

FRATTINI SUBGROUPS OF 3-MANIFOLD GROUPS

BY

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ABSTRACT. In this paper it is shown that if the Frattini subgroup of the fundamental group of a compact, orientable, irreducible, sufficiently large 3-manifold is nontrivial then the 3-manifold is a Seifert fibered space. We show further that the Frattini subgroup of the group of a Seifert fibered space is trivial or cyclic. As a corollary to our work we prove that every knot group has trivial Frattini subgroup.

1. Introduction. The importance of Seifert fibered spaces in the study of 3-manifolds has become increasingly apparent [13]. In [25] it was proved that if a sufficiently large 3-manifold has a nontrivial center then the manifold is a Seifert fibered space. In this paper we establish a parallel result for the Frattini subgroup. The Frattini subgroup $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G . Equivalently, $\Phi(G)$ is the set of nongenerators of G , that is, $x \in \Phi(G)$ if and only if whenever $\langle x, x_1, x_2, \dots \rangle = G$ then $\langle x_1, x_2, \dots \rangle = G$.

In [18] L. Neuwirth asked the question: What can be said of the Frattini subgroup of a knot group? K. Murasugi [17] showed that the Frattini subgroups of composite and alternating knot groups are trivial. Furthermore, he conjectured that the Frattini subgroup of any knot group is trivial. In [1] and [2] Allenby and Tang proved that the Frattini subgroups of cable and fibered knot groups are trivial. As a corollary to our work we settle Neuwirth's question by confirming Murasugi's conjecture, that is:

For every knot K in S^3 the Frattini subgroup of $\pi_1(S^3 - K)$ is trivial.

More generally, we prove that many 3-manifold groups have trivial Frattini subgroups. For such 3-manifolds this means that every nontrivial simple closed curve x is important in the sense that other simple closed curves can somehow be chosen to complete x to a generating set from which x cannot be

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deleted. More precisely, we show that:

If G is the fundamental group of a compact, orientable, irreducible, sufficiently large 3-manifold and if $\Phi(G) \neq 1$, then G is the fundamental group of a Seifert fibered space.

To complete the picture some rather intricate computations are used to show that:

If G is the fundamental group of a Seifert fibered space, then $\Phi(G)$ is trivial or cyclic.

(We in fact produce explicit generators for $\Phi(G)$.) The techniques involve applying known theorems about Frattini subgroups of generalized free products and HNN extensions to the special types of such constructions induced on the fundamental group of a sufficiently large 3-manifold by a hierarchy.

Some definitions will be given, but the reader is assumed to be familiar with such terms as *incompressible 2-manifold, sufficiently large 3-manifold, irreducible 3-manifold, Seifert fibered space, etc.* A good reference for these basic concepts is [8]. For Seifert fibered spaces, in particular, the reader may wish to consult [20].

As for notation we use $*(A, B: H)$ to denote the generalized free product of A and B amalgamated along H and $*(H, t: tAt^{-1} = B)$ to denote the HNN extension of H relative to the isomorphic subgroups A, B of H .

The letter I will denote the unit interval. By a *double twisted I -bundle* over a closed nonorientable 2-manifold F , we mean the 3-manifold that is the union of two orientable I -bundles over F .

For each integer n we define an integer $\varphi(n)$ as follows. If $|n| = p_1^{k_1} \cdot \dots \cdot p_m^{k_m}$ where p_i are distinct primes, we define $\varphi(n) = p_1 \cdot \dots \cdot p_m$. Observe that $\varphi(n)|n$ and that $\varphi(n)$ generates $\Phi(\mathbb{Z}_n)$. We define $\varphi(0) = 0$ and $\varphi(\pm 1) = 1$.

2. Reduction to Seifert fibered spaces and bundles. The topological constructions to follow are motivated by the following two theorems.

THEOREM 2.1 (A. WHITTEMORE [28]). *Let $G = *(A, B: H)$ with $A \neq H \neq B$. If there is an $x \in G$ such that $xHx^{-1} \cap H = 1$, then $\Phi(G) = 1$.*

THEOREM 2.2 (ALLENBY AND TANG [2]). *Let $G = *(H, t: tAt^{-1} = B)$ be a countable HNN group with $A \neq A \cap B \neq B$. Then $\Phi(G) \subseteq A \cap B$.*

LEMMA 2.3. *Let F be a closed, incompressible 2-manifold in a compact, orientable, irreducible 3-manifold M such that F does not cobound a product with any component of $\text{bd } M$. Let $p: N \rightarrow M$ be the covering map of M associated with $\pi_1(F)$. Then there are infinitely many components of $N - p^{-1}(F)$ each of whose closures contains at least two components of $p^{-1}(F)$.*

PROOF. Suppose the conclusion of the lemma does not hold. That is, suppose there are only n components of $N - p^{-1}(F)$ each of whose closures contains at least two components of $p^{-1}(F)$. Consider first the possibility that F does not separate M . Then the closure of each component of $N - p^{-1}(F)$ contains at least two components of $p^{-1}(F)$ since it is a covering space of M cut at F . Furthermore, the number of components of $N - p^{-1}(F)$ is infinite since there are infinitely many double cosets $\pi_1(F)g\pi_1(M - F)$ in $\pi_1(M)$.

Thus it must be the case that F separates M into two components M_1 and M_2 . The Seifert-van Kampen Theorem gives $\pi_1(M)$ as a generalized free product $\ast(\pi_1(M_1), \pi_1(M_2); \pi_1(F))$. Since neither M_1 nor M_2 is a product, the inclusion induced maps $i_\ast: \pi_1(F) \rightarrow \pi_1(M_j)$ ($j = 1, 2$) are not onto [26]. So there are elements of $\pi_1(M)$ of arbitrary length with respect to the generalized free product structure of $\pi_1(M)$. In particular, there is an x in $\pi_1(M)$ of length at least $n + 2$.

Let $\ast \in F$ be the base point of M , \ast_0 the base point of N and F_0 the component of $p^{-1}(F)$ containing \ast_0 . Since $i_\ast: \pi_1(F_0) \rightarrow \pi_1(N)$ is an isomorphism, the following facts are evident.

- (i) Each component F_j of $p^{-1}(F)$ separates N into two components whose closures we denote by A_j^0 and A_j^1 . (The notation is chosen so that $\ast_0 \in A_j^0$.)
- (ii) The inclusion induced map $i_\ast: \pi_1(F_j) \rightarrow \pi_1(A_j^1)$ is onto for each j .

Let \tilde{x} be the lifting of x based at \ast_0 . Using (ii) above it is possible to construct a path \tilde{y} in N with the following properties.

- (a) \tilde{y} is homotopic with endpoints fixed to \tilde{x} .
- (b) \tilde{y} meets no component of $p^{-1}(F)$ more than once.

Observe then that if R is a component of $N - p^{-1}(F)$ whose closure contains only one component of $p^{-1}(F)$, then $\tilde{y} \cap R$ is either empty or an endpoint of \tilde{y} . This together with (i) above shows that there are at most $n + 1$ points in $\tilde{y} \cap p^{-1}(F)$. Hence $p(\tilde{y})$ has length at most $n + 1$ which contradicts the fact that $p(\tilde{y})$ is homotopic to x and x has length $n + 2$.

LEMMA 2.4. *Let M be a closed, orientable, irreducible 3-manifold and F a closed incompressible 2-manifold in M . If $x\pi_1(F)x^{-1} \cap \pi_1(F) \neq 1$ for each x in $\pi_1(M)$, then M contains an incompressible torus.*

PROOF. Let $p: N \rightarrow M$ be the covering space of M associated with $\pi_1(F)$. Choose k a positive integer, $k > \exp(g + b + 1)^2$ where g is the genus of F and $b = \beta_1(F)$. Let $\ast \in F$ be the base point of M and $\ast_0 \in p^{-1}(\ast)$ the base point of N . By Lemma 2.3 there is a path x in N with the following properties.

- (i) The initial point of x is \ast_0 .
- (ii) The terminal point \ast_k of x lies in a component F_k of $p^{-1}(F)$.
- (iii) There are $k - 1$ components F_1, F_2, \dots, F_{k-1} of $p^{-1}(F)$ such that

$x \cap F_i$ is a single transverse intersection point.

According to our hypothesis $\pi_1(F_k) \neq 1$. So let \tilde{y}_k be a nontrivial loop in F_k based at $*$. The isomorphism $i_*: \pi_1(F_0) \rightarrow \pi_1(N)$ gives a deformation of the loop $x\tilde{y}_kx^{-1}$ into F_0 . From the intersection of this deformation with $\{F_0, F_1, \dots, F_{k-1}\}$ one gets loops $\tilde{y}_0 \subset F_0, \tilde{y}_1 \subset F_1, \dots, \tilde{y}_{k-1} \subset F_{k-1}$ each of which is freely homotopic in N to \tilde{y}_k . Upon composing with p , one obtains noncontractible loops y_0, y_1, \dots, y_k on F such that y_i is freely homotopic in M to y_j for each i, j . If for some $i \neq j, y_i$ is freely homotopic on F to y_j , a singular map of a torus into M is obtained. Then F. Waldhausen's torus-annulus theorem [4], [27] gives the required incompressible torus. If, on the other hand, y_i is not freely homotopic on F to y_j for any $i \neq j$, the torus is promised by [6].

LEMMA 2.5. *Let M be a compact, orientable, irreducible 3-manifold and T an incompressible torus boundary component of M . If $x\pi_1(T)x^{-1} \cap \pi_1(T) \neq 1$ for each x in $\pi_1(M)$, then M is a Seifert fibered space.*

PROOF. Let $p: N \rightarrow M$ be the covering space of M associated with $\pi_1(T)$. According to [23], N is homeomorphic to $S^1 \times S^1 \times [0, 1]$ less a compact subset of $S^1 \times S^1 \times 1$. If $N = S^1 \times S^1 \times [0, 1]$, then $\pi_1(N) = Z \oplus Z$ is of finite index in $\pi_1(M)$. It follows that M is either $S^1 \times S^1 \times [0, 1]$ or the orientable $[0, 1]$ bundle over the Klein bottle. In either case we have our result.

Suppose then that N is noncompact. If $S^1 \times S^1 \times 0$ is the only component of $p^{-1}(T)$, then p is onto and M is $S^1 \times S^1 \times [0, 1]$. Otherwise, each component of $p^{-1}(T)$, other than $S^1 \times S^1 \times 0$, is an incompressible annular subset of $S^1 \times S^1 \times 1$ (for no component of $p^{-1}(T)$ is simply connected since $x\pi_1(T)x^{-1} \cap \pi_1(T) \neq 1$). The center curve of each such component is homotopic in N to a fixed curve \tilde{t} on $S^1 \times S^1 \times 0$. Put $t = p(\tilde{t})$. Then $x\pi_1(T)x^{-1} \cap \pi_1(T) \subseteq \langle t \rangle$ since loops in $x\pi_1(T)x^{-1} \cap \pi_1(T)$ correspond precisely to loops in a component of $p^{-1}(T)$. We have then that $1 \neq xs^qt^n x^{-1} = s^l t^m \in \langle t \rangle$ for some integers q, l, m, n and $s, t \in \pi_1(T)$. It follows that $q = l = 0$ and that $xt^n x^{-1} = t^m$. Furthermore, according to [12] $m = \pm n$ and so $xt^n x^{-1} = t^{\pm n}$. Thus for each $x \in \pi_1(M)$ we have an n_x such that $xt^{n_x} x^{-1} = t^{\pm n_x}$. Since $\pi_1(M)$ is finitely generated, there is an integer k such that $\langle t^k \rangle$ is normal in $\pi_1(M)$. The lemma follows now from [13].

LEMMA 2.6. *Let M be a Seifert fibered space other than $S^1 \times S^1 \times I$ and the orientable I -bundle over the Klein bottle. Let t be a simple closed curve on a component T of $\text{bd } M$ such that t generates a normal subgroup of $\pi_1(M)$. Let $x \in \pi_1(M), x \notin \pi_1(T)$. If $t_1, t_2 \in \pi_1(T)$ and $xt_1 x^{-1} = t_2$, then there is an integer n such that $t_1 = t^n$ and $t_2 = t^{\pm n}$.*

This lemma is little more than an algebraic formulation of Lemma 2.8 of [13]. The proof is omitted.

LEMMA 2.7. *Let M be a compact, orientable, irreducible 3-manifold and T an incompressible torus in M that separates M . If $x\pi_1(T)x^{-1} \cap \pi_1(T) \neq 1$ for each $x \in \pi_1(M)$, then M is Seifert fibered or M is a double twisted I -bundle over the Klein bottle.*

PROOF. Let T separate M into components M_1, M_2 . By Lemma 2.5 both M_1 and M_2 are Seifert fibered. If either M_1 or M_2 is $S^1 \times S^1 \times I$, or if both M_1 and M_2 are I -bundles over the Klein bottle, the conclusion of the lemma is immediate. The proof proceeds then assuming that M_1 is neither $S^1 \times S^1 \times I$ nor an I -bundle over the Klein bottle and that M_2 is not $S^1 \times S^1 \times I$.

Let t be a simple closed curve on T that generates a normal subgroup of $\pi_1(M_1)$. We aim to show that $\langle t^k \rangle$ is normal in $\pi_1(M)$ for some integer k . If for each $y \in \pi_1(M_2)$ we have $y \in \pi_1(T)$, the result is immediate. Suppose then that there exists $y \in \pi_1(M_2), y \notin \pi_1(T)$. Choose $x \in \pi_1(M_1), x \notin \pi_1(T)$. By hypothesis there exist elements t_1, t_2 in $\pi_1(T)$ such that $(xyx)t_1(xy x)^{-1} = t_2$. Thus $yxxt_1x^{-1}y^{-1} = x^{-1}t_2x$. By the uniqueness of normal forms in generalized free products we see that t_2 and $x^{-1}t_2x$ both belong to T as do both t_1 and xt_1x^{-1} . Thus by Lemma 2.6 there exist integers m, n such that $t_2 = t^n, x^{-1}t_2x = t^{\pm n}$ and $t_1 = t^m, xt_1x^{-1} = t^{\pm m}$. Thus $yt^{\pm n}y^{-1} = t^{\pm n}$. A theorem of Jaco [12] now gives $m = \pm n$. Thus $yt^ny^{-1} = t^{\pm n}$.

We have then for each $y \in \pi_1(M_2)$ an n_y such that $yt^{n_y}y^{-1} = t^{\pm n_y}$. Since $\pi_1(M_2)$ is finitely generated, there is an integer k such that $\langle t^k \rangle$ is normal in $\pi_1(M)$. The conclusion of the lemma is now given by [13].

LEMMA 2.8. *If M is a compact, orientable, irreducible, sufficiently large 3-manifold and if $\Phi(\pi_1(M)) \neq 1$, then one of the following is true.*

- (i) M is a Seifert fibered space.
- (ii) M is a 2-manifold bundle over S^1 .
- (iii) M is a double twisted I -bundle.

PROOF. Choose a cutting surface F for M according to the following list of preferences.

- (1) F is a disk.
- (2) F is an annulus.
- (3) F is a torus.
- (4) F is a nonseparating bounded surface other than an annulus.
- (5) F is a closed 2-manifold other than a torus.

Case 1. F is a disk. In this case $\pi_1(M) = Z$ or a nontrivial free product, and so $\Phi(\pi_1(M)) = 1$ [10].

Case 2. F is a torus or annulus and F does not separate. Let N denote M cut

at F and let F_1, F_2 be copies of F in $\text{bd } N$. Then $\pi_1(M)$ is an HNN extension $\ast(\pi_1(N), t: t\pi_1(F_1)t^{-1} = \pi_1(F_2))$. By Theorem 2.2, $\Phi(\pi_1(M))$ lies in the finitely generated abelian group $\pi_1(F)$. Thus $\Phi(\pi_1(M))$ is a finitely generated normal subgroup of infinite index in $\pi_1(M)$. We conclude that (i) holds if $\Phi(\pi_1(M))$ is cyclic and that either (ii) or (iii) holds if $\Phi(\pi_1(M))$ is free abelian of rank 2 [8].

Case 3. F is a torus and F separates M . If $x\pi_1(F)x^{-1} \cap \pi_1(F) = 1$ for any x in $\pi_1(M)$, then it follows directly from Theorem 2.1 that $\Phi(\pi_1(M)) = 1$. Otherwise, Lemma 2.7 gives either (i) or (iii) [9].

Case 4. F is an annulus and F separates. Once again, if $x\pi_1(F)x^{-1} \cap \pi_1(F) = 1$ for any $x \in \pi_1(M)$, then $\Phi(\pi_1(M)) = 1$. Suppose then that $x\pi_1(F)x^{-1} \cap \pi_1(F) \neq 1$ for each $x \in \pi_1(M)$. Let t denote the generator of $\pi_1(F)$. Then for each x in $\pi_1(M)$ there are integers n_x, m_x such that $xt^{n_x}x^{-1} = t^{m_x}$. From [12] we get $n_x = \pm m_x$ for each x . Since $\pi_1(M)$ is finitely generated, there is an integer k such that $\langle t^k \rangle$ is normal in $\pi_1(M)$. Then [13] gives that M is a Seifert fibered space.

Case 5. F is a nonseparating, nonannular, bounded surface. According to our list of preferences we may assume in this case that M contains no essential torus or annulus. Theorem 2.2 gives $\Phi(\pi_1(M)) \subseteq \pi_1(F)$. Suppose there is an element x in $\Phi(\pi_1(M))$ that is not freely homotopic on F to any loop in $\text{bd } F$. Let t be a simple closed curve in M that meets F in a single transverse intersection point. Consider the set $K^+ = \{t^n x t^{-n} | n = 0, 1, 2, \dots\}$. Observe first of all that the normality of $\Phi(\pi_1(M))$ gives $K^+ \subset F$. There must be integers n and $k, n \neq k$, such that $t^n x t^{-n}$ is freely homotopic on F to $t^k x t^{-k}$. Otherwise, [6] gives an essential torus or annulus in M . This gives a singular map g of a torus into M . The map g cannot be essential. Otherwise, [27] gives an essential torus or annulus. Thus $t^n x t^{-n}$ must be homotopic on F to a loop in $\text{bd } F$.

A similar analysis on $K^- = \{t^{-m} x t^m | m = 0, 1, 2, \dots\}$ gives a positive integer m such that $t^{-m} x t^m$ is freely homotopic on F to a loop in $\text{bd } F$. Thus we get a singular map $g: S^1 \times [0, 1] \rightarrow M$ with $g(S^1 \times 0) = t^{-m} x t^m, g(S^1 \times \frac{1}{2}) = x$ and $g(S^1 \times 1) = t^n x t^{-n}$. The map g must be essential since x does not deform through F into $\text{bd } F$. An essential annulus in M is then given by [27].

It must be the case then that each element in $\Phi(\pi_1(M))$ is freely homotopic on F to a loop in $\text{bd } F$. Furthermore, there can be at most one such boundary component x involved. For if x and y are distinct curves in $\text{bd } F$ and if $x', y^s \in \Phi(\pi_1(M))$, then $x'y^s$ is an element of $\Phi(\pi_1(M))$ that is not freely homotopic on F to a loop in $\text{bd } F$. Thus each element of $\Phi(\pi_1(M))$ is conjugate in $\pi_1(F)$ to a power of x . Since F is not an annulus, $\pi_1(F)$ is free of rank at least two. Let $\langle x, y_1, y_2, \dots \rangle$ denote a set of free generators for

$\pi_1(F)$. If $x' \in \Phi(\pi_1(M))$, then

$$y_1 x' y_1^{-1} y_1^2 x' y_1^{-2} = y_1 x' y_1 x' y_1^{-2} \in \Phi(\pi_1(M)).$$

But this element cannot be conjugate to a power of x in the free group $\langle x, y_1, y_2, \dots \rangle$. We conclude that $\Phi(\pi_1(M)) = 1$.

Case 6. F is a closed 2-manifold which is not a torus. In this case it must be so that M is closed and contains no essential torus. If F does not separate, then a much simplified version of Case 5 gives the result. If F separates, then Lemma 2.4 gives an x in $\pi_1(M)$ such that $x\pi_1(F)x^{-1} \cap \pi_1(F) = 1$. We conclude from Theorem 2.1 that $\Phi(\pi_1(M)) = 1$. This completes the proof of Lemma 2.8.

At this point it seems appropriate to state the following theorem whose proof awaits the calculations made in §§4 and 5. The reader is asked to note that the conclusion of the theorem does indeed follow from Lemma 2.8 and the results in §§4 and 5. A separate proof will not be given.

THEOREM 2.9. *If M is a compact, orientable, irreducible, sufficiently large 3-manifold and if $\Phi(\pi_1(M)) \neq 1$, then M is a Seifert fibered space. Furthermore, $\Phi(\pi_1(M))$ is infinite cyclic.*

COROLLARY 2.10. *If M is a compact, orientable, irreducible 3-manifold and if M contains a nonannular, nonseparating, bounded incompressible surface F , then $\Phi(\pi_1(M)) = 1$.*

PROOF. Theorem 2.2 gives immediately that $\Phi(\pi_1(M)) \subseteq \pi_1(F)$. Then according to Theorem 2.9, $\Phi(\pi_1(M))$ is a cyclic normal subgroup of $\pi_1(F)$. Since $\pi_1(F)$ is free of rank at least two, it follows that $\Phi(\pi_1(M)) = 1$. This completes the proof.

If K is a knot in S^3 , then its complement $M = S^3 - K$ contains a spanning surface F . Thus $S^3 - K$ satisfies the hypotheses of Corollary 2.10, and we obtain as a special case of our work an answer to Neuwirth's question [18].

THEOREM 2.11. *If $G = \pi_1(S^3 - K)$ is the group of a knot K in S^3 , then $\Phi(G) = 1$.*

3. Group theoretic lemmas. We study here the Frattini subgroups of several seemingly artificial classes of groups. This has the advantage of avoiding the presentation of many calculations that are at worst superficially different.

LEMMA 3.1. *Let $G = A * B$ with $A \neq 1 \neq B$. If $1 \neq g \in G$ and if $G \neq Z_2 * Z_2$, then there is an x in G such that $x\langle g \rangle x^{-1} \cap \langle g \rangle = 1$.*

The proof is an easy exercise in the normal forms of elements of free products and is left to the reader.

LEMMA 3.2. *Let $G = *(A, B: H)$ with $A \neq H \neq B$ and H is cyclic. If A or B is a nontrivial free product other than $Z_2 * Z_2$, then $\Phi(G) = 1$.*

PROOF. According to Lemma 3.1 there is an $x \in G$ such that $xHx^{-1} \cap H = 1$. The conclusion is given by Theorem 2.1.

LEMMA 3.3. *Let G be a split extension of H by K where $H \cong Z \oplus Z$. If $\Phi(G) \subset H$, in particular, if $\Phi(K) = 1$, then $\Phi(G)$ is not free abelian of rank two.*

PROOF. Let $H = \langle x, y \mid [x, y] = 1 \rangle$ and suppose $\Phi(G) = \langle x^s y^t, x^u y^v \rangle$ is free abelian of rank two. Let $l = sv - tu$. Then $x^l = (x^s y^t)^v (x^u y^v)^{-t} \in \Phi(G)$, and $l \neq 0$ since $x^s y^t, x^u y^v$ are free generators. Also $y^l = (x^s y^t)^{-u} (x^u y^v)^s \in \Phi(G)$. Choose a prime p larger than $|l|$. Then $\langle x^l, y^l, x^p, y^p, K \rangle = G$, but $\langle x^p, y^p, K \rangle \neq G$ since $\langle x^p, y^p \rangle$ is a proper characteristic subgroup of $\langle x, y \rangle$.

LEMMA 3.4. *Let G be a group and let $S_i, 0 \leq i \leq n$, be simple groups. Let $h: G \rightarrow \bigoplus_{i=0}^n S_i$ be a homomorphism and let $p_j: \bigoplus_{i=0}^n S_i \rightarrow S_j$ be the natural projection. If $p_j \circ h$ is onto for each j , then $h(G)$ is isomorphic to $S_{i_1} \oplus \dots \oplus S_{i_k}$ for some $0 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n$.*

PROOF. The proof is by induction on n . For $n = 0$ there is nothing to prove. Suppose then that $n > 0$, and let $p: \bigoplus_{i=0}^n S_i \rightarrow \bigoplus_{i=1}^n S_i$ be the natural projection. By the inductive hypothesis there is an isomorphism f of $(p \circ h)(G)$ onto $S_{i_1} \oplus \dots \oplus S_{i_k}$ for some $1 \leq i_1 \leq \dots \leq i_k \leq n$. Let $H = \{x \in S_0 \mid (x, 1) \in h(G)\}$. Suppose $h(g) = (x, 1)$ and $s \in S_0$. By hypothesis there is an $s_1 \in \bigoplus_{i=1}^n S_i$ and a $g_1 \in G$ such that $h(g_1) = (s, s_1)$. Then $h(g_1 g g_1^{-1}) = (sxs^{-1}, 1)$, and so $sxs^{-1} \in H$. Thus H is a normal subgroup of S_0 , and so $H = 1$ or $H = S_0$. If $H = 1$, then $h(G) \cap \ker p = 1$. Thus $h(G) \cong (p \circ h)(G) \cong S_{i_1} \oplus \dots \oplus S_{i_k}$. If, on the other hand, $H = S_0$, it is not difficult to see that the map $F: h(G) \rightarrow S_0 \oplus S_{i_1} \oplus \dots \oplus S_{i_k}$ defined by $F(y) = (p_0(y), f \circ p(y))$ is an isomorphism.

LEMMA 3.5. *Let N be a finitely generated normal subgroup of the group G such that N is residually free of rank two. Then $N \cap \Phi(G) = 1$. Hence, if $\Phi(G/N) = 1$, then $\Phi(G) = 1$.*

PROOF. Suppose $1 \neq x \in N \cap \Phi(G)$. Since N is residually free, it is residually a finite nonabelian simple group (for example, $\text{PSL}(2, p)$ [14]). Let $f: N \rightarrow S_0$ be a map of N onto a finite nonabelian simple group such that $f(x) \neq 1$. Let K_0 be the kernel of f . Now let K_0, K_1, \dots, K_n be the distinct copies of K_0 which arise on mapping K_0 under all the automorphisms of N . This set of K_i is finite since N , being finitely generated, has at most finitely many subgroups of index $|N : K_0|$. Clearly, $K = \bigcap_{i=0}^n K_i$ is a characteristic subgroup of N and is of finite index in N . Furthermore, since we have the

mappings $N \rightarrow N/K \rightarrow N/K_i$, it follows from Lemma 3.4 that N/K is a direct sum of finitely many (isomorphic) finite simple groups.

Now K is characteristic in N and N is normal in G so K is normal in G . Thus we obtain a natural projection $f_1: G \rightarrow G/K$ such that $f_1(x) \in \Phi(G)K/K \subseteq N/K \cap \Phi(G/K)$. Since N/K is finite, the centralizer C/K of N/K in G/K is of finite index in G/K . Furthermore, $C/K \cap N/K = 1$ since N/K has trivial center. But C/K is normal in G/K (since N/K is normal in G/K) and so we may consider the projection $f_2: G/K \rightarrow G/K/C/K \cong G/C$. Clearly, G/C is a finite group and $f_2(f_1(x))$ is a nontrivial element of $\Phi(G/C)$ —a nilpotent group [22]. Thus $f_2(N/K)$ is a direct product of nonabelian simple groups whose intersection with the nilpotent normal subgroup $\Phi(G/C)$ is nontrivial. This manifest impossibility proves the lemma.

DEFINITION 3.6. Let $A(m : n)$ denote a group with presentation of the following form:

$$A(m : n) = \langle x_1, x_2, \dots, x_m, z \mid x_i z x_i^{-1} = z^{\pm 1}, w(x_1, \dots, x_m) = z^n \rangle.$$

We call $A(m : n)$ *acceptable* if the following properties hold.

- (i) $\langle z \rangle$ is infinite cyclic.
- (ii) $A(m : n)/\langle z \rangle = B(m)$ is Hopfian.
- (iii) $\Phi(B(m)) = 1$.

We say that an acceptable group $A(m : n)$ is *stable* if given any subgroup H of $A(m : n)$ that projects naturally onto $B(m)$, there are elements y_1, \dots, y_m of H such that $\langle y_1, \dots, y_m \rangle$ projects onto $B(m)$ and $w(y_1, \dots, y_m) = z^n$. An acceptable group $A(m : n)$ is *regularly unstable* if given any k there are elements y_1, \dots, y_m of $A(m : n)$ such that $\langle y_1, \dots, y_m \rangle$ projects onto $B(m)$ and $w(y_1, \dots, y_m) = z^{n+2k}$.

LEMMA 3.7. *If $A(m : n)$ is acceptable, then $\Phi(A(m : n)) \subseteq \langle z^{\varphi(n)} \rangle$.*

PROOF. First of all $\Phi(B(m)) = 1$, and so $\Phi(A(m : n)) \subseteq \langle z \rangle$. Suppose $\varphi(n)$ does not divide s . Then $\bar{z}^s \notin \Phi(Z_n)$. Thus there is an integer t such that $\langle \bar{z}^s, \bar{z}^t \rangle = Z_n$ but $\langle \bar{z}^t \rangle \neq Z_n$. Consider $S = \langle z^s, z^t, x_1, \dots, x_m \rangle$. Now $w(x_1, \dots, x_m) = z^n, z^s, z^t \in S$ and so $z \in S$. Hence

$$S = \langle z^s, z^t, x_1, \dots, x_m \rangle = A(m : n).$$

Let $R = \langle z^t, x_1, \dots, x_m \rangle$, and consider $R \cap \langle z \rangle$. Now each element of R can be written as $z^c x$ where $x \in \langle x_1, \dots, x_m \rangle$ and $c \in \mathbb{Z}$. Thus $R \cap \langle z \rangle \subseteq \langle z^t, z^n \rangle$. But $\langle z^t, z^n \rangle \neq \langle z \rangle$ since \bar{z}^t does not generate Z_n . It follows that $R \neq A(m : n)$ and so $z^s \notin \Phi(A(m : n))$.

LEMMA 3.8. *If $A(m : n)$ is acceptable and if y_1, \dots, y_m are elements of $A(m : n)$ such that $H = \langle y_1, \dots, y_m \rangle$ projects onto $B(m)$ and $w(y_1, \dots, y_m) = z^t$, then $H \cap \langle z \rangle = \langle z^t \rangle$.*

PROOF. It is clearly enough to show that $H \cap \langle z \rangle \subseteq \langle z^l \rangle$. Since $z^l \in H$, there is an inclusion induced map $i: H/\langle z^l \rangle \rightarrow H/H \cap \langle z \rangle$. In $H/\langle z^l \rangle$ the relation $w(\bar{y}_1, \dots, \bar{y}_m) = \bar{1}$ holds. Thus there is an epimorphism $e: B(m) \rightarrow H/\langle z^l \rangle$ defined by $e(\bar{x}_i) = \bar{y}_i$. Let $f: H/H \cap \langle z \rangle \rightarrow B(m)$ be the restriction of the natural projection of $A(m : n)$ onto $B(m)$. Finally, put $F = f \circ i \circ e: B(m) \rightarrow B(m)$. Now $F(\{\bar{x}_1, \dots, \bar{x}_m\}) = f(\{\bar{y}_1, \dots, \bar{y}_m\})$ is a generating set for $B(m)$, and so F is onto. Since $B(m)$ is Hopfian, F is an isomorphism. It follows that i is one-to-one and, consequently, that $H \cap \langle z \rangle = \langle z^l \rangle$.

LEMMA 3.9. *If $A(m : n)$ is regularly unstable, then $\Phi(A(m : n)) = 1$.*

PROOF. By Lemma 3.7 it suffices to show that for every nonzero integer d , $z^{d\varphi(n)} \notin \Phi(A(m : n))$. To this end choose c so that $2c + 1$ is a prime p larger than $d\varphi(n)$. Let y_1, \dots, y_m be elements of $A(m : n)$ that project onto $B(m)$ and $w(y_1, \dots, y_m) = z^{n+2cn}$. Then $\langle z^{d\varphi(n)}, z^p, y_1, \dots, y_m \rangle = A(m : n)$. Consider $R = \langle z^p, y_1, \dots, y_m \rangle$. Now each element of R can be written as $z^{sp}y$ where $y \in \langle y_1, \dots, y_m \rangle$. By Lemma 3.8, $\langle y_1, \dots, y_m \rangle \cap \langle z \rangle = \langle z^{n+2cn} \rangle$. It follows that each element of $R \cap \langle z \rangle$ is of the form $z^{sp}(z^{n+2cn})^t = (z^p)^{s+nt}$. Thus $R \neq A(m : n)$, and so $z^{d\varphi(n)} \notin \Phi(A(m : n))$.

LEMMA 3.10. *If $A(m : n)$ is stable, then $\Phi(A(m : n)) = \langle z^{\varphi(n)} \rangle$.*

PROOF. By Lemma 3.7 it suffices to show that $z^{\varphi(n)} \in \Phi(A(m : n))$. So suppose that $\langle z^{\varphi(n)}, y_1, \dots, y_r \rangle = A(m : n)$. Then $\langle y_1, \dots, y_r \rangle$ projects onto $B(m)$, and since $A(m : n)$ is stable $z^n \in \langle y_1, \dots, y_r \rangle$. Let $\langle y_1, \dots, y_r \rangle \cap \langle z \rangle = \langle z^k \rangle$. Then $\langle z^{\varphi(n)}, z^k \rangle = \langle z \rangle$. It follows that k is relatively prime to $\varphi(n)$ and so also to n . Hence $\langle z^n, z^k \rangle = \langle z \rangle$. We have then that $z \in \langle y_1, \dots, y_r \rangle$, and so $\langle y_1, \dots, y_r \rangle = A(m : n)$. Hence $z^{\varphi(n)} \in \Phi(A(m : n))$, and the proof of Lemma 3.10 is complete.

4. Seifert fibered spaces. This section begins in earnest the calculation of Frattini subgroups of 3-manifold groups. Consider first of all the planar discontinuous groups

$$\begin{aligned}
 &P(l, r, g: \alpha_1, \alpha_2, \dots, \alpha_r) \\
 &= \left\langle f_1, \dots, f_l, q_1, \dots, q_r, a_1, b_1, \dots, a_g, b_g \mid q_i^{\alpha_i} = 1, \right. \\
 &\quad \left. \left(\prod_{i=1}^l f_i \right) \left(\prod_{i=1}^r q_i \right) \left(\prod_{i=1}^g [a_i, b_i] \right) = 1 \right\rangle
 \end{aligned}$$

and

$$Q(l, r, g: \alpha_1, \alpha_2, \dots, \alpha_r) = \left\langle f_1, \dots, f_l, q_1, \dots, q_r, c_1, \dots, c_g \mid q_i^{\alpha_i} = 1, \left(\prod_{i=1}^l f_i \right) \left(\prod_{i=1}^r q_i \right) \left(\prod_{i=1}^g c_i^2 \right) = 1 \right\rangle.$$

We assume that $\alpha_i > 1$ for each i .

LEMMA 4.1. *Let $G = P(l, r, g: \alpha_1, \dots, \alpha_r)$ or $Q(l, r, g: \alpha_1, \dots, \alpha_r)$. If G is infinite, then $\Phi(G) = 1$.*

PROOF. The reader is asked to observe the following for infinite $G = P$ or Q .

- (i) $G(l, 1, g)$ and $G(0, 0, g)$ are one relator groups.
- (ii) $G(l > 1, 0, g)$ and $G(l > 1, r > 2, g)$ are nontrivial free products except for $G(2, 0, 0)$ and $Q(1, 0, 1)$ which are infinite cyclic.
- (iii) If G is $P(0, r > 2, g > 1)$, $Q(0, r > 2, g > 2)$, $Q(0, r > 3, 1)$, $Q(0, 2, 1: \text{some } \alpha_i > 2)$, $G(0, r > 5, 0)$ or $G(0, r > 4, 0: \text{some } \alpha_i > 2)$, then G is a nontrivial free product with amalgamation $*(A, B : H)$ where H is cyclic and A is a nontrivial free product other than $Z_2 * Z_2$.

We get $\Phi(G) = 1$ for (i) from [2], for (ii) from [10] and for (iii) from Lemma 3.2.

The remaining cases must be handled separately.

Case 1. $G = Q(0, 2, 1: 2, 2) = \langle q_1, q_2, c \mid q_1^2 = q_2^2 = q_1q_2c^2 = 1 \rangle$. For each odd prime p let $N(p) = \langle c^{2p} \rangle$. Observe that $N(p)$ is normal in G and that $G/N(p) = *(A, B : H)$ where

$$A = \langle \bar{q}_1, \bar{q}_2 \mid \bar{q}_1^2 = \bar{q}_2^2 = (\bar{q}_1\bar{q}_2)^p = \bar{1} \rangle, \quad B = \langle \bar{c} \mid \bar{c}^{2p} = \bar{1} \rangle \quad \text{and} \quad H = \langle \bar{c}^2 \rangle.$$

Now H is a finite cyclic normal subgroup of $G/N(p)$, and so

$$\Phi(G/N(p)) = \langle H \cap \Phi(A), H \cap \Phi(B) \rangle$$

[24]. But A is dihedral of order $2p$, and B is cyclic of order $2p$. Thus $\Phi(A) = \Phi(B) = 1$, and so $\Phi(G/N(p)) = 1$. It follows that $\Phi(G) \subseteq \bigcap_p N(p) = 1$.

Case 2. $G = P(0, 4, 0 : 2, 2, 2, 2) = \langle q_1, q_2, q_3, q_4 \mid q_1^2 = q_2^2 = q_3^2 = q_4^2 = q_1q_2q_3q_4 = 1 \rangle$. Observe that G is isomorphic to

$$\langle x, y, z \mid [x, y] = 1, zxz^{-1} = x^{-1}, yzy^{-1} = y^{-1}, z^2 = 1 \rangle$$

(by the mapping $x \mapsto q_2q_3, y \mapsto q_3q_1, z \mapsto q_3$). Now $G/\langle x \rangle \cong Z_2 * Z_2$ and $G/\langle y \rangle \cong Z_2 * Z_2$. Thus $\Phi(G) \subseteq \langle x \rangle \cap \langle y \rangle = 1$.

Case 3. $G = P(0, 3, 0 : \alpha_1, \alpha_2, \alpha_3) = \langle q_1, q_2, q_3 | q_1^{\alpha_1} = q_2^{\alpha_2} = q_3^{\alpha_3} = q_1q_2q_3 = 1 \rangle$ with $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 < 1$. Then G has a normal subgroup N of finite index such that N is isomorphic to the fundamental group of a compact, orientable 2-manifold of genus ≥ 2 [11]. According to [3], N is residually free. Lemma 3.5 then gives $N \cap \Phi(G) = 1$. Since N is a normal subgroup of finite index, it follows that $\Phi(G)$ is finite. If $\Phi(G) \neq 1$, then by [11] $\Phi(G)$ is cyclic and $\Phi(G) = \langle wq_i^n w^{-1} \rangle$ for some $w \in G$ and some i . Now $q_i^n, q_j q_i^n q_j^{-1} \in \Phi(G)$ since $\Phi(G)$ is normal. Thus $q_i^n = wq_i^{ns} w^{-1}, q_j q_i^n q_j^{-1} = wq_i^{nt} w^{-1}$ for some $s, t \in \mathbb{Z}$. Hence, $q_j q_i^{ns} q_j^{-1} = q_i^{nt}$ and $q_i^{nt} \neq 1$ since $q_i^n \neq 1$, and so $\langle q_i \rangle \cap q_j \langle q_i \rangle q_j^{-1} \neq 1$. By [11] $q_j \in \langle q_i \rangle$. It follows that G is a finite cyclic group which is a contradiction.

Case 4. $G = P(0, 3, 0 : \alpha_1, \alpha_2, \alpha_3)$ with $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 = 1$. In this case $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 3), (4, 4, 2)$ or $(6, 3, 2)$, and G has a normal subgroup of finite index which is free abelian of rank two. In particular,

$$G = \langle q_1, q_2, q_3 | q_1^6 = q_2^3 = q_3^2 = q_1q_2q_3 = 1 \rangle$$

is isomorphic to

$$\langle x, y, z | zxz^{-1} = y, yz^{-1} = x^{-1}y, [x, y] = 1, z^6 = 1 \rangle$$

by the mapping $x \mapsto q_1^{-2}q_2, y \mapsto q_1^{-1}q_2q_1^{-1}, z \mapsto q_1$. We have immediately that $\Phi(G) \subseteq \langle x, y \rangle$ and by Lemma 3.3 that $\Phi(G)$ is not free abelian of rank two. So suppose that $\Phi(G)$ is infinite cyclic. Let $\Phi(G) = \langle x^u y^v \rangle$; then $(x^u y^v)^{\pm 1} = zx^u y^v z^{-1} = x^{-u} y^{u+v}$. Hence $u = \mp v$ and $v = \pm(u + v)$ which implies that $u = v = 0$. Thus $\Phi(G) = 1$. Similarly, if

$$G = \langle q_1, q_2, q_3 | q_1^3 = q_2^3 = q_3^3 = q_1q_2q_3 = 1 \rangle,$$

which is isomorphic to

$$\langle x, y, z | zxz^{-1} = y, yz^{-1} = x^{-1}y^{-1}, [x, y] = 1, z^3 = 1 \rangle$$

by the mapping $z \mapsto q_1, x \mapsto q_1^{-1}q_2, y \mapsto q_2q_1^{-1}$, it can be shown that $\Phi(G) = 1$. Finally, if

$$G = \langle q_1, q_2, q_3 | q_1^4 = q_2^4 = q_3^2 = q_1q_2q_3 = 1 \rangle,$$

then G is isomorphic to

$$\langle x, y, z | zxz^{-1} = y, yz^{-1} = x^{-1}, [x, y] = 1, z^4 = 1 \rangle$$

by the mapping $z \mapsto q_2, x \mapsto q_1q_2^{-1}, y \mapsto q_2q_1q_2^{-2}$. Now

$$G/\langle x, y \rangle = \langle \bar{z} | \bar{z}^4 = \bar{1} \rangle \cong Z_4 \quad \text{and} \quad \Phi(Z_4) = \langle \bar{z}^2 | (\bar{z}^2)^2 = \bar{1} \rangle$$

so $\Phi(G) \subseteq \langle x, y, z^2 \rangle$. Suppose that $x^u y^v z^2 \in \Phi(G)$. Then

$$x(x^u y^v z^2)x^{-1} = x^{u+2} y^v z^2, \quad y(x^u y^v z^2)y^{-1} = x^u y^{v+2} z^2 \in \Phi(G)$$

and so

$$x^2 = (x^{u+2}y^{vz^2})(x^uy^vz^2)^{-1}, \quad y^2 = (x^uy^{v+2z^2})(x^uy^vz^2)^{-1} \in \Phi(G).$$

Let p be an odd prime. Then $\langle x^2, y^2, x^p, z \rangle = G$, but $\langle x^p, z \rangle \neq G$ since $\langle x^p, y^p \rangle$ is a proper characteristic subgroup of $\langle x, y \rangle$. Thus $\Phi(G) \subseteq \langle x, y \rangle$. Now by Lemma 3.3, $\Phi(G)$ is not free abelian of rank two. Suppose $x^uy^v \in \Phi(G)$. Then $x^{-v}y^u = z(x^uy^v)z^{-1} \in \Phi(G)$, and so

$$x^{u^2+v^2} = (x^uy^v)^u(x^{-v}y^u)^{-v}, \quad y^{u^2+v^2} = (x^uy^v)^v(x^{-v}y^u)^u \in \Phi(G).$$

It follows that $u = v = 0$ so that $\Phi(G) = 1$ as required. This completes the proof of Lemma 4.1.

Now if M is a Seifert fibered space, then there are three possible presentations for $\pi_1(M)$ [20]. For relatively prime integers α_i, β_i we have:

(4.2) M has nonempty boundary.

$$\begin{aligned} \pi_1(M) = \langle x_1, \dots, x_k, q_1, \dots, q_r, h | x_i h x_i^{-1} = h^{\pm 1}, \\ q_i h q_i^{-1} = h, q_i^{\alpha_i} h^{\beta_i} = 1 \rangle. \end{aligned}$$

(4.3) M is closed with orientable Seifert surface.

$$\begin{aligned} \pi_1(M) = O(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r) \\ = \left\langle a_1, b_1, \dots, a_g, b_g, q_1, \dots, q_r, h | a_i h a_i^{-1} = h^{\pm 1}, b_i h b_i^{-1} = h^{\pm 1}, \right. \\ \left. q_i h q_i^{-1} = h, q_i^{\alpha_i} h^{\beta_i} = 1, \left(\prod_{i=1}^r q_i \right) \left(\prod_{i=1}^g [a_i, b_i] \right) = h^b \right\rangle. \end{aligned}$$

(4.4) M is closed with nonorientable Seifert surface.

$$\begin{aligned} \pi_1(M) = N(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r) \\ = \left\langle c_1, \dots, c_g, q_1, \dots, q_r, h | c_i h c_i^{-1} = h^{\pm 1}, \right. \\ \left. q_i h q_i^{-1} = h, q_i^{\alpha_i} h^{\beta_i} = 1, \left(\prod_{i=1}^r q_i \right) \left(\prod_{i=1}^g c_i^2 \right) = h^b \right\rangle. \end{aligned}$$

With each of these presentations, we associate the following integers.

$$\begin{aligned} (4.5) \quad \hat{\alpha} = \alpha_1 \alpha_2 \cdots \alpha_r, \hat{\alpha}_i = \beta_i \hat{\alpha} / \alpha_i, \rho = \gcd(\hat{\alpha}, \hat{\alpha}_1, \dots, \hat{\alpha}_r), \\ \xi = b \hat{\alpha} + \hat{\alpha}_1 + \cdots + \hat{\alpha}_r, \xi^* = \xi / \rho, \lambda = \varphi(\xi^*) / \gcd(\hat{\alpha} / \rho, \varphi(\xi^*)). \end{aligned}$$

Observe that in each of the presentations (4.2)–(4.4), $\langle h \rangle$ is a normal subgroup. Furthermore, $\pi_1(M) / \langle h \rangle$ is one of the groups $P(l, r, g), Q(l, r, g)$. If $\pi_1(M)$ is infinite, it is torsion free [5]. Thus, if $\pi_1(M) / \langle h \rangle$ is finite, then $\pi_1(M)$ is infinite cyclic [8]. If $\pi_1(M) / \langle h \rangle$ is infinite, we have from Lemma 4.1

that $\Phi(\pi_1(M)/\langle h \rangle) = 1$. In either case we get

LEMMA 4.6. *If M is Seifert fibered with infinite fundamental group, then $\Phi(\pi_1(M)) \subseteq \langle h \rangle$.*

THEOREM 4.7. *If M is Seifert fibered and if $\text{bd } M \neq \emptyset$, then $\Phi(\pi_1(M)) = 1$.*

PROOF. Note first that for the infinite dihedral group $D_\infty = \langle w, z \mid zwz^{-1} = w^{-1}, z^2 = 1 \rangle$ we have $\Phi(D_\infty) = 1$ since $D_\infty \cong Z_2 * Z_2$. Now in the presentation (4.2) for $\pi_1(M)$ if $r = 0$, define a map $f: \pi_1(M) \rightarrow D_\infty$ by $f(x_i) = w$ if $x_ihx_i^{-1} = h$, $f(x_i) = z$ if $x_ihx_i^{-1} = h^{-1}$, and $f(h) = w$. If $r > 0$, define $f: \pi_1(M) \rightarrow D_\infty$ by $f(x_i) = w$ if $x_ihx_i^{-1} = h$, $f(x_i) = z$ if $x_ihx_i^{-1} = h^{-1}$, $f(q_i) = w^{-\hat{\alpha}_i/\rho}$ and $f(h) = w^{\hat{\alpha}/\rho}$. (See 4.5.) Now $\{\hat{\alpha}/\rho, \hat{\alpha}_1/\rho, \dots, \hat{\alpha}_r/\rho\}$ consists of relatively prime integers. So the image of f is either D_∞ or an infinite cyclic subgroup of D_∞ . In either case $\Phi(\pi_1(M)) \subseteq \ker f$. But $\langle h \rangle \cap \ker f = 1$. Lemma 4.6 now gives the desired conclusion.

Next we consider 3-manifolds which are Seifert fibered over an orientable surface other than the 2-sphere. So we assume $g \geq 1, r \geq 0$ in the presentation (4.3) for $\pi_1(M)$.

THEOREM 4.8. *If $\pi_1(M) = O(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$ is infinite and if*

- (i) $a_iha_i^{-1} = h^{-1}$ or $b_ihb_i^{-1} = h^{-1}$ for some i , then $\Phi(\pi_1(M)) = 1$, whereas if
- (ii) $a_iha_i^{-1} = h$ and $b_ihb_i^{-1} = h$ for each i , then $\Phi(\pi_1(M)) = \langle h^\lambda \rangle$, where $\lambda = \varphi(\xi^*)/\text{gcd}(\hat{\alpha}/\rho, \varphi(\xi^*))$.

The proof proceeds in several steps. First we consider the case $r = 0$.

LEMMA 4.9. $\Phi(O(0, g : b)) = \langle h^{\varphi(b)} \rangle$ if h is central in $O(0, g : b)$, and $\Phi(O(0, g : b)) = 1$ otherwise.

PROOF. Observe that as a consequence of Lemma 4.1, $O(0, g : b)$ is an acceptable group since $\langle h \rangle$ is infinite cyclic and $P(0, 0, g)$ is Hopfian [3].

Case 1. h is central in $O(0, g : b)$. Let S be a subgroup of $O(0, g : b)$ that projects naturally onto $O(0, g : b)/\langle h \rangle$. Then there are integers $k_1, \dots, k_g, l_1, \dots, l_g$ such that $a_ih^{k_i}, b_ih^{l_i} \in S$. Since h is central,

$$\prod_{i=1}^g [a_ih^{k_i}, b_ih^{l_i}] = \prod_{i=1}^g [a_i, b_i] = h^b.$$

It follows that $O(0, g : b)$ is stable, and, consequently from Lemma 3.10, that $\Phi(O(0, g : b)) = \langle h^{\varphi(b)} \rangle$.

Case 2. h is not central in $O(0, g : b)$. Let k be a positive integer. Observe the following facts.

- (i) If $a_iha_i^{-1} = h^{-1}$ and $b_ihb_i^{-1} = h$, then $[a_i, h^{-k}b_i] = h^{2k}[a_i, b_i]$.
- (ii) If $a_iha_i^{-1} = h$ and $b_ihb_i^{-1} = h^{-1}$, then $[h^ka_i, b_i] = h^{2k}[a_i, b_i]$.
- (iii) If $a_iha_i^{-1} = h^{-1}$ and $b_ihb_i^{-1} = h^{-1}$, then $[h^ka_i, b_i] = h^{2k}[a_i, b_i]$.

(iv) For each i , $[[a_i, b_i], h] = 1$.

It is clear from (i) through (iv) that it is possible to choose elements $y_1, \dots, y_g, z_1, \dots, z_g$ of $O(0, g : b)$ so that $\langle y_1, \dots, y_g, z_1, \dots, z_g \rangle$ projects onto $O(0, g : b)/\langle h \rangle$ and that

$$\prod_{i=1}^g [y_i, z_i] = h^{2k} \prod_{i=1}^g [a_i, b_i] = h^{b+2k}.$$

Thus $O(0, g : b)$ is regularly unstable, and so the result follows from Lemma 3.9.

To complete the proof of Theorem 4.8 we consider the groups $O(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$ with $r > 1$ which we abbreviate by $O(r, g : b)$. Define a map $f: O(r, g : b) \rightarrow O(0, g : \xi^*)$ by $f(a_i) = a_i, f(b_i) = b_i, f(q_i) = h^{-\hat{\alpha}_i/\rho}, f(h) = h^{\hat{\alpha}/\rho}$. Since $\{\hat{\alpha}/\rho, \hat{\alpha}_1/\rho, \dots, \hat{\alpha}_r/\rho\}$ consists of relatively prime integers, f is onto. If h is not central in $O(r, g : b)$, then we have from Lemma 4.9 that $\Phi(O(0, g : \xi^*)) = 1$ and so $\Phi(O(r, g : b)) \subseteq \ker f$. But $(\ker f) \cap \langle h \rangle = 1$, and so $\Phi(O(r, g : b)) = 1$.

If h is central in $O(r, g : b)$, then Lemma 4.9 gives that

$$\Phi(O(r, g : b)) \subseteq \langle h \rangle \cap f^{-1}(\langle h^{\varphi(\xi^*)} \rangle) = \langle h^\lambda \rangle.$$

It remains to show then that $h^\lambda \in \Phi(O(r, g : b))$. So suppose that $\langle h^\lambda, z_1, \dots, z_n \rangle = O(r, g : b)$. Let $R = \langle z_1, \dots, z_n \rangle$, and let $R \cap \langle h \rangle = \langle h^u \rangle$. Now R projects onto $O(r, g : b)/\langle h \rangle$, and hence there are integers k_i, l_i, m_i such that $a_i h^{k_i}, b_i h^{l_i}, q_i h^{m_i} \in R$. Now

$$h^b \left(\prod_{i=1}^r q_i \right)^{-1} = \prod_{i=1}^g [a_i, b_i] = \prod_{i=1}^g [a_i h^{k_i}, b_i h^{l_i}] \in R$$

and so

$$h^{b+\sum_{i=1}^r m_i} = h^b \left(\prod_{i=1}^r q_i \right)^{-1} \cdot \left(\prod_{i=1}^r q_i h^{m_i} \right) \in R.$$

Also from $q_i^\alpha = h^{-\beta_i}$ we see that $h^{\alpha_i m_i - \beta_i} = (q_i h^{m_i})^{\alpha_i} \in R$. Thus $b + \sum_{i=1}^r m_i \equiv 0 \pmod u$ and $\alpha_i m_i - \beta_i \equiv 0 \pmod u$. From these congruences it can be shown that $\xi \equiv 0 \pmod u$.

Suppose now that p is a prime that divides both $\hat{\alpha}$ and u . Then p divides α_i for some i . But $\alpha_i m_i - \beta_i \equiv 0 \pmod u$, and so p divides β_i . This contradicts the fact that α_i and β_i are relatively prime, and so we conclude that $\gcd(\hat{\alpha}, u) = 1$. It follows that $\gcd(\rho, u) = 1$ and hence that $\xi^* \equiv 0 \pmod u$.

Since h is central in $O(r, g : b)$, each element of $\langle h^\lambda, R \rangle = O(r, g : b)$ can be written as $h^{c\lambda} w$ where $w \in R$ and $c \in \mathbb{Z}$. In particular, it follows that $\langle h \rangle = \langle h^\lambda, h^u \rangle$ and, consequently, that $\gcd(\lambda, u) = 1$.

Now if p is a prime that divides u , then p also divides ξ^* and so also $\varphi(\xi^*)$.

Since $\gcd(\lambda, u) = 1$ and $\lambda = \varphi(\xi^*)/\gcd(\hat{\alpha}/\rho, \varphi(\xi^*))$, p must divide $\gcd(\hat{\alpha}/\rho, \varphi(\xi^*))$ and, hence, $\hat{\alpha}$. But this contradicts the fact that $\gcd(\hat{\alpha}, u) = 1$. Thus we must have that $u = 1$ and, consequently, that $R = O(r, g : b)$. It follows that $h^\lambda \in \Phi(O(r, g : b))$, which completes the proof of Theorem 4.8.

Now for 3-manifolds M which are Seifert fibered over the 2-sphere, h is central in $\pi_1(M)$, and we have

THEOREM 4.10. *If $\pi_1(M) = O(r, 0 : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$ is infinite, then $\Phi(\pi_1(M)) = \langle h^\lambda \rangle$.*

SKETCH OF PROOF. Again Lemma 4.6 gives that $\Phi(O(r, 0 : b)) \subseteq \langle h \rangle$. Define a map $f: O(r, 0 : b) \rightarrow O(0, 0 : \xi^*) = Z_{\xi^*}$ by $f(q_i) = h^{-\hat{\alpha}/\rho}$, $f(h) = h^{\hat{\alpha}/\rho}$. Then

$$\Phi(O(r, 0 : b)) \subseteq \langle h \rangle \cap f^{-1}(\langle h^{\varphi(\xi^*)} \rangle) = \langle h^\lambda \rangle.$$

Thus we need only show that $h^\lambda \in \Phi(O(r, 0 : b))$. This can be done as in the proof of Theorem 4.8.

Finally, if M is Seifert fibered over a nonorientable surface with $g > 1$ crosscaps and $r \geq 0$, then we have

THEOREM 4.11. *If $\pi_1(M) = N(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$ is infinite and if*

(i) $c_i hc_i^{-1} = h$ for some i , then $\Phi(\pi_1(M)) = 1$,

whereas if

(ii) $c_i hc_i^{-1} = h^{-1}$ for each i , then $\Phi(\pi_1(M)) = \langle h^\lambda \rangle$.

The proof involves the same techniques used in proving Theorem 4.8 and is omitted.

REMARK 4.12. It is interesting to note for the Seifert fibered spaces M with fundamental groups $O(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$ that $\Phi(\pi_1(M)) \neq 1$ if and only if $Z(\pi_1(M)) \neq 1$ and that $\Phi(\pi_1(M)) \subseteq Z(\pi_1(M))$. However, for the groups $\pi_1(M) = N(r, g : b, \alpha_1, \beta_1, \dots, \alpha_r, \beta_r)$, $\Phi(\pi_1(M)) \neq 1$ in some cases but $Z(\pi_1(M)) = 1$. On the other hand, for the Seifert fibered spaces M with boundary, $\Phi(\pi_1(M)) = 1$ but $Z(\pi_1(M))$ may be nontrivial.

5. Surface bundles over S^1 and double twisted I -bundles. If M is a surface bundle over S^1 , then $\pi_1(M)$ has a normal subgroup H isomorphic to the fundamental group of a compact, orientable 2-manifold and $\pi_1(M)/H \cong Z$. If M is a double twisted I -bundle, H is again a normal subgroup of $\pi_1(M)$ and $\pi_1(M)/H \cong Z_2 * Z_2$.

THEOREM 5.1. *If M is a bundle over S^1 with fiber a compact, orientable 2-manifold other than a torus, or if M is a double twisted I -bundle over a closed, nonorientable 2-manifold other than a Klein bottle, then $\Phi(\pi_1(M)) = 1$.*

PROOF. If M is either a sphere bundle over S^1 or a disk bundle over S^1 , then $\pi_1(M) = Z$, and if M is an annulus bundle over S^1 , then $\pi_1(M)$ is either $Z \oplus Z$ or $\langle x, y | yxy^{-1} = x^{-1} \rangle$. If M is a double twisted I -bundle over a projective plane, then $\pi_1(M) = Z_2 * Z_2$. Since these groups all have trivial Frattini subgroups, we proceed assuming that H is either free of rank greater than one or the fundamental group of a closed, orientable 2-manifold of genus greater than one. According to [3], H is residually free of rank two. Furthermore, $1 = \Phi(Z) = \Phi(Z_2 * Z_2) = \Phi(\pi_1(M)/H)$. Hence from Lemma 3.5 we conclude that $\Phi(\pi_1(M)) = 1$ which completes the proof of Theorem 5.1.

Now suppose M is a torus bundle over S^1 . Then

$$\pi_1(M) = \langle x, y, z | [x, y] = 1, zxz^{-1} = x^p y^q, zyz^{-1} = x^r y^s \rangle$$

where $ps - qr = \pm 1$.

LEMMA 5.2. Let $A \in GL_2(Z)$ with characteristic polynomial $p(\lambda) = (\lambda - a)(\lambda - c)$ where a and c are integers. If $a \neq c$, then A is similar over Z to $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. If $a = c$, then A is similar over Z to $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for some integer b .

PROOF. If $a \neq c$, then A is similar over Q to $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ and there are eigenvectors $x, y \in Q \oplus Q$ with $Ax = ax$, $Ay = cy$. Choose $x_1, y_1 \in Z \oplus Z$ where x_1, y_1 are nonzero integral multiples of x and y . Then $Ax_1 = ax_1$, $Ay_1 = cy_1$. Let $x_1 = kx_2$ where $k \in Z$ and x_2 is primitive in $Z \oplus Z$ (that is, $x_2 \neq k'x'_2$ for any $x'_2 \in Z \oplus Z$, $k' \in Z$). Then $Akx_2 = Ax_1 = ax_1 = akx_2$, and so $Ax_2 = ax_2$. Similarly, there exists $y_2 \in Z \oplus Z$ such that y_2 is primitive and $Ay_2 = cy_2$. Then $\{x_2, y_2\}$ is a basis for $Z \oplus Z$ with $Ax_2 = ax_2$, $Ay_2 = cy_2$, and so A is similar over Z to $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$.

If $a = c$, then A is similar over Q to $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ or to $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. The first case is handled just as above. In the second case we choose $x_1, y_1 \in Z \oplus Z$ with $Ax_1 = ax_1 + y_1$, $Ay_1 = ay_1$. Suppose that $x_1 = kx_2$ where x_2 is a primitive in $Z \oplus Z$ and $k \in Z$. Then $Akx_2 = Ax_1 = ax_1 + y_1 = akx_2 + y_1$, hence $k(Ax_2 - ax_2) = y_1$. Put $y_2 = Ax_2 - ax_2$. Then $Ax_2 = ax_2 + y_2$. Also $ky_2 = y_1$. Hence $Aky_2 = Ay_1 = ay_1 = ak y_2$, and so $Ay_2 = ay_2$. Now let $y_2 = by_3$ where y_3 is a primitive element of $Z \oplus Z$ and $b \in Z$. Then $Ax_2 = ax_2 + y_2 = ax_2 + by_3$ and $Ay_3 = ay_3$. Furthermore, $\{x_2, y_3\}$ is a basis for $Z \oplus Z$, and so A is similar over Z to $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

THEOREM 5.3. Let M be a torus bundle over S^1 .

(i) If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is similar over Z to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \neq 0$, then $\pi_1(M) \cong \langle X, Y, Z | [X, Y] = 1, ZXZ^{-1} = XY^b, ZYZ^{-1} = Y \rangle$ and $\Phi(\pi_1(M)) = \langle Y^{\Phi(b)} \rangle$.

(ii) If $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is similar over Z to $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ with $b \neq 0$, then $\pi_1(M) \cong \langle X, Y, Z | [X, Y] = 1, ZXZ^{-1} = X^{-1}Y^b, ZYZ^{-1} = Y^{-1} \rangle$ and $\Phi(\pi_1(M)) = \langle Y^{\Phi(b)} \rangle$.

(iii) In all other cases $\Phi(\pi_1(M)) = 1$.

PROOF. It is clear that $\Phi(\pi_1(M)) \subseteq \langle x, y \rangle$ and by Lemma 3.3 that $\Phi(\pi_1(M))$ is not free abelian of rank two. So suppose that $\Phi(\pi_1(M))$ is infinite cyclic. Let $\Phi(\pi_1(M)) = \langle x^u y^v \rangle$. Since $\Phi(\pi_1(M))$ is normal, we get

$$(x^u y^v)^{\pm 1} = z(x^u y^v)z^{-1} = x^{pu+rv} y^{qu+sv}.$$

So $\pm u = pu + rv$ and $\pm v = qu + sv$. Hence

$$0 = uv - vu = (pu + rv)v - (qu + sv)u = rv^2 + (p - s)uv - qu^2.$$

Now at least one of u, v is nonzero since $x^u y^v \neq 1$. Suppose without loss of generality that $v \neq 0$. If $r \neq 0$, then

$$v = \frac{(s - p)u \pm u\sqrt{(p - s)^2 + 4qr}}{2r}.$$

Since v is an integer,

$$(p - s)^2 + 4qr = (p + s)^2 - 4(ps - qr) = (p + s)^2 \mp 4$$

must be a perfect square. We conclude that $p + s = \pm 2$ and $ps - qr = 1$ or $p + s = 0$ and $ps - qr = -1$. Thus the characteristic polynomial $P(\lambda)$ of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is $(\lambda \pm 1)^2$ or $\lambda^2 - 1$. If $r = 0$, we immediately have this result.

Case 1. $P(\lambda) = \lambda^2 - 2\lambda + 1$. Then Lemma 5.2 gives $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ similar to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ where $b \in Z$. Thus a presentation for $\pi_1(M)$ is

$$\begin{aligned} \langle X, Y, Z | [X, Y] = 1, ZXZ^{-1} = XY^b, ZYZ^{-1} = Y \rangle \\ = \langle X, Y, Z | XYX^{-1} = Y, ZYZ^{-1} = Y, [Z, X] = Y^b \rangle = O(0, 1 : b) \end{aligned}$$

(see (4.3)). Since Y is central, we get from Theorem 4.8 that $\Phi(\pi_1(M)) = \langle Y^{\varphi(b)} \rangle$.

Case 2. $P(\lambda) = \lambda^2 + 2\lambda + 1$. Then $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is similar to $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, and so $\pi_1(M)$ has the presentation

$$\langle X, Y, Z | [X, Y] = 1, ZXZ^{-1} = X^{-1}Y^b, ZYZ^{-1} = Y^{-1} \rangle.$$

Putting $c_1 = XZ, c_2 = Z^{-1}, h = Y$ we have that

$$\pi_1(M) \cong \langle c_1, c_2, h | c_1 h c_1^{-1} = h^{-1}, c_2 h c_2^{-1} = h^{-1}, c_1^2 c_2^2 = h^b \rangle = N(0, 2 : b).$$

Thus from Theorem 4.11 we get $\Phi(\pi_1(M)) = \langle Y^{\varphi(b)} \rangle$.

Case 3. $P(\lambda) = \lambda^2 - 1$. By Lemma 5.2, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus $\pi_1(M) \cong \langle X, Y, Z | [X, Y] = 1, ZXZ^{-1} = X, ZYZ^{-1} = Y^{-1} \rangle \cong O(0, 1 : 0)$, and so $\Phi(\pi_1(M)) = 1$ by Theorem 4.8. This completes the proof of Theorem 5.3.

Now suppose M is a double twisted I -bundle over the Klein bottle. Then

$$\pi_1(M) = \langle a, b, x, y | bab^{-1} = a^{-1}, yxy^{-1} = x^{-1}, a = x^p y^{2q}, b^2 = x^r y^{2s} \rangle$$

where $ps - rq = \pm 1$ [7].

THEOREM 5.4. *If M is a double twisted I -bundle over the Klein bottle, then*

- (i) $\Phi(\pi_1(M)) = \langle b^{\varphi(2p)} \rangle$ if $s = 0$,
- (ii) $\Phi(\pi_1(M)) = \langle x^{\varphi(r)} \rangle$ if $q = 0$,
- (iii) $\Phi(\pi_1(M)) = \langle y^{\varphi(2q)} \rangle$ if $r = 0$,
- (iv) $\Phi(\pi_1(M)) = \langle y^{\varphi(2s)} \rangle$ if $p = 0$,
- (v) $\Phi(\pi_1(M)) = 1$ if none of p, q, r, s is equal to zero.

PROOF. Let $H = \langle a, b^2 \rangle$; then we also have $H = \langle x, y^2 \rangle$ since $x = (a^s b^{-2q})^{\pm 1}$, $y^2 = (a^{-r} b^{2p})^{\pm 1}$. Now H is normal in $\pi_1(M)$, and $\pi_1(M)/H \cong Z_2 * Z_2$. It follows that $\Phi(\pi_1(M)) \subseteq H$. Now $\Phi(\pi_1(M))$ cannot be free abelian of rank two. For suppose $\Phi(\pi_1(M)) = \langle x^c y^{2d}, x^u y^{2v} \rangle$. Let $l = cv - du$. Then

$$x^l = (x^c y^{2d})^v (x^u y^{2v})^{-d} \in \Phi(\pi_1(M)),$$

and so $l \neq 0$ since $x^c y^{2d}, x^u y^{2v}$ are free generators. Also,

$$y^{2l} = (x^c y^{2d})^{-u} (x^u y^{2v})^c \in \Phi(\pi_1(M)).$$

Let m be a prime larger than $|2l|$. Then $\langle b^m, x^m, y^m, x^l, y^{2l} \rangle = \pi_1(M)$ since it contains x and y , hence also $a = x^p y^{2q}$ and $b^2 = x^r y^{2s}$, hence also b . Let $R = \langle b^m, x^m, y^m \rangle$. Every element of R is of the form $b^{m\alpha} y^{m\beta} x^{m\gamma} \cdot \dots \cdot x^{m\eta}$. Choose coset representatives of $\langle a, b \rangle \text{ mod } H$ to be $1, b^m$ and of $\langle x, y \rangle \text{ mod } H$ to be $1, y^m$. Now $(b^m)^{2k}$ is of the form $B = (x^m)^{rk} (y^{2m})^{sk} \in H$ and, similarly, $(y^m)^{2k}$ is of the form $Y = (a^m)^{-rk} (b^{2m})^{pk} \in H$. Conjugation of B by y^m , or Y by b^m , simply changes x to x^{-1} , respectively, a to a^{-1} . Thus every element of R can be expressed in the form $(b^m) y^m \cdot \dots \cdot b^m (y^m) w$ where $w \in \langle x^m, y^{2m} \rangle$. By our choice of coset representatives we see that R cannot include x . Hence $R = \langle b^m, x^m, y^m \rangle \neq \pi_1(M)$ which contradicts the fact that $x^l, y^{2l} \in \Phi(\pi_1(M))$.

So suppose now that $\Phi(\pi_1(M))$ is infinite cyclic. Let $\Phi(\pi_1(M)) = \langle x^u y^{2v} \rangle$. Then since $\Phi(\pi_1(M))$ is normal,

$$y(x^u y^{2v})y^{-1} = x^{-u} y^{2v} \in \Phi(\pi_1(M)).$$

Thus

$$x^{2u} = (x^u y^{2v})(x^{-u} y^{2v})^{-1} \in \Phi(\pi_1(M)),$$

and

$$y^{4v} = x^{-2u} (x^u y^{2v})^2 \in \Phi(\pi_1(M)).$$

Since $\Phi(\pi_1(M))$ is infinite cyclic, precisely one of u and v must be zero.

Case 1. $u \neq 0, v = 0$. Then

$$x^u = (a^{su} b^{-2qu})^{\pm 1} \in \Phi(\pi_1(M)),$$

and

$$bx^ub^{-1} = (a^{-su}b^{-2qu})^{\pm 1} \in \Phi(\pi_1(M)).$$

Hence $a^{2su} \in \Phi(\pi_1(M))$, and $b^{4qu} \in \Phi(\pi_1(M))$. Thus we get $s = 0$ or $q = 0$ but not both since $x^u \neq 1$.

If $s = 0$, then

$$\pi_1(M) = \langle a, b, x, y | bab^{-1} = a^{-1}, yxy^{-1} = x^{-1}, a = x^p y^{2q}, b^2 = x^r \rangle$$

where $q = \pm 1, r = \pm 1$. Eliminating a and x from the presentation we get

$$\pi_1(M) = \langle b, y | (y^{2qb})^2 = b^{-4pr+2}, yb^2y^{-1} = b^{-2} \rangle.$$

Then $\pi_1(M)$ is isomorphic to

$$N(2, 1 : rp - 1, (2, 1), (2, 1))$$

$$= \langle q_1, q_2, c_1, h | q_i h q_i^{-1} = h, c_1 h c_1^{-1} = h^{-1}, q_i^2 h = 1, q_1 q_2 c_1^2 = h^{rp-1} \rangle$$

by the mapping $h \mapsto b^2, q_1 \mapsto y^{2qb}^{2rp-1}, q_2 \mapsto b^{-1}, c_1 \mapsto y^{-q}$. Thus by Theorem 4.11,

$$\Phi(\pi_1(M)) = \langle h^{\varphi(2p)/2} \rangle = \langle b^{\varphi(2p)} \rangle.$$

If $q = 0$, then

$$\pi_1(M) = \langle a, b, x, y | bab^{-1} = a^{-1}, yxy^{-1} = x^{-1}, a = x^p, b^2 = x^r y^{2s} \rangle$$

where $p = \pm 1, s = \pm 1$. Now $\pi_1(M)$ is isomorphic to

$$N(0, 2 : pr) = \langle c_1, c_2, h | c_i h c_i^{-1} = h^{-1}, c_1^2 c_2^2 = h^{pr} \rangle$$

by the mapping $c_1 \mapsto b, c_2 \mapsto y^{-s}, h \mapsto x^p$. Thus

$$\Phi(\pi_1(M)) = \langle h^{\varphi(r)} \rangle = \langle x^{\varphi(r)} \rangle$$

by Theorem 4.11.

Case 2. $u = 0, v \neq 0$. As in Case 1 the reader may check that $r = 0$ or $p = 0$, but not both. If $r = 0$, then

$$\pi_1(M) = \langle a, b, x, y | bab^{-1} = a^{-1}, yxy^{-1} = x^{-1}, a = x^p y^{2q}, b^2 = y^{2s} \rangle$$

where $p = \pm 1, s = \pm 1$. This group is isomorphic to

$$O(4, 0 : psq - 2, (2, 1), (2, 1), (2, 1), (2, 1))$$

$$= \langle q_1, q_2, q_3, q_4, h | [q_i, h] = 1, q_i^2 h = 1, q_1 q_2 q_3 q_4 = h^{psq-2} \rangle.$$

If $s = +1$, the mapping $h \mapsto y^{2s}, q_1 \mapsto by^{-2s}, q_2 \mapsto y^{-2s+1}, q_3 \mapsto xy^{-2s+1}, q_4 \mapsto xby^{2psq-2}$ defines an isomorphism. If $s = -1$, the mapping $h \mapsto y^{2s}, q_1 \mapsto by^{-2s}, q_2 \mapsto y, q_3 \mapsto xy, q_4 \mapsto xby^{2psq-2}$ defines an isomorphism. Thus by Theorem 4.8,

$$\Phi(\pi_1(M)) = \langle h^{\varphi(2q)/2} \rangle = \langle y^{\varphi(2q)} \rangle.$$

Finally, notice that the case $p = 0$ is the same as the case $s = 0$ with a and x , b and y , interchanged. Therefore, $\Phi(\pi_1(M)) = \langle y^{\varphi(2^s)} \rangle$, and the proof of Theorem 5.4 is complete.

6. Finite groups of 3-manifolds. By Corollary 8.7 of [5] each finite group which occurs as the fundamental group of a 3-manifold also occurs as the fundamental group of a closed, orientable 3-manifold. J. Milnor [16] has listed all the finite groups which can possibly occur as the fundamental group of a closed orientable 3-manifold. Each group which is known to occur also occurs as the fundamental group of a Seifert fibered space. Using the following lemmas we show that in each case the Frattini subgroup is trivial or finite cyclic.

LEMMA 6.1. *Let G be a group. Then $\Phi(G) \supseteq Z(G) \cap G'$.*

The proof is an easy exercise which appears in §7.3 of [22].

LEMMA 6.2. *If H is a normal subgroup of a group G and if $\Phi(H)$ is finitely generated, then $\Phi(G) \supseteq \Phi(H)$ [22].*

THEOREM 6.3. *The Frattini subgroups of all the known finite fundamental groups of 3-manifolds are as follows.*

- (i) $\Phi(1) = 1.$
- (ii) $P_{120} = \langle x, y | x^2 = (xy)^3 = y^5, x^4 = 1 \rangle,$
 $\Phi(P_{120}) = \langle x^2 | x^4 = 1 \rangle \cong Z_2.$
- (iii) $P_{48} = \langle x, y | x^2 = (xy)^3 = y^4, x^4 = 1 \rangle,$
 $\Phi(P_{48}) = \langle x^2 | x^4 = 1 \rangle \cong Z_2.$
- (iv) $Q_{8n} = \langle x, y | x^2 = (xy)^2 = y^{2n} \rangle, \quad n > 1,$
 $\Phi(Q_{8n}) = \langle y^{\varphi(4n)} \rangle \cong Z_{4n/\varphi(4n)}.$
- (v) $D_{2^k(2n+1)} = \langle x, y | x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} \rangle,$
 $k \geq 2, n \geq 1,$
 $\Phi(D_{2^k(2n+1)}) = \langle x^2, y^{\varphi(2n+1)} \rangle \cong Z_{2^{k-1}(2n+1)/\varphi(2n+1)}.$

$$(vi) \quad P_{8 \cdot 3^k} = \langle x, y, z | x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 \rangle,$$

$$k \geq 1,$$

$$\Phi(P_{8 \cdot 3^k}) = \langle x^2, z^3 \rangle \cong Z_{2 \cdot 3^{k-1}}.$$

$$(vii) \quad Z_n = \langle x | x^n = 1 \rangle, \quad n \geq 2,$$

$$\Phi(Z_n) = \langle x^{\varphi(n)} \rangle \cong Z_{n/\varphi(n)}.$$

(viii) *The direct sum of any of the above groups with a cyclic group of relatively prime order. The Frattini subgroup is the direct sum of the Frattini subgroups of the factors and is cyclic.*

PROOF. Since the groups are finite, (viii) follows from (i)–(vii) [22]. For (i) and (vii) the result is obvious. So we proceed to consider the remaining groups.

(ii) The binary dodecahedral group P_{120} is perfect, that is, $P_{120} = P'_{120}$. Furthermore, $Z(P_{120}) = \langle x^2 | x^4 = 1 \rangle$ and

$$P_{120}/Z(P_{120}) \cong \langle \bar{x}, \bar{y} | \bar{x}^2 = (\bar{xy})^3 = \bar{y}^5 = \bar{1} \rangle \cong A_5.$$

Since A_5 is simple, $\Phi(A_5) = 1$ and so $\Phi(P_{120}) \subseteq Z(P_{120})$. By Lemma 6.1, $\Phi(P_{120}) \supseteq Z(P_{120}) \cap P'_{120} = Z(P_{120})$. Hence

$$\Phi(P_{120}) = Z(P_{120}) = \langle x^2 | x^4 = 1 \rangle.$$

(iii) For P_{48} , the binary octahedral group, we have $Z(P_{48}) = \langle x^2 | x^4 = 1 \rangle$ and

$$P_{48}/Z(P_{48}) \cong \langle \bar{x}, \bar{y} | \bar{x}^2 = (\bar{xy})^3 = \bar{y}^4 = \bar{1} \rangle \cong S_4.$$

As is known [22], $\Phi(S_4) = 1$, and so $\Phi(P_{48}) \subseteq Z(P_{48}) = \langle x^2 | x^4 = 1 \rangle$. Furthermore, $x^2 \in Z(P_{48})$, and since $P_{48}/P'_{48} \cong \langle \bar{x} | \bar{x}^2 = \bar{1} \rangle$, we also have $x^2 \in P'_{48}$. Hence $x^2 \in Z(P_{48}) \cap P'_{48}$, and so by Lemma 6.1, $x^2 \in \Phi(P_{48})$. Therefore,

$$\Phi(P_{48}) = Z(P_{48}) = \langle x^2 | x^4 = 1 \rangle.$$

(iv) In Q_{8n} , $H = \langle y \rangle$ is normal, and $\Phi(H) = \langle y^{\varphi(4n)} \rangle$ is finitely generated (in fact, finite). Hence by Lemma 6.2, $\langle y^{\varphi(4n)} \rangle \subseteq \Phi(Q_{8n})$. Also, $\Phi(H)$ is characteristic in H , and H is normal in Q_{8n} . So $\Phi(H)$ is normal in Q_{8n} . Consider

$$Q_{8n}/\Phi(H) \cong \langle \bar{x}, \bar{y} | \bar{x}^2 = \bar{1}, \bar{y}^{\varphi(4n)} = \bar{1}, \bar{x}\bar{y}\bar{x}^{-1} = \bar{y}^{-1} \rangle \cong D_{2\varphi(4n)},$$

the dihedral group of order $2\varphi(4n)$. Let $\varphi(4n) = p_1 \cdot \dots \cdot p_s$ where p_i are distinct primes. Then $\langle \bar{y}^{p_i}, \bar{x} \rangle$ is of index p_i in $D_{2\varphi(4n)}$, hence is maximal.

Similarly, $\langle \bar{y} \rangle$ is of index 2 in $D_{2\varphi(4n)}$ and so is maximal. Therefore

$$\Phi(D_{2\varphi(4n)}) \subseteq \bigcap_{i=1}^s \langle \bar{y}^{p_i}, \bar{x} \rangle \cap \langle \bar{y} \rangle = \bar{1}.$$

Hence $\Phi(Q_{8n}) \subseteq \Phi(H)$ so that

$$\Phi(Q_{8n}) = \langle y^{\varphi(4n)} \rangle \cong Z_{4n/\varphi(4n)}.$$

(v) Now consider $D_{2^k(2n+1)}$. Since $x^2 \in Z(D_{2^k(2n+1)})$, we have that

$$H = \langle x^2, y \rangle \cong Z_{2^{k-1}(2n+1)}$$

and H is normal in $D_{2^k(2n+1)}$. Also, $\Phi(H)$ is finite. Hence by Lemma 6.2,

$$\Phi(H) = \langle x^4, y^{\varphi(2n+1)} \rangle \subseteq \Phi(D_{2^k(2n+1)}).$$

Let $N = \langle x^2, y^{\varphi(2n+1)} \rangle$. Then N is normal in $D_{2^k(2n+1)}$, and

$$D_{2^k(2n+1)}/N \cong \langle \bar{x}, \bar{y} | \bar{x}^2 = \bar{1}, \bar{y}^{\varphi(2n+1)} = \bar{1}, \bar{x}\bar{y}\bar{x}^{-1} = \bar{y}^{-1} \rangle \cong D_{2\varphi(2n+1)}.$$

As in (iv) $\Phi(D_{2\varphi(2n+1)}) = 1$, and so $\Phi(D_{2^k(2n+1)}) \subseteq N$. It remains to show that $x^2 \in \Phi(D_{2^k(2n+1)})$. Let M be a maximal subgroup of $D_{2^k(2n+1)}$. Suppose $x^2 \notin M$. Then $D_{2^k(2n+1)} = M\langle x^2 \rangle$. In particular, $x = m(x^2)^\lambda$ for some $m \in M$, $\lambda \in Z$. Therefore, $x^{1-2\lambda} \in M$. But $x^4 \in M$. Hence $x \in M$, and so $x^2 \in M$. Thus $x^2 \in \Phi(D_{2^k(2n+1)})$, and so

$$\Phi(D_{2^k(2n+1)}) = \langle x^2, y^{\varphi(2n+1)} \rangle \cong Z_{2^{k-1}(2n+1)/\varphi(2n+1)}.$$

(vi) Finally, in $P_{8 \cdot 3^k}$ let $z_1 = z^3$. Then $z_1 \in Z(P_{8 \cdot 3^k})$. Hence

$$\begin{aligned} H &= \langle x, y, z_1 | x^2 = (xy)^2 = y^2, z_1x = xz_1, z_1y = yz_1, z_1^{3^k-1} = 1 \rangle \\ &\cong Q_8 \oplus Z_{3^k-1}, \end{aligned}$$

and H is normal in $P_{8 \cdot 3^k}$. Also $\Phi(H) = \langle x^2, z_1^3 \rangle \cong Z_{2 \cdot 3^{k-2}}$ for $k \geq 2$. (If $k = 1$, then $H = Q_8$ and $\Phi(H) = \langle x^2 \rangle \cong Z_2$.) By Lemma 6.2, $\Phi(H) \subseteq \Phi(P_{8 \cdot 3^k})$. Now let $N = \langle x^2, z_1 \rangle$. Then N is normal in $P_{8 \cdot 3^k}$ and

$$\begin{aligned} P_{8 \cdot 3^k}/N &\cong \langle \bar{x}, \bar{y}, \bar{z} | \bar{x}^2 = (\bar{x}\bar{y})^2 = \bar{y}^2 = \bar{1}, \bar{z}\bar{x}\bar{z}^{-1} = \bar{y}, \bar{z}\bar{y}\bar{z}^{-1} = \bar{x}\bar{y}, \bar{z}^3 = \bar{1} \rangle \\ &\cong K_4 \rtimes Z_3 \cong A_4. \end{aligned}$$

But since A_4 is normal in S_4 and $\Phi(A_4)$ is finite, $\Phi(A_4) \subseteq \Phi(S_4) = 1$. Hence $\Phi(P_{8 \cdot 3^k}/N) = 1$, and so $\Phi(P_{8 \cdot 3^k}) \subseteq N$. It remains to show that $z_1 \in \Phi(P_{8 \cdot 3^k})$ for $k \geq 2$. (If $k = 1$, then $z_1 = 1$ so we already have the result.) Let M be a maximal subgroup of $P_{8 \cdot 3^k}$. Suppose $z_1 \notin M$. Then $P_{8 \cdot 3^k} = M\langle z_1 \rangle$. In particular, $z = mz_1^\lambda$ for some $m \in M$, $\lambda \in Z$. Therefore, $z_1 = z^3 = m^3z_1^{3\lambda} \in M$, which is a contradiction. Hence, $z^3 \in \Phi(P_{8 \cdot 3^k})$, and so

$$\Phi(P_{8 \cdot 3^k}) = \langle x^2, z^3 \rangle \cong Z_{2 \cdot 3^{k-1}}.$$

This completes the proof of Theorem 6.3.

According to [16] and [15] the only other finite group which can possibly be the fundamental group of a 3-manifold is the group

$$Q(8n, k, l) = \langle x, y, z | x^2 = (xy)^2 = y^{2n}, z^{kl} = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} \rangle$$

where $8n, k, l$ are pairwise relatively prime positive integers, $r \equiv -1 \pmod k$, $r \equiv 1 \pmod l$, n is odd and $n > k > l > 1$ or the direct sum of this group with a cyclic group of relatively prime order. Since the groups are finite, the Frattini subgroup of the direct sum is the direct sum of the Frattini subgroups of the factors. Thus it only remains to determine the Frattini subgroup of $Q(8n, k, l)$.

THEOREM 6.4. *The group $Q(8n, k, l)$ has Frattini subgroup*

$$\Phi(Q(8n, k, l)) = \langle y^{\varphi(4n)}, z^{\varphi(kl)} \rangle \cong Z_{4nkl/\varphi(4nkl)}.$$

PROOF. Since $\langle z \rangle$ is normal in $Q(8n, k, l)$ and since $\Phi(\langle z \rangle) = \langle z^{\varphi(kl)} \rangle$ is finite,

$$\langle z^{\varphi(kl)} \rangle \subseteq \Phi(Q(8n, k, l)).$$

Similarly, $\langle y^2 \rangle$ is normal in $Q(8n, k, l)$, and so

$$\Phi(\langle y^2 \rangle) = \langle y^{\varphi(2n)} \rangle \subseteq \Phi(Q(8n, k, l)).$$

Let $N = \langle y^{\varphi(4n)}, z^{\varphi(kl)} \rangle$. Note that $\varphi(2n) = \varphi(4n)$. Then N is normal in $Q(8n, k, l)$, and

$$\begin{aligned} \overline{Q(8n, k, l)} &= Q(8n, k, l)/N \\ &\cong \langle \bar{x}, \bar{y}, \bar{z} | \bar{x}^2 = (\bar{xy})^2 = \bar{y}^{\varphi(4n)} = \bar{z}^{\varphi(kl)} = \bar{1}, \\ &\hspace{15em} \bar{x}\bar{z}\bar{x}^{-1} = \bar{z}^r, \bar{y}\bar{z}\bar{y}^{-1} = \bar{z}^{-1} \rangle \\ &\cong Z_{\varphi(kl)} \rtimes D_{2\varphi(4n)}. \end{aligned}$$

Since $\Phi(D_{2\varphi(4n)}) = 1$, we have $\Phi(\overline{Q(8n, k, l)}) \subseteq \langle \bar{z} \rangle$. Now let $\varphi(kl) = q_1 \cdot \dots \cdot q_t$ where q_i are distinct primes. Then $\langle \bar{z}^{q_i}, \bar{x}, \bar{y} \rangle$ is of index q_i in $\overline{Q(8n, k, l)}$, and so it is maximal. Therefore,

$$\Phi(\overline{Q(8n, k, l)}) \subseteq \bigcap_{i=1}^t \langle \bar{z}^{q_i}, \bar{x}, \bar{y} \rangle \cap \langle \bar{z} \rangle = \bar{1}.$$

Hence $\Phi(Q(8n, k, l)) \subseteq N$ so that

$$\Phi(Q(8n, k, l)) = \langle y^{\varphi(4n)}, z^{\varphi(kl)} \rangle \cong Z_{4nkl/\varphi(4nkl)}$$

as required.

7. Comments on the general problem. If $\pi_1(M)$ is infinite and M is not compact, orientable or sufficiently large, little more can be said.

In particular, we cannot do the nonsufficiently large case. However, all the known nonsufficiently large 3-manifolds with infinite fundamental groups are Seifert fibered spaces. The Seifert fibered spaces with Seifert surface a 2-sphere with three singular fibers of type (α_i, β_i) where $\sum_{i=1}^3 1/\alpha_i < 1$ and $\xi \neq 0$ have infinite fundamental group and are not sufficiently large. Since $\xi \neq 0$, we have $\lambda \neq 0$, and so from Theorem 4.10 it follows that all the known nonsufficiently large 3-manifolds with infinite fundamental groups have nontrivial Frattini subgroups. The problem when M is almost sufficiently large is no easier since properties of the Frattini subgroup are not generally inherited by subgroups, quotient groups or extensions.

If M is noncompact and $G = \pi_1(M)$ is finitely generated, then G also occurs as the fundamental group of a compact 3-manifold [21] so $\Phi(G)$ is given by our work. However, if $\pi_1(M)$ is not finitely generated, then opposite extremes can occur for the Frattini subgroup. For example, let F_∞ be an infinitely generated free group. Then F_∞ occurs as the fundamental group of a noncompact 3-manifold, and $\Phi(F_\infty) = 1$. On the other hand, the complement of a solenoid in S^3 is a noncompact 3-manifold with fundamental group the additive group of rationals Q . The additive group of rationals is the union of an ascending chain of proper infinite cyclic subgroups. Thus it has no maximal subgroups, and hence $\Phi(Q) = Q$. Furthermore, each subgroup of Q is the fundamental group of some 3-manifold, and there is a large variety of infinitely generated subgroups of Q . The determination of the Frattini subgroups of all these groups is generally impossible.

If M is nonorientable, then M has a double cover \tilde{M} which is orientable and $\tilde{G} = \pi_1(\tilde{M})$ is normal of index two in $G = \pi_1(M)$. We know that if $\Phi(\tilde{G})$ is finitely generated, then $\Phi(\tilde{G}) \subseteq \Phi(G)$. Hence, if $\Phi(\tilde{G})$ is nontrivial, then $\Phi(G)$ is nontrivial. But, in general, we do not know how to handle the nonorientable case again mainly because properties of the Frattini subgroup are not hereditary.

REFERENCES

1. R. B. J. T. Allenby and C. Y. Tang, *The Frattini subgroup of certain generalized free products and knot groups* (preprint).
2. ———, *On the Frattini subgroups of generalized free products*, J. Algebra **52** (1978), 460–474.
3. G. Baumslag, *On generalized free products*, Math. Z. **78** (1962), 423–438.
4. J. W. Cannon and C. D. Feustel, *Essential annuli and Möbius bands in M^3* , Trans. Amer. Math. Soc. **215** (1976), 219–239.
5. D. B. A. Epstein, *Projective planes in 3-manifolds*, Proc. London Math. Soc. **11** (1961), 469–484.
6. B. Evans, *The conjugacy problem for 3-manifold groups*, Ann. of Math. (submitted).
7. B. Evans and L. Moser, *Solvable fundamental groups of compact 3-manifolds*, Trans. Amer. Math. Soc. **168** (1972), 189–210.

8. J. Hempel, *3-manifolds*, Ann. of Math. Studies, no. 86, Princeton Univ. Press, Princeton, N. J., 1976.

9. J. Hempel and W. Jaco, *Fundamental groups of 3-manifolds which are extensions*, Ann. of Math. (2) **95** (1972), 86–98.

10. G. Higman and B. H. Neumann, *On two questions of $\text{li\hat{o}}$* , J. London Math. Soc. **29** (1954), 84–88.

11. A. H. M. Hoare, A. Karass and D. Solitar, *Subgroups of infinite index in Fuchsian groups*, Math. Z. **125** (1972), 59–69.

12. W. Jaco, *Roots, relations and centralizers in 3-manifold groups*, Proc. Geometric Topology Conf., Lecture Notes in Math., vol. 438, Springer-Verlag, New York, 1975, pp. 283–310.

13. W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*. I, II, III (preprint).

14. R. A. Katz and W. Magnus, *Residual properties of free groups*, Comm. Pure Appl. Math. **22** (1969), 1–13.

15. R. Lee, *Semicharacteristic classes*, Topology **12** (1973), 183–199.

16. J. W. Milnor, *Groups which act on S^n without fixed points*, Amer. J. Math. **79** (1957), 623–630.

17. K. Murasugi, *Knot theory*, Lecture notes, Univ. of Toronto, 1971.

18. L. P. Neuwirth, *Knot groups*, Ann. of Math. Studies, no. 56, Princeton Univ. Press, Princeton, N. J., 1965.

19. ———, *The status of some problems related to knot groups*, Proc. Conf. Topology, Lecture Notes in Math., vol. 375, Springer-Verlag, New York, 1974.

20. P. Orlik, *Seifert manifolds*, Lecture Notes in Math., vol. 291, Springer-Verlag, New York, 1972.

21. G. P. Scott, *Compact submanifolds of 3-manifolds*, J. London Math. Soc. **7** (1973), 246–250.

22. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

23. J. Simon, *Compactification of covering spaces of compact 3-manifolds*, Michigan Math. J. **23** (1976), 245–256.

24. C. Y. Tang, *On the Frattini subgroups of generalized free products with cyclic amalgamations*, Canad. Math. Bull. **15** (1972), 569–573.

25. F. Waldhausen, *Gruppen mit Zentrum und dreidimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517.

26. ———, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

27. ———, *On the determination of some bounded 3-manifolds by their fundamental groups alone*, Proc. Internat. Sympos. Topology and Appl., Herceg-Novci, Yugoslavia, 1968: Beograd, 1969, pp. 331–332.

28. A. Whittemore, *On the Frattini subgroup*, Trans. Amer. Math. Soc. **141** (1969), 323–333.

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