) = (4s^2 - 25)p_1(s)/24,$$

while if $|w_2| \geq 1$ and $\arg w_2 = 3\pi/2$, then in (3.10) we may put

$$\begin{aligned} T(z_2) &= 2^{-1} C_1(x_2) \{ 1 + O(|w_2|^{-1}) \}, \\ S(z_2) &= C_2(x_2) \{ 1 + O(|w_2|^{-1}) \}, \\ T'(z_2) &= 2^{-1} C_3(x_2) \{ 1 + O(|w_2|^{-1}) \}, \\ S'(z_2) &= -C_4(x_2) \{ 1 + O(|w_2|^{-1}) \}, \end{aligned} \quad (3.12)$$

where $\kappa C_j(x_2)$ equals $z_2^{-1/4} \exp\{i((-1)^j w_2 + \pi/4)\}$ for $j = 1, 2$ and $z_2^{1/4} \exp\{i((-1)^j w_2 - \pi/4)\}$ for $j = 3, 4$. Finally, we remark that in all of the above formulae (as well as in the formulae to follow) the constant implied in any one of the O symbols is *independent* of x_1, x_2, t , and λ .

Let x'_1 and $x'_2 (> x'_1)$ denote the points of $[c, d]$ at which $|w(x)| = 1$ and

put $P_j = P(x_j)$ for $j = 1, 2$. Then in light of the foregoing results we may now argue with (3.7) and the Gronwall lemma to show that

$$v(x_2, x_1) = P_1^{-1/4}P_2^{-1/4} \left[z_1^{1/4}z_2^{1/4}V(z_2, z_1) + O(\lambda^{-1/2}E_{12}) \right], \quad (3.13)$$

and similarly we can show that

$$v'(x_2, x_1) = P_1^{-1/4}P_2^{1/4} \left[z_1^{1/4}z_2^{-1/4}V'(z_2, z_1) + O(\lambda^{-1/3}E_{12}) \right], \quad (3.14)$$

where $E_{12} = \exp\{\text{Im}(w_2 - w_1)\}$, and it is to be understood that in (3.13)–(3.14) we are to replace $P_j^{\pm 1/4}$ by $(z_j')^{\pm 1/2}$ ($z_j' = z'(x_j)$) and $z_j^{\pm 1/4}$ by 1 when $x_1' < x_j < x_2'$ for $j = 1, 2$. Analogous results for u and u' can also be obtained in a similar manner.

3.3. *Further developments.* Due to future requirements we are now going to develop further the formulae (3.13)–(3.14). Accordingly, let K be a number not less than 1 chosen large enough so that the constant implied in any one of the O symbols appearing in (3.8)–(3.12) and in the formulae for U and U' analogous to (3.8)–(3.10) does not exceed K . Choose $R > 10^2K$ and let α be an arbitrary, but fixed positive number. Assuming henceforth that $\lambda > \max\{(R/\alpha)^{16/3}, (10^4(\alpha + R)/M^*)^4\}$ (in which case it follows from a simple argument involving (3.3) that $w(d) - w(1)$, $|w(c) - w(c^*)|$, and hence $w(d)$ and $|w(c)|$ all exceed $10^3(\alpha\lambda^{3/16} + R)$), let $x_1^\#, x_2^\# (> x_1^\#)$ denote the points of $[c, d]$ at which $|w(x)| = R$ and $x_3^\#, x_4^\# (> x_3^\#)$ the points of $[c, d]$ at which $|w(x)| = \alpha\lambda^{3/16}$. Then standard arguments involving the use of (3.7)–(3.12) and techniques similar to those used in [4, pp. 6–11], [6, Theorem 3.4] show that:

(1) when $x_1 \geq x_4^\#$ or when $x_2 \leq x_3^\#$,

$$v(x_2, x_1) = P_1^{-1/4}P_2^{-1/4} \left[\sin \theta_{12} \{ 1 + O(|w|^{-2}) + o(\lambda^{-1/2}) \} \right. \\ \left. + \cos \theta_{12} \{ \nu w_1^{-1} - \nu w_2^{-1} - \theta_{12}^* + o(\lambda^{-1/2}) \} \right], \quad (3.15)$$

$$v'(x_2, x_1) = P_1^{-1/4}P_2^{1/4} \left[\cos \theta_{12} \{ 1 + O(\lambda^{-3/8}) \} \right. \\ \left. + \sin \theta_{12} \{ O(\lambda^{-3/16}) \} \right], \quad (3.16)$$

where ν and $|w|$ are as in (3.8),

$$\theta_{12} = \theta(x_2, x_1), \quad \theta_{12}^* = \theta^*(x_2, x_1), \quad \theta(s_2, s_1) = \int_{s_1}^{s_2} P^{1/2}(s) ds,$$

and

$$\theta^*(s_2, s_1) = 2^{-1} \int_{s_1}^{s_2} Q(s)P^{-1/2}(s) ds$$

(see (3.7));

(2) when $x_1 > x_3^\#$ and $x_2 \geq x_4^\#$,

$$v(x_2, x_1) = \kappa \rho_1 (z_1')^{-1/2} P_2^{-1/4} \left[\sin W_{12}^* \{1 + O(w_2^{-2}) + o(\lambda^{-1/2})\} - \cos W_{12}^* \{ \nu w_2^{-1} + \theta^*(x_2, \xi) + o(\lambda^{-1/2}) \} \right], \quad (3.17)$$

where W_{12}^* is defined above (see (3.9)) and $\xi = \max\{x_1, x^*\}$;

(3) when $x_3^\# < x_1 \leq x_1^\#$ and $x_2 \leq x^*$,

$$v(x_2, x_1) = \kappa^2 P_1^{-1/4} (z_2')^{-1/2} z_1^{1/4} S(z_1) \times \left[T(z_2) \{1 + o(\lambda^{1/2})\} - S(z_2) T(z_1) / S(z_1) \right]; \quad (3.18)$$

(4) when $x_1 \leq x_3^\#$ and $E_{12}^* = \kappa |P_{11}|^{-1/4} (z_2')^{-1/2} \exp\{|w_1|\}$,

$$\begin{aligned} v(x_2, x_1) &= E_{12}^* \left[T(z_2) \{1 - i\nu w_1^{-1} + i\theta_{12}^* + O(|w_1|^{-2}) + o(\lambda^{-1/2})\} \right. \\ &\quad \left. + S(z_2) \{O(\exp\{-2|w_1|\})\} \right] \\ &\quad \text{if } x_2 \leq x^*, \\ &= E_{12}^* \left[T(z_2) \{1 - i\nu w_1^{-1} + i\theta^*(x^*, x_1) + O(|w_1|^{-2}) + o(\lambda^{-1/2})\} \right. \\ &\quad \left. - S(z_2) \{\theta^*(x_2, x^*) + |w_1|^{-3} + h(z_2)\} \right] \\ &\quad \text{if } x_2 > x^*, \end{aligned} \quad (3.19)$$

where $h(z_2)$ is $O(\lambda^{-2/3})$ for $x_2 < x_2^\#$, $(x_2 - x^*)^{1/2} O(\lambda^{-1/2})$ for $x_2^\# \leq x_2 < x_4^\#$, and $O(\lambda^{-29/48}) + (x_2 - x^*)^{1/2} o(\lambda^{-1/2})$ for $x_2 \geq x_4^\#$;

(5) when $x_1 \leq x_3^\#$, $x_2 \geq x_4^\#$, and $E_{12}^\dagger = \kappa |P_{11}|^{-1/4} P_2^{1/4} \exp\{|w_1|\}$,

$$\begin{aligned} v'(x_2, x_1) &= E_{12}^\dagger \left[z_2^{-1/4} T'(z_2) \{1 - i\nu w_1^{-1} + i\theta^*(x^*, x_1) \right. \\ &\quad \left. + O(|w_1|^{-2}) + o(\lambda^{-1/2}) \right] \\ &\quad - z_2^{-1/4} S'(z_2) \{ \theta^*(x_2, x^*) + o(\lambda^{-1/2}) \} \\ &\quad + z_2^{1/4} T(z_2) \{ o(\lambda^{-3/8}) \} + z_2^{1/4} S(z_2) \{ o(\lambda^{-1/2}) \} \right], \quad (3.20) \end{aligned}$$

as $\lambda \rightarrow \infty$, with each of these formulae holding uniformly in x_1 , x_2 , and t . Finally, we remark that further results for v' as well as analogous results for u and u' can also be obtained in a similar fashion.

3.4. Zeros. Guided by future requirements we are now going to derive some facts concerning the zeros of $v(x_2, x_1)$. Accordingly, we first note from [7, pp. 98–99] that the zeros of $T(z)$ and $T'(z)$ all lie on the positive real axis, and if

we denote these zeros by τ_n and τ'_n , respectively, $n \geq 1$, arranged in increasing order of magnitude, then $0 < \tau'_1 < \tau_1 < \tau'_2 < \tau_2 < \dots$, and

$$2\tau_n^{3/2}/3 = (n - 1/4)\pi + O(n^{-1}), \quad 2(\tau'_n)^{3/2}/3 = (n - 3/4)\pi + O(n^{-1})$$

as $n \rightarrow \infty$. It can also be verified that the real zeros of $S(z)$ and $S'(z)$ are all positive and if we denote these zeros by s_n and s'_n , respectively, $n \geq 1$, arranged in increasing order of magnitude, then $0 < s_1 < s'_1 < s_2 < s'_2 < \dots, s_1 < \tau_1 < s_2 < \tau_2 < \dots, \tau'_1 < s'_1 < \tau'_2 < s'_2 < \dots$, and

$$2s_n^{3/2}/3 = (n - 3/4)\pi + O(n^{-1}), \quad 2(s'_n)^{3/2}/3 = (n - 1/4)\pi + O(n^{-1})$$

as $n \rightarrow \infty$. By arguing with these results, the definition of $\gamma(z)$, and (3.12), we may now show that when z runs from $-\infty$ to ∞ , $\gamma(z)$ strictly decreases from $\pi/2$ to $-\infty$, taking on the value $-(n - 1)\pi$ at $z = s_n$ and $-(2n - 1)\pi/2$ at $z = \tau_n$ for $n \geq 1$. Moreover, $d\gamma(z)/dz = -(1/\kappa\rho(z))^2$. Similarly, we can show that $\gamma^*(z)$ strictly increases from $-\pi/2$ to $-\pi/3$ when z runs from $-\infty$ to 0 and strictly decreases from $-\pi/3$ to $-\infty$ when z runs from 0 to ∞ , taking on the value $-\pi\pi$ at $z = s'_n$ and $-(2n - 1)\pi/2$ at $z = \tau'_n$ for $n \geq 1$. We also have $d\gamma^*(z)/dz = -z/(\kappa\rho^*(z))^2$. Since $\gamma(z) - \gamma^*(z)$ is never equal to a multiple of π and $\gamma(\tau_1) = -\pi/2$, $-3\pi/2 < \gamma^*(\tau_1) < -\pi/2$, it therefore follows that $0 < \gamma(z) - \gamma^*(z) < \pi$ for all real z .

Next let m denote the smallest value of n for which $\tau'_n \geq R^*$ ($= (3R/2)^{2/3}$, where R is defined at the beginning of subsection 3.3). Then it follows from the foregoing results and (3.11) that $m > 15$ and

$$p(n - 1/2) < \tau_n < p(n), \quad p(n - 1) < \tau'_n < p(n - 1/2) \quad \text{for } n \geq m,$$

where $p(s) = (3\pi s/2)^{2/3}$. Observe also from (3.11) that

$$z^{1/4}|T(z)| > 1/2\kappa \quad \text{in } p(n - 1) \leq z \leq p(n - 1/2)$$

and

$$z^{-1/4}|T'(z)| > 1/2\kappa \quad \text{in } p(n - 1/2) \leq z \leq p(n) \text{ for } n \geq m.$$

Now put

$$\begin{aligned} z_0^\dagger &= 0, & z_1^\dagger &= (\tau'_1 + \tau_1)/2, & z_{2n}^\dagger &= (\tau'_{n+1} + \tau_n)/2, \\ z_{2n+1}^\dagger &= (\tau'_{n+1} + \tau_{n+1})/2 \quad \text{for } n = 1, \dots, (m - 2), \\ z_{2n}^\dagger &= p(n) \quad \text{and} \quad z_{2n+1}^\dagger = p(n + 1/2) \quad \text{for } n \geq (m - 1). \end{aligned}$$

Then there exists the positive constant β such that $|T(z)| \geq \beta$ (resp. $|T'(z)| \geq \beta$) in each of the intervals $z_{2n}^\dagger \leq z \leq z_{2n+1}^\dagger$ (resp. $z_{2n+1}^\dagger \leq z \leq z_{2n+2}^\dagger$) for $n = 0, \dots, (m - 2)$. Note from (3.11) that if $\zeta(z) = 1$ for $0 \leq z < p(m - 1)$ and $\zeta(z) = z$ for $z \geq p(m - 1)$, then the absolute values of $\zeta^{1/4}(z)T(z)$, $\zeta^{1/4}(z)S(z)$, $\zeta^{-1/4}(z)T'(z)$, and $\zeta^{-1/4}(z)S'(z)$ in the interval $0 \leq z < \infty$ all remain less than some bound independent of z .

(a) We shall now employ the above results to deduce some facts concerning the zeros of $v(x_2, x_1)$ when x_1 is held fixed at a value satisfying $x_1 \leq x_3^\#$ (see subsection 3.3). Indeed, by arguing with (3.8) and (3.14) for the interval $x_1 \leq x_2 \leq x_1^\#$ and with (3.12), (3.14), and the formula

$$V'(z_2, z_1) = \kappa^2 \rho_1 \rho_2^* \sin(\gamma_1 - \gamma_2^*)$$

for the interval $x_1^\# < x_2 \leq x^*$ (where $\rho_j^* = \rho^*(z_j)$, $\gamma_j^* = \gamma^*(z_j)$ for $j = 1, 2$), it is easy to see that in $[x_1, x^*]$, $v(x_2, x_1)$ can only vanish when $x_2 = x_1$ if λ is sufficiently large (that is, if λ exceeds a certain fixed positive number which of course does not depend upon x_1 or t). Turning to the case where $x_2 > x^*$, let n^\dagger denote the largest value of n for which $n\pi \leq w(d)$ and x_n^\dagger , $n = 0, \dots, 2n^\dagger$, the points of $[c, d]$ satisfying $z(x_n^\dagger) = z_n^\dagger$ (it is not difficult to verify that $n^\dagger - m > 10^4$ and $x_{2n^\dagger}^\dagger > 1$). Then assuming λ sufficiently large, we may argue with (3.10) and (3.13)–(3.14) to verify that $v(x_2, x_1)$ vanishes, but $v'(x_2, x_1)$ does not vanish, in each of the intervals $(x_{2n-1}^\dagger, x_{2n}^\dagger)$, $n = 1, \dots, n^\dagger$, while in each of the intervals $[x_{2n}^\dagger, x_{2n+1}^\dagger]$, $n = 0, \dots, (n^\dagger - 1)$, $v(x_2, x_1) \neq 0$. Hence if λ is sufficiently large, if $x^* < x' \leq x_{2n^\dagger}^\dagger$, and if $v(x_2, x_1)$ has precisely k zeros in (x_1, x') and vanishes at $x_2 = x'$, then $x_{2k+1}^\dagger < x' < x_{2k+2}^\dagger$, $z_{2k+1}^\dagger < z(x') < z_{2k+2}^\dagger$, and when $k \geq m - 1$, $(k + 1/2)\pi < w(x') < (k + 1)\pi$.

(b) We are now going to establish some facts concerning the zeros of $v(x_2, x_1)$ when x_1 is held fixed at a value satisfying $x_3^\# < x_1 < x_4^\#$; and to this end we shall firstly investigate the behaviour of $V(z, z_1)$ and $V'(z, z_1)$ in the interval $I = \{z | z_1 \leq z < \infty\}$. Accordingly, put $\tau_0 = -\infty$ and suppose that for some $j \geq 0$ either (i) $\tau_j < z_1 \leq s_{j+1}$, or (ii) $s_{j+1} < z_1 \leq \tau_{j+1}$. Note that for case (i), $-j\pi \leq \gamma_1 < -(j - 1/2)\pi$, while for case (ii), $-(j + 1/2)\pi \leq \gamma_1 < -j\pi$; and a simple calculation also shows that the absolute value of $\gamma_1 \lambda^{-3/16}$ in $x_3^\# < x_1 < x_4^\#$ remains less than some bound independent of x_1 and λ . If we now express V, V' in the form

$$\begin{aligned} V(z, z_1) &= \kappa^2 \rho_1 \rho(z) \sin(\gamma_1 - \gamma(z)), \\ V'(z, z_1) &= \kappa^2 \rho_1 \rho^*(z) \sin(\gamma_1 - \gamma^*(z)), \end{aligned} \quad (3.21)$$

and denote by t_n , $n \geq 0$ (resp. t'_n , $n \geq 1$), the points of I at which $\gamma(z) = \gamma_1 - n\pi$ (resp. $\gamma^*(z) = \gamma_1 - n\pi$), then it follows that $V(z, z_1)$ (resp. $V'(z, z_1)$) has a simple zero at each of the points t_n (resp. t'_n) and vanishes nowhere else in I , and that $z_1 = t_0 < t'_1 < t_1 < t'_2 < t_2 < \dots$, $t'_1 > 0$, $\tau_{j+n} < t_n \leq s_{j+n+1}$ for $n \geq 0$ if $\tau_j < z_1 \leq s_{j+1}$, and $s_{j+n+1} < t_n \leq \tau_{j+n+1}$ for $n \geq 0$ if $s_{j+1} < z_1 \leq \tau_{j+1}$. Since it is clear from the definition of m above that

$$p(n - 1) < s_n < p(n - 1/2) < \tau_n < p(n) \quad \text{for } n \geq m - 15,$$

it therefore follows that if m^\dagger denotes the smallest value of n for which

$t'_n \geq R^*$ and $m^* = (m^\dagger - \min\{m^\dagger, 10\})$, then

$$j + m^\dagger \geq m - 2, \quad m^\dagger < (3 + R/\pi),$$

$$p(j + n - 1/2) < t_n < p(j + n + 1/2) \quad \text{if } \tau_j < z_1 \leq s_{j+1},$$

$$p(j + n) < t_n < p(j + n + 1) \quad \text{if } s_{j+1} < z_1 \leq \tau_{j+1},$$

and hence

$$(n - 3/4)\pi < ((2/3)t_n^{3/2} + \gamma_1 - \pi/4) < (n + 3/4)\pi \quad \text{for } n \geq m^*.$$

Since $((m^* - 1)\pi - \gamma_1 + \pi/4) > (m - 14)\pi > R/2$, it follows from (3.9) and the interlacing of the t_n and t'_n that

$$p^*(n - 1/4) < t_n < p^*(n + 1/4) \quad \text{for } n \geq m^*,$$

$$p^*(n - 3/4) < t'_n < p^*(n - 1/4) \quad \text{for } n \geq m^* + 1,$$

where $p^*(s) = [3((s + 1/4)\pi - \gamma_1)/2]^{2/3}$. Observe also from (3.9) that

$$z^{1/4}|V(z, z_1)| > \kappa\rho_1/2 \quad \text{in } p^*(n - 3/4) \leq z < p^*(n - 1/4)$$

for $n \geq m^* + 1$

and

$$z^{-1/4}|V'(z, z_1)| > \kappa\rho_1/2 \quad \text{in } p^*(n - 1/4) \leq z \leq p^*(n + 1/4) \text{ for } n \geq m^*.$$

Lastly, let n^* denote the largest value of n for which $((n + 1/2)\pi - \gamma_1) \leq w(d)$ and $x(n^*)$ the point of $[c, d]$ satisfying $z(x(n^*)) = p^*(n^* + 1/4)$ (it is not difficult to verify that $n^* - m^\dagger > 10^4$ and $x(n^*) > 1$). Then by appealing to the above properties of $\gamma(z)$ and $\gamma^*(z)$, to (3.21), and proceeding in a manner similar to that in part (a) above, it is not difficult to verify that *if λ is sufficiently large, if $x_1 < x' \leq x(n^*)$, if $v(x_2, x_1)$ has precisely k zeros in (x_1, x') and vanishes at $x_2 = x'$, and if $k \geq m^*$, then*

$$(k + 3/4)\pi < (w(x') + \gamma_1 - \pi/4) < (k + 5/4)\pi.$$

4. Preliminary results. We are now going to use the results of §3 to derive some estimates for the eigenvalues of the system (1.1)–(1.4). These estimates will enable us in §5 to establish asymptotic formulae for the eigenvalues and eigenfunctions of the above system. Accordingly, recall from [6, §2] that by an eigenvalue of (1.1)–(1.4) we mean a pair of numbers, (λ^*, μ^*) , such that for $\lambda = \lambda^*$ and $\mu = \mu^*$, (1.1) and (1.3) have nontrivial solutions satisfying (1.2) and (1.4), respectively. If $y_1(x_1, \lambda^*, \mu^*)$ and $y_2(x_2, \lambda^*, \mu^*)$ denote these solutions, respectively, then the product, $\prod_{i=1}^2 y_i(x_i, \lambda^*, \mu^*)$, is called an eigenfunction of the system (1.1)–(1.4) corresponding to (λ^*, μ^*) . Important results pertaining to the eigenvalues and eigenfunctions of (1.1)–(1.4) were recorded in the reference just cited, and in particular, it was noted that the eigenvalues were all real and could be represented in the form $(\lambda_{j,k}, \mu_{j,k})$, $j, k = 0, 1, 2, \dots$, where, with ϕ_i ($i = 1, 2$) denoting the solution of $(1.2i - 1)$

satisfying $\phi_i(0, \lambda, \mu) = \sin \alpha_i$, $\phi_i'(0, \lambda, \mu) = \cos \alpha_i$, $\phi_1(x_1, \lambda_{j,k}, \mu_{j,k})$ has precisely j zeros in $0 < x_1 < 1$ and $\phi_2(x_2, \lambda_{j,k}, \mu_{j,k})$ has precisely k zeros in $0 < x_2 < 1$.

Notation. (1) Let

$$h_1(t) = \int_0^1 (A_1(x_1) - tB_1(x_1))^{1/2} dx_1 \quad \text{for } -\infty < t \leq b_1,$$

$$h_2(t) = \int_0^1 (-A_2(x_2) + tB_2(x_2))^{1/2} dx_2 \quad \text{for } a_2 \leq t < \infty$$

(see §2 for terminology used here and below; also throughout this section we assume that fractional powers of positive quantities have their positive values).

(2) Let $g(t) = h_2(t)/h_1(t)$ for $a_2 \leq t < b_1$, $g(b_1) = h_2(b_1)/h_1(b_1)$ if $h(b_1) \neq 0$, and $g(b_1) = \infty$ otherwise. We remark that besides the properties of g listed in [6, Subsection 3.0], we also have, as a consequence of the conditions imposed in this paper upon A_2 and B_2 , the further properties that $g'(t)$ exists and is continuous and positive in $a_2 \leq t < b_1$ (here and below $' = d/dt$).

(3) Let $t_1 = a_2 + \delta/2$ and t_2 be any number satisfying $t_1 < t_2 < b_1$.

(4) Let $\theta_1^* = \tan^{-1}g(a_2)$ and $\theta_i = \tan^{-1}g(t_i)$ for $i = 1, 2$, where the principal branch of the inverse tangent is taken (observe that $0 < \theta_1^* < \theta_1 < \theta_2 < \pi/2$).

(5) Let Ω and Ω_1 denote the sectors in the (x, y) -plane defined by the inequalities $\theta_1 \leq \theta \leq \theta_2$ and $0 \leq \theta < \theta_1$, respectively, where θ denotes the angle which a ray emanating from the origin makes with the positive x -axis.

(6) Let

$$H(t) = \int_{x(t)}^1 (-A_2(x_2) + tB_2(x_2))^{1/2} dx_2,$$

$$G(t) = H(t)/h_1(t) \quad \text{for } a_1 \leq t \leq a_2 + \delta.$$

It is clear that in this interval $G(t)$ is strictly increasing, $G(a_1) = 0$, $G(a_2) = g(a_2)$, and since $H(t) = (2/3)(1 - x(t))^{3/2}\Psi(1, t)$, we also see that $G(t)$ is of class C^1 in $[a_1, a_2 + \delta]$ and of class C^2 in $(a_1, a_2 + \delta]$. Furthermore, it is readily verified that $G'(t) > 0$ in $(a_1, a_2 + \delta]$ and $G'(a_2) = g'(a_2)$.

(7) Let $G_1(t) = (3G(t)/2)^{2/3}$ for $a_1 \leq t \leq a_2 + \delta$. Observe that in this interval $G_1(t)$ is of class C^2 , $G_1'(t) > 0$, and $G_1(a_1) = 0$.

(8) Put

$$a = g'(a_2) = G'(a_2), \quad a^* = 2(G_1'(a_1))^{3/2}/3,$$

$$b = \left[1 + a(g(t_1) - g(a_2))^{-1} + (a + a^*)/g(a_2) \right]^6,$$

and let Ω_i^* , $1 \leq i \leq 5$, denote that subset of Ω_1 which is composed of points

(x, y) for which $x > b$ and:

$$\begin{aligned} (\tan \theta_1^* + ax^{-5/12}) &\leq y/x < \tan \theta_1 \quad \text{if } i = 1, \\ |\tan \theta_1^* - y/x| &< ax^{-5/12} \quad \text{if } i = 2, \\ (\tan \theta_1^* - ax^{-1/6}) &< y/x \leq (\tan \theta_1^* - ax^{-5/12}) \quad \text{if } i = 3, \\ a^*x^{-5/8} &\leq y/x \leq (\tan \theta_1^* - ax^{-1/6}) \quad \text{if } i = 4, \end{aligned}$$

and lastly, $0 \leq y/x < a^*x^{-5/8}$ if $i = 5$.

The sectors Ω and Ω_1 defined above are precisely the sectors introduced in §1; and as stated there, asymptotic formulae were derived in [6] for $\lambda_{j,k}$ and $\mu_{j,k}$ as $j \rightarrow \infty$, $(j, k) \in \Omega$. To derive corresponding formulae for Ω_1 (under the conditions assumed in this paper), it is essential that we first obtain estimates for $\lambda_{j,k}$, $\mu_{j,k}$ for (j, k) in this sector. Accordingly, we observe from [6, Equation (4.2)] that we already have some information at our disposal, for we know that there exist the positive numbers M_1, M_2 such that

$$M_1^2 < \lambda_{j,k}/j^2 < M_2^2 \tag{4.1}$$

when $(j, k) \in \Omega_1$ and j is sufficiently large. Putting $\eta_{j,k} = \mu_{j,k}/\lambda_{j,k}$, we may argue precisely as we did in [8, Lemma 1] to show also that

LEMMA 4.1. *If j is sufficiently large, then $a_1 < \eta_{j,0} < \eta_{j,1} < \dots < \eta_{j,k^*} < a_2 + \delta$, where $k^* = k^*(j)$ denotes the greatest integer less than $j \tan \theta_1$.*

We now come to our main result, namely,

THEOREM 4.1. *If j is sufficiently large, then*

$$\begin{aligned} (a_2 + 2^{-1}j^{-5/12}) &< \eta_{j,k} < (a_2 + \delta) \quad \text{if } (j, k) \in \Omega_1^*, \\ |\eta_{j,k} - a_2| &< 2j^{-5/12} \quad \text{if } (j, k) \in \Omega_2^*, \\ (a_2 - 2j^{-1/6}) &< \eta_{j,k} < (a_2 - 2^{-1}j^{-5/12}) \quad \text{if } (j, k) \in \Omega_3^*, \\ (a_1 + 2^{-1}j^{-5/12}) &< \eta_{j,k} < (a_2 - 2^{-1}j^{-1/6}) \quad \text{if } (j, k) \in \Omega_4^*, \end{aligned}$$

and

$$a_1 < \eta_{j,k} < (a_1 + 2j^{-5/12}) \quad \text{if } (j, k) \in \Omega_5^*.$$

PROOF. We shall only prove the theorem for the case $\alpha_i = 0, \beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Moreover, we shall henceforth suppose that $(j, k) \in \Omega_1$ and that $j > [b + 2\delta^{-1} + 4(a_2 - a_1)^{-1}]^6$ and is large enough to ensure the validity of (4.1) and the assertion of Lemma 4.1. Also, in this proof we shall frequently introduce the order symbol O in relation to functions which depend upon some or all of the variables x_2, j, k ; and it is to be understood that in each case the constant implied in the O symbol is independent of any of these variables.

We are now going to employ the results of §3 to obtain estimates for k/j for various values of $\eta_{j,k}$, and the assertion of the theorem will follow from these estimates. Accordingly, in that section let us now take $\lambda = \lambda_{j,k}$ and $t = \eta_{j,k}$. Observe from (3.2)–(3.3) that (here and below we use the notation of §§2 and 3)

$$P(x) = \lambda B(x)(t - r(x)) = \lambda(x - x^*)\Psi_2(x, t),$$

$$w(x) = \int_{x^*}^x P^{1/2}(s) ds = (2/3)\lambda^{1/2}(x - x^*)^{3/2}\Psi(x, t), \quad (4.2)$$

for $c \leq x \leq d$, where $x^* = x(t)$ and $P^{1/2}$, $(x - x^*)^{3/2}$ are to be interpreted according to the convention given in the statements preceding (3.3). Then with β a constant satisfying $1/2 < \beta \leq 2$ and putting

$$e_i = M_i N_i \quad (4.3)$$

for $i = 1, 2$ (see §2), let us firstly suppose that $\eta_{j,k} \geq t^\dagger = a_2 + \beta j^{-5/12}$. For this case it is clear from (4.1)–(4.2) that

$$w(0) > (2/3)\lambda^{1/2}(-x(t^\dagger))^{3/2}\Psi(0, t^\dagger)$$

$$> M_1 j(t^\dagger - a_2)^{3/2}\Phi(0, t^\dagger) > e_1 j^{3/8}/4,$$

and hence it follows from (3.15)–(3.16), (4.1)–(4.2) that as $j \rightarrow \infty$ (throughout this proof this limit symbol is to be interpreted as: when j exceeds a certain fixed positive number and $\eta_{j,k}$ satisfies the given supposition),

$$\phi_2(x_2) = P^{-1/4}(0)P^{-1/4}(x_2)$$

$$\times [\sin \theta(x_2, 0)\{1 + O(j^{-3/4})\} + \cos \theta(x_2, 0)\{O(j^{-3/8})\}],$$

$$\phi_2'(x_2) = P^{-1/4}(0)P^{1/4}(x_2)$$

$$\times [\cos \theta(x_2, 0)\{1 + O(j^{-3/4})\} + \sin \theta(x_2, 0)\{O(j^{-3/8})\}],$$

for $0 \leq x_2 \leq 1$, where here (and in the sequel) $\phi_2(x_2) = \phi_2(x_2, \lambda_{j,k}, \mu_{j,k})$ and all other expressions are defined in §3. A standard argument now shows that as $j \rightarrow \infty$,

$$\int_0^1 P_2^{1/2}(x_2, \lambda_{j,k}, \mu_{j,k}) dx_2 = (k + 1)\pi + O(j^{-3/8}),$$

while from Lemma 4.1 and [6, Theorem 3.4] we also have

$$\int_0^1 P_1^{1/2}(x_1, \lambda_{j,k}, \mu_{j,k}) dx_1 = (j + 1)\pi + O(j^{-1}), \quad (4.4)$$

and thus in light of (4.1)–(4.2), it follows that $g(\eta_{j,k}) = k/j + O(j^{-1})$. Since $g(\eta_{j,k}) \geq g(t^\dagger)$, we therefore conclude that there exists the positive integer j_1

such that $k/j > (g(a_2) + 2^{-1}a\beta j^{-5/12})$ when $\eta_{j,k} > (a_2 + \beta j^{-5/12})$ and $j > j_1$.

Suppose next that $|\eta_{j,k} - a_2| \leq \beta j^{-5/12}$. Then as above we can show that $|w(0)| < 4e_2 j^{3/8}$ and hence it follows from (3.17), (4.1)–(4.2) that

$$0 = \phi_2(1) = \kappa\rho_0(z'(0))^{-1/2}P^{-1/4}(0) \times \left[\sin W_{12}^* \{1 + O(j^{-1})\} + \cos W_{12}^* \{O(j^{-1})\} \right] \text{ as } j \rightarrow \infty,$$

where now $W_{12}^* = (w(1) + \gamma_0 - \pi/4)$, $\rho_0 = \rho(z(0))$, and $\gamma_0 = \gamma(z(0))$ (see §3 for terminology). From the results given in part (b) of subsection 3.4 it is a simple matter to deduce that

$$w(1) = ((k + 1)\pi - \gamma_0 + \pi/4 + O(j^{-1})) \text{ as } j \rightarrow \infty,$$

and thus in light of (4.1)–(4.2), the bound for γ_0 given in the statements preceding (3.21), and (4.4), we conclude that

$$w(1) = \lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k + 1)\pi \{1 + O(j^{-5/8})\}$$

and

$$G(\eta_{j,k}) = k/j + O(j^{-5/8})$$

as $j \rightarrow \infty$. Hence it follows that there exists the positive integer j_2 such that

$$|g(a_2) - k/j| < 2a\beta j^{-5/12}$$

when $|a_2 - \eta_{j,k}| \leq \beta j^{-5/12}$ and $j > j_2$.

Consider now the case where

$$(a_2 - \beta j^{-1/6}) \leq \eta_{j,k} \leq (a_2 - \beta j^{-5/12}).$$

Then for this case we may show that

$$e_1 j^{3/8}/4 < |w(0)| < 4e_2 j^{3/4}, \quad \arg w(0) = 3\pi/2,$$

and hence it follows from (3.11), (3.19), and (4.1)–(4.2) that

$$0 = \phi_2(1) = |P(0)|^{-1/4}P^{-1/4}(1)e^{i\arg w(0)} \times \left[\sin W_1 \{1 + O(j^{-3/8})\} + \cos W_1 \{O(j^{-1})\} \right]$$

as $j \rightarrow \infty$, where $W_1 = w(1) + \pi/4$. From the results given in part (a) of subsection 3.4, it is now easy to see that as $j \rightarrow \infty$,

$$w(1) = \lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k + 3/4)\pi + O(j^{-1}),$$

and hence (see (4.4))

$$G(\eta_{j,k}) = k/j + O(j^{-1}).$$

We therefore conclude that there exists the positive integer j_3 such that

$$2^{-1}j^{-5/12} < (g(a_2) - k/j)/a\beta < 2j^{-1/6}$$

whenever $\eta_{j,k}$ satisfies the restrictions given above and $j \geq j_3$.

Turning to the case where

$$(a_1 + \beta j^{-5/12}) < \eta_{j,k} < (a_2 - \beta j^{-1/6}),$$

we may show that here

$$|w(0)| > e_1 j^{3/4}/4, \quad \arg w(0) = 3\pi/2, \quad \text{and} \quad w(1) > e_1 j^{3/8}/4.$$

By arguing as in the previous case it is readily verified that $G(\eta_{j,k}) = k/j + O(j^{-1})$ as $j \rightarrow \infty$, and hence it follows that there exists the positive integer j_4 such that

$$2^{-1}a^*\beta^{3/2}j^{-5/8} < k/j < (g(a_2) - 2^{-1}a\beta j^{-1/6})$$

whenever $\eta_{j,k}$ satisfies the above bounds and $j \geq j_4$.

Finally, suppose that

$$\eta_{j,k} < (a_1 + \beta j^{-5/12}).$$

Then it is not difficult to verify that

$$w(1) < 4e_2 j^{3/8}, \quad |w(0)| > e_1 (2^{-1}(a_2 - a_1))^{3/2} j, \quad \arg w(0) = 3\pi/2,$$

and hence it follows from (3.19), (4.1)–(4.2) that as $j \rightarrow \infty$,

$$0 = \phi_2(1) = \kappa |P(0)|^{-1/4} (z'(1))^{-1/2} e^{i|w(0)|} \\ \times [T(z(1))\{1 + O(j^{-1})\} + S(z(1))\{O(j^{-1})\}].$$

From the results of subsection 3.4 (see in particular part (a)) it now follows that as $j \rightarrow \infty$,

$$z(1) = \lambda_{j,k}^{1/3} (3H(\eta_{j,k})/2)^{2/3} = \tau_{k+1} + O(j^{-1}),$$

and hence

$$G_1(\eta_{j,k}) = \tau_{k+1} (j\pi)^{-2/3} + O(j^{-1}).$$

Thus we conclude that there exists the positive integer j_5 such that

$$2\tau_{k+1}^{3/2}/3j\pi < 2a^*\beta^{3/2}j^{-5/8}$$

when $\eta_{j,k} < (a_1 + \beta j^{-5/12})$ and $j \geq j_5$.

By taking $\beta = \frac{1}{2}$ in the above calculations and assuming that $j > \max\{j_i\}$, $i = 1, \dots, 5$, we see that

$$k/j < (g(a_2) + aj^{-5/12}) \quad \text{whenever} \quad |\eta_{j,k} - a_2| < 2^{-1}j^{-5/12},$$

$$k/j < (g(a_2) - aj^{-5/12}/4)$$

$$\text{whenever} \quad (a_1 + 2^{-1}j^{-5/12}) < \eta_{j,k} < (a_2 - 2^{-1}j^{-5/12}),$$

and in light of the definition of Ω_1^* and the results given in subsection 3.4 concerning the zeros of $T(z)$, we also see that there exists the positive number

j_0 such that $(k + \frac{1}{2})/j < a^*j^{-5/8}$ whenever $\eta_{j,k} < (a_1 + 2^{-1}j^{-5/12})$, $(j, k) \in \Omega_1^*$, and $j > j_0$. The assertion of the theorem concerning the values of $\eta_{j,k}$ for $(j, k) \in \Omega_1^*$ follows immediately from these results and we may argue in the same way with each of the sets Ω_i^* , $i = 2, \dots, 5$, to complete the proof of the theorem.

5. Asymptotic formulae.

5.0. *Introduction.* We are now going to employ the foregoing results to derive asymptotic formulae for the eigenvalues and eigenfunctions of the system (1.1)–(1.4) for $(j, k) \in \Omega_1$ (see subsections 5.1–5.5). Accordingly, in this section we will suppose that $(j, k) \in \cup_{i=1}^5 \Omega_i^*$ and that j is large enough to ensure that (4.1) and the assertions of Lemma 4.1 and Theorem 4.1 are valid. Furthermore, it will always be assumed henceforth that fractional powers of positive quantities have their positive values. Finally, in the sequel careful attention must be paid to the notation given in §§2, 4, and subsection 3.0.

Notation. (1) Let χ_i ($i = 1, 2$) denote the solution of (1.2i – 1) satisfying $\chi_i(1, \lambda, \mu) = \sin \beta_i$, $\chi'_i(1, \lambda, \mu) = \cos \beta_i$, and put

$$\psi_{j,k}^\dagger(x_1, x_2) = \prod_{i=1}^2 \chi_i(x_i, \lambda_{j,k}, \mu_{j,k}),$$

$$\psi_{j,k}^*(x_1, x_2) = \prod_{i=1}^2 \phi_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

(see §4). Also let

$$\|F\| = \left(\iint_{I^2} \Delta(x_1, x_2) |F(x_1, x_2)|^2 dx_1 dx_2 \right)^{1/2}$$

for any function F that is square-integrable in I^2 .

(2) For $a_1 < t \leq a_2 + \delta$, $0 < \xi_1 < x_1 \leq 1$, $c \leq \xi_2 \leq x_2 \leq 1$, and $0 < \omega_i < \pi$, $i = 1, 2$, let

$$X_1(x_1, \xi_1, t) = \int_{\xi_1}^{x_1} P_1^{1/2}(s_1, 1, t) ds_1,$$

$$X_2(x_2, \xi_2, t) = \int_{\xi_2}^{x_2} (|t - r(s_2)|B_2(s_2))^{1/2} ds_2,$$

$$X_1^*(x_1, t) = \int_{x_1}^1 B_1(s_1)P_1^{-1/2}(s_1, 1, t) ds_1,$$

$$X_2^*(x_2, t) = \int_{x_2}^1 B_2(s_2)(|s_2 - x(t)|\Psi_2(s_2, t))^{-1/2} ds_2,$$

$$X_1^\dagger(x_1, \omega_1, t) = P_1^{-1/2}(x_1, 1, t) \\ \times [\cot \omega_1 + P_1'(x_1, 1, t)/4P_1(x_1, 1, t)], \quad ' = d/dx_1,$$

$$X_2^\dagger(x_2, \omega_2, t) = [\cot \omega_2 + \Phi_1'(x_2, t)/2\Phi_1(x_2, t)], \quad ' = d/dx_2,$$

$$Q_1(x_1, t) = q_1(x_1) + P_1^{1/4}(x_1, 1, t)[d^2P_1^{-1/4}(x_1, 1, t)/dx_1^2],$$

$$Q_2(x_2, t) = q_2(x_2) + \Phi_1^{1/2}(x_2, t)[d^2\Phi_1^{-1/2}(x_2, t)/dx_2^2],$$

$$X_1^\#(x_1, \xi_1, t) = -2^{-1} \int_{\xi_1}^{x_1} Q_1(s_1, t) P_1^{-1/2}(s_1, 1, t) ds_1,$$

$$X_2^\#(x_2, \xi_2, t) = -2^{-1} \int_{\xi_2}^{x_2} Q_2(s_2, t) (|s_2 - x(t)| \Psi_2(s_2, t))^{-1/2} ds_2,$$

$$c_1^*(t) = X_1^\dagger(0, \alpha_1, t) - X_1^\dagger(1, \beta_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 \neq 0 \text{ and } \beta_1 \neq \pi, \\ = X_1^\dagger(0, \alpha_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 \neq 0 \text{ and } \beta_1 = \pi, \\ = -X_1^\dagger(1, \beta_1, t) + X_1^\#(1, 0, t) \quad \text{if } \alpha_1 = 0 \text{ and } \beta_1 \neq \pi, \\ = X_1^\#(1, 0, t) \quad \text{if } \alpha_1 = 0 \text{ and } \beta_1 = \pi.$$

$$c_2^*(x_2, t) = -7(72(t - a_1)^{3/2}\Phi(1, t))^{-1} \\ - X_2^\dagger(1, \beta_2, t)(B_2(1)(t - a_1))^{-1/2} + X_2^\#(1, x_2, t) \quad \text{if } \beta_2 \neq \pi, \\ = 5(72(t - a_1)^{3/2}\Phi(1, t))^{-1} + X_2^\#(1, x_2, t) \quad \text{if } \beta_2 = \pi,$$

$$Y_i^*(x_i, t) = (c_i^*(t)/h_1(t))X_i(1, x_i, t) \quad \text{for } i = 1, 2,$$

(see §4 for the definition of h_1),

$$Y_1^\dagger(x_1, t) = -Y_1^*(x_1, t) - X_1^\dagger(1, \beta_1, t) + X_1^\#(1, x_1, t) \quad \text{if } \beta_1 \neq \pi, \\ = Y_1^*(x_1, t) - X_1^\#(1, x_1, t) \quad \text{if } \beta_1 = \pi,$$

$$Y_2^\dagger(x_2, t) = \mp Y_2^*(x_2, t) \pm c_2^*(x_2, t), \quad Y^*(x_2, t) = \mp 5/72\Phi(x_2, t),$$

where the upper signs are taken if $\beta_2 \neq \pi$ and the

lower signs if $\beta_2 = \pi$,

$$Y_1(x_1, j, k) = N_{j,k}^{-1}Y_1^\dagger(x_1, \eta_{j,k})$$

and

$$Z_i(x_i, j, k) = N_{j,k}X_i(1, x_i, \eta_{j,k}) \quad \text{for } i = 1, 2,$$

where

$$N_{j,k} = (j + \nu_1)\pi/h_1(\eta_{j,k}),$$

$\nu_i = 0$ if $\alpha_i \neq 0$ and $\beta_i \neq \pi$, $\nu_i = \frac{1}{2}$ if $\alpha_i \neq 0$ and $\beta_i = \pi$ or if $\alpha_i = 0$ and $\beta_i \neq \pi$, $\nu_i = 1$ if $\alpha_i = 0$ and $\beta_i = \pi$, and lastly, when $x_2 > x(\eta_{j,k})$ we shall

also put

$$Y_2(x_2, j, k) = N_{j,k}^{-1} \left[(\eta_{j,k} - r(x_2))^{-3/2} Y^\#(x_2, \eta_{j,k}) + Y_2^\dagger(x_2, \eta_{j,k}) \right].$$

(3) For $a_1 < t < a_2 + \delta$, $0 < x_1 < 1$, and $c < x_2 < 1$, let

$$c_{21}^\dagger(t) = -7/72\Phi(0, t) \quad \text{and} \quad c_{22}^\dagger(t) = X_2^\dagger(0, \alpha_2, t)/B_2^{1/2}(0) \quad \text{if } \alpha_2 \neq 0,$$

$$c_{21}^\dagger(t) = 5/72\Phi(0, t) \quad \text{and} \quad c_{22}^\dagger(t) = 0 \quad \text{if } \alpha_2 = 0,$$

$$y_1^*(x_1, t) = C^*(t)X_1(x_1, 0, t),$$

$$y_2^*(x_2, t) = C^*(t)|r(x_2) - t|^{3/2}\Phi(x_2, t),$$

where $C^*(t) = c_1^*(t)/h_1(t)$,

$$y_1^\dagger(x_1, t) = -y_1^*(x_1, t) + X_1^\dagger(0, \alpha_1, t) + X_1^\#(x_1, 0, t) \quad \text{if } \alpha_1 \neq 0,$$

$$= y_1^*(x_1, t) - X_1^\#(x_1, 0, t) \quad \text{if } \alpha_1 = 0,$$

$$y_1(x_1, j, k) = N_{j,k}^{-1}y_1^\dagger(x_1, \eta_{j,k}),$$

$$Z_1^*(x_1, j, k) = N_{j,k}X_1(x_1, 0, \eta_{j,k}),$$

$$y^*(x_2, t) = -5/72\Phi(x_2, t),$$

$$\nu^* = 1/4 \quad \text{if } \beta_2 \neq \pi, \quad \nu^* = 3/4 \quad \text{if } \beta_2 = \pi,$$

and lastly, when $x_2 < x(\eta_{j,k})$ we shall also put

$$\begin{aligned} Z_2^*(x_2, j, k) &= N_{j,k}X_2(x(\eta_{j,k}), x_2, \eta_{j,k}) \\ &= N_{j,k}(r(x_2) - \eta_{j,k})^{3/2}\Phi(x_2, \eta_{j,k}). \end{aligned}$$

Finally, we remark that the notation given in part (3) above will only be utilized in subsections 5.3–5.5.

5.1. *Formulae in Ω_1^* .* In this subsection we suppose that $(j, k) \in \Omega_1^*$ (see Theorem 4.1). Then it follows from the definitions of Ω_1^* and $g(t)$ given in §4 that

$$(g(a_2) + 2^{-1}aj^{-5/12}) < (k + \nu_2)/(j + \nu_1) < g(a_2 + \delta)$$

when j is sufficiently large. Hence assuming j large, we shall denote by $t_{j,k}$ the solution of the equation $g(t) = (k + \nu_2)/(j + \nu_1)$; and it is easy to see that $(a_2 + 4^{-1}j^{-5/12}) < t_{j,k} < (a_2 + \delta)$.

For $a_2 < t < a_2 + \delta$, let

$$c_{21}^*(t) = 7/72\Phi(0, t) \quad \text{and} \quad c_{22}^*(t) = X_2^\dagger(0, \alpha_2, t)/B_2^{1/2}(0) \quad \text{if } \alpha_2 \neq 0,$$

$$c_{21}^*(t) = -5/72\Phi(0, t) \quad \text{and} \quad c_{22}^*(t) = 0 \quad \text{if } \alpha_2 = 0,$$

$$c_{23}^*(t) = c_2^*(0, t), \quad d_i^*(t) = L(t)X_i^*(0, t)$$

and

$$d_i^\dagger(t) = td_i^*(t) + (-1)^{i-1}L(t)h_i(t) \quad \text{for } i = 1, 2$$

(where $L(t) = 1/h_1^2(t)g'(t)$),

$$\begin{aligned} c_1(t) &= c_1^*(t)d_2^*(t), & d_1(t) &= c_1^*(t)d_2^\dagger(t), \\ c_{2m}(t) &= c_{2m}^*(t)d_1^*(t), & d_{2m}(t) &= c_{2m}^*(t)d_1^\dagger(t) \quad \text{for } m = 1, 2, 3, \\ c(t) &= c_1(t) + c_{23}(t), & d(t) &= d_1(t) + d_{23}(t), \end{aligned}$$

and

$$D(t) = \iint_{I^2} \Delta(x_1, x_2)(|x_2 - x(t)|P_1(x_1, 1, t)\Psi_2(x_2, t))^{-1/2} dx_1 dx_2.$$

For $0 < x_i < 1$, $i = 1, 2$, let us define the expressions $E_m (= E_m(x_1, x_2, j, k))$, σ_m, σ_m^* , $m = 1, \dots, 4$, by putting:

$$\begin{aligned} E_1 &= 1, & E_2 &= Y_2(x_2, j, k), & E_3 &= Y_1(x_1, j, k), & E_4 &= 0, \\ \sigma_1 &= 1, & \sigma_1^* &= 0, & \text{and } \sigma_m &= 0, \\ \sigma_m^* &= 1 \quad \text{for } m \neq 1 \text{ if } \beta_i \neq \pi \text{ for } i = 1, 2; \end{aligned} \tag{1}$$

$$\begin{aligned} E_1 &= Y_2(x_2, j, k), & E_2 &= 1, & E_3 &= 0, & E_4 &= Y_1(x_1, j, k), \\ \sigma_2 &= 1, & \sigma_2^* &= 0, & \text{and } \sigma_m &= 0, \\ \sigma_m^* &= 1 \quad \text{for } m \neq 2 \text{ if } \beta_1 \neq \pi \text{ and } \beta_2 = \pi; \end{aligned} \tag{2}$$

$$\begin{aligned} E_1 &= Y_1(x_1, j, k), & E_2 &= 0, & E_3 &= 1, & E_4 &= Y_2(x_2, j, k), \\ \sigma_3 &= 1, & \sigma_3^* &= 0, & \text{and } \sigma_m &= 0, \\ \sigma_m^* &= 1 \quad \text{for } m \neq 3 \text{ if } \beta_1 = \pi \text{ and } \beta_2 \neq \pi; \end{aligned} \tag{3}$$

$$\begin{aligned} E_1 &= 0, & E_2 &= Y(x_1, j, k), & E_3 &= Y_2(x_2, j, k), & E_4 &= 1, \\ \sigma_4 &= 1, & \sigma_4^* &= 0, & \text{and } \sigma_m &= 0, \\ \sigma_m^* &= 1 \quad \text{for } m \neq 4 \text{ if } \beta_i = \pi \text{ for } i = 1, 2. \end{aligned} \tag{4}$$

Then putting $D_{j,k} = D(\eta_{j,k})$ and $\psi_{j,k}(x_1, x_2) = \psi_{j,k}^\dagger(x_1, x_2)/\|\psi_{j,k}^\dagger\|$, we have

THEOREM 5.1. *It is the case that*

$$\begin{aligned} \lambda_{j,k} &= M_{j,k}^2 \left[1 + M_{j,k}^{-2} \langle (t_{j,k} - a_2)^{-3/2} c_{21}(t_{j,k}) \right. \\ &\quad \left. + (t_{j,k} - a_2)^{-1/2} c_{22}(t_{j,k}) + c(t_{j,k}) \rangle o(j^{-2}) \right], \\ \mu_{j,k} &= M_{j,k}^2 \left[t_{j,k} + M_{j,k}^{-2} \langle (t_{j,k} - a_2)^{-3/2} d_{21}(t_{j,k}) \right. \\ &\quad \left. + (t_{j,k} - a_2)^{-1/2} d_{22}(t_{j,k}) + d(t_{j,k}) \rangle + o(j^{-2}) \right], \end{aligned}$$

$$\begin{aligned}
 \psi_{j,k}(x_1, x_2) = & 2D_{j,k}^{-1/2} [(\eta_{j,k} - r(x_2))P_1(x_1, 1, \eta_{j,k})B_2(x_2)]^{-1/4} \\
 & \times \left[\prod_{i=1}^2 \cos Z_i(x_i, j, k) \{ E_1(x_1, x_2, j, k) \right. \\
 & \qquad \qquad \qquad \left. + \sigma_1 O(j^{-3/4}) + \sigma_1^* o(j^{-1}) \right\} \\
 & + \cos Z_1(x_1, j, k) \sin Z_2(x_2, j, k) \\
 & \times \{ E_2(x_1, x_2, j, k) + \sigma_2 O(j^{-3/4}) + \sigma_2^* o(j^{-1}) \} \\
 & + \sin Z_1(x_1, j, k) \cos Z_2(x_2, j, k) \\
 & \times \{ E_3(x_1, x_2, j, k) + \sigma_3 O(j^{-3/4}) + \sigma_3^* o(j^{-1}) \} \\
 & + \prod_{i=1}^2 \sin Z_i(x_i, j, k) \{ E_4(x_1, x_2, j, k) \\
 & \qquad \qquad \qquad \left. + \sigma_4 O(j^{-3/4}) + \sigma_4^* o(j^{-1}) \} \right] \tag{5.1}
 \end{aligned}$$

as $j \rightarrow \infty$, $(j, k) \in \Omega_1^*$, where $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k})$. This last result holds uniformly for $0 < x_i < 1$, $i = 1, 2$.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then we know from the proof of Theorem 4.1 that when j is sufficiently large (3.15)–(3.16) apply in full force, and consequently, by arguing with these equations as we did in the proof just cited, it is easy to see that as $j \rightarrow \infty$,

$$\lambda_{j,k}^{1/2} h_2(\eta_{j,k}) = (k + 1)\pi + \lambda_{j,k}^{-1/2} C(\eta_{j,k}) + o(j^{-1}), \tag{5.2}$$

where

$$C(t) = (t - a_2)^{-3/2} c_{21}^*(t) + c_{23}^*(t),$$

while from [6, Theorem 3.4] we also have

$$\lambda_{j,k}^{1/2} h_1(\eta_{j,k}) = (j + 1)\pi + \lambda_{j,k}^{-1/2} c_1^*(\eta_{j,k}) + o(j^{-1}), \tag{5.3}$$

and hence

$$g(\eta_{j,k}) = (k + 1)/(j + 1) + O(j^{-11/8}).$$

From this last result and (5.3) it follows that

$$\eta_{j,k} = t_{j,k} + O(j^{-11/8}) \quad \text{and} \quad \lambda_{j,k} = M_{j,k}^2 (1 + O(j^{-11/8})) \quad \text{as } j \rightarrow \infty.$$

Substituting these expressions for $\lambda_{j,k}$ and $\eta_{j,k}$ in (5.2)–(5.3), we obtain

$$g(\eta_{j,k}) = (k + 1)/(j + 1) + C_1(t_{j,k})/((j + 1)\pi)^2 + o(j^{-2}),$$

where

$$C_1(t) = C(t)h_1(t) - c_1^*(t)h_2(t),$$

and hence

$$\eta_{j,k} = t_{j,k} + C_1(t_{j,k})/((j + 1)\pi)^2 g'(t_{j,k}) + o(j^{-2})$$

as $j \rightarrow \infty$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows immediately from this result, (5.3), and the relation $d_1^*(t)h_2(t) + d_2^*(t)h_1(t) = 2$.

From (5.3) and (3.15) it is easy to see that

$$\begin{aligned} \chi_2(x_2, \lambda_{j,k}, \mu_{j,k}) &= K_2(j, k)P_2^{-1/4}(x_2, 1, \eta_{j,k}) \\ &\quad \times \left[\sin Z_2(x_2, j, k) \{1 + O(j^{-3/4})\} \right. \\ &\quad \left. + \cos Z_2(x_2, j, k) \{Y_2(x_2, j, k) + o(j^{-1})\} \right] \quad (5.4) \end{aligned}$$

as $j \rightarrow \infty$, uniformly in $0 < x_2 \leq 1$, where

$$K_i(j, k) = 1/N_{j,k}P_i^{1/4}(1, 1, \eta_{j,k}) \quad \text{for } i = 1, 2,$$

while by arguing in a manner similar to that in [6, Theorem 3.4] we can also verify that

$$\begin{aligned} \chi_1(x_1, \lambda_{j,k}, \mu_{j,k}) &= K_1(j, k)P_1^{-1/4}(x_1, 1, \eta_{j,k}) \\ &\quad \times \left[\sin Z_1(x_1, j, k) \{1 + o(j^{-1})\} \right. \\ &\quad \left. + \cos Z_1(x_1, j, k) \{Y_1(x_1, j, k) + o(j^{-1})\} \right] \quad (5.5) \end{aligned}$$

as $j \rightarrow \infty$, uniformly in $0 < x_1 \leq 1$. A simple argument now shows that

$$\|\psi_{j,k}^\dagger\|^{-1} = \left(2/D_{j,k}^{1/2} \prod_{i=1}^2 K_i(j, k) \right) \{1 + O(j^{-3/4})\}$$

as $j \rightarrow \infty$, and hence the assertion of our theorem concerning $\psi_{j,k}$ follows immediately.

5.2. *Formulae in Ω_2^* .* In this subsection we suppose that $(j, k) \in \Omega_2^*$ (see Theorem 4.1) and for $c < x_2 < d, r(d) < t < r(c)$, and $s > 0$, let

$$\zeta_1 = \zeta_1(x_2, s, t) = s^{1/3}(t - r(x_2))\Phi_2(x_2, t),$$

$$\zeta_2 = \zeta_2(x_2, s, t) = ((s + \nu_1)\pi/h_1(t))^{2/3}(t - r(x_2))\Phi_2(x_2, t),$$

$$R(x_2, s, t) = \kappa\rho^*(\zeta_2),$$

$$J_i(x_2, s, t) = \gamma(\zeta_i) \quad \text{and} \quad J_i^*(x_2, s, t) = \gamma^*(\zeta_i) \quad \text{for } i = 1, 2,$$

where all terms are defined in §§2 and 3.

Let $\beta^\dagger = 4[3\pi N_2/h_1((b_1 + a_2)/2)]^{2/3}$, where N_2 is defined in §2. Then we observe from the properties of $\gamma(z)$, $\gamma^*(z)$ given in subsection 3.4 and from (3.11)–(3.12) that there exists a positive constant β such that both $|\gamma(z)|$ and $|\gamma^*(z)|$ do not exceed $\beta j^{3/8}$ when $|z| < \beta^\dagger j^{1/4}$. Furthermore, it is clear from the definitions of Ω_2^* and $G(t)$ given in §4 that there exists the positive integer j_1^\dagger such that when $j > j_1^\dagger$,

$$G((a_1 + a_2)/2) < f_-(j, k) < f_+(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t_\pm(j, k) - a_2| < 4j^{-5/12},$$

where

$$f_\pm(j, k) = ((k + \nu_2)\pi \pm (\beta j^{3/8} + \pi/4))/(j + \nu_1)\pi$$

and $G(t_\pm(j, k)) = f_\pm(j, k)$. Hence if we assume that $j > j_1^\dagger$ and, for $i = 1, \dots, 6$, define $t_i(j, k)$ recursively by means of the formulae

$$G(t_i(j, k)) = ((k + \nu_2)\pi + F_{i-1}(j, k))/(j + \nu_1)\pi,$$

where $F_0(j, k) = 0$ and for $i \geq 1$, $F_i(j, k)$ equals $-(J_2^*(0, j, t_i(j, k)) + \pi/4)$ if $\alpha_2 \neq 0$ and $-(J_2(0, j, t_i(j, k)) - \pi/4)$ if $\alpha_2 = 0$, then it is readily verified that each $t_i(j, k)$ exists,

$$G((a_1 + a_2)/2) < G(t_i(j, k)) < G(a_2 + 3\delta/4),$$

and

$$|t_i(j, k) - a_2| < 4j^{-5/12}.$$

Next put

$$F(j, k) = ((j + \nu_1)\pi/h_1(t_6))^{-2/3} X_2^\dagger(0, \alpha_2, t_6)/\Phi_1(0, t_6) R^2(0, j, t_6)$$

if $\alpha_2 \neq 0$,

$$= 0 \quad \text{if } \alpha_2 = 0,$$

where $t_6 = t_6(j, k)$. Then it is clear from (3.11)–(3.12) that $F(j, k) = O(j^{-2/3})$ as $j \rightarrow \infty$, and hence it follows that there exists the integer $j_1 > j_1^\dagger$

such that when $j \geq j_1$,

$$G((a_1 + a_2)/2) < f_{-}^{\dagger}(j, k) < f_{+}^{\dagger}(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t_{\pm}^{\dagger}(j, k) - a_2| < 4j^{-5/12},$$

where

$$f_{\pm}^{\dagger}(j, k) = f_{\pm}(j, k) \pm |F(j, k)| / (j + \nu_1)\pi \quad \text{and} \quad G(t_{\pm}^{\dagger}(j, k)) = f_{\pm}^{\dagger}(j, k).$$

Assuming henceforth that $j \geq j_1$ and putting

$$f(j, k) = ((k + \nu_2)\pi + F_6(j, k) + F(j, k)) / (j + \nu_1)\pi,$$

it now follows that

$$G((a_1 + a_2)/2) < f(j, k) < G(a_2 + 3\delta/4)$$

and

$$|t_7(j, k) - a_2| < 4j^{-5/12}$$

(where $G(t_7(j, k)) = f(j, k)$), while if we define $F_7(j, k)$ to be that quantity which is obtained by putting $t_i(j, k) = t_7(j, k)$ in the above formula for F_i and let

$$f^*(j, k) = ((k + \nu_2)\pi + F_7(j, k) + F(j, k)) / (j + \nu_1)\pi,$$

then

$$G((a_1 + a_2)/2) < f^*(j, k) < G(a_2 + 3\pi/4).$$

Finally, we shall denote by $t_{j,k}^*$ the solution of the equation $G(t) = f^*(j, k)$; observe that $|t_{j,k}^* - a_2| < 4j^{-5/12}$.

Referring to the notation given at the beginning of this and subsection 5.0, respectively, let

$$c_{21}^*(t) = c_2^*(x(t), t),$$

$$d_1^*(t) = L(t)X_1^*(0, t),$$

$$d_2^*(t) = L(t)X_2^*(x(t), t),$$

$$d_1^{\dagger}(t) = td_1^*(t) + L(t)h_1(t), \quad d_2^{\dagger}(t) = td_2^*(t) - L(t)H(t)$$

(where $L(t) = 1/h_1^2(t)G'(t)$),

$$c(t) = c_1^*(t)d_2^*(t) + c_{21}^*(t)d_1^*(t),$$

and

$$d(t) = c_1^*(t)d_2^{\dagger}(t) + c_{21}^*(t)d_1^{\dagger}(t).$$

Putting

$$A_{j,k} = X_2^{\dagger}(0, a_2, \eta_{j,k}) / \Phi_1(0, \eta_{j,k}) R^2(0, j, \eta_{j,k}),$$

we shall also let

$$\begin{aligned}
 J_{j,k} &= J_2^*(0, j, \eta_{j,k}), \\
 a_{j,k} &= -A_{j,k} \cos J_{j,k}, \text{ and } b_{j,k} = A_{j,k} \sin J_{j,k} \text{ if } \alpha_2 \neq 0, \\
 J_{j,k} &= J_2(0, j, \eta_{j,k}) - \pi \text{ and } a_{j,k} = b_{j,k} = 0 \text{ if } \alpha_2 = 0, \\
 \zeta(x_2, j, k) &= \zeta_2(x_2, j, \eta_{j,k}), \\
 T_{j,k}(x_2) &= T(\zeta(x_2, j, k)), \quad S_{j,k}(x_2) = S(\zeta(x_2, j, k)),
 \end{aligned}$$

and

$$\begin{aligned}
 D_i(j, k) &= N_{j,k}^{-1/3} \iint_{I^2} \Delta(x_1, x_2) \zeta'(x_2, j, k) W_i(x_2, j, k) \\
 &\quad \times P_1^{-1/2}(x_1, 1, \eta_{j,k}) \Phi_1^{-2}(x_2, \eta_{j,k}) dx_1 dx_2,
 \end{aligned}$$

for $i = 1, 2$, where T and S are defined in §3, $' = d/dx_2$, $W_1(x_2, j, k) = W^2(x_2, j, k)$,

$$W_2(x_2, j, k) = W(x_2, j, k)(a_{j,k} T_{j,k}(x_2) - b_{j,k} S_{j,k}(x_2)),$$

and

$$W(x_2, j, k) = (T_{j,k}(x_2) \sin J_{j,k} - S_{j,k}(x_2) \cos J_{j,k}).$$

It is easy to see from (3.11)–(3.12) and [4, p. 86] that $D_1(j, k)$, $1/D_1(j, k)$, and $|D_2(j, k)|$ all remain less than some positive number independent of j and k for all large j .

Lastly, for large j we shall put

$$\begin{aligned}
 B_{j,k} &= N_{j,k}^{-2/3} D_1^{-1}(j, k) D_2(j, k), \\
 B_{j,k}^* &= N_{j,k}^{-2/3} (a_{j,k} - D_1^{-1}(j, k) D_2(j, k) \sin J_{j,k}), \\
 B_{j,k}^\dagger &= N_{j,k}^{-2/3} (b_{j,k} - D_1^{-1}(j, k) D_2(j, k) \cos J_{j,k}),
 \end{aligned}$$

and for $0 < x_1 < 1$, $x(\eta_{j,k}) < x_2 < 1$, define the expressions $E_m (= E_m(x_1, x_2, j, k))$, σ_m , σ_m^* , $E_m^* (= E_m^*(x_1, j, k))$, $E_m^\dagger (= E_m^\dagger(x_1, j, k))$, ϵ_m , and ϵ_m^* , $m = 1, \dots, 4$, by putting:

$$\begin{aligned}
 E_1 &= 1 - B_{j,k}, \quad E_2 = Y_2(x_2, j, k), \quad E_3 = Y_1(x_1, j, k), \quad E_4 = 0, \\
 \sigma_1 &= 1, \quad \sigma_1^* = 0, \quad \text{and } \sigma_m = 0, \\
 \sigma_m^* &= 1 \text{ for } m \neq 1 \text{ if } \beta_i \neq \pi \text{ for } i = 1, 2;
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 E_1 &= Y_2(x_2, j, k), \quad E_2 = 1 - B_{j,k}, \quad E_3 = 0, \quad E_4 = Y_1(x_1, j, k), \\
 \sigma_2 &= 1, \quad \sigma_2^* = 0, \quad \text{and } \sigma_m = 0, \\
 \sigma_m^* &= 1 \text{ for } m \neq 2 \text{ if } \beta_1 \neq \pi \text{ and } \beta_2 = \pi;
 \end{aligned} \tag{2}$$

$$\begin{aligned}
E_1 &= Y_1(x_1, j, k), \quad E_2 = 0, \quad E_3 = 1 - B_{j,k}, \quad E_4 = Y_2(x_2, j, k), \\
\sigma_3 &= 1, \quad \sigma_3^* = 0, \quad \text{and} \quad \sigma_m = 0, \\
\sigma_m^* &= 1 \quad \text{for } m \neq 3 \text{ if } \beta_1 = \pi \text{ and } \beta_2 \neq \pi;
\end{aligned} \tag{3}$$

$$\begin{aligned}
E_1 &= 0, \quad E_2 = Y_1(x_1, j, k), \quad E_3 = Y_2(x_2, j, k), \quad E_4 = 1 - B_{j,k}, \\
\sigma_4 &= 1, \quad \sigma_4^* = 0, \quad \text{and} \quad \sigma_m = 0, \\
\sigma_m^* &= 1 \quad \text{for } m \neq 4 \text{ if } \beta_i = \pi \text{ for } i = 1, 2;
\end{aligned} \tag{4}$$

$$\begin{aligned}
E_1^* &= \sin J_{j,k} + B_{j,k}^*, \quad E_2^* = -\cos J_{j,k} - B_{j,k}^\dagger, \\
E_3^* &= Y_1(x_1, j, k) \sin J_{j,k}, \quad E_4^* = -Y_1(x_1, j, k) \cos J_{j,k}, \\
E_1^\dagger &= -E_2^\dagger = 1, \quad E_3^\dagger = -E_4^\dagger = Y_1(x_1, j, k), \\
\varepsilon_1 &= \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 1, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 0 \quad \text{if } \beta_1 \neq \pi;
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
E_1^* &= Y_1(x_1, j, k) \sin J_{j,k}, \quad E_2^* = -Y_1(x_1, j, k) \cos J_{j,k}, \\
E_3^* &= \sin J_{j,k} + B_{j,k}^*, \quad E_4^* = -\cos J_{j,k} - B_{j,k}^\dagger, \\
E_1^\dagger &= -E_2^\dagger = Y_1(x_1, j, k), \quad E_3^\dagger = -E_4^\dagger = 1, \\
\varepsilon_1 &= \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 0, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 1 \quad \text{if } \beta_1 = \pi.
\end{aligned} \tag{6}$$

It is clear from the proof of Theorem 4.1 and (4.2) that if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and suppose that j is sufficiently large, then

$$|w(0)| < 4e_2 j^{3/8}, \quad w(1) > e_1 ((a_2 - a_1)/2)^{3/2} j,$$

where the e_i are defined in (4.3). Assuming j large, we shall now denote by $\xi_{j,k}$ the point of $[c, d]$ at which $r(x_2) = \eta_{j,k} - (4e_2/e_1)^{2/3} j^{-5/12}$; and by arguing with (4.2) it is not difficult to verify that

$$w(x_2) \geq 4e_2 j^{3/8} \quad \text{for } x_2 \geq \xi_{j,k}$$

(from which it follows that $\xi_{j,k} > 0$) and that

$$|w(x_2)| < 4e_1^{-1} e_2^2 j^{3/8} \quad \text{for } 0 < x_2 < \xi_{j,k}.$$

Then putting

$$D_{j,k}^* = D_1(j, k), \quad D_{j,k} = 2\kappa^2 D_{j,k}^*,$$

and

$$\psi_{j,k}(x_1, x_2) = \psi_{j,k}^\dagger(x_1, x_2) / \|\psi_{j,k}^\dagger\|,$$

we have

THEOREM 5.2. *It is the case that*

$$\begin{aligned} \lambda_{j,k} &= M_{j,k}^2 [1 + M_{j,k}^{-2} c(t_{j,k}^*) + o(j^{-2})], \\ \mu_{j,k} &= M_{j,k}^{-2} [t_{j,k}^* + M_{j,k}^{-2} d(t_{j,k}^*) + o(j^{-2})], \end{aligned} \quad (5.6)$$

$\psi_{j,k}(x_1, x_2)$ is given by (5.1) for $0 \leq x_1 \leq 1$, $\xi_{j,k} \leq x_2 \leq 1$, provided that throughout (5.1) we replace the expression $O(j^{-3/4})$ by $o(j^{-2/3})$,

$$\begin{aligned} \psi_{j,k}(x_1, x_2) &= E(x_1, x_2, j, k) \\ &\times \left[\cos Z_1(x_1, j, k) T_{j,k}(x_2) \{ E_1^*(x_1, j, k) + \varepsilon_1 o(j^{-2/3}) + \varepsilon_1^* o(j^{-1}) \} \right. \\ &\quad + \cos Z_1(x_1, j, k) S_{j,k}(x_2) \{ E_2^*(x_1, j, k) + \varepsilon_2 o(j^{-2/3}) + \varepsilon_2^* o(j^{-1}) \} \\ &\quad + \sin Z_1(x_1, j, k) T_{j,k}(x_2) \{ E_3^*(x_1, j, k) + \varepsilon_3 o(j^{-2/3}) + \varepsilon_3^* o(j^{-1}) \} \\ &\quad \left. + \sin Z_1(x_1, j, k) S_{j,k}(x_2) \{ E_4^*(x_1, j, k) + \varepsilon_4 o(j^{-2/3}) + \varepsilon_4^* o(j^{-1}) \} \right] \\ &\quad \text{for } 0 < x_1 < 1, \max\{0, x(\eta_{j,k})\} < x_2 < \xi_{j,k}, \end{aligned}$$

$$\begin{aligned} \psi_{j,k}(x_1, x_2) &= E(x_1, x_2, j, k) \\ &\times \left[\cos Z_1(x_1, j, k) T_{j,k}(x_2) \sin J_{j,k} \right. \\ &\quad \times \{ E_1^\dagger(x_1, j, k) + \varepsilon_1 o(j^{-1/2}) + \varepsilon_1^* o(j^{-1}) \} \\ &\quad + \cos Z_1(x_1, j, k) S_{j,k}(x_2) \cos J_{j,k} \\ &\quad \times \{ E_2^\dagger(x_1, j, k) + \varepsilon_2 o(j^{-1/2}) + \varepsilon_2^* o(j^{-1}) \} \\ &\quad + \sin Z_1(x_1, j, k) T_{j,k}(x_2) \sin J_{j,k} \\ &\quad \times \{ E_3^\dagger(x_1, j, k) + \varepsilon_3 o(j^{-1/2}) + \varepsilon_3^* o(j^{-1}) \} \\ &\quad + \sin Z_1(x_1, j, k) S_{j,k}(x_2) \cos J_{j,k} \\ &\quad \times \{ E_4^\dagger(x_1, j, k) + \varepsilon_4 o(j^{-1/2}) + \varepsilon_4^* o(j^{-1}) \} \left. \right] \\ &\quad \text{for } 0 < x_1 < 1, 0 < x_2 < x(\eta_{j,k}) \text{ if } x(\eta_{j,k}) > 0, \end{aligned} \quad (5.7)$$

as $j \rightarrow \infty$, $(j, k) \in \Omega_2^*$, where $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^*)$ and

$$\begin{aligned} &(-1)^{k+1} E(x_1, x_2, j, k) \\ &= N_{j,k}^{1/6} (2/D_{j,k}^*)^{1/2} P_1^{-1/4}(x_1, 1, \eta_{j,k}) \Phi_1^{-1/2}(x_2, \eta_{j,k}). \end{aligned} \quad (5.8)$$

These results for $\psi_{j,k}(x_1, x_2)$ hold uniformly in x_1 and x_2 .

REMARK. Before proving the theorem, let us make the following clarifications. By arguing as we did in the proof of Theorem 4.1, it is easily seen that $x(\eta_{j,k}) > 0$ for at least one (but not all) k when j is sufficiently large.

(5.7) is, of course, only intended to apply to that subset of Ω_2^* composed of points (j, k) for which j exceeds a certain fixed positive number and $x(\eta_{j,k}) > 0$. Writing ζ for $\zeta(x_2, j, k)$, η for $\eta_{j,k}$ and x^* for $x(\eta)$, observe also that $\zeta = N_{j,k}^{2/3}(\eta - r(x_2))\Phi_2(x_2, \eta)$, while from (4.2) we have $(\eta - r(x_2)) = (x_2 - x^*)\Psi_1(x_2, \eta)$. These equations together with the convention described in the statements preceding (3.3) determine $(2/3)\zeta^{3/2}$. By taking $z_2 = \zeta$ and $w_2 = (2/3)\zeta^{3/2}$ in (3.11)–(3.12), the behaviour of $T_{j,k}$ and $S_{j,k}$ may be ascertained. In particular, it is important to observe in (5.7) that the absolute value of $S_{j,k}(x_2)\cos J_{j,k}$ remains less than some bound independent of x_2, j , and k . This observation follows immediately from the definitions of the terms involved.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0$, $\beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then from the discussion immediately preceding this theorem it is clear that (3.17) applies in full force, and by arguing as we did in the proof of Theorem 4.1 it may readily be verified that

$$\lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k+1)\pi + c^*(\lambda_{j,k}, \eta_{j,k}) + \lambda_{j,k}^{-1/2}c_{21}^*(\eta_{j,k}) + o(j^{-1}) \quad (5.9)$$

as $j \rightarrow \infty$, where $c^*(s, t) = -J_1(0, s, t) + \pi/4$ (see the beginning of this subsection). Thus in light of (5.3) and the arguments given in the proof of Theorem 4.1, we therefore conclude that

$$G(\eta_{j,k}) = ((k+1)/(j+1)) + O(j^{-5/8}) \quad \text{as } j \rightarrow \infty.$$

From this result and (5.3) it now follows that as $j \rightarrow \infty$,

$$\eta_{j,k} = t_1 + O(j^{-5/8}), \quad \lambda_{j,k} = ((j+1)\pi/h_1(t_1))^2 \{1 + O(j^{-5/8})\}$$

(where $t_1 = t_1(j, k)$), and a simple argument also shows that

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_1(j, k) + O(j^{1/6})$$

(we refer to the discussion concerning $t_{j,k}^*$ given at the beginning of this subsection for terminology). If we substitute this latter expression for c^* in (5.9) and make use of (5.3), then we conclude that

$$G(\eta_{j,k}) = ((k+1)\pi + F_1(j, k))/(j+1)\pi + O(j^{-5/6})$$

as $j \rightarrow \infty$, and hence it follows that

$$\eta_{j,k} = t_2 + O(j^{-5/6}), \quad \lambda_{j,k} = ((j+1)\pi/h_1(t_2))^2 \{1 + O(j^{-5/6})\},$$

and

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_2(j, k) + O(j^{-1/24})$$

as $j \rightarrow \infty$, where $t_2 = t_2(j, k)$. By substituting this latter expression for c^* in (5.9) and repeating the above arguments, it is now easy to deduce that

$$\eta_{j,k} = t_i + O(j^{-m_i}), \quad \lambda_{j,k} = ((j+1)\pi/h_1(t_i))^2 \{1 + O(j^{-m_i})\},$$

and

$$c^*(\lambda_{j,k}, \eta_{j,k}) = F_i(j, k) + O(j^{-(m_i-19/24)})$$

as $j \rightarrow \infty$ for $i = 1, \dots, 7$, where $t_i = t_i(j, k)$ and $m_i = (5i + 10)/24$. Thus, as $j \rightarrow \infty$, $G(\eta_{j,k}) = f^*(j, k) + O(j^{-2})$, and hence

$$\eta_{j,k} = t_{j,k}^* + O(j^{-2}), \quad \lambda_{j,k} = M_{j,k}^2(1 + O(j^{-2})).$$

A substitution of these expressions for $\eta_{j,k}$ and $\lambda_{j,k}$ in (5.3) and (5.9) shows that

$$G(\eta_{j,k}) = f^*(j, k) + ((j + 1)\pi)^{-2}C(t_{j,k}^*) + o(j^{-2}),$$

and hence

$$\eta_{j,k} = t_{j,k}^* + C(t_{j,k}^*) / ((j + 1)\pi)^2 G'(t_{j,k}^*) + o(j^{-2})$$

as $j \rightarrow \infty$, where $C(t) = c_{21}^*(t)h_1(t) - c_1^*(t)H(t)$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows immediately from this latter result, (5.3), and the relation $d_1^*(t)H(t) + d_2^*(t)h_1(t) = 2$.

Turning next to $\chi_{j,k}(x_2) = \chi_2(x_2, \lambda_{j,k}, \mu_{j,k})$, it follows from (5.3) and (3.15) that (5.4) remains valid as $j \rightarrow \infty$ for $\xi_{j,k} \leq x_2 \leq 1$, and with this result holding uniformly in x_2 . To obtain approximations for $\chi_{j,k}$ when $x_2 < \xi_{j,k}$ we must turn again to the work of §3, and taking now $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, we shall indicate the dependence on j and k of the functions P , w , z , θ^* defined there by writing $\omega_{j,k}(x_2)$ for $\omega(x_2)$ and $\theta_{j,k}^*(x_2, x_1)$ for $\theta^*(x_2, x_1)$, where ω is any one of the functions P , w , or z . Then putting

$$L_{j,k}(x_2) = (-1)^k \kappa(z'_{j,k}(x_2))^{-1/2} P_{j,k}^{-1/4}(1),$$

$$T_{j,k}^*(x_2) = T(z_{j,k}(x_2)), \quad S_{j,k}^*(x_2) = S(z_{j,k}(x_2)),$$

$$\Gamma_{j,k} = \gamma(z_{j,k}(0)), \quad \text{and} \quad I_{j,k}(x_2) = \theta_{j,k}^*(x_2, x(\eta_{j,k})),$$

we may argue with (3.17) and (5.9) to show that as $j \rightarrow \infty$,

$$\begin{aligned} \chi_{j,k}(x_2) &= L_{j,k}(x_2) \left[T_{j,k}^*(x_2) \{ \sin \Gamma_{j,k} - I_{j,k}(x_2) \cos \Gamma_{j,k} + o(\lambda_{j,k}^{-1/2}) \} \right. \\ &\quad \left. - S_{j,k}^*(x_2) \{ \cos \Gamma_{j,k} + I_{j,k}(x_2) \sin \Gamma_{j,k} + o(\lambda_{j,k}^{-1/2}) \} \right] \\ &\quad \text{for } \max\{0, x(\eta_{j,k})\} \leq x_2 \leq 1, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &= L_{j,k}(x_2) \left[T_{j,k}^*(x_2) \{ \sin \Gamma_{j,k} + \lambda_{j,k}^{-1/2} I_{j,k}^*(x_2) \} \right. \\ &\quad \left. - S_{j,k}^*(x_2) \{ \cos \Gamma_{j,k} + \lambda_{j,k}^{-1/2} I_{j,k}^\dagger(x_2) \} \right] \\ &\quad \text{for } 0 < x_2 < x(\eta_{j,k}) \quad \text{if } x(\eta_{j,k}) > 0, \end{aligned} \tag{5.11}$$

where (5.10) holds uniformly in x_2 and both $I_{j,k}^*$ and $I_{j,k}^\dagger$ are $o(1)$ as $j \rightarrow \infty$, uniformly in x_2 (see the remark immediately preceding the statement of this theorem).

We now assert that as $j \rightarrow \infty$,

$$\chi_{j,k}(x_2) = L_{j,k}(x_2) \left[T_{j,k}^*(x_2) \sin \Gamma_{j,k} \{1 + o(\lambda_{j,k}^{-1/4})\} - S_{j,k}^*(x_2) \cos \Gamma_{j,k} \{1 + o(\lambda_{j,k}^{-1/4})\} \right] \quad (5.12)$$

for $0 \leq x_2 < x(\eta_{j,k})$ if $x(\eta_{j,k}) > 0$, with this result holding uniformly in x_2 . To prove this assertion let $0 < \varepsilon < 1$, choose the positive integer j_0 large enough so that $|I_{j,k}^*(x_2)| + |I_{j,k}^\dagger(x_2)| < \varepsilon$ in $0 \leq x_2 < x(\eta_{j,k})$ whenever $x(\eta_{j,k}) > 0$ and $j \geq j_0$, and under these latter conditions consider each of the cases: (i) $|w_{j,k}(0)| \leq \omega_{j,k}(\varepsilon) (= 4^{-1} \log\{\lambda_{j,k}^{1/2}/\varepsilon\})$, (ii) $|w_{j,k}(0)| > \omega_{j,k}(\varepsilon)$. For case (i) we may argue with (3.12) and (5.11) to show that when j is sufficiently large, (5.12) is valid in $0 \leq x_2 < x(\eta_{j,k})$, provided that in (5.12) each of the expressions $o(\lambda_{j,k}^{-1/4})$ is replaced by an expression of the form $C_{j,k}(x_2)$, where $|C_{j,k}(x_2)| < \beta \varepsilon^{1/2} \lambda_{j,k}^{-1/4}$ and β denotes a positive constant. A similar result also holds for case (ii), which can easily be seen by arguing with (3.12), (5.11) and the relation

$$\chi_{j,k}(x_2) = \chi_{j,k}(x_1^\#) \phi_{j,k}(x_2) / \phi_{j,k}(x_1^\#),$$

where $x_1^\# (= x_1^\#(j, k))$ denotes the point of $[c, d]$ defined in the statements preceding (3.15),

$$\phi_{j,k}(x_2) = \phi_2(x_2, \lambda_{j,k}, \mu_{j,k}) = v(x_2, 0),$$

and $v(x_2, 0)$ is given by (3.18). Since ε is arbitrary, the assertion follows.

By arguing with (5.3), (5.10), and (5.12) we can next show that as $j \rightarrow \infty$,

$$\begin{aligned} \chi_{j,k}(x_2) &= L_{j,k}^*(x_2) \left[T_{j,k}(x_2) \{ \sin J_{j,k} + O(j^{-1}) \} \right. \\ &\quad \left. - S_{j,k}(x_2) \{ \cos J_{j,k} + O(j^{-1}) \} \right] \\ &\quad \text{for } \xi_{j,k} \leq x_2 \leq 1, \\ &= L_{j,k}^*(x_2) \left[T_{j,k}(x_2) \{ \sin J_{j,k} + o(j^{-1}) \} \right. \\ &\quad \left. - S_{j,k}(x_2) \{ \cos J_{j,k} + o(j^{-1}) \} \right] \\ &\quad \text{for } \max\{0, x(\eta_{j,k})\} \leq x_2 < \xi_{j,k}, \\ &= L_{j,k}^*(x_2) \left[T_{j,k}(x_2) \sin J_{j,k} \{ 1 + o(j^{-1/2}) \} \right. \\ &\quad \left. - S_{j,k}(x_2) \cos J_{j,k} \{ 1 + o(j^{-1/2}) \} \right] \\ &\quad \text{for } 0 \leq x_2 < x(\eta_{j,k}) \text{ if } x(\eta_{j,k}) > 0, \quad (5.13) \end{aligned}$$

uniformly in x_2 , where

$$L_{j,k}^*(x_2) = K_2^*(j, k)\Phi_1^{-1/2}(x_2, \eta_{j,k}),$$

$$K_2^*(j, k) = (-1)^{k+1}\kappa N_{j,k}^{-5/6}P_2^{-1/4}(1, 1, \eta_{j,k}),$$

and all other terms are defined in the statements preceding this theorem. From these results and (5.5) it may readily be verified that

$$\|\psi_{j,k}^\dagger\|^{-1} = N_{j,k}^{1/6}K_1^{-1}(j, k)|K_2^*(j, k)|^{-1}(2/D_{j,k}^*)^{1/2}\{1 + o(j^{-2/3})\}$$

as $j \rightarrow \infty$ (where K_1 is defined in the statements preceding (5.5)), and the assertion of our theorem concerning $\psi_{j,k}$ now follows from (5.4), (5.5), and (5.13).

5.3. *Formulae in Ω_3^* .* Throughout this subsection we assume that $(j, k) \in \Omega_3^*$ (see Theorem 4.1). Then referring to the notation of subsection 5.0 (see, in particular, part (3)), it follows from the definition of Ω_3^* that

$$(G(a_2) - 2aj^{-1/6}) < (k + \nu^*) / (j + \nu_1) < (G(a_2) - 2^{-1}aj^{-5/12})$$

when j is sufficiently large. Assuming j large, we shall now denote by $t_{j,k}^*$ the solution of the equation $G(t) = (k + \nu^*) / (j + \nu_1)$; and it is not difficult to verify that

$$(a_2 - 3j^{-1/6}) < t_{j,k}^* < (a_2 - 4^{-1}j^{-5/12})$$

(see part (6) of the notation of §4 for the properties of G).

In this subsection we shall define the functions $c_{21}^*(t)$, $d_i^*(t)$ and $d_i^\dagger(t)$ ($i = 1, 2$), $c(t)$, $d(t)$, $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, and $S_{j,k}(x_2)$ precisely as we did in subsection 5.2, put $D_{j,k}^* = N_{j,k}^{-1/3}I_{j,k}$, where

$$I_{j,k} = \int \int_{I^2} \Delta(x_1, x_2)\zeta'(x_2, j, k)T_{j,k}^2(x_2)P_1^{-1/2}(x_1, 1, \eta_{j,k})$$

$$\times \Phi_1^{-2}(x_2, \eta_{j,k}) dx_1 dx_2, \quad (5.14)$$

($' = d/dx_2$), and for $t < a_2$ and $x_2 < x(t)$ (here and below special attention must be paid to part (3) of the notation of subsection 5.0) let

$$c^\dagger(t) = c_{21}^\dagger(t)(a_2 - t)^{-3/2},$$

$$y_2^\dagger(x_2, t) = (y^*(x_2, t)(r(x_2) - t)^{-3/2} + c^\dagger(t)),$$

$$y_{21}^\dagger(x_2, t) = y_2^\dagger(x_2, t) - c^\dagger(t), \quad y_{22}^\dagger(x_2, t) = y_2^\dagger(x_2, t) + c^\dagger(t),$$

and

$$y_{2i}(x_2, j, k) = N_{j,k}^{-1}y_{2i}^\dagger(x_2, \eta_{j,k}) \quad \text{for } i = 1, 2.$$

It is easy to see from (3.11)–(3.12) and the results given in [4, p. 86] that $D_{j,k}^*$ and $1/D_{j,k}^*$ remain less than some positive number independent of j or k for all large j . We shall also define the expressions $E_m(x_1, x_2, j, k)$ (for $0 < x_1 < 1$, $x(\eta_{j,k}) < x_2 < 1$), σ_{m-} , and σ_m^* ($m = 1, \dots, 4$) precisely as we did in

subsection 5.1 and for $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq x(\eta_{j,k})$ define the expressions $E^\dagger(x_1, j, k)$, $E^\#(x_1, j, k)$, $E_m^*(x_1, x_2, j, k)$, ε_m , ε_m^* ($m = 1, \dots, 4$), and ε by putting:

$$\begin{aligned} E^\dagger &= 1, \quad E^\# = y_1(x_1, j, k), \quad E_1^* = 1 + y_{21}(x_2, j, k), \\ E_2^* &= -1 + y_{22}(x_2, j, k), \quad E_3^* = -E_4^* = y_1(x_1, j, k), \\ \varepsilon_1 &= \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 1, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 0 \quad \text{if } \alpha_1 \neq 0, \end{aligned} \quad (1)$$

$$\begin{aligned} E^\dagger &= y_1(x_1, j, k), \quad E^\# = 1, \quad E_1^* = -E_2^* = y_1(x_1, j, k), \\ E_3^* &= 1 + y_{21}(x_2, j, k), \quad E_4^* = -1 + y_{22}(x_2, j, k), \\ \varepsilon_1 &= \varepsilon_2 = \varepsilon_3^* = \varepsilon_4^* = 0, \quad \text{and} \quad \varepsilon_1^* = \varepsilon_2^* = \varepsilon_3 = \varepsilon_4 = 1 \quad \text{if } \alpha_1 = 0, \end{aligned} \quad (2)$$

and

$$\varepsilon = 1 \quad \text{if } \alpha_2 \neq 0, \quad \varepsilon = 0 \quad \text{if } \alpha_2 = 0. \quad (3)$$

Finally, if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and assume that j is sufficiently large, then we may argue as we did in the proof of Theorem 4.1 to show that

$$4^{-1}e_1j^{3/8} < |w(0)| < 4e_2j^{3/4} \quad \text{and} \quad w(1) > e_1((a_2 - a_1)/2)^{3/2}j,$$

where the e_i are defined in (4.3). Assuming j large, we shall now denote by $\xi_{j,k}^-$ and $\xi_{j,k}^+$ the points of $[c, d]$ at which

$$r(x_2) = \eta_{j,k} + e^*j^{-5/12} \quad \text{and} \quad r(x_2) = \eta_{j,k} - e^*j^{-5/12},$$

respectively, where $e^* = (e_1/4e_2)^{2/3}$; it is clear that

$$0 < \xi_{j,k}^- < x(\eta_{j,k}) < \xi_{j,k}^+ < 1, \quad |w(x_2)| < 4^{-1}e_1j^{3/8}$$

for $\xi_{j,k}^- < x_2 < \xi_{j,k}^+$, and $|w(x_2)| > (e_1^2/4e_2)j^{3/8}$ for $x_2 \leq \xi_{j,k}^-$ and $x_2 \geq \xi_{j,k}^+$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k} = \psi_{j,k}^*(x_1, x_2) / \|\psi_{j,k}^*\|,$$

and recalling the definition of $M_{j,k}$ given in Theorem 5.2, we have

THEOREM 5.3. *It is the case that $\lambda_{j,k}$, $\mu_{j,k}$ are given by (5.6), $(-1)^{j+k}\psi_{j,k}(x_1, x_2)$ is given by the right-hand side of (5.1) for $0 \leq x_1 \leq 1$, $\xi_{j,k}^+ \leq x_2 \leq 1$,*

$$\psi_{j,k}(x_1, x_2) = E(x_1, x_2, j, k)$$

$$\times \left[\cos Z_1^*(x_1, j, k) T_{j,k}(x_2) \{ E^\dagger(x_1, j, k) + \varepsilon_1 O(j^{-3/4}) + \varepsilon_1^* o(j^{-1}) \} \right.$$

$$+ \cos Z_1^*(x_1, j, k) S_{j,k}(x_2) f_1(x_1, x_2, j, k)$$

$$+ \sin Z_1^*(x_1, j, k) T_{j,k}(x_2) \{ E^\#(x_1, j, k) + \varepsilon_3 O(j^{-3/4}) + \varepsilon_3^* o(j^{-1}) \} \left. \right]$$

$$+ \sin Z_1^*(x_1, j, k) S_{j,k}(x_2) f_2(x_1, x_2, j, k) \quad \left. \right]$$

$$\text{for } 0 \leq x_1 \leq 1, \quad \xi_{j,k}^- < x_2 < \xi_{j,k}^+, \quad (5.15)$$

$$\begin{aligned} \psi_{j,k}(x_1, x_2) &= D_{j,k}^{-1/2} [(r(x_2) - \eta_{j,k})P_1(x_1, 1, \eta_{j,k})B_2(x_2)]^{-1/4} \\ &\times \left[\cos Z_1^*(x_1, j, k) \exp\{-Z_2^*(x_2, j, k)\} \right. \\ &\quad \times \{E_1^*(x_1, x_2, j, k) + \varepsilon_1 O(j^{-3/4}) + \varepsilon_1^* o(j^{-1})\} \\ &\quad + (-1)^e \cos Z_1^*(x_1, j, k) \exp\{-Z_2^\dagger(x_2, j, k)\} \\ &\quad \times \{E_2^*(x_1, x_2, j, k) + \varepsilon_2 O(j^{-3/4}) + \varepsilon_2^* o(j^{-1})\} \\ &\quad + \sin Z_1^*(x_1, j, k) \exp\{-Z_2^*(x_2, j, k)\} \\ &\quad \times \{E_3^*(x_1, x_2, j, k) + \varepsilon_3 O(j^{-3/4}) + \varepsilon_3^* o(j^{-1})\} \\ &\quad + (-1)^e \sin Z_1^*(x_1, j, k) \exp\{-Z_2^\dagger(x_2, j, k)\} \\ &\quad \left. \times \{E_4^*(x_1, x_2, j, k) + \varepsilon_4 O(j^{-3/4}) + \varepsilon_4^* o(j^{-1})\} \right] \\ &\quad \text{for } 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq \xi_{j,k}^-, \quad (5.16) \end{aligned}$$

as $j \rightarrow \infty$, $(j, k) \in \Omega_j^*$, where $E(x_1, x_2, j, k)$ is given by the right-hand side of (5.8), $f_p(x_1, x_2, j, k)$ equals $o(j^{-1})$ for $x_2 \geq x(\eta_{j,k})$ and $\exp\{-2Z_2^*(0, j, k)\}O(1)$ for $x_2 < x(\eta_{j,k})$, $p = 1, 2$, and $Z_2^\dagger(x_2, j, k) = 2Z_2^*(0, j, k) - Z_2^*(x_2, j, k)$. These results for $\psi_{j,k}(x_1, x_2)$ hold uniformly in x_1 and x_2 .

REMARK. It is clear from the remark immediately following the statement of Theorem 5.2 and the definitions of the terms involved that

$$(2/3)\zeta^{3/2}(x_2, j, k) = -iZ_2^*(x_2, j, k) \quad \text{for } 0 \leq x_2 < x(\eta_{j,k}).$$

Observe also that Z_2^* strictly decreases in this interval. Hence it follows from (3.12) that the expression $S_{j,k}(x_2)f_p(x_1, x_2, j, k)$ given in (5.15) is $o(\exp\{-Z_2^*(0, j, k)\})$ as $j \rightarrow \infty$ uniformly in $\xi_{j,k}^- < x_2 < x(\eta_{j,k})$ for $p = 1, 2$.

PROOF. We shall only prove the theorem for the case $\alpha_i = 0, \beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then by arguing with (3.11), (3.19) as we did in the proof of Theorem 4.1, it is easy to see that

$$\lambda_{j,k}^{1/2}H(\eta_{j,k}) = (k + 3/4)\pi + \lambda_{j,k}^{-1/2}c_{21}^*(\eta_{j,k}) + o(j^{-1}) \quad (5.17)$$

as $j \rightarrow \infty$. In light of (5.3), it follows that as $j \rightarrow \infty$,

$$G(\eta_{j,k}) = ((k + 3/4)/(j + 1)) + O(j^{-2}),$$

and hence

$$\eta_{j,k} = t_{j,k}^* + O(j^{-2}), \quad \lambda_{j,k} = M_{j,k}^2 \{1 + O(j^{-2})\}.$$

Substituting these expressions for $\eta_{j,k}$ and $\lambda_{j,k}$ in (5.3) and (5.17), we obtain

$$G(\eta_{j,k}) = (k + 3/4)/(j + 1) + C(t_{j,k}^*)/((j + 1)\pi)^2 + o(j^{-2}),$$

where $C(t) = c_{21}^*(t)h_1(t) - c_1^*(t)H(t)$, and hence

$$\eta_{j,k} = t_{j,k}^* + C(t_{j,k}^*) / ((j+1)\pi)^2 G'(t_{j,k}^*) + o(j^{-2})$$

as $j \rightarrow \infty$. The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows from this latter result, (5.3), and the relation $d_1^*(t)H(t) + d_2^*(t)h_1(t) = 2$.

For convenience of notation we shall henceforth put

$$\phi_i^*(x_i, j, k) = \phi_i(x_i, \lambda_{j,k}, \mu_{j,k}), \quad \chi_i^*(x_i, j, k) = \chi_i(x_i, \lambda_{j,k}, \mu_{j,k})$$

for $i = 1, 2$. Then by arguing with (3.19) and (5.3) it is easy to see that as $j \rightarrow \infty$,

$$\begin{aligned} \phi_2^*(x_2, j, k) &= K_2^*(j, k) \Phi_1^{-1/2}(x_2, \eta_{j,k}) \\ &\times [T_{j,k}(x_2) \{1 + N_{j,k}^{-1} c^\dagger(\eta_{j,k}) + F_{j,k}(x_2)\} + S_{j,k}(x_2) F_{j,k}^*(x_2)] \end{aligned}$$

for $0 \leq x_2 \leq 1$, where

$$K_2^*(j, k) = \kappa N_{j,k}^{-5/6} |P_2(0, 1, \eta_{j,k})|^{-1/4} \exp\{Z_2^*(0, j, k)\},$$

$F_{j,k}(x_2) = O(j^{-3/4})$, and $F_{j,k}^*(x_2)$ equals $O(j^{-1})$ for $\xi_{j,k}^+ \leq x_2 \leq 1$, $o(j^{-1})$ for $x(\eta_{j,k}) \leq x_2 < \xi_{j,k}^+$, and $O(\exp\{-2Z_2^*(0, j, k)\})$ for $0 \leq x_2 < x(\eta_{j,k})$, with these results for $F_{j,k}$ and $F_{j,k}^*$ holding uniformly in x_2 . Arguments similar to those used in [6, Theorem 3.4] show that

$$\begin{aligned} \phi_1^*(x_1, j, k) &= K_1^*(j, k) P_1^{-1/4}(x_1, 1, \eta_{j,k}) \\ &\times [\sin Z_1^*(x_1, j, k) \{1 + o(j^{-1})\} \\ &\quad + \cos Z_1^*(x_1, j, k) \{y_1(x_1, j, k) + o(j^{-1})\}] \end{aligned}$$

as $j \rightarrow \infty$, uniformly in $0 \leq x_1 \leq 1$, where $K_1^*(j, k) = 1/N_{j,k} P_1^{1/4}(0, 1, \eta_{j,k})$, and hence it is now not difficult to verify that

$$\|\psi_{j,k}^*\|^{-1} = N_{j,k}^{1/6} (2/D_{j,k}^*)^{1/2} \left(\prod_{i=1}^2 K_i^*(j, k) \right)^{-1} \{1 - N_{j,k}^{-1} c^\dagger(\eta_{j,k}) + O(j^{-3/4})\}$$

as $j \rightarrow \infty$. By arguing with (3.15), (5.3), we may next deduce that as $j \rightarrow \infty$,

$$\begin{aligned} \phi_2^*(x_2, j, k) &= K_2^*(j, k) |P_2(x_2, 1, \eta_{j,k})|^{-1/4} \\ &\times [\exp\{-Z_2^*(x_2, j, k)\} \{1 + N_{j,k}^{-1} y_2^\dagger(x_2, \eta_{j,k}) + O(j^{-3/4})\} \\ &\quad + \exp\{-Z_2^\dagger(x_2, j, k)\} \{-1 + N_{j,k}^{-1} y_2^\dagger(x_2, \eta_{j,k}) + O(j^{-3/4})\}] \end{aligned}$$

for $0 \leq x_2 \leq \xi_{j,k}^-$, uniformly in x_2 , where

$$K_2^*(j, k) = (2N_{j,k} |P_2(0, 1, \eta_{j,k})|^{1/4})^{-1} \exp\{Z_2^*(0, j, k)\}.$$

Finally, we also know that for $i = 1, 2$,

$$\phi_i^*(x_i, j, k) = k_i(j, k) \chi_i^*(x_i, j, k) \quad \text{for } 0 \leq x_i \leq 1,$$

where $k_i(j, k) = -d\phi_i^\#(1, j, k)/dx_i$ and $\chi_2^\#$ is given by (5.4) for $\xi_{j,k}^+ < x_2 < 1$ and $\chi_1^\#$ by (5.5) for $0 < x_1 < 1$ (with the first formula holding uniformly in x_2 and the second uniformly in x_1). Moreover, by arguing with (3.11), (3.20), and (5.17), it is not difficult to deduce that

$$k_2(j, k) = (-1)^k |P_2(0, 1, \eta_{j,k})|^{-1/4} P_2^{1/4}(1, 1, \eta_{j,k}) \times \exp\{Z_2^*(0, j, k)\} \{1 + N_{j,k}^{-1} c^\dagger(\eta_{j,k}) + O(j^{-3/4})\}$$

as $j \rightarrow \infty$, while a standard argument shows that

$$k_1(j, k) = (-1)^j P_1^{-1/4}(0, 1, \eta_{j,k}) P_1^{1/4}(1, 1, \eta_{j,k}) \{1 + o(j^{-1})\}$$

as $j \rightarrow \infty$. With these results it is now a simple matter to verify the assertion of the theorem concerning $\psi_{j,k}$.

5.4. *Formulae in Ω_4^* .* In this subsection we assume that $(j, k) \in \Omega_4^*$ (see Theorem 4.1). Then it may readily be verified that

$$2^{-1} a^* j^{-5/8} < (k + \nu^*) / (j + \nu_1) < (G(a_2) - 2^{-1} a j^{-1/6})$$

when j is sufficiently large (see subsection 5.0 for terminology). Assuming j large, we shall denote by $t_{j,k}^*$ the solution of the equation $G(t) = (k + \nu^*) / (j + \nu_1)$; and it is not difficult to show that $(a_1 + 4^{-1} j^{-5/12}) < t_{j,k}^* < (a_2 - 4^{-1} j^{-1/6})$.

Referring to the notation of §§2, 4, and subsection 5.0 (see part (2)), we shall now put

$$\begin{aligned} c_{21}^*(t) &= -7/72\Phi(1, t), & c_{22}^*(t) &= -B_2^{-1/2}(1)X_2^\dagger(1, \beta_2, t) \text{ if } \beta_2 \neq \pi, \\ c_{21}^*(t) &= 5/72\Phi(1, t) \text{ and } c_{22}^*(t) &= 0 \text{ if } \beta_2 = \pi, \\ c_{23}^*(t) &= X_2^\#(1, x(t), t), \\ G^*(t) &= G'(t)(1 - x(t))^{-1/2} \text{ for } a_1 < t \leq a_2 + \delta, \\ G^*(a_1) &= -x'(a_1)h_1^{-1}(a_1)\Psi(1, a_1) \end{aligned}$$

(where $' = d/dt$; observe also from the definition of $G(t)$ in §4 that $G^*(t)$ is continuous and positive in $[a_1, a_2 + \delta]$),

$$\begin{aligned} d_1^*(t) &= L(t)X_1^\dagger(0, t)\Psi_1^{1/2}(1, t), & d_2^*(t) &= L(t)X_2^\dagger(x(t), t)\Psi_1^{1/2}(1, t), \\ d_1^\dagger(t) &= t d_1^*(t) + L(t)h_1(t)\Psi_1^{1/2}(1, t), \\ d_2^\dagger(t) &= t d_2^*(t) - L(t)H(t)\Psi_1^{1/2}(1, t) \end{aligned}$$

(where $L(t) = 1/h_1^2(t)G^*(t)$),

$$\begin{aligned} c_1(t) &= c_1^*(t)d_2^*(t), & d_1(t) &= c_1^*(t)d_2^\dagger(t), \\ c_{2m}(t) &= c_{2m}^*(t)d_1^*(t), & d_{2m}(t) &= c_{2m}^*(t)d_1^\dagger(t) \text{ for } m = 1, 2, 3, \\ c(t) &= c_1(t) + c_{23}(t) \text{ and } d(t) &= d_1(t) + d_{23}(t). \end{aligned}$$

Referring next to the terminology of (3.11), let

$$\begin{aligned}\kappa^\# &= -\beta^\dagger + \nu^\dagger \alpha^\dagger + 3^{-1}(\nu^\dagger)^3 \quad \text{and} \quad \kappa^\dagger = \nu^\dagger \quad \text{if} \quad \beta_2 \neq \pi, \\ \kappa^\# &= \beta - \nu\alpha - 3^{-1}\nu^3 \quad \text{and} \quad \kappa^\dagger = -\nu \quad \text{if} \quad \beta_2 = \pi,\end{aligned}$$

and in the interval $a_1 < t \leq a_2 + \delta$ put

$$G^\dagger(t) = 2^{-1}(1 - x(t))^{1/2}G''(t)$$

(observe from the definition of G in §4 that G^\dagger is bounded in this interval),

$$c_1^\#(t) = d_{21}(t) - tc_{21}(t),$$

$$\begin{aligned}c_2^\#(t, s) &= \kappa^\# \Phi^{-3}(1, t) - (c_1^\#(t))^2 \Psi_1^{1/2}(1, t) h_1(t) G^\dagger(s) \\ &\quad + 3\kappa^\dagger c_1^\#(t) / 2\Phi(1, t),\end{aligned}$$

$$c^\#(t, s) = d_1^\#(t) c_2^\#(t, s) \quad \text{and} \quad d^\#(t, s) = d_1^\dagger(t) c_2^\#(t, s),$$

where $a_1 < s \leq a_2 + \delta$ and the terminology of the previous paragraph is utilized.

Lastly, paying special attention to part (3) of the notation of subsection 5.0, let

$$\begin{aligned}c^\dagger(t) &= c_{21}^\dagger(t)(a_2 - t)^{-3/2} + c_{22}^\dagger(t)(a_2 - t)^{-1/2} \\ &\quad + X_2^\#(x(t), 0, t) + y_2^*(0, t)\end{aligned}$$

for $t < a_2$, and when $t < a_2$ and $x_2 < x(t)$, put

$$\begin{aligned}y_2^\dagger(x_2, t) &= y^\#(x_2, t)(r(x_2) - t)^{-3/2} - X_2^\#(x(t), x_2, t) \\ &\quad - y_2^*(x_2, t) + c^\dagger(t),\end{aligned}$$

$$y_{21}^\dagger(x_2, t) = y_2^\dagger(x_2, t) - c^\dagger(t), \quad y_{22}^\dagger(x_2, t) = y_2^\dagger(x_2, t) + c^\dagger(t),$$

and

$$y_{2i}(x_2, j, k) = N_{j,k}^{-1} y_{2i}^\dagger(x_2, \eta_{j,k}) \quad \text{for} \quad i = 1, 2.$$

Then in this subsection we shall define $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, $S_{j,k}(x_2)$, $E_m(x_1, x_2, j, k)$, σ_m , σ_m^* ($m = 1, \dots, 4$), $E^\dagger(x_1, j, k)$, $E^\#(x_1, j, k)$, and ε precisely as we did in subsection 5.3. We shall also define the E_m^* , $m = 1, \dots, 4$, in terms of the y_{2i} and y_1 precisely as we did in subsection 5.3, except we are now to use our new definition of the y_{2i} here; and as for the ε_m and ε_m^* defined in that subsection, we shall henceforth take $\varepsilon_m = 0$, $\varepsilon_m^* = 1$, $m = 1, \dots, 4$, regardless of whether α_1 is zero or not. Finally, let

$$D_{j,k}^* = N_{j,k}^{-1/3} (\eta_{j,k} - a_1)^{-1/2} I_{j,k},$$

where $I_{j,k}$ is given by (5.14). It may readily be verified that $D_{j,k}^*$ and $1/D_{j,k}^*$ remain less than some positive number independent of j and k for all large j .

If in §3 we take $\lambda = \lambda_{j,k}$ and $t = \eta_{j,k}$, then we may argue as we did in the

proof of Theorem 4.1 to show that

$$|w(0)| > 4^{-1}e_1j^{3/4} \quad \text{and} \quad w(1) > 4^{-1}e_1j^{3/8},$$

where the e_i are defined in (4.3). We shall now denote by $\xi_{j,k}^-$ and $\xi_{j,k}^+$ the points of $[c, d]$ at which

$$r(x_2) = \eta_{j,k} + e^*j^{-1/6} \quad \text{and} \quad r(x_2) = \eta_{j,k} - e^*j^{-5/12},$$

respectively, where $e^* = (e_1/4e_2)^{2/3}$; and it is clear that $0 < \xi_{j,k}^- < x(\eta_{j,k}) < \xi_{j,k}^+ < 1$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k}(x_1, x_2) = \psi_{j,k}^*(x_1, x_2) / \|\psi_{j,k}^*\|,$$

and recalling the definitions of $E(x_1, x_2, j, k)$, the $f_i(x_1, x_2, j, k)$, and $Z_2^\dagger(x_2, j, k)$ given in Theorem 5.3, we may argue as we did in that theorem to show that

THEOREM 5.4. *It is the case that*

$$\begin{aligned} \lambda_{j,k} = & M_{j,k}^2 \left[1 + M_{j,k}^{-2} \langle (t_{j,k}^* - a_1)^{-2} c_{21}(t_{j,k}^*) \right. \\ & + (t_{j,k}^* - a_1)^{-1} c_{22}(t_{j,k}^*) + (t_{j,k}^* - a_1)^{-1/2} c(t_{j,k}^*) \rangle \\ & + (t_{j,k}^* - a_1)^{-5} M_{j,k}^{-4} c^\#(t_{j,k}^*, s_{j,k}) \\ & \left. + (t_{j,k}^* - a_1)^{-1/2} O(j^{-53/24}) + o(j^{-2}) \right], \\ \mu_{j,k} = & M_{j,k}^2 \left[t_{j,k}^* + M_{j,k}^{-2} \langle (t_{j,k}^* - a_1)^{-2} d_{21}(t_{j,k}^*) \right. \\ & + (t_{j,k}^* - a_1)^{-1} d_{22}(t_{j,k}^*) + (t_{j,k}^* - a_1)^{-1/2} d(t_{j,k}^*) \rangle \\ & + (t_{j,k}^* - a_1)^{-5} M_{j,k}^{-4} d^\#(t_{j,k}^*, s_{j,k}) \\ & \left. + (t_{j,k}^* - a_1)^{-1/2} O(j^{-53/24}) + o(j^{-2}) \right], \end{aligned}$$

$(-1)^{j+k}(\eta_{j,k} - a_1)^{1/4} \psi_{j,k}(x_1, x_2)$ is given by the right-hand side of (5.1) for $0 < x_1 < 1$, $\xi_{j,k}^+ < x_2 < 1$, $(\eta_{j,k} - a_1)^{1/4} \psi_{j,k}(x_1, x_2)$ is given by the right-hand side of (5.15) for $0 < x_1 < 1$, $\xi_{j,k}^- < x_2 < \xi_{j,k}^+$, and by the right-hand side of (5.16) for $0 < x_1 < 1$, $0 < x_2 < \xi_{j,k}^-$, as $j \rightarrow \infty$, $(j, k) \in \Omega_4^*$, where $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^*)$ and $s_{j,k} = t_{j,k}^* + O(j^{-7/6})$. These results for $\psi_{j,k}(x_1, x_2)$ hold uniformly in x_1 and x_2 .

REMARK. It is important to observe from the definitions of terms involved that in the above formulae for $\lambda_{j,k}$ and $\mu_{j,k}$, the expressions $(t_{j,k}^* - a_1)^{-1/2} c(t_{j,k}^*)$ and $(t_{j,k}^* - a_1)^{-1/2} d(t_{j,k}^*)$, which depend upon the q_i , are both $O(1)$ as $j \rightarrow \infty$. These expressions, when multiplied by $M_{j,k}^{-2}$, are, of course, absorbed into the error terms when k/j is small.

5.5. *Formulae in Ω_5^* .* In this subsection we assume that $(j, k) \in \Omega_5^*$ (see Theorem 4.1). Then recalling that in subsection 3.4 we denoted the zeros of $T(z)$ and $T'(z)$ by τ_n and τ'_n , respectively, $n \geq 1$, we shall now put

$$c_{21}^*(t) = -X_2^\dagger(1, \beta_2, t)/\Phi_1(1, t) \quad \text{and} \quad \Lambda_k = \tau'_{k+1} \quad \text{if} \quad \beta_2 \neq \pi,$$

$$c_{21}^*(t) = 0 \quad \text{and} \quad \Lambda_k = \tau_{k+1} \quad \text{if} \quad \beta_2 = \pi$$

(see subsection 5.0). In light of the properties of Λ_k given in subsection 3.4, it is clear that $j^{-1/4}\Lambda_k$ remains less than some positive number independent of j and k for all large j . Assuming j large, we shall denote by $t_{j,k}^\dagger$ the solution of the equation $G_1(t) = \Lambda_k/((j + \nu_1)\pi)^{2/3}$; and it is not difficult to verify that $a_1 < t_{j,k}^\dagger < a_1 + 3j^{-5/12}$ (see part (7) of the notation of §4 for the properties of G_1).

In what follows we let

$$c(t) = c_{21}^*(t)L(t)X_1^*(0, t), \quad d(t) = tc(t) + c_{21}^*(t)L(t)h_1(t),$$

where $L(t) = 1/h_1^{5/3}(t)G_1'(t)$, $' = d/dt$, and define $\zeta(x_2, j, k)$, $T_{j,k}(x_2)$, $S_{j,k}(x_2)$, $E^\dagger(x_1, j, k)$, $E^\#(x_1, j, k)$, ε , $E_m^*(x_1, x_2, j, k)$, ε_m , ε_m^* , $m = 1, \dots, 4$, precisely as we did in subsection 5.4. We shall also put $D_{j,k}^* = \Lambda_k^{-1/2}I_{j,k}$, where $I_{j,k}$ is given by (5.14). It is clear from (3.11)–(3.12) and [4, p. 86] that $D_{j,k}^*$ and $1/D_{j,k}^*$ remain less than some positive number independent of j and k for all large j .

Finally, if in §3 we take $\lambda = \lambda_{j,k}$, $t = \eta_{j,k}$, and assume that j is sufficiently large, then we may argue as we did in the proof of Theorem 4.1 to show that

$$w(1) < 4e_2j^{3/8} \quad \text{and} \quad |w(0)| > e_1((a_2 - a_1)/2)^{3/2}j,$$

where the e_i are defined in (4.3). Assuming j large, we shall now denote by $\xi_{j,k}$ the point of $[c, d]$ at which $r(x_2) = \eta_{j,k} + e_1^{-2/3}j^{-1/6}$; and it is clear that $0 < \xi_{j,k} < x(\eta_{j,k})$. Then putting

$$D_{j,k} = 2\kappa^2 D_{j,k}^*, \quad \psi_{j,k}(x_1, x_2) = \psi_{j,k}^*(x_1, x_2)/\|\psi_{j,k}^*\|,$$

and recalling the definitions of $E(x_1, x_2, j, k)$, the $f_i(x_1, x_2, j, k)$, and $Z_2^\dagger(x_2, j, k)$ given in Theorem 5.3, we have

THEOREM 5.5. *It is the case that*

$$\lambda_{j,k} = M_{j,k}^2 \left[1 + M_{j,k}^{-4/3} \Lambda_k^{-1} c(t_{j,k}^\dagger) + O(j^{-2}) \right],$$

$$\mu_{j,k} = M_{j,k}^2 \left[t_{j,k}^\dagger + M_{j,k}^{-4/3} \Lambda_k^{-1} d(t_{j,k}^\dagger) + O(j^{-2}) \right],$$

$\Lambda_k^{1/4} N_{j,k}^{-1/6} \psi_{j,k}(x_1, x_2)$ is given by the right-hand side of (5.15) for $0 \leq x_1 \leq 1$, $\xi_{j,k} < x_2 \leq 1$, and by the right-hand side of (5.16) for $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq \xi_{j,k}$ as $j \rightarrow \infty$, $(j, k) \in \Omega_5^*$, where $M_{j,k} = (j + \nu_1)\pi/h_1(t_{j,k}^\dagger)$. These results for $\psi_{j,k}(x_1, x_2)$ hold uniformly in x_1 and x_2 .

PROOF. We shall only prove the theorem for the case $\alpha_i = 0, \beta_i = \pi$ for $i = 1, 2$; the other cases can be similarly treated. Then by arguing with (3.19) as we did in the proof of Theorem 4.1, it is easy to see that

$$\lambda_{j,k}^{1/3} (3H(\eta_{j,k})/2)^{2/3} = \Lambda_k + O(j^{-4/3}) \quad \text{as } j \rightarrow \infty.$$

In light of (5.3) it follows that as $j \rightarrow \infty$,

$$G_1(\eta_{j,k}) = \Lambda_k((j+1)\pi)^{-2/3} + O(j^{-2}),$$

and hence

$$\eta_{j,k} = t_{j,k}^\dagger + O(j^{-2}).$$

The assertion of the theorem concerning $\lambda_{j,k}$ and $\mu_{j,k}$ follows from this result and (5.3), while the assertion concerning $\psi_{j,k}$ can be proved by using arguments similar to those used in the proof of Theorem 5.3.

6. Conclusions and final remarks. We are now in a position to furnish a negative answer to the question regarding the uniform boundedness of the eigenfunctions of the system (1.1)–(1.4) (see §1); for it is clear from the foregoing results that the eigenfunctions, $\psi_{j,k}(x_1, x_2)$, are not uniformly bounded in I^2 for $(j, k) \in \Omega_1$. We do note however that

THEOREM 6.1. *The absolute values of the $j^{-1/3}\psi_{j,k}(x_1, x_2)$ for $(j, k) \in \Omega_1$ and $(x_1, x_2) \in I^2$ remain less than some bound independent of x_1, x_2, j , and k .*

It is important to observe from Theorem 5.5 that this result is false if $1/3$ is replaced by $1/3 - \epsilon, \epsilon > 0$.

A second point worth mentioning deals with the dependence of the eigenvalues on the q_i . Indeed, it is clear from the above results that if in (1.2i - 1), $1 \leq i \leq 2$, we replace $q_i(x_i)$ by $q_i^*(x_i)$ and denote the eigenvalues of the system (1.1)–(1.4), with the q_i replaced by the q_i^* , by $(\lambda_{j,k}^*, \mu_{j,k}^*), j, k = 0, 1, \dots$, then $\lambda_{j,k}^* - \lambda_{j,k}$ and $\mu_{j,k}^* - \mu_{j,k}$ are $O(1)$ as $j \rightarrow \infty, (j, k) \in \Omega_1$.

We wish also to state that further developments of the formulae given in Theorems 5.1–5.5 are possible if the coefficients in the differential equations (1.1), (1.3) are suitably defined; we refer to the remark given at the end of §3 in [6] and to the work in [4, pp. 6–11] for further clarification.

As stated in §2, Theorems 5.1–5.5 are valid only for the case $r'(x_2) < 0$ in $c \leq x_2 \leq d$. However this is no restriction; for if $r'(x_2) > 0$ in this interval, then the substitutions $x_2 = 1 - s_2, \alpha_2^* = \pi - \beta_2, \beta_2^* = \pi - \alpha_2$, and a re-labelling of terms in (1.3)–(1.4) reduces the system (1.1)–(1.4) to a form where the results of Theorems 5.1–5.5 may again be utilized.

Finally, recalling the definition of θ_2 given in §4, let Ω_2 denote the sector in the (x, y) -plane defined by the inequalities $\theta_2 < \theta \leq \pi/2$. Then it is not difficult to verify that results analogous to those above hold for the case

$(j, k) \in \Omega_2$ if we choose θ_2 appropriately and assume that A_1, B_1 are of class C^3 in some interval containing the interval $0 < x_1 < 1$ in its interior and that $A_1' B_1 - A_1 B_1' \neq 0$ in $0 < x_1 < 1$.

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