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A GENERAL STONE-GELFAND DUALITY

BY

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ABSTRACT. We give a simple characterization of full subcategories of equational categories. If $\mathcal{A}$ is one such and $\mathcal{B}$ is the category of topological spaces, we consider a pair of adjoint functors $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ which are represented by objects I and J in the sense that the underlying sets of $U(A)$ and $F(B)$ are $\mathcal{A}(A, I)$ and $\mathcal{B}(B, J)$. (One may take I and J to have the same underlying set.) Such functors always establish a duality between Fix FU and Fix UF. We study conditions under which one can conclude that FU and UF are reflectors into Fix FU and Fix UF, that Fix FU = Image F = the limit closure of I in $\mathcal{A}$ and that Fix UF = Image U = the limit closure of J in $\mathcal{B}$. For example, this happens if (1) $\mathcal{A}$ is a limit closed subcategory of an equational category, (2) J is compact Hausdorff and has a basis of open sets of the form $\{x \in J \mid \alpha(I)(x) \neq \beta(I)(x)\}$, where α and β are unary $\mathcal{A}$-operations, and (3) there are quaternary operations ξ and η such that, for all $x \in J^4$, $\xi(I)(x) = \eta(I)(x)$ if and only if $x_1 = x_2$ or $x_3 = x_4$. (The compactness of J may be dropped, but then one loses the conclusion that Fix FU is the limit closure of I.) We also obtain a quite different set of conditions, a crucial one being that J is compact and that every f in $\mathcal{B}(J^n, J)$, n finite, can be uniformly approximated arbitrarily closely by $\mathcal{A}$-operations on I. This generalizes the notion of functional completeness in universal algebra. The well-known dualities of Stone and Gelfand are special cases of both situations and the generalization of Stone duality by Hu is also subsumed.

0. Introduction. We continue here our study of duality, begun in [**16**], [**17**]. We now study dualities between full reflective subcategories of the category of topological spaces and full reflective subcategories of a category $\mathcal{A}$ which is, in a wide sense, of algebraic type. In fact, $\mathcal{A}$ can be described as any full subcategory of a category which is equational in the sense of Linton. However, for the most part, we shall not assume a knowledge of Linton's theory [**19**].

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In particular, we investigate dualities of subcategories of the category of compact Hausdorff spaces. This may be considered as a full subcategory of the category of uniform spaces, and we make considerable use of this fact in §3.

Our main results are in §§3 and 4. In §1 we discuss the class of categories we propose to consider and give examples. In §2 we collect some needed results on adjoint functors.

The Stone duality has been generalized by Hu **[8]**, **[9]** (see also **[17]**). He obtains a duality from a Birkhoff algebra satisfying a condition called "primality". Our §3 may be viewed as a generalization of Hu's work from finite discrete algebras to compact topological algebras. The Gelfand duality then appears as another special case.

In §4 we study dualities based on topological algebras which need not be compact. In addition to the previous examples, we have as an example the duality between real compact topological spaces and their rings of continuous real-valued functions.

1. Operational categories. Consider a category $\mathcal{C}$ with a functor H: $\mathcal{C} \to$ Sets. For any natural number n, an *n-ary operation* is a natural transformation $H^n \to H$, where $H^n(A) = (H(A))^n$.

More generally, for any set X, an *X-ary operation* is a natural transformation ω: $H^X \to H$. A function f: $H(A) \to H(A')$ is said to be an *H-homomorphism* from A to A' if it preserves all operations, that is, if

$$\begin{array}{ccc} H(A)^X & \xrightarrow{f^X} & H(A')^X \\ \omega(A)\downarrow & & \downarrow\omega(A') \\ H(A) & \xrightarrow[f]{} & H(A') \end{array}$$

commutes for all operations ω. Clearly, $H(g)$ is a homomorphism for all $g \in \mathcal{C}(A, A')$.

It is customary to call $(\mathcal{C}, H)$ a *concrete* category if H is faithful. We shall call $(\mathcal{C}, H)$ an *operational* category if it is concrete and if every homomorphism from A to A' is of the form $H(g)$ for some g: $A \to A'$.

If $\mathcal{C}_1$ is a full subcategory of $\mathcal{C}$ and H_1 is the restriction of H to H_1, then clearly $(\mathcal{C}_1, H_1)$ is operational if $(\mathcal{C}, H)$ is.

If $\mathcal{C}$ is the category of all Birkhoff algebras of some given similarity type and H is the usual underlying set functor, then $(\mathcal{C}, H)$ is operational. More generally, $(\mathcal{C}, H)$ could be any equational category in the sense of Linton or a full subcategory of an equational category.

The last example is actually quite general. For any set-valued functor H: $\mathcal{C} \to$ Sets, Linton **[19]** has defined the *equational category* of all H-algebras, call it H-Alg, with forgetful functor H', and the comparison functor Φ:

$\mathcal{Q} \to H$-Alg, such that $H'\Phi = H$. It is not difficult to see that $(\mathcal{Q}, H)$ is operational if and only if Φ is full and faithful. Thus, an operational category is just any concrete category equivalent (as a concrete category) to a full subcategory of an equational category. Linton calls the category of H-algebras *varietal* if H' has a left adjoint. This is so iff for each set X, there is only a set of operations $H^X \to H$, e.g. it is so if H has a left adjoint.

It is not difficult to see that, if $(\mathcal{Q}, H)$ is an operational category, then H reflects limits, in particular, H reflects isomorphisms. (The latter fact can also be seen directly by observing that, if f: $H(A) \to H(A')$ is a homomorphism and if it has an inverse f^{-1}, then f^{-1} is also a homomorphism.)

In view of the above, many well-known concrete categories are not operational. For example, the concrete categories of topological and uniform spaces are not operational.

In view of the work of the Prague school **[13]**, it appears that, under certain set-theoretical assumptions, for every concrete category $(\mathcal{Q}, H)$ there exists a functor H': $\mathcal{Q} \to$ Sets such that $(\mathcal{Q}, H')$ is operational.

The concrete category of compact Hausdorff spaces is known to be varietal **[17]**, hence it is operational. A direct proof of this latter fact may be of interest. Let $\mathcal{Q}$ be compact Hausdorff spaces with H the usual underlying set functor. Let X be any set and $\overline{X}$ the discrete topological space on X. Any point p of the Stone-Čech compactification $\beta\overline{X}$ determines an operation ω_p: $H^X \to H$ as follows. For each A in $\mathcal{Q}$ and $f \in H(A)^X$ let f^*: $\beta(\overline{X}) \to A$ be the unique continuous function extending the obviously continuous function f: $\overline{X} \to A$. Let $\omega_p(A)(f) = f^*(p)$. It is easily seen that ω_p is an operation. A homomorphism g from A to B will preserve, in particular, all operations ω_p. This means that for all continuous functions f: $\overline{X} \to A$ (i.e. all functions f: $X \to H(A)$) we have $gf^* = (gf)^*$ so that gf^* is continuous. We want to deduce that g is continuous, that is that $C' = g^{-1}(C)$ is closed in A for every closed subset C of B. Since gf^* is continuous, we do know that $(f^*)^{-1}(C')$ is closed in $\beta(\overline{X})$. Take $X = H(A)$ and $f = 1_{H(A)}$, then f^*: $\beta(\overline{X}) \to A$ is a surjection as well as a continuous function between compact Hausdorff spaces. Therefore, $C' = f^*((f^*)^{-1}(C'))$ is indeed closed.

Another interesting example is given by the category of normed vector spaces (real or complex) with norm-decreasing linear mappings as morphisms. For any object A, let $H(A)$ be the closed unit ball of A and, for any morphism f, let $H(f)$ be the obvious restriction of f. Among the operations are the following:

(1) for each scalar λ with $|\lambda| \leqslant 1$, an operation λ^*: $H \to H$ with $\lambda^*(A)(a) = \lambda a$;

(2) the midpoint operation μ defined by $\mu(A)(a, a') = \frac{1}{2}(a + a')$.

It is easily shown that, if a function f: $H(A) \to H(A')$ preserves these operations, then there is a unique linear mapping $g\colon A \to A'$ such that $g(a) = f(a)$ for all $a \in H(A)$. In fact, $g(x) = \|x\| f(\|x\|^{-1}x)$ for $x \neq 0$. The mapping g is norm-decreasing, since $g(H(A)) \subseteq H(A')$. Thus g is a morphism from A to A' and $H(g) = f$. H is clearly faithful, so the concrete category of normed vector spaces is operational.

The concrete category of normed algebras (real or complex) with norm-decreasing morphisms is therefore also operational, with the added operation of multiplication (remembering that $\|a \cdot a'\| \leqslant \|a\| \cdot \|a'\|$).

The categories of Banach spaces and Banach algebras, being full subcategories of the above categories, are also operational with regard to the unit ball functor.

2. Adjoint functors and representing objects. Consider a pair of functors

$$\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$$

with F left adjoint to U and adjunctions $\eta\colon \mathrm{id} \to UF$ in $\mathfrak{B}$ and $\varepsilon\colon \mathrm{id} \to FU$ in $\mathfrak{A}$. They always induce a duality between certain (possibly empty) subcategories of $\mathfrak{A}$ and $\mathfrak{B}$. However all the interesting dualities seem to arise from adjoint pairs of a very special type, in which the dual subcategories are just the images of U and F, and are reflective subcategories of $\mathfrak{A}$, $\mathfrak{B}$ with reflectors FU, UF and reflection maps $\varepsilon(A)$, $\eta(B)$ respectively.

More precisely, one has the following theorem, most of which was first proved by Isbell **[10]**, **[11]**. A proof may also be found in our previous paper **[16]**. The equivalences (1) $\Leftrightarrow$ (5), (3) $\Leftrightarrow$ (6) are well known.

Let $\mathrm{Fix}(UF, \eta)$ (respectively Image U) be the full subcategory of $\mathfrak{B}$ containing those objects B for which $\eta(B)$ is an isomorphism (respectively, B is isomorphic to some $U(A)$).

THEOREM 2.0. (a) *U and F restrict to an equivalence*

$$\mathrm{Fix}(FU, \varepsilon)^{\mathrm{op}} \rightleftarrows \mathrm{Fix}(UF, \eta)$$

i.e. a duality between $\mathrm{Fix}(FU, \varepsilon)$ *and* $\mathrm{Fix}(UF, \eta)$.

(b) *The following conditions are equivalent.*

(1) ηUF *is an isomorphism* (*i.e.* $\mathrm{Fix}(UF, \eta) = \mathrm{Image}\ UF$);

(2) ηU *is an isomorphism* (*i.e.* $\mathrm{Fix}(UF, \eta) = \mathrm{Image}\ U$);

(3) εFU *is an isomorphism* (*i.e.* $\mathrm{Fix}(FU, \varepsilon) = \mathrm{Image}\ FU$);

(4) εF *is an isomorphism* (*i.e.* $\mathrm{Fix}(FU, \varepsilon) = \mathrm{Image}\ F$);

(5) *UF is* (*after restricting its codomain*) *a reflector from* $\mathfrak{B}$ *into* Image UF *with reflection maps* $\eta(B)$;

(6) *FU is* (*after restricting its codomain*) *a reflector from* $\mathfrak{A}$ *into* Image FU, *with reflection maps* $\varepsilon(A)$.

Condition (1) is expressed by saying that (UF, η), or just UF, is an *idempotent triple*. Actually the "triple" is $(UF, \eta, U\varepsilon F)$, εF being the uniquely determined inverse of $F\eta$.

COROLLARY. *Let* $\mathcal{A}_1$, $\mathcal{B}_1$ *be full subcategories of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively which contain* Image F *and* Image U *respectively. Then* F *and* U *restrict to adjoint functors* $\mathcal{A}_1^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathcal{B}_1$ *and* UF *is idempotent if and only if* U_1F_1 *is idempotent.*

PROOF. F and U restrict to $\mathcal{A}_1^{\mathrm{op}} \underset{F'}{\overset{U'}{\rightleftarrows}} \mathcal{B}$ and $U'F' = UF$, $\eta' U'F' = \eta UF$. Thus UF is idempotent if and only if $U'F'$ is idempotent, hence if and only if $F'U'$ is idempotent. By a similar argument $F'U'$ is idempotent if and only if U_1F_1 is idempotent.

It was first emphasized by Lawvere that an adjoint pair F, U can be regarded as an "object sitting in both $\mathcal{A}$ and $\mathcal{B}$". The most obvious aspect of this is that, if $\mathcal{A}$ and $\mathcal{B}$ are concrete categories, i.e. are given with faithful functors $H\colon \mathcal{A} \to$ Sets and $K\colon \mathcal{B} \to$ Sets, and H, K are representable by objects A_0, B_0 respectively, then

$$HF(B) \cong \mathcal{A}(A_0, F(B)) \cong \mathcal{B}(B, U(A_0)),$$
$$KU(A) \cong \mathcal{B}(B_0, U(A)) \cong \mathcal{A}(A, F(B_0)),$$

so F, U are determined by the objects $I = F(B_0)$, $J = U(A_0)$. Moreover

$$H(I) = HF(B_0) \cong \mathcal{B}(B_0, J) \cong K(J),$$

so I and J have canonically isomorphic underlying sets.

In §§3 and 4 we will study pairs F, U when $\mathcal{B}$ is the category of topological spaces (or compact Hausdorff spaces) and $(\mathcal{A}, H)$ is concrete, particularly when $(\mathcal{A}, H)$ is operational. We do not wish to restrict H to be representable, but we will consider only pairs F, U with HF representable.

In this preliminary section we consider more general situations. We first give a useful condition for UF to be idempotent (Proposition 2.1), then recall a description of η and ε (Proposition 2.2), then give a description of the reflective subcategory Image U when UF is idempotent and $(\mathcal{A}, H)$ is operational (Proposition 2.3). We then study further the existence of adjoint pairs and their relation to pairs of objects (Propositions 2.4–2.7).

PROPOSITION 2.1. *Suppose* $(\mathcal{A}, H)$ *is any concrete category* (*that is*, H: $\mathcal{A} \to$ Sets *is a faithful functor*) *and* $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ *is an adjoint pair such that* $HF \cong \mathcal{B}(-, J)$. *Then* UF *is idempotent if and only if the adjunction* η: id $\to UF$ *is such that* $\mathcal{B}(\eta(B), J)$ *is an injective mapping for each* B *in* $\mathcal{B}$, *that is, for each* $f\colon B \to J$ *there is at most one* $g\colon UF(B) \to J$ *such that* $g\eta(B) = f$.

PROOF. If the condition holds, then $HF\eta(B)$ is a monomorphism for each B in $\mathfrak{B}$. Since H is faithful, $F\eta(B)$ is also a monomorphism. Now $(F\eta(B))(\varepsilon F(B)) = 1_{F(B)}$, where ε is the other adjunction. Thus $\varepsilon F(B)$ is an isomorphism. By Theorem 2.0, UF is idempotent.

Conversely, suppose UF is idempotent. Then UF is a reflector into Fix UF with reflection maps $\eta(B)$. Clearly, it will suffice to show that $J \varepsilon \operatorname{Fix} UF$, that is, that $\eta(J)$ is an isomorphism. Now $\varepsilon F(J)$ is an isomorphism with inverse $F\eta(J)$. Thus $\mathfrak{B}(\eta(J), J) \cong HF\eta(J)$ is a bijection, hence there is an h: $UF(J) \to J$ such that $h\eta(J) = 1_J$. Then $\eta(J)h\eta(J) = \eta(J)$ and, since $\eta(J)$ is a reflection map, this implies that $\eta(J)h = 1_{UF(J)}$. Therefore $\eta(J)$ is an isomorphism, and our proof is complete.

The proof becomes easier when H is representable, $H \cong \mathfrak{A}(A_0, -)$, for then $J \cong U(A_0)$. Now if UF is idempotent then Fix UF = Image U so J is in Fix UF.

In §§3 and 4 we will need to calculate $K\eta$ and $H\varepsilon$, e.g. to apply Proposition 2.1. The following result is equivalent to 0.2 and 0.3 of [7]. We have lightened a hypothesis slightly and stated the theorem more clearly.

PROPOSITION 2.2. *Assume given* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$ *such that*:

(1) $(\mathfrak{A}, H)$ *and* $(\mathfrak{B}, K)$ *are concrete categories with* $K \cong \mathfrak{B}(B_0, -)$;

(2) F *is left adjoint to* U *and* $HF \cong \mathfrak{B}(-, J)$.

Then:

(a) *If* $I \cong F(B_0)$ *then* $KU \cong \mathfrak{B}(B_0, U(-)) \cong \mathfrak{A}(-, F(B_0)) \cong \mathfrak{A}(-, I)$ *and* $H(I) \cong HF(B_0) \cong \mathfrak{B}(B_0, J) \cong K(J)$.

For simplicity of exposition we make the assumption (not really necessary) that the given isomorphisms $HF \to \mathfrak{B}(-, J)$, $KU \to \mathfrak{A}(-, I)$ *and* $H(I) \to K(J)$ *are identities.*

(b) *For any* $a \in H(A)$, $(H\varepsilon(A))(a)$ *is in* $HFU(A) = \mathfrak{B}(U(A), J)$, *and has as underlying set map* $KU(A) \to K(J)$, *i.e.* $\mathfrak{A}(A, J) \to H(I)$, *the evaluation* $\hat{a}$, $\hat{a}(f) = H(f)(a)$.

If $\mathfrak{B}$ *has products (necessarily preserved by* K*), then the morphism* $\alpha(A)$: $U(A) \to J^{H(A)}$, *such that* $\pi(a)\alpha(A) = (H\varepsilon(A))(a)$ *for all* a *in* $H(A)$, *has as underlying set map* $KU(A) \to K(J^{H(A)})$, *i.e.* $\mathfrak{A}(A, I) \to H(I)^{H(A)}$, *the function which takes* f *to* $H(f)$.

Hence $\alpha(A)$ *is a monomorphism.*

(c) *For any* b *in* $K(B)$, $(K\eta(B))(b)$ *is in* $KUF(B) = \mathfrak{A}(F(B), I)$ *and has as underlying set map* $HF(B) \to H(I)$, *i.e.* $\mathfrak{B}(B, J) \to K(J)$, *the evaluation* $\hat{b}$, *where* $\hat{b}(g) = K(g)(b)$.

If $\mathfrak{A}$ *has products and* H *preserves them, then the map* $\beta(B)$: $F(B) \to I^{K(B)}$, *such that* $\pi(b)\beta(B) = (K\eta(B))(b)$ *for all* b *in* $K(B)$, *has as underlying set map* $HF(B) \to H(I^{K(B)})$, *i.e.* $\mathfrak{B}(B, J) \to K(J)^{K(B)}$, *the function which takes* f *to* $K(f)$.

Hence $\beta(B)$ is a monomorphism.

PROOF. (a) is obvious. To prove (b) consider the isomorphisms

$$K \xrightarrow{\sim} \mathfrak{B}(B_0, -), \quad \mathfrak{B}(B, U(-)) \xrightarrow{*} \mathfrak{A}(-, F(B)), \quad F(B_0) \xrightarrow{\theta} I$$

and verify that, for any a in $H(A)$, $a' = H\varepsilon(A)(a)$ and f in $\mathfrak{A}(A, I)$,

$$\begin{aligned}(K(a')(f))\hat{} &= \mathfrak{B}(B_0, a')(\hat{f}) = a'\hat{f} = HF(\hat{f})(a') \\ &= H(\hat{f}^*)(a) = \big(H(\theta\hat{f}^*)(a)\big)\hat{},\end{aligned}$$

whence $K(a')(f) = H(\theta\hat{f}^*)(a) = H(f)(a)$, since $\theta\hat{f}^* = f$.

To prove (c), one verifies that, for any $b \in K(B)$, $b' = (K\eta(B))(b)$ and $g \in \mathfrak{B}(B, J)$,

$$b' = \theta\big(\eta(B)\hat{b}\big)^* = \theta F(\hat{b}), \quad H(b')(g) = H(\theta)HF(\hat{b})(g) = H(\theta)(g\hat{b}),$$

whence $(H(b')(g))\hat{} = g\hat{b} = (K(g)(b))\hat{}$.

The assumptions that $HF = \mathfrak{B}(-, J)$ etc. can be made true for any interesting choice of $(\mathfrak{A}, H)$ and $(\mathfrak{B}, K)$ by choosing I properly and replacing F and U by equivalent functors. $(\mathfrak{A}, H)$ and $(\mathfrak{B}, K)$ need only have the *unique transfer property*, i.e. every bijection $H(A) \to X$, where X is any set, is $H(f)$ for a unique f in $\mathfrak{A}$, and similarly for $(\mathfrak{B}, K)$. This property also implies a tacit assumption in (b) and (c), that $J^{H(A)}$ can be chosen so that $K(J^{H(A)}) = K(J)^{H(A)}$ is the canonical power Sets$(H(A), K(J))$ and similarly for $I^{K(B)}$. Of course, this assumption is only needed for simplicity of statement.

PROPOSITION 2.3. *Suppose $(\mathfrak{A}, H)$ is operational, $\mathfrak{B}$ is complete and $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$ is an adjoint pair with $HF \cong \mathfrak{B}(-, J)$. Then:*

(a) *Each $U(A)$ is the equalizer of a pair of morphisms between powers of J.*

(b) *If UF is idempotent then the reflective subcategory* Image *U is the limit closure $\mathfrak{L}_{\mathfrak{B}}(J)$ of J in $\mathfrak{B}$.* Image *U can also be described as the full subcategory $\mathfrak{E}$ of $\mathfrak{B}$ containing those objects which are equalizers of some pairs of morphisms between powers of J.*

REMARK. It follows from (b) and [16] that, if UF is idempotent, then $UF(B)$ is the joint equalizer of all pairs of maps $J^{\mathfrak{B}(B,J)} \rightrightarrows J$ which are equalized by the canonical map $B \to J^{\mathfrak{B}(B,J)}$. Thus, in the terminology of [14], we have an example of *localization*.

PROOF. For the moment we postpone the proof of (a). To prove (b) we recall that $\mathfrak{L}_{\mathfrak{B}}(J)$ is, by definition, the smallest limit closed full subcategory of $\mathfrak{B}$ containing J. In view of (a), Image $U \subseteq \mathfrak{E}$, and clearly $\mathfrak{E} \subseteq \mathfrak{L}_{\mathfrak{B}}(J)$. Thus, we need only show that $\mathfrak{L}_{\mathfrak{B}}(J) \subseteq$ Fix UF = Image U. Since Fix UF is a reflective subcategory, it is limit closed in $\mathfrak{B}$, so we need only verify that $J \in$ Fix UF. This was shown while proving Proposition 2.1.

We now return to the proof of (a), which is an obvious consequence of the following proposition. This proposition is closely related to Proposition 2.5, and some remarks on their history are given after 2.5.

PROPOSITION 2.4. *Suppose* $(\mathcal{A}, H)$ *is operational and* $\mathcal{B}$ *is complete. If* $\mathcal{A}^{\mathrm{op}} \xleftarrow{F} \mathcal{B}$ *is a functor with* $HF \cong \mathcal{B}(-, J)$, *then* F *has a right adjoint* U *and each* $U(A)$ *is the equalizer of a pair of maps between powers of* J.

PROOF. Let ω be any operation $H^X \to H$. The natural transformation

$$\mathcal{B}(-, J^X) \xrightarrow{\sim} \mathcal{B}(-, J)^X \xrightarrow{\sim} (HF)^X \xrightarrow{\omega F} HF \xrightarrow{\sim} \mathcal{B}(-, J)$$

is $\mathcal{B}(-, \omega^*)$ for some $\omega^* \varepsilon \mathcal{B}(J^X, J)$, by Yoneda's lemma. For simplicity of notation, let us assume $HF = \mathcal{B}(-, J)$. Let $\langle - \rangle$ denote the canonical natural isomorphism $\mathcal{B}(-, J)^X \xrightarrow{\sim} \mathcal{B}(-, J^X)$. Then, for any $h \in \mathcal{B}(B, J)^X$,

$$\omega F(B)(h) = \omega^* \langle h \rangle. \tag{1}$$

For each operation ω and each mapping $a\colon X \to H(A)$ we have a pair of morphisms $J^{H(A)} \to J$, namely $\pi(\omega(A)(a))$, the canonical projection corresponding to $\omega(A)(a)$ in $H(A)$, and $\omega^* J^a\colon J^{H(A)} \xrightarrow{J^a} J^X \xrightarrow{\omega^*} J$. Let $j(A)\colon U(A) \to J^{H(A)}$ be the joint equalizer of all these pairs of morphisms $J^{H(A)} \rightrightarrows J$. This exists, because these pairs of morphisms form a set Y, even though the pairs (ω, a) from which they are constructed may form a proper class. Thus $j(A)$ is the equalizer of a single pair of morphisms $J^{H(A)} \rightrightarrows J^Y$.

For any $g \varepsilon \mathcal{A}(A, A')$ we define $U(g)$ as the unique morphism such that the following square commutes:

$$\begin{tikzcd}
U(A') \arrow[r, "j(A')"] \arrow[d, dotted, "U(g)"'] & J^{H(A')} \arrow[d, "J^{H(g)}"] & \\
U(A) \arrow[r, "j(A)"'] & J^{H(A)} \arrow[r, shift left, "\omega^* J^a"] \arrow[r, shift right, "\pi(\omega(A)(a))"'] & J
\end{tikzcd}$$

The existence of $U(g)$ follows from the fact that

$$\begin{aligned}
\omega^* J^a J^{H(g)} j(A') &= \omega^* J^b j(A') \\
&= \pi(\omega(A')(b)) j(A') = \pi(\omega(A)(a)) J^{H(g)} j(A'),
\end{aligned}$$

where $b = H(g)a = H(g)^X(a)$. It is easily seen that U is a functor.

We shall now establish a one-to-one correspondence between $\mathcal{B}(B, U(A))$ and $\mathcal{A}(A, F(B))$. In view of the construction of $U(A)$, the morphisms f: $B \to U(A)$ are in bijective correspondence with the morphisms g: $B \to J^{H(A)}$

such that

$$\omega^* J^a g = \pi(\omega(A)(a))\, g \tag{2}$$

for all ω: $H^X \to H$ and a: $X \to H(A)$.

We have the bijection $\langle - \rangle$: $\mathfrak{B}(B, J)^{H(A)} \to \mathfrak{B}(B, J^{H(A)})$, $g = \langle h \rangle$ being the unique morphism such that $\pi(p)g = h(p)$ for all $p \in H(A)$. Thus the set of morphisms g satisfying (2) corresponds bijectively to the set of h: $H(A) \to \mathfrak{B}(B, J)$ such that

$$\omega^*\langle ha \rangle = h(\omega(A)(a)) \tag{3}$$

for all pairs (ω, a). By equation (1) above, this is equivalent to

$$\omega F(B)(ha) = h(\omega(A)(a)). \tag{4}$$

Recalling that h is a function $H(A) \to \mathfrak{B}(B, J) = HF(B)$, we see that h satisfies (4) for all $a \in H(A)^X$ if and only if the square (5) commutes.

$$\begin{array}{ccc} H(A)^X & \xrightarrow{\omega(A)} & H(A) \\ {\scriptstyle h^X}\downarrow & & \downarrow{\scriptstyle h} \\ HF(B)^X & \xrightarrow[\omega F(B)]{} & HF(B) \end{array} \tag{5}$$

This is true for all operations ω if and only if h is a homomorphism. Since $(\mathfrak{A}, H)$ is operational, $\mathfrak{B}(B, U(A))$ thus corresponds bijectively to $\mathfrak{A}(A, F(B))$. It may be checked that this correspondence is natural in A and B, and so F is left adjoint to U.

REMARKS. (1) We could have constructed a functor U from an arbitrary class Ω of operations ω: $H^X \to H$ instead of the class of all operations. The proof that F is left adjoint to U goes through in the same way provided that, for all A and A' in $\mathfrak{A}$, every function f: $H(A) \to H(A')$ which preserves all operations in Ω is $H(g)$ for some g in $\mathfrak{A}(A, A')$.

(2) For any B in $\mathfrak{B}$, $\eta(B)$: $B \to UF(B)$ corresponds by adjointness to the identity map $F(B) \to F(B)$. An inspection of the proof of Proposition 2.4 shows that

$$g(B) = \left(B \xrightarrow{\eta(B)} UF(B) \xrightarrow{jF(B)} J^{\mathfrak{B}(B, J)}\right)$$

is such that $\pi(u)g(B) = u$ for all u in $\mathfrak{B}(B, J)$. We defined $jF(B)$ as the joint equalizer of certain pairs of morphisms into J. Clearly these pairs are also equalized by $g(B)$. But if UF is idempotent then every pair $J^{\mathfrak{B}(B,J)} \rightrightarrows J$ equalized by $g(B)$ is also equalized by $jF(B)$, since $\eta(B)$ is a reflection map into Image U and J is in Image U by Proposition 2.3(b). Thus $jF(B)$ is the joint equalizer of all pairs $J^{\mathfrak{B}(B,J)} \rightrightarrows J$ which are equalized by $g(B)$. This establishes a connection with our former paper [14]. In the terminology of

[14], J is κ-injective and UF is the localization functor determined by J (assuming that UF is idempotent).

How can adjoint pairs $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ be constructed? Sometimes very easily in an ad hoc way. For example, let $\mathcal{A}$ be the category of Banach algebras (real or complex) with norm reducing homomorphisms and let $H(A)$ be the closed unit ball of A. We know that $(\mathcal{A}, H)$ is operational. Let $(\mathcal{B}, K)$ be the concrete category of topological spaces. Let I be any Banach algebra and let J be $H(I)$ with the metric topology. Define F: $\mathcal{B} \to \mathcal{A}^{\mathrm{op}}$ by $F = C^*(-, I)$, that is, for any topological space B, $F(B)$ is the Banach algebra of bounded continuous functions f: $B \to I$ with the sup norm and pointwise operations. Clearly $HF(B) = \mathcal{B}(B, J)$ and $K(J) = H(I)$.

F has an obvious right adjoint U. Let $\hat{a}$: $\mathcal{A}(A, I) \to H(I)$ be the evaluation map at $a \in H(A)$, that is, $\hat{a}(s) = H(s)(a)$ for all $s \in \mathcal{A}(A, I)$. Take $U(A)$ to be $\mathcal{A}(A, I)$ with the weak topology induced by the functions $\hat{a}$ and the topology of J on $H(I)$. Of course, this U must coincide with the right adjoint constructed in Proposition 2.4.

There are similar examples for the categories of normed algebras, Banach spaces and normed vector spaces.

We return to the idea that an adjoint pair $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ is an "object sitting in both $\mathcal{A}$ and $\mathcal{B}$". This is rather clearly true in a very precise sense if $\mathcal{B}$ is the category Top of topological spaces and $\mathcal{A}$ is the category of all algebras defined by certain operations and equations (in the sense of Birkhoff or of Lawvere and Linton). The pair (F, U) then corresponds, up to equivalence, to a set made into both a topological space J and an algebra I such that all algebra operations are continuous. The algebra $F(B)$ is $\mathcal{B}(B, J)$ with pointwise operations, $U(A)$ is $\mathcal{A}(A, I)$ topologized as a subspace of $J^{H(A)}$.

It appears necessary here in constructing U to use the fact that any subset of a topological space has a natural topology. This is not really necessary because $\mathcal{A}(A, I)$ is a very special subset of $H(I^{H(A)}) = K(J^{H(A)})$ (where H, K denote as usual the forgetful functors). An algebra homomorphism $A \to I$ is a function $H(A) \to H(I) = K(J)$ which preserves certain operations, and this amounts to satisfying certain equations. Thus $U(A)$ is a subobject of $J^{H(A)}$ which can be obtained as an equalizer in $\mathcal{B}$, and this is precisely the construction of Proposition 2.4. Thus Top can be replaced by any complete category, which need not even be concrete. The "object in both $\mathcal{A}$ and $\mathcal{B}$" will then appear as an "algebra in $\mathcal{B}$"; i.e., an object J in $\mathcal{B}$ provided with a map $J^X \to J$ in $\mathcal{B}$ for each X-ary operation of $\mathcal{A}$, these maps satisfying the defining equations of $\mathcal{A}$. If $\mathcal{B}$ is concrete with a limit preserving forgetful functor, this reduces to the previous description (or rather an obvious generalization of it).

The correspondence between adjoint pairs and algebras in $\mathfrak{B}$ is in fact an equivalence of categories. The morphisms $(F, U) \to (F_1, U_1)$ in the category of adjoint pairs may be defined as natural transformations from F to F_1, where F, F_1 are regarded as functors $\mathfrak{B}^{\mathrm{op}} \to \mathfrak{A}$. These determine by adjointness natural transformations $U \to U_1$.

We recall some of Linton's definitions [**19**], [**20**]. An equational theory is a product preserving functor T: $\mathrm{Sets}^{\mathrm{op}} \to \mathfrak{T}$ which is one-one on objects and product preserving. $\mathfrak{T}$ may be an illegitimate category, i.e. $\mathfrak{T}(V, W)$ may be a proper class rather than a set. A T-algebra in $\mathfrak{B}$ is a functor P: $\mathfrak{T} \to \mathfrak{B}$ such that PT preserves products. A homomorphism between T-algebras is a natural transformation. One obtains a category $\mathfrak{B}^T$ of T-algebras in $\mathfrak{B}$, and evaluation at $T(1)$ is a faithful functor $\mathfrak{B}^T \to \mathfrak{B}$. Thus Sets^T with this "forgetful functor" is a concrete category. Sets^T is called an *equational* category. It is called *varietal* if its forgetful functor has a left adjoint, which will certainly be the case if $\mathfrak{T}$ is a legitimate category.

PROPOSITION 2.5. *Let $\mathfrak{A}$ be an equational category* Sets^T *with forgetful functor H, and let $\mathfrak{B}$ be complete. Then the category $\mathfrak{P}$ of adjoint pairs $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$ with HF representable is equivalent to $\mathfrak{B}^T$.*

REMARK. If $\mathfrak{A}$ is varietal then HF is representable for every adjoint pair F, U. If moreover $\mathfrak{A}$ has a rank then this theorem is a special case of Theorem 2.5 of Pultr [**25**]. There is also a general theorem in Isbell [**11**, p. 572]. Freyd's paper [**4**] contains special cases of 2.4 and 2.5 in which $\mathfrak{A}$ is assumed to be an algebraic category in the sense of Lawvere.

PROOF. The category $\mathfrak{P}$ is easily seen to be equivalent to the category of adjoint pairs (F, U) such that $HF = \mathfrak{B}(-, J)$ (not just $HF \cong \mathfrak{B}(-, J)$) for some J in $\mathfrak{B}$. It follows from Proposition 2.4 that this is equivalent to the category $\mathfrak{F}$ of functors F: $\mathfrak{B} \to \mathfrak{A}^{\mathrm{op}}$ such that $HF = \mathfrak{B}(-, J)$. We now define an equivalence $\mathfrak{F} \underset{M}{\overset{L}{\rightleftarrows}} \mathfrak{B}^T$.

For any object P of $\mathfrak{B}^T$ let $M(P) = F$, where

$$F(B) = \mathfrak{B}(B, P(-)) \colon \mathfrak{T} \to \mathrm{Sets}, \qquad F(f) = \mathfrak{B}(f, P(-)),$$

for any object B and map f in $\mathfrak{B}$. Then $F(B)T$ preserves products because PT does, so $F(B) \in \mathfrak{A}$. Also $HF(B) = F(B)(T(1)) = \mathfrak{B}(B, PT(1))$, so $F \in \mathfrak{F}$. For any α: $P_1 \to P_2$ in $\mathfrak{B}^T$, $M(\alpha)$: $M(P_1) \to M(P_2)$ is defined as $\mathfrak{B}(B, \alpha(-))$. Clearly M is a functor $\mathfrak{B}^T \to \mathfrak{F}$.

Conversely, given F in $\mathfrak{F}$, we define $P = L(F)$. Let $P(T(X))$ be some choice of J^X. We have canonical bijections

$$\begin{aligned} e(F, B, X) \colon F(B)(T(X)) &\xrightarrow{\sim} \big(F(B)(T(1))\big)^X \\ &= HF(B)^X = \mathfrak{B}(B, J)^X \xrightarrow{\sim} \mathfrak{B}(B, J^X) \end{aligned}$$

which are easily seen to be natural in F, B and X.

For any t: $T(X) \to T(Y)$ in $\mathfrak{T}$ we define $P(t)$ as the unique map $PT(X) \to PT(Y)$ which makes the diagram (6) commute.

$$\begin{array}{ccc} F(B)(T(X)) & \underset{e}{\overset{\sim}{\to}} & \mathfrak{B}(B, PT(X)) \\ F(B)(t)\downarrow & & \downarrow\mathfrak{B}(B, P(t)) \\ F(B)(T(Y)) & \underset{e}{\overset{\sim}{\to}} & \mathfrak{B}(B, PT(Y)) \end{array} \tag{6}$$

Such a map exists by the Yoneda lemma, since the induced map $\mathfrak{B}(B, PT(X)) \to \mathfrak{B}(B, PT(Y))$ is natural in B. Clearly P is a functor $\mathfrak{T} \to \mathfrak{B}$ and $PT = J^{(-)}$: $\text{Sets}^{\text{op}} \to \mathfrak{B}$, which is right adjoint to $\mathfrak{B}(-, J)$ and hence preserves products. Thus $P \in \mathfrak{B}^T$.

We define L on maps in the obvious way, again using the bijections e. L becomes a functor $\mathfrak{F} \to \mathfrak{B}^T$. It is clear that we can choose L (i.e. choose the powers J^X) so that $LM = \text{id}$. Also, from the diagram (6), it is clear that e is a natural equivalence $\text{id} \to ML$. Thus $\mathfrak{F} \underset{M}{\overset{L}{\rightleftarrows}} \mathfrak{B}^T$ is an equivalence of categories.

If $\mathfrak{T}$ is legitimate, L can be defined more simply. We then have the Yoneda functor Y: $\mathfrak{T} \to (\text{Sets}^T)^{\text{op}}$, with $Y(V)(W) = \mathfrak{T}(V, W)$. Clearly $Y(V)$ is in Sets^T. The composition YT: $\text{Sets}^{\text{op}} \to (\text{Sets}^T)^{\text{op}}$ is right adjoint to H and so preserves limits. Thus, if U is the right adjoint to F: $\mathfrak{B} \to \mathfrak{A}^{\text{op}}$, we can define $L(F)$ to be UY: $\mathfrak{T} \to \mathfrak{B}$, and it is in $\mathfrak{B}^T$.

The construction in 2.5 of $P(t)$: $J^X \to J^Y$, where $P = L(F)$, is nearly the same as the construction of ω^* in 2.4. Take $Y = 1$. A map $t \in \mathfrak{T}(T(X), T(1))$ induces $\bar{t} = \omega$: $H^X \to H$, defined by $\omega(A) = A(t)$. (Note that $AT(X) = AT(1)^X = H(A)^X$.) Comparing diagram (6) of 2.5 with the diagram (7) below,

$$\begin{array}{ccccc} F(B)(T(X)) & \overset{\sim}{\to} & HF(B)^X & \overset{\sim}{\to} & \mathfrak{B}(B, J^X) \\ \downarrow F(B)(t) & & \downarrow\omega F(B) & & \downarrow\mathfrak{B}(B, \omega^*) \\ F(B)(T(1)) & \overset{=}{\to} & HF(B) & \overset{\sim}{\to} & \mathfrak{B}(B, J) \end{array} \tag{7}$$

we see that $P(t) = \omega^*$.

Let us now consider the following situation.

(A0) $\mathfrak{A} = \text{Sets}^T$ with forgetful functor H, $(\mathfrak{B}, K)$ is a concrete category with unique transfer, $\mathfrak{B}$ has limits and K preserves them.

The assumption that $(\mathfrak{B}, K)$ has unique transfer (defined just after Proposition 2.2) is purely for convenience of notation. It implies that $\mathfrak{B}$ has canonical products which are carried by K to the canonical products in Sets.

In this situation one has a more concrete description of T-algebras in $\mathfrak{B}$. Consider the pairs (I, J) such that:

(A1) I is a functor $\mathfrak{T} \to \text{Sets}$ such that IT takes canonical products in Sets^{op} to canonical products (thus $I \in \mathfrak{A}$), $J \in \mathfrak{B}$ and $H(I) = K(J)$;

(A2) for each t in $\mathfrak{T}(T(X), T(Y))$ there is a (necessarily unique) t^+ in $\mathfrak{B}(J^X, J^Y)$ such that $K(t^+) = I(t)$. Here J^X, J^Y are canonical powers, so that

$$K(J^X) = K(J)^X = H(I)^X = IT(1)^X = IT(X).$$

The class of pairs satisfying (A1) and (A2) is made into a category, a morphism $(I, J) \to (I', J')$ being a pair (t_1, t_2) such that $t_1 \in \mathfrak{A}(I, I')$, $t_2 \in \mathfrak{B}(J, J')$ and $H(t_1) = K(t_2)$.

Proposition 2.6. *Assume* (A0). *Then the category of adjoint pairs* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$ *with HF representable is equivalent to the category of pairs (I, J) satisfying* (A1) *and* (A2).

The adjoint pair (F, U) associated to (I, J) can be chosen so that $KU = \mathfrak{A}(-, I)$, $HF = \mathfrak{B}(-, J)$.

Proof. We need only show that the category of (I, J) is equivalent to $\mathfrak{B}^T$, and this last is clearly equivalent to the full subcategory $\mathfrak{E}$ of $\mathfrak{B}^T$ consisting of those P: $\mathfrak{T} \to \mathfrak{B}$ such that PT preserves canonical products. For any such P let $I = KP$ and $J = PT(1)$. Then (A1) and (A2) are clearly satisfied, with $t^+ = P(t)$. A natural transformation α: $P_1 \to P_2$ induces $\alpha_1 = K\alpha$: $I_1 \to I_2$ and $\alpha_2 = \alpha(1)$: $J_1 \to J_2$, and $H(\alpha_1) = K(\alpha_2)$. Clearly one has a functor from $\mathfrak{E}$ to the category of (I, J). Conversely, given (I, J) satisfying (A1) and (A2), define P: $\mathfrak{T} \to \mathfrak{B}$ by $P(T(X)) = J^X$ (the canonical power) and $P(t) = t^+$. Then $P \in \mathfrak{E}$. Given (t, t'): $(I_1, J_1) \to (I_2, J_2)$, define t: $P_1 \to P_2$ by $t(T(X)) = (t')^X$: $J_1^X \to J_2^X$. Then one obtains a functor to $\mathfrak{E}$ from the category of (I, J). It is easily seen that our two functors give an isomorphism of categories.

The algebra P: $\mathfrak{T} \to \mathfrak{B}$ obtained from (I, J) determines, as in Proposition 2.5, an F: $\mathfrak{B} \to \mathfrak{A}^{\mathrm{op}}$ such that $HF = \mathfrak{B}(-, J)$. If U is a right adjoint to F constructed as in Proposition 2.4, we know that $j(A)$: $U(A) \to J^{H(A)}$ is the joint equalizer of all pairs

$$\begin{tikzcd} J^{H(A)} \arrow[rr, "\pi_J(\omega(A)(a))"] \arrow[dr, "J^a"'] & & J \\ & J^X \arrow[ur, "\omega^*"'] & \end{tikzcd}$$

where ω: $H^X \to H$ and $a \in H(A)^X$. In fact we need only consider the ω of the form $\bar{t}$, $t \in \mathfrak{T}(T(X), T(1))$, since these suffice to determine homomorphisms. Recalling that $(\bar{t})^* = P(t)$ and that K preserves limits, we see that $K(j(A))$: $KU(A) \to K(J)^{H(A)}$ is the joint equalizer of all pairs

$$\begin{tikzcd} K(J)^{H(A)} \arrow[rr, "\pi_{K(J)}(\omega(A)(a))"] \arrow[dr, "K(J)^a"'] & & K(J) \\ & K(J)^X \arrow[ur, "KP(t)"'] & \end{tikzcd}$$

Since $K(J) = H(I)$ and $KP(t) = I(t)$, $Kj(A)$ is the joint equalizer of all pairs $(\pi_{H(I)}(\omega(A)(a)), I(t)H(I)^a)$. That is, the image of $Kj(A)$ consists of all f: $H(A) \to H(I)$ such that $I(t)(fa) = f(A(t)(a))$ for all sets X, all $a\colon X \to H(A)$ and all $t\colon T(X) \to T(1)$. This is precisely the set of H-homomorphisms $A \to I$, which corresponds bijectively with $\mathfrak{A}(A, I)$. Thus we have bijections $KU(A) \to \mathfrak{A}(A, I)$ which are easily seen to be natural in A.

U is determined by F (hence by (I, J)) only up to equivalence. Since $KU \cong \mathfrak{A}(-, I)$ and $(\mathfrak{B}, K)$ has unique transfer, we may choose U so that $KU = \mathfrak{A}(-, I)$.

This completes the proof.

We can obtain a correspondence between adjoint pairs (F, U) and pairs (I, J) also in another situation, in which we assume less about $\mathfrak{A}$ and more about $\mathfrak{B}$. Suppose

(B0) $(\mathfrak{A}, H)$ is operational and has unique transfer, $\mathfrak{A}$ is complete and H preserves limits. $(\mathfrak{B}, K)$ satisfies analogous conditions.

We then consider all (I, J) satisfying:

(B1) I and J are objects of $\mathfrak{A}$ and $\mathfrak{B}$ respectively and $H(I) = K(J)$;

(B2) every H-operation on I is a K-homomorphism, i.e., for every ω: $H^X \to H$ there is a (necessarily unique) ω^+ in $\mathfrak{B}(J^X, J)$ such that $\omega(I) = K(\omega^+)$.

The class of all such (I, J) is made into a category as before.

We observe that condition (B2) can be given the symmetric form: every H-operation on I commutes with every K-operation on J. More precisely, the following diagram commutes for every $\alpha\colon H^X \to H$ and $\beta\colon K^Y \to K$:

$$\begin{array}{ccc} \left(H(I)^X\right)^Y \cong \left(K(J)^Y\right)^X & \overset{\beta(J)^X}{\rightarrow} & K(J)^X \cong H(I)^X \\ \alpha(I)^Y\downarrow & & \downarrow\alpha(I) \\ H(I)^Y \cong K(J)^Y & \underset{\beta(J)}{\rightarrow} & K(J) \cong H(I) \end{array}$$

Because of the symmetry this is also equivalent to: every K-operation on J corresponds to an H-homomorphism.

PROPOSITION 2.7. *Assume* (B0). *Then*:

(a) *the category* $\mathfrak{Q}$ *of pairs* (I, J) *satisfying* (B1) *and* (B2) *is equivalent to a full subcategory* $\mathfrak{P}'$ *of the category* $\mathfrak{P}$ *of adjoint pairs* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{B}$ *such that HF is representable*;

(b) *the adjoint pair* (F, U) *associated to* (I, J) *can be chosen so that* $HF = \mathcal{B}(-, J)$ *and* $KU = \mathcal{A}(-, I)$;

(c) *if* K *is representable then* $\mathcal{P}' = \mathcal{P}$.

PROOF. Following Linton [20], form the theory $\mathcal{T}_H$ whose objects are sets, $\mathcal{T}_H(X, Y)$ being the class of natural transformations $H^X \to H^Y$. T_H is the obvious functor $\mathrm{Sets}^{\mathrm{op}} \to \mathcal{T}_H$. Let $(\mathcal{A}_1, H_1)$ be the concrete category of T_H-algebras in Sets and let Φ: $\mathcal{A} \to \mathcal{A}_1$ be Linton's comparison functor. Thus

$$\Phi(A)(X) = H(A)^X, \qquad \Phi(A)(\omega) = \omega(A),$$

and Φ is full and faithful. It is easily seen that $\mathcal{D}$ is isomorphic to a full subcategory $\mathcal{D}'$ of the category $\mathcal{D}_1$ of pairs (I_1, J) satisfying (A1) and (A2) with $\mathcal{A}$ replaced by $\mathcal{D}_1$. The pair (I, J) in $\mathcal{D}$ corresponds to $(\Phi(I), J)$ in $\mathcal{D}'$, and (I_1, J) in $\mathcal{D}_1$ is in $\mathcal{D}'$ iff $I_1 = \Phi(I)$ for some I in $\mathcal{A}$. By Proposition 2.6, $\mathcal{D}_1$ is equivalent to the category $\mathcal{P}_1$ of adjoint pairs $\mathcal{A}_1^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathcal{B}$ with H_1F representable (and, necessarily, KU_1 representable also). It is clear that $\mathcal{D}'$ is equivalent to the full subcategory of $\mathcal{P}_1$ which contains $\mathcal{A}_1^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathcal{B}$ iff KU_1 is represented by an object $I_1 = \Phi(I)$. This subcategory of $\mathcal{P}_1$ is equivalent to a full subcategory $\mathcal{P}'$ of the category $\mathcal{P}$. A pair $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ in $\mathcal{P}$ is in $\mathcal{P}'$ iff the right adjoint U_1 of $F_1 = \Phi F$ is such that KU_1 is represented by an object $I_1 = \Phi(I)$.

The only detail which perhaps should be given is this: suppose $\mathcal{A}_1^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathcal{B}$ is an adjoint pair with $KU_1 = \mathcal{A}(-, \Phi(I))$. Then by Proposition 2.2 each $F_1(B)$ is the equalizer of a pair of morphisms $\Phi(I)^X \rightrightarrows \Phi(I)^Y$. Let $F(B)$ be the equalizer in $\mathcal{A}$ of the corresponding morphisms $I^X \rightrightarrows I^Y$. Then $\Phi F(B)$ is canonically isomorphic to $F_1(B)$, since $H = H_1\Phi$ preserves limits and H_1 reflects them. F is defined on maps in the obvious way and becomes a functor $\mathcal{B} \to \mathcal{A}$ with $\Phi F \cong F_1$ and right adjoint $U = U_1\Phi$.

This completes the proof of part (a). Part (b) follows from Proposition 2.6, and (c) is clear.

We observe that the assumption that an operational category $(\mathcal{A}, H)$ has unique transfer is clearly equivalent to: every T_H-algebra in Sets which is isomorphic to some $\Phi(A)$ is itself $\Phi(A')$ for some A'. Thus every operational $(\mathcal{A}, H)$ is equivalent to an operational category with unique transfer.

3. Compact objects and duality. Let "Com" denote the category of compact topological spaces (assumed to be Hausdorff). We shall study dualities arising from functor pairs $\mathcal{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ such that:

(C1) $(\mathcal{A}, H)$ is a concrete category, $\mathcal{A}$ has limits and H preserves them;

(C2) F is left adjoint to U with adjunctions η and ε;

(C3) $(\mathcal{A}, H)$ has unique transfer (this assumption is not really necessary and is made merely to simplify the notation).

Let K be the underlying set functor of Com (we often denote $K(B)$ simply by B, as is usual). Clearly $KU \cong \mathcal{A}(-, I)$ where $I = F(B_0)$, B_0 being the one point space. We recall that (Com, K) is varietal [**19**]. Thus Proposition 2.6 applies and shows that $HF \cong \mathrm{Com}(-, J)$ for some J. Clearly $H(I) \cong K(J)$. Replacing J by a homeomorphic space we may assume without loss of generality that $H(I) = K(J)$. Similarly, replacing U by an equivalent functor, we may assume $KU = \mathcal{A}(-, I)$. By (C3), we may also replace F by an equivalent functor so that $HF = \mathrm{Com}(-, J)$.

We gave a proof in §1 that (Com, K) is operational (which of course also follows from the fact that it is varietal). If $(\mathcal{A}, H)$ is also operational and has unique transfer, then Proposition 2.7 applies and tells us that an adjoint pair (F, U) can be constructed from any (I, J) satisfying (B1) and (B2).

Com is a full reflective subcategory of Top, the category of topological spaces. It can also be considered as a full reflective subcategory of Uni, the category of uniform spaces. Thus the adjoint pair (U, F) gives rise to adjoint pairs $\mathcal{A}^{\mathrm{op}} \rightleftarrows \mathrm{Top}$ and $\mathcal{A}^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathrm{Uni}$. We shall need the latter pair. Letting K_1 be the underlying set functor $\mathrm{Uni} \to \mathrm{Sets}$, we have $K_1 U_1 = \mathcal{A}(-, I)$ and $HF_1 = \mathrm{Uni}(-, J)$, if F_1 and U_1 are properly chosen.

We shall obtain necessary and sufficient conditions for UF, FU to be idempotent, provided a further assumption is satisfied by J:

(C4) J is injective with regard to the inclusion maps $B \to J^X$ for all sets X and all closed subspaces B of J^X.

For example, this is true if $J = [-1, 1]$ or J is a finite discrete space.

We need two definitions. (I, J) always denotes the pair of objects associated to (F, U). Let $\mathcal{A}_1$ denote a full subcategory of $\mathcal{A}$ containing Image F, and $H_1 = H|\mathcal{A}_1$. We call (I, J) *functionally complete* with regard to $\mathcal{A}_1$ if, for every set X and every f in $\mathrm{Com}(J^X, J)$, there is an H_1-operation ω: $(H_1)^X \to H_1$ such that $K(f) = \omega(I)$. When $\mathcal{A}$ is a category of Birkhoff algebras and J is finite (hence discrete), this property is equivalent to what has been called "strictly functionally complete" in the literature (see [**8**], which gives references to earlier papers). Well-known examples of such algebras are finite prime fields in the category of rings, and any finite field F in the category of F-algebras.

We call (I, J) *Weierstrass* with regard to $\mathcal{A}_1$ if, for every integer $n \geqslant 0$, every f in $\mathrm{Com}(J^n, J)$ can be uniformly approximated arbitrarily closely by H_1-operations. That is, for each vicinity V of the diagonal of J^2, there is an n-ary H_1-operation ω such that, for all x in $H(I)^n$, $(K(f)(x), \omega(I)(x)) \in V$.

It is clear that if (I, J) is functionally complete or Weierstrass with regard to $\mathcal{C}$, then it has the same property with regard to any smaller $\mathcal{C}_1$, in particular with regard to Image F.

Why do we only look at finite n in the definition of a Weierstrass object?

LEMMA 3.1. *Let B be any uniform space, V any vicinity for B and X any set. Then every f in* $\mathrm{Uni}(B^X, B)$ *can be V-uniformly approximated by some composition $B^X \xrightarrow{p} B^n \xrightarrow{g} B$ for some finite n, with g uniformly continuous and $p = \langle \pi(x_1), \ldots, \pi(x_n) \rangle$, where $\pi(x)\colon B^X \to B$ is the projection corresponding to $x \in X$. That is, $(f(s), gp(s)) \in V$ for all s in B^X.*

PROOF. Since f is uniformly continuous, there is a vicinity W for B^X such that

$$(s, t) \in W \Rightarrow (f(s), f(t)) \in V.$$

Now any vicinity W for B^X contains a vicinity

$$W' = \left\{(s, t) \in (B^X)^2 | (s(x_i), t(x_i)) \in V' \text{ for } i = 1, \ldots, n\right\},$$

where V' is some vicinity for B and $x_1, \ldots, x_n$ are distinct elements of X. Then $(x_1, \ldots, x_n)$ determines $p\colon B^X \to B^n$ as above. Define $\varphi\colon B^n \to B^X$ by

$$\varphi(b_1, \ldots, b_n)(x) = \begin{cases} b_i & \text{if } x = x_i, \\ b_1 & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

Then φ is uniformly continuous and $(\varphi p)(s)(x_i) = \pi(x_i)(s) = s(x_i)$, hence $(s, (\varphi p)(s)) \in W'$ for all s in B^X. Thus $g = f\varphi$ is uniformly continuous and

$$\forall_{s \in B^X} (f(s), (gp)(s)) = (f(s), (f\varphi p)(s)) \in V.$$

This completes the proof.

For any uniform spaces B and C, we denote by $\mathbf{Uni}(B, C)$ the uniform space consisting of the set $\mathrm{Uni}(B, C)$ with the *uniform uniformity*, a basis of which is the set of all W^B, where W is a vicinity for C and

$$W^B = \left\{(f, g) \in (\mathrm{Uni}(B, C))^2 | \forall_{b \in B} (f(b), g(b)) \in W\right\}.$$

PROPOSITION 3.2. *Suppose* $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfies* (C1), (C2), (C3) *and* (C4) *and the associated* (I, J) *is* (a) *Weierstrass or* (b) *functionally complete with regard to $\mathcal{C}$. Let E be a subset of $H(I^X) = H(I)^X = K(J)^X$ which is closed under* (a) *all finitary H-operations on I^X,* (b) *all H-operations on I^X. Let B be the set X endowed with the weak uniformity determined by E and J, i.e., the smallest uniformity $\mathcal{U}$ such that all e in E are uniformly continuous maps $(X, \mathcal{U}) \to J$. Then E is* (a) *dense in* $\mathbf{Uni}(B, J)$, *or* (b) *equal to* $\mathbf{Uni}(B, J)$.

REMARK. It is clear (assuming the proposition to be true) that it will suffice if (I, J) is Weierstrass or functionally complete with regard to Image F.

PROOF. (a) B has a basis of vicinities of the form

$$W(V, e) = \{(x, x') \in X^2 | (e_i(x), e_i(x')) \in V \text{ for } i = 1, \ldots, n\},$$

where V is any vicinity for J and $e = \langle e_1, \ldots, e_n \rangle: X \to J^n$ with $e_1, \ldots, e_n \in E$.

Uni(B, J) has a basis of vicinities of the form V^B defined above. Now let $f \in \mathrm{Uni}(B, J)$. We want to show that it can be V-approximated by an element of E, for any given vicinity V for J. There exists a vicinity W for B such that $(f \times f)(W) \subseteq V$, hence there is a vicinity V' for J and $e_V = (e_1, \ldots, e_{n(V)})$ with $e_1, \ldots, e_{n(V)} \in E$ such that

$$(x, x') \in W(V', e_V) \Rightarrow (f(x), f(x')) \in V,$$

that is,

$$(e_V(x), e_V(x')) \in V'^n \Rightarrow (f(x), f(x')) \in V.$$

Consider $\prod_V J^{n(V)} = J^Z$, where $Z = \sum_V n(V)$, V running over all vicinities for J, with projections $\pi(z): J^Z \to J$ (for $z \in Z$) and $\pi_V: J^Z \to J^{n(V)}$. Also let $e \in \mathrm{Uni}(X, J^Z)$ be determined by $\pi_V e = e_V$, then $\pi(z)e \in E$ for all $z \in Z$.

Now

$$(e(x) = e(x')) \Rightarrow \forall_V (e_V(x) = e_V(x')) \Rightarrow \forall_V \big((f(x), f(x')) \in V\big).$$

Since J is separated,

$$e(x) = e(x') \Rightarrow f(x) = f(x'),$$

hence there is a function g: Image $e \to J$ such that $g(e(x)) = f(x)$ for all x in X. Moreover, it is easily verified that g is uniformly continuous. Hence it can be extended to a uniformly continuous function $\overline{\text{Image } e} \to J$, and this, by (C4), can be extended to $h \in \mathrm{Com}(J^Z, J)$. Now (I, J) is Weierstrass with regard to $\mathcal{Q}$, and it follows easily from Lemma 3.1 that h can be V-approximated by $\omega(I)$ for some Z-ary H-operation ω, that is, $(h(t), \omega(I)(t)) \in V$ for all t in J^Z. Hence we obtain $(f(x), \omega(I)(e(x))) \in V$ for all x in X, recalling that $f(x) = ge(x) = he(x)$.

We apply Proposition 2.2 to the adjoint pair $\mathcal{Q}^{\mathrm{op}} \underset{F_1}{\overset{U_1}{\rightleftarrows}} \mathrm{Uni}$. By part (c) we conclude that, for each x in $X = K(B)$, the evaluation map $\hat{x}$: $\mathrm{Uni}(B, J) \to K(J)$, i.e. $HF_1(B) \to H(I)$, is an H-homomorphism from $F_1(B)$ to I, whence the diagram (1) commutes.

$$\begin{array}{ccc} \mathrm{Uni}(B, J)^X & \xrightarrow{\omega F_1(B)} & \mathrm{Uni}(B, J) \\ (\hat{x})^Z \downarrow & & \downarrow \hat{x} \\ H(I)^Z & \xrightarrow[\omega(I)]{} & H(I) \end{array} \qquad (1)$$

Let $\bar{e}$ in $\mathrm{Uni}(B, J)^X$ correspond to e: $B \to J^X$ and let $e' = \omega F_1(B)(\bar{e})$. Then

$(\hat{x})^Z(\bar{e}) = e(x)$, so

$$\omega(I)(e(x)) = \hat{x}\big(\omega F_1(B)(\bar{e})\big) = \hat{x}(e') = e'(x).$$

$F_1(B)$ is a subobject of I^X by Proposition 2.2. Now $E \subseteq HF_1(B) \subseteq H(I^X)$ and E is closed under H-operations on I^X. Hence E is closed under operations on $F_1(B)$, so $e' \in E$.

We recall that $(f(x), e'(x)) \in V$ for all x in X, that is, $(f, e') \in V^B$. This completes the proof in case (a).

(b) The proof is an obvious modification of that of (a). Instead of approximating $h: J^Z \to J$ by $\omega(I)$, we can find ω such that $h = \omega(I)$. We then arrive finally at the conclusion $f = e'$, with e' in E.

Before stating a corollary to Proposition 3.2, we need another definition.

A basis $\mathfrak{W}$ of the uniformity of J will be called *stable* over $\mathfrak{B}_1 \subseteq$ Com if, for all W in $\mathfrak{W}$, B in $\mathfrak{B}_1$ and f, g in $HF(B) = \mathrm{Com}(B, J)$, we have:

$$\big(\forall_{b \in B}(f(b), g(b)) \in W\big) \Rightarrow \forall_{s \in \mathfrak{A}(F(B), I)}(H(s)(f), H(s)(g)) \in W.$$

For example, when J is discrete the basis $\mathfrak{W}$ containing only the diagonal of J^2 is stable over Com. The existence of a stable basis implies that each $H(s)$: $\mathrm{Com}(B, J) = HF(B) \to H(I) = K(J)$ is a uniformly continuous map from $\mathbf{Uni}(B, J)$ to J, for each B in $\mathfrak{B}_1$.

COROLLARY 3.3. *In case* (a), *the image of* $H\varepsilon(A)$ *is dense in* $HFU(A) = \mathrm{Com}(U(A), J)$, *in the uniform uniformity*, *for all* A *in* $\mathfrak{A}$. *If in addition the uniformity of* J *has a basis stable over the image of* U (*or* UF *or* UFU, *etc.*) *then* FU *is idempotent. In the case* (b), $H\varepsilon(A)$ *is a surjection, hence* FU *is idempotent.*

In case (b), $H \in (A)$ *is a surjection, hence* FU *is idempotent.*

PROOF. (a) We apply Proposition 3.2 with $X = KU(A) = \mathfrak{A}(A, I)$. Let E be the image of $H\varepsilon(A)$. This is a subset of $HFU(A) = \mathrm{Com}(U(A), J)$, so can be considered as a subset of $H(I)^X = H(I^X)$. It is closed under H-operations on $FU(A)$, hence under H-operations on I^X. By Proposition 2.2 it is precisely the set of evaluation functions $\hat{a}$: $\mathfrak{A}(A, I) \to H(I)$, where $\hat{a} \in H(A)$. Now, also by Proposition 2.2, there is a monomorphism θ: $U(A) \to J^{H(A)}$ such that $\pi(a)\theta = \hat{a}$ for all a in $H(A)$. The weak uniformity on $KU(A)$ induced by the functions $\hat{a}$ is therefore the weak uniformity induced by $K(\theta)$. But in Com every monomorphism is an embedding, so this weak uniformity is the original uniformity of $U(A)$. Thus, by Proposition 3.2, E is dense in $\mathbf{Uni}(U(A), J)$, i.e. in $\mathrm{Com}(U(A), J)$ with the uniform uniformity.

Suppose now that the uniformity of J has a basis $\mathfrak{W}$ stable over Image U. For any A in $\mathfrak{A}$ and s in $\mathfrak{A}(FU(A), I)$, $H(s)$ is a uniformly continuous map to J from $\mathrm{Com}(U(A), J)$ with the uniform uniformity. Since E is dense in $\mathrm{Com}(U(A), J)$, $H(s)$ is determined by $H(s)H\varepsilon(A)$, hence s is determined by

$s\varepsilon(A)$. By Proposition 2.1, FU is idempotent.

Suppose we only assume that $\mathfrak{U}$ is stable over Image UF. By applying the above result to (Image $F)^{\mathrm{op}} \underset{F'}{\overset{U'}{\rightleftarrows}}$ Image U (obtained by restriction of U and F), we see that $F'U'$ is idempotent, whence FU is idempotent, by the Corollary to Theorem 2.0. By repeated application of 2.0 we see that it suffices if $\mathfrak{U}$ is stable over Image UFU, etc.

(b) Since the image of $H\varepsilon(A)$ is closed under all H-operations on $FU(A)$, it coincides with $HFU(A)$, so again $H(s)$ is determined by $H(s)H\varepsilon(A)$.

THEOREM 3.4. *If* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfy* (C1) *to* (C4) *then the following statements are equivalent*:

(1) *UF and FU are idempotent*;

(2) *(I, J) is functionally complete with regard to* Image F;

(3) *(I, J) is Weierstrass with regard to* Image F *and the uniformity of J has a basis stable over the image of U* (*or of UF or UFU, etc.*)

PROOF. That (2) ⇒ (1) and (3) ⇒ (1) follows from Corollary 3.3. We shall now show that (1) implies (2) and (3).

Suppose UF is idempotent. Then $\eta U(A)$: $U(A) \to UFU(A)$ is an isomorphism for each A in $\mathfrak{A}$. Thus, by Proposition 2.2, each s in $KUFU(A) = \mathfrak{A}(FU(A), I)$ is an evaluation map $\hat{b}$ for some b in $U(A)$. That is, $H(s)(f) = f(b) = \hat{b}(f)$. Therefore every basis of the uniformity of J is stable over Image U.

U and F induce an equivalence (Image $F)^{\mathrm{op}} \underset{F'}{\overset{U'}{\rightleftarrows}}$ Image U. Let $H' = H|$Image F. Since $H'F' = \mathfrak{B}(-, J)$ and (F', U') is an equivalence of categories, it is clear that every natural transformation $\mathfrak{B}(-, J)^X \to \mathfrak{B}(-, J)$ is $\omega F'$ for some ω: $(H')^X \to H'$. It follows that for every t in $\mathfrak{B}(J^X, J)$ there is an ω such that $\omega F' = \mathfrak{B}(-, t)$. In particular, $\omega F'(B_0) = \mathfrak{B}(B_0, t) \cong K(t)$, whence $\omega(I) = K(t)$. Thus (I, J) is functionally complete (and, a fortiori, Weierstrass) with regard to Image F.

The proof is now complete.

The next two results, while not actually needed later, seem of some interest.

PROPOSITION 3.5. *Suppose* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfy* (C1) *to* (C4) *and the uniformity of J has a basis consisting of all sets*

$$W_{\alpha,\beta} = \{(i, j) \in J^2 | \alpha(I)(i, j) = \beta(I)(i, j)\}$$

where α and β are given binary operations $H^2 \to H$. Then this basis is stable over Image U.

PROOF. Suppose $f, g \in \mathrm{Com}(U(A), J)$ and

$$\forall_{b \in U(A)} (f(b), g(b)) \in W_{\alpha,\beta},$$

i.e. $\forall_{b \in U(A)} \alpha(I)(f(b), g(b)) = \beta(I)(f(b), g(b))$. Recalling that $\hat{b}$ is an H-homomorphism we may deduce that

$$\alpha FU(A)(f, g) = \beta FU(A)(f, g),$$

so, for all $s \in \mathcal{Q}(FU(A), I)$,

$$s(\alpha FU(A)(f, g)) = s(\beta FU(A)(f, g)),$$

i.e.

$$\alpha(I)(s(f), s(g)) = \beta(I)(s(f), s(g))$$

or

$$(s(f), s(g)) \in W_{\alpha, \beta}.$$

The proof is now complete.

PROPOSITION 3.6. *Suppose* $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfies* (C1) *to* (C3). *Let* $H_1 = H|\mathrm{Image}\, F$ *and let* E *be the set of all* f *in* $\mathrm{Com}(J^X, J)$ *such that* $K(f) = \omega(I)$ *for some* ω: $H_1^X \to H_1$. *If the uniformity of* J *has a basis stable over* Com, *then* E *is closed in* $\mathrm{Com}(J^X, J)$ *in the uniform uniformity.*

PROOF. We recall that, as in the proof of Proposition 2.4, for any ω: $H_1^X \to H_1$ there is a unique ω^* in $\mathrm{Com}(J^X, J)$ such that

$$\omega F = \left(\mathrm{Com}(-, J)^X \xrightarrow{\sim} \mathrm{Com}(-, J^X) \xrightarrow{\mathrm{Com}(-, \omega^*)} \mathrm{Com}(-, J)\right)$$

and that $K(\omega^*) = \omega(I)$.

Each f in $\mathrm{Com}(J^X, J)$ determines a natural transformation $\bar{f}$: $H_1 F^X \to H_1 F$, $\bar{f} = (\mathrm{Com}(-, J)^X \xrightarrow{\sim} \mathrm{Com}(-, J^X) \xrightarrow{\mathrm{Com}(-, f)} \mathrm{Com}(-, J))$. It is clear that $f = \omega^*$ iff $\bar{f} = \omega F$. Thus E is the set of all f such that $\bar{f}$ is of the form ωF for some ω: $H_1^X \to H_1$, and clearly this is true iff the diagram

$$\begin{array}{ccc} \mathrm{Com}(B, J)^X & \xrightarrow{\bar{f}(B)} & \mathrm{Com}(B, J) \\ H(\theta)^X \downarrow & & \downarrow H(\theta) \\ \mathrm{Com}(B_1, J)^X & \xrightarrow[\bar{f}(B_1)]{} & \mathrm{Com}(B_1, J) \end{array}$$

commutes for all B, B_1 in Com and all θ in $\mathcal{Q}(F(B), F(B_1))$.

Suppose f is in the closure of E, $u \in \mathrm{Com}(B, J)^X$, $v = \bar{f}(B_1)H(\theta)^X(u)$ and $w = H(\theta)\bar{f}(B)(u)$. We need to show that $v = w$, and it will suffice to show that (v, w) is in every vicinity of the diagonal for $\mathrm{Com}(B_1, J)$. This is easily shown to follow from the facts: (1) f can be uniformly approximated arbitrarily closely by elements g of E (for which $\bar{g}$ must be of the form ωF), whence $\bar{f}$ is approximated by ωF; (2) if V is any element of the stable basis $\mathcal{V}$

then $H(\theta)$ carries two V-close functions in $\mathrm{Com}(B, J)$ to V-close functions in $\mathrm{Com}(B_1, J)$.

This completes the proof.

This result could clearly be used to give a different proof of (3) $\Rightarrow$ (1) in Theorem 3.4 (one must use also 3.1, (2) $\Rightarrow$ (1) of 3.4 and the corollary to 2.0).

If UF and FU are idempotent, what can we say about the dual categories Image U and Image F? We need a few lemmas, which we shall collect under one heading.

LEMMA 3.7. *Given a compact space J which satisfies* (C4), *let $\mathcal{C} = \mathcal{C}(J)$ be the category of all compact* (*Hausdorff*) *spaces which are homeomorphic to closed subspaces of powers of J. Then*:

(a) *Every C in $\mathcal{C}$ is an equalizer of some pair of morphisms $J^X \rightrightarrows J^Y$.*

(b) *$\mathcal{C}$ is a full reflective subcategory of* Com *with surjective reflection maps.*

(c) *$\mathcal{C}$ is the limit closure of J in* Com.

(d) *$(\mathcal{C}^{\mathrm{op}}, L)$ is varietal, where $L = \mathfrak{B}(-, J)$.*

PROOF. (a) Suppose C is a closed subspace of J^X. We need only show that, for s in J^X but not in C, there exist $f, f' \in \mathrm{Com}(J^X, J)$ which are equal on C yet have $f(s) \neq f'(s)$. In fact, given any $i, j \in J$, we may define $h\colon B \cup \{s\} \to J$ by putting $h(b) = i$ for $b \in B$ and $h(s) = j$. Then h is continuous and, by (C4), has a continuous extension $f\colon J^X \to J$. Replacing j by $j' \neq j$, we similarly obtain $f'\colon J^X \to J$. Then f and f' agree on B but differ on s.

The argument assumes that J has at least two distinct points. If J has only one point or if J is empty, the result holds trivially.

(b) is well known.

(c) That $\mathcal{C}$ is contained in the limit closure of J follows from (a). The converse inclusion holds, because $J \in \mathcal{C}$ and $\mathcal{C}$ is limit closed by (b).

(d) To show that $(\mathcal{C}^{\mathrm{op}}, L)$ is varietal, we need only (by **[14]**) check that:

(i) $\mathcal{C}$ is complete and has cokernel pairs,

(ii) $\mathcal{C}$ is the limit closure of J in $\mathcal{C}$,

(iii) J is injective with regard to all regular monos of $\mathcal{C}$,

(iv) every coequivalence relation in $\mathcal{C}$ is a cokernel pair.

Now (i) is true because $\mathcal{C}$ is a full reflective subcategory of Com.

(ii) follows from (c), since $\mathcal{C}$-limits are just Com-limits of objects in $\mathcal{C}$.

(iii) follows from (C4) and the observation that any regular mono $C' \to C$ in $\mathcal{C}$ is a regular mono in Com, that is, a subspace inclusion, and C is a subspace of some power of J.

(iv) Let $u, v\colon B \rightrightarrows C$ be a coequivalence relation in $\mathcal{C}$. In fact, we need merely assume that u, v are jointly epi and that there is a morphism $t\colon C \to B$ such that $tu = tv = 1_B$. Let $k\colon K \to B$ be the equalizer of u and v in $\mathcal{C}$, hence in Com. Let $r, s\colon B \rightrightarrows D$ be the cokernel pair of k in Com. Then there is a unique map $w\colon D \to C$ such that $wr = u$ and $ws = v$. Since r, s are jointly epi in Com, D is the union of $r(B)$ and $s(B)$. Also $twr = tws = 1_B$ and it follows easily that w is injective. Thus w is a homeomorphism of D with a closed subspace of C. It follows as in part (a) above that w is a regular mono of $\mathcal{C}$. But w is epi in $\mathcal{C}$ because u, v are jointly epi in $\mathcal{C}$. Therefore w is an isomorphism and so (u, v) is the cokernel pair of k in Com, hence in $\mathcal{C}$.

REMARK. In the special case when J is a closed interval and $\mathcal{C} = \text{Com}$, so that $\mathcal{C}^{\text{op}}$ is the category of commutative C^*-algebras, the result (d) was obtained by Joan Pelletier [23]. If J is finite with more than one element, then $\mathcal{C}$ is the category of Boolean spaces and $\mathcal{C}^{\text{op}}$ can be described in many ways as a category of Birkhoff algebras, the best known description being that as the category of Boolean algebras.

THEOREM 3.8. *Suppose* $\mathcal{A}^{\text{op}} \underset{F}{\overset{U}{\rightleftarrows}} \text{Com}$ *satisfy* (C1) *to* (C4), $(\mathcal{A}, H)$ *is operational and* UF, FU *are idempotent. Then*:

(1) Image U *is the limit closure of* J *in* Com (*or* Top *or* Uni) *and its objects are precisely the equalizers of pairs of morphisms* $J^X \rightrightarrows J^Y$.

(2) *The objects of* Image U *are the compact Hausdorff spaces cogenerated by* J, *that is, homeomorphic to subspaces of powers of* J.

(3) $\eta(B)$: $B \to UF(B)$ *is surjective for every compact Hausdorff space* B.

(4) Image F *is the limit closure of* I *in* A *and its objects are precisely the equalizers of pairs of morphisms* $I^X \rightrightarrows I^Y$.

(5) (Image F, H') *is varietal, where* H' *is the restriction of* H, *and the free object generated by the set* X *is* $F(J^X)$.

(6) *If* $(\mathcal{A}, H)$ *is varietal, then* Image F *contains the coequalizer in* $\mathcal{A}$ *of any congruence relation in* Image F.

REMARK. The surjectivity of $\eta(B)$ in (3), even only for B in Image U, is (by Proposition 2.1) a sufficient condition for UF to be idempotent.

PROOF. (1) simply recalls Proposition 2.3.

(2) and (3) follow from Lemma 3.7, noting that $\eta(B)$ is the reflection map from B into Image U.

(4) is again Proposition 2.3.

(5) follows immediately from Lemma 3.7.

(6) Suppose $\mathcal{A}'$ (in our case Image F) is any full subcategory of $\mathcal{A}$, such that $(\mathcal{A}', H|\mathcal{A}')$ is varietal. Then $\mathcal{A}'$ is closed under coequalizers of congruence relations. This is well known for Birkhoff varieties and may easily be derived for Linton varieties from the following two facts:

(1) every congruence relation in $\mathcal{A}'$ is a kernel pair in $\mathcal{A}'$, hence is the kernel pair of its coequalizer in $\mathcal{A}'$;

(2) if the diagram $A_1 \underset{v}{\overset{u}{\rightrightarrows}} A_2 \xrightarrow{s} A_3$ in $\mathcal{A}'$ is exact, i.e. (u, v) is the kernel pair of s and s is the coequalizer of (u, v) in $\mathcal{A}'$, then the diagram is carried by H

to an exact diagram in Sets and hence the original diagram is exact also in $\mathcal{Q}$ (since $(\mathcal{Q}, H)$ is operational).

THEOREM 3.9. *Suppose* $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfies* (C1) *to* (C4) *and* $(\mathcal{Q}, H)$ *is operational.*

(a) *If* (I, J) *is functionally complete with regard to* $\mathcal{Q}$ *then* Image F *is subobject closed in* $\mathcal{Q}$ *and hence consists precisely of all subobjects in* $\mathcal{Q}$ *of powers of* I. *Also* Image F *is the equational closure of* I *in* $\mathcal{Q}$, *that is, it contains precisely those* A *in* $\mathcal{Q}$ *for which, if* ω, ω': $H^X \to H$ *are operations such that* $\omega(I) = \omega'(I)$, *then also* $\omega(A) = \omega'(A)$. *Hence, if* $p \in \mathcal{Q}(F(B), A)$ *and* $H(p)$ *is surjective then* A *is in* Image F.

(b) *If* (I, J) *is Weierstrass with regard to* $\mathcal{Q}$ *then* Image F *contains all subobjects in* $\mathcal{Q}$ *of any* $F(B)$ *whose underlying sets are closed in the uniform uniformity of* $HF(B) = \mathrm{Com}(B, J)$. *Hence* Image F *consists precisely of such subobjects of powers of* I.

PROOF. (a) Suppose (I, J) is functionally complete with regard to $(\mathcal{Q}, H)$. By Proposition 3.2 any subset E of $H(I^X)$ which is closed under all H-operations on I^X is $HF_1(B)$ for some uniform space B, hence is $HF(B')$ for some B' in Com (the reflection of B). Clearly this remains true if we replace I^X by any $F(B'')$.

Now suppose that m: $A \to F(B'')$ is a monomorphism. Then $H(m)$: $H(A) \to HF(B'')$ is also a monomorphism, since H preserves limits. Thus there is some B' such that $HF(B') = \mathrm{Image}\, H(m)$, the inclusion $HF(B') \to HF(B'')$ being an H-homomorphism $F(B') \to F(B'')$. Since $(\mathcal{Q}, H)$ is operational, A is isomorphic to $F(B')$. That is, A is in Image F.

Since any $F(B)$ is a subobject of a power of I, Image F consists precisely of all such subobjects. It is then clear that Image F is contained in the equational closure of I. To show the converse, let H' be the restriction of H to Image F. As in the proof of Proposition 2.7, form the theory $(\mathfrak{T}_{H'}, T_{H'})$, the category $\mathcal{Q}'$ of T-algebras and the comparison functor Φ: Image $F \to \mathcal{Q}'$. (Image F, H') is varietal by Theorem 3.8, hence Φ is an equivalence **[19]**.

Suppose A is in the equational closure of I in $\mathcal{Q}$. Then we define an H'-algebra A^*: $\mathfrak{T}_{H'} \to$ sets. Let $A^*(X) = H(A)^X$. To define A^* on maps t: $(H')^X \to (H')^Y$, it suffices to consider the special case $Y = 1$. Thus, suppose t: $H'^X \to H'$. Then $t(I)$: $H(I)^X \to H(I)$ and, by functional completeness, there exists an H-operation ω_t: $H^X \to H$ such that $t(I) = \omega_t(I)$. Now $\omega_t(A)$ is uniquely determined by t; for if also $t(I) = \omega_t'(I)$, then $\omega_t(A) = \omega_t'(A)$, since A is in the equational closure of I. We put $A^*(t) = \omega_t(A)$. We then define A^* on maps $(H')^X \to (H')^Y$ so that $A^* T_{H'}$ is product preserving.

Since Φ is an equivalence, there is a natural isomorphism e: $A^* \xrightarrow{\sim} \Phi F(B)$

for some B. Then $e(1)$: $H(A) \xrightarrow{\sim} HF(B)$ is easily seen to be an H-homomorphism. Hence $A \cong F(B)$. The proof is now complete.

An interesting consequence of Proposition 3.2 is the following variation on a classical theme.

THEOREM 3.10. *Suppose* $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfies* (C1) *to* (C4) *and* (I, J) *is* (a) *Weierstrass or* (b) *functionally complete with regard to* $\mathcal{C}$. *Let* B *be a compact space and* E *a subset of* $\mathrm{Com}(B, J) = HF(B)$ *which separates points of* B *and is closed under* (a) *finitary* H-*operations, or* (b) *all* H-*operations on* $F(B)$. *Then* E *is* (a) *dense in* $\mathrm{Com}(B, J)$ *in the uniform uniformity, or* (b) *equal to* $\mathrm{Com}(B, J)$.

PROOF. Suppose B has underlying set X and uniformity $\mathcal{V}$. Let $\mathcal{W}$ be the smallest uniformity on X which will make all mappings $e \in E$ uniformly continuous from $(X, \mathcal{W})$ to J. Then $\mathcal{W} \subseteq \mathcal{V}$, since $E \subseteq \mathrm{Uni}(B, J)$. Also $\mathcal{W}$ is separated, since J is separated and E separates points of X. Since $\mathcal{V}$ is compact, $\mathcal{W}$ is separated and $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{W}$ and $\mathcal{V}$ must induce the same topology. Since this topology is compact Hausdorff, there is only one uniformity which induces it, hence $\mathcal{W} = \mathcal{V}$.

The conclusion of the theorem now follows from Proposition 3.2.

The results of this section generalize certain known results on primal algebras. To recapture these results, let us take $(\mathcal{C}, H)$ to be an algebraic category. (To say that $(\mathcal{C}, H)$ is *algebraic* means that it is varietal and that all operations may be obtained by composition from projections and finitary operations.) Equivalently, $\mathcal{C}$ is a primitive class of Birkhoff algebras with the obvious forgetful functor H.

EXAMPLE 1. Let I be any algebra in $\mathcal{C}$ whose underlying set $H(I)$ is finite. Then there is a unique compact Hausdorff space J with underlying set $H(I)$, namely the discrete space. For each finitary operation ω: $H^n \to H$, $\omega(I)$: $H(I)^n \to H(I)$ is clearly a continuous mapping $J^n \to J$. Since $\mathcal{C}$ is algebraic, the infinitary operations are simply projections onto finite products followed by finitary operations. Hence also, for any infinitary operations ω: $H^X \to H$, $\omega(I)$: $H(I)^X \to H(I)$ will be continuous. Thus (I, J) is a compact (Hausdorff) algebra and, by Proposition 2.6, determines an adjoint pair $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$.

Before proceeding any further, we must check that assumption (C4) is satisfied.

LEMMA 3.11. *A discrete uniform space* J *is injective with regard to subspace inclusions* $B \to J^X$ *in* Uni *for all sets* X.

PROOF. Suppose $f \in \mathrm{Uni}(B, J)$ where B is a subspace of J^X in Uni. Then there is a vicinity W for B such that

$$\forall_{b,b' \in B}((b, b') \in W \Rightarrow f(b) = f(b')).$$

We can choose

$$W = \{(b, b') \in B^2 | b(x_i) = b'(x_i) \quad \text{for } i = 1, \ldots, n\}$$

for some finite set $\{x_1, \ldots, x_n\} \subseteq X$. Let p: $J^X \to J^n$ be the obvious projection, which is clearly uniformly continuous. Then

$$\forall_{b,b' \in B}(p(b) = p(b') \Rightarrow f(b) = f(b')).$$

Therefore, there is a function g; Image $p \to J$ such that $gp = f$, and this can be extended to a function h: $J^n \to J$. Since J^n is discrete, $h \in \text{Uni}(J^n, J)$. Thus hp is a uniformly continuous extension of f over J^X.

Since Com is a full reflective subcategory of Uni, this gives the result we want.

We shall now continue our discussion of the example where J is finite and discrete.

In view of Lemma 3.1 the condition that (I, J) is functionally complete reduces to the statement that, for each finite n, every function g: $H(I)^n \to H(I)$ has the form $g = \omega(I)$ for some operation ω: $H^n \to H$. (For any uniformly continuous functions f: $H(I)^X \to H(I)$ can be approximated by a composition $J^X \xrightarrow{p} J^n \xrightarrow{g} J$ and, since J is discrete, $f = gp$.) Thus, our "functionally complete" reduces to the "strictly functional complete" of the literature **[8]**. If, moreover, I has at least two points, it is called "primal".

We now apply Corollary 3.3 and deduce that a primal algebra induces a duality between Image F and Image U. By Theorem 3.8, Image U consists of the closed subspaces of powers of J, which is a finite discrete space with more than one element. As is well known, this is precisely the category of *Boolean spaces*, that is, compact spaces with a basis for open sets consisting of sets which are both open and closed. By Theorem 3.8, Image F is the equational closure of I in $\mathcal{Q}$. Thus we recapture the following result of Hu **[8]**.

THEOREM 3.12 (HU). *Let* $(\mathcal{Q}, H)$ *be an algebraic category and* I *a primal algebra in* $\mathcal{Q}$. *Then there is a duality between Boolean spaces and the equational closure of* I *in* $\mathcal{Q}$. *Moreover, the latter consists precisely of the subalgebras of powers of* I.

Let us just mention two well-known examples of primal algebras.

Let $\mathcal{Q}$ be the category of rings and I any prime field $\mathbf{Z}/(p)$. Then I is primal and its equational closure consists of all *p-rings*, that is, rings A such that $px = 0$ and $x^P = x$ for all $x \in A$.

Let I be any finite field, say of $q = p^n$ elements, and $\mathcal{Q}$ the category of I-algebras. Then I, regarded as an object of $\mathcal{Q}$, is primal and its equational closure consists of all I-algebras A such that $x^q = x$ for all $x \in A$.

REMARK. Hu actually obtained a (more complicated) duality from a "locally primal" algebra, which is more general than a primal algebra.

Hu obtained certain known properties of primal algebras as consequences of his duality theorem. Some of these may be generalized to our set up.

PROPOSITION 3.12. *Suppose* $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ *satisfy* (C1) *to* (C4), $(\mathcal{Q}, H)$ *is operational and UF and FU are idempotent. Let A be any object in* Image F. *Then*

(a) *every reflexive relation on A is a congruence relation*;

(b) *any two congruence relations are permutable, that is, their relative products* $\theta * \theta'$ *and* $\theta' * \theta$ *are equal*;

(c) *the congruence relations on A form a distributive lattice, in fact, a complete Brouwerian lattice*;

(d) *a proper congruence relation on A is maximal if and only if it is the kernel pair of some morphism* $A \to I$, *hence* $U(A) = \mathcal{Q}(A, I)$ *may be regarded as the set of maximal proper congruence relations on* A.

PROOF. (a) was established incidentally while proving Lemma 3.7(d), since Image F is the dual of the category $\mathcal{C}$ considered there.

(b) is a well known and easy consequence of (a).

(c) Since Image F is varietal, any congruence relation on $A = F(B)$ is the kernel pair of its coequalizer in Image F. Hence there is a bijection between the set of congruence relations on A and the set of regular epis from A in Image F (more precisely, of sets of equivalence classes of regular epis). By duality, the set of regular epis from A corresponds to the set of regular monos into $U(A)$ in Image U, equivalently in Com. This is just the set of closed subspaces of $U(A)$. The bijection between congruence relations on A and closed subspaces of $U(A)$ is obviously order reversing. The lattice of closed subspaces of $U(A)$ is clearly distributive, in fact, it is known to be the dual of a complete Brouwerian lattice [3].

(d) The minimal nonempty closed subspaces of $U(A)$ are points, i.e. maps $B_0 \to U(A)$, where B_0 is the one point space. These correspond to morphisms $A \to I$ under the duality, since $F(B_0) = I$. (The empty subspace of $U(A)$ corresponds to the improper congruence relation on A.)

The real and complex Gelfand dualities are also special cases of the dualities considered in this section.

EXAMPLE 2. Let $\mathcal{Q}$ be the category of real Banach algebras (with unit) with norm reducing homomorphisms, and let H be the unit ball functor. Let I be R, the algebra of real numbers and let J be $H(I) = [-1, 1]$ with the metric topology. As discussed in §2, we have a corresponding adjoint pair $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ with $F(B) = C^*(B, R)$, $U(A) = \mathcal{Q}(A, R)$ topologized as a subspace of $J^{H(A)}$.

$U(A)$ is closed in $J^{H(A)}$ (by Proposition 2.3 or a simple ad hoc argument), hence is compact. Thus we have an adjoint pair $\mathcal{C}^{\mathrm{op}} \rightleftarrows \mathrm{Com}$, which we still denote by U, F, and which satisfies (C1) to (C4). J satisfies (C4) by Tietze's theorem.

The statement that (I, J) is Weierstrass with regard to Image F is nearly the classical Weierstrass theorem (in n variables). Not quite, because a polynomial in n variables maps R^n into R, but need not carry J^n into J. Let us consider the functor $\tilde{H}$: Image $F \to$ Sets which takes every algebra to its entire underlying set (not just its unit ball). Thus $\tilde{H}F(B) = \mathrm{Top}(B, R)$ for all B in Com. Clearly an $\tilde{H}$-operation which preserves unit balls defines an H_1-operation, where $H_1 = H|\mathrm{Image}\ F$. Now there is (as will be shown) a unary $\tilde{H}$-operation ω such that

$$\omega(F(B))(f) = \inf(1, \sup(f, -1))$$

for all f: $B \to R$. It follows that, if p is any real polynomial in n variables, then ωp is an $\tilde{H}$-operation, where

$$\omega p(F(B))(f) = \omega(F(B))(p(f_1, \ldots, f_n))$$

for all $f_1, \ldots, f_n$: $B \to R$. Also ωp preserves unit balls, whence it is also an H_1-operation. By the Weierstrass theorem in n variables, any continuous map $J^n \to J$ can, for each $\varepsilon > 0$, be ε-approximated by $p(I)$ for some polynomial p, and it is then clearly ε-approximated by the H_1-operation $\omega p(I)$.

A stable basis for the (metric) uniformity of J is given by

$$\{V_\varepsilon | \varepsilon > 0\} \quad \text{where } V_\varepsilon = \left\{(x_1, x_0) \in J^2 \,|\, |x_1 - x_0| \leqslant \varepsilon\right\}.$$

In fact, for any $f, g \in HF(B)$, the condition that $(f(b), g(b)) \in V_\varepsilon$ for all b in B says that $\|f - g\| \leqslant \varepsilon$. Now any s in $\mathcal{C}(F(B), R)$ is norm reducing so $|s(f) - s(g)| \leqslant \varepsilon$, i.e. $(s(f), s(g)) \in V_\varepsilon$.

Thus, by Corollary 3.3, UF and FU are idempotent and establish a duality $(\mathrm{Image}\ F)^{\mathrm{op}} \cong \mathrm{Image}\ U$. It is clear from Proposition 3.7, part (2), that Image U = Com, using the easily proved fact that any compact (Hausdorff) space can be embedded in a power of J. The dual category Image F is well known to consist of the C^*-algebras. Our approach does not seem to contribute anything to the proof of this.

We return to the (well-known) proof that ω is an $\tilde{H}$-operation. Clearly one need only show that the binary sup and inf are $\tilde{H}$-operations. For any topological spaces X and Y, any homomorphism φ: $\mathrm{Top}(X, R) \to \mathrm{Top}(Y, R)$ preserves squares and hence preserves nonnegativity. For any f in $\mathrm{Top}(X, R)$ let f_+, f_- be the unique nonnegative functions in $\mathrm{Top}(X, R)$ such that $f = f_+ - f_-$ and $f_+ f_- = 0$. These conditions are preserved by any homomorphism φ. Thus $+$ is an $\tilde{H}$-operation, and binary sups and infs are easily defined in terms of it.

Clearly $\|f_+\| \leqslant 1$ if $\|f\| \leqslant 1$, hence the binary sup and inf are H_1-operations. This will be needed in §4.

EXAMPLE 3. We change Example 2 by replacing "real" by "complex" in the definition of $\mathcal{Q}$ and replacing R by the complex algebra $\mathcal{C}$ of complex numbers. As before, we have $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Com}$ satisfying (C1) to (C4). J is now the unit disc in the complex algebra $\mathcal{C}$.

As in Example 2, the uniformity of J has a basis stable over Com. We will show that every continuous $g\colon \mathcal{C}^n \to \mathcal{C}$ is $\omega(\mathcal{C})$ for some $\omega\colon H_1^n \to H_1$, where $H_1 = H|\mathrm{Image}\ F$. We begin by considering the functor $\tilde{H}$: Image $F \to$ Sets which takes every algebra to its underlying set (not just its unit ball). Then $\tilde{H}F(B) = \mathrm{Top}(B, \mathcal{C})$ for all B in Com. To every continuous $g\colon \mathcal{C}^n \to \mathcal{C}$ and B in Com we associate

$$\bar{g}(F(B))\colon \mathrm{Top}(B, \mathcal{C})^n \to \mathrm{Top}(B, \mathcal{C}), \quad \bar{g}(F(B))(u) = \left(B \xrightarrow{\langle u \rangle} \mathcal{C}^n \xrightarrow{g} \mathcal{C}\right).$$

We want to show that this defines an $\tilde{H}$-operation, i.e. that the diagram

$$\begin{array}{ccc} \mathrm{Top}(B_1, \mathcal{C})^n & \xrightarrow{\tilde{H}(s)^n} & \mathrm{Top}(B_2, \mathcal{C})^n \\ \bar{g}(F(B_1))\downarrow & & \downarrow \bar{g}(F(B_2)) \\ \mathrm{Top}(B_1, \mathcal{C}) & \xrightarrow[\tilde{H}(s)]{} & \mathrm{Top}(B_2, \mathcal{C}) \end{array} \tag{2}$$

commutes for every s in $\mathcal{Q}(F(B_1), F(B_2))$. We recall an argument due to Arens [**21**, p. 88] which shows that conjugation is an $\tilde{H}$-operation. Clearly one need only show that, for every s in $\mathcal{Q}(F(B), \mathcal{C})$ and real valued f in $\tilde{H}F(B) = \mathrm{Top}(B, \mathcal{C})$, $\tilde{H}(s)(f)$ is a real number. Let λ be an arbitrary real number. Putting $\tilde{H}(s)(f) = a + bi$ with a and b real we have

$$\begin{aligned} \|f\|^2 + \lambda^2 \geqslant \|f + i\lambda\|^2 &\geqslant \left|\tilde{H}(s)(f + i\lambda)\right|^2 \\ &= a^2 + b^2 + 2b\lambda + \lambda^2 \geqslant 2b\lambda + \lambda^2. \end{aligned}$$

Hence $\|f\|^2 \geqslant 2b\lambda$ for all λ, whence $b = 0$.

It follows immediately that, if $p\colon \mathcal{C}^n \to \mathcal{C}$ is such that $p(x_1 + iy_1, \dots, x_n + iy_n)$ is a polynomial in $x_1, y_1, \dots, x_n, y_n$, then $\bar{p}$ is an $\tilde{H}$-operation. Now, by the Weierstrass theorem, any continuous $g\colon \mathcal{C}^n \to \mathcal{C}$ can be ε-approximated by a polynomial p on any compact subspace of $\mathcal{C}^n$. By the argument used in Proposition 3.6 we can show that the diagram (2) commutes for any g (we use the fact that, for any f in $\mathrm{Top}(B_1, \mathcal{C})$, the union of $f(B_1)$ and $\hat{H}(s)(f)(B_2)$ is a compact subspace of $\mathcal{C}$).

Now every g in $\mathrm{Com}(J^n, J)$ extends to a continuous $g_1\colon \mathcal{C}^n \to J$, which defines an $\tilde{H}$-operation $\bar{g}_1$. Clearly $\bar{g}_1(F(B))$ maps $HF(B)^n$ to $HF(B)$ so it defines an H_1-operation ω, and clearly $\omega(\mathcal{C}) = g$.

It now follows from Corollary 3.3 that UF and FU are idempotent. As in

Example 2, Image U is clearly equal to Com.

4. Topological objects and duality. In this section we consider adjoint pairs $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ such that:

(D1) $(\mathfrak{A}, H)$ is a concrete category, $\mathfrak{A}$ is complete and H preserves limits;

(D2) F is left adjoint to U with adjunctions η and ε and $HF = \mathfrak{B}(-, J)$, where J is a Hausdorff space.

As in §3, we may assume $KU = \mathfrak{A}(-, J)$ and $H(I) = K(J)$ where $I = F(B_0)$, B_0 being the one point space.

If $(\mathfrak{A}, H)$ is equational then such an adjoint pair can be constructed from a suitable (I, J), as in §2.

An object A of $\mathfrak{A}$ will be called *Stonean* (with regard to (F, U)) if $U(A)$ has a basis of open sets

$$\mathcal{V}(A) = \{V(a, a') | a, a' \in H(A)\},$$

where

$$V(a, a') = \{s \in U(A) | H(s)(a) \neq H(s)(a')\}.$$

We recall that $H(s)(a) = \hat{a}(s) = (H\varepsilon(A)(a))(s)$ by Proposition 2.2. Since each $(H\varepsilon(A))(a)$ is a continuous map $U(A) \to J$ and J is Hausdorff, $V(a, a')$ is always open in $U(A)$.

PROPOSITION 4.1. *Suppose* (U, F) *satisfies* (D1) *and* (D2) *and every object in* Image F *is Stonean. Then*:

(a) *UF is idempotent, so that U and F induce a duality*

$$(\text{Image } F)^{\mathrm{op}} \cong \text{Image } U;$$

(b) *if $(\mathfrak{A}, H)$ is operational, then* Image U *can be described as the limit closure of J in* Top, *also as the full subcategory of* Top *whose objects are the equalizers of pairs of morphisms $J^X \rightrightarrows J^Y$*;

(c) *if also J is compact, then* Image F *can be described as in Proposition* 3.9.

PROOF. (a) By Proposition 2.1, we need only show that, for any topological space B, the image of $\eta(B)$ is dense in $UF(B)$. Let s_0 be any point of $UF(B)$ and W any neighborhood of s_0. Since $F(B)$ is Stonean,

$$s_0 \in V(f, g) = \{s \in UF(B) | s(f) \neq s(g)\} \subseteq W$$

for some $f, g \in HF(B) = \mathrm{Top}(B, J)$. Since $s_0(f) \neq s_0(g)$, f and g are distinct. Hence $f(b) \neq g(b)$ for some $b \in B$, that is, $\eta(B)(b)(f) \neq \eta(B)(b)(g)$. Thus $\eta(B)(b) \in V(f, g) \subseteq W$, and so the image of $\eta(B)$ is dense.

(b) follows from Proposition 2.3.

(c) If J is compact, we see from (b) that $U(A)$ is compact for each A in $\mathfrak{A}$. Now $F(B) \cong FUF(B)$, so Image $F = \mathrm{Image}(F|\mathrm{Com})$ and we can apply Proposition 3.7.

We shall give some sufficient conditions for all objects of Image F (or a larger category) to be Stonean. Let $\mathcal{Q}'$ be a full subcategory of $\mathcal{Q}$ containing Image F, and $H' = H|\mathcal{Q}'$. We say that (I, J) satisfies (S1) or (S2) with regard to $\mathcal{Q}'$ under the following conditions:

(S1) If q is an element of $J^4 \setminus P$, where

$$P = \{(x_1, x_2, x_3, x_4) \in J^4 | x_1 = x_2 \text{ or } x_3 = x_4\},$$

then there are H'-operations ξ, η: $H^4 \to H$ such that

$$\xi(I)(q) \neq \eta(I)(q) \quad \text{but} \quad \xi(I)|P = \eta(I)|P.$$

(S2) J has a subbasis of open sets consisting of sets of the form

$$V_{\alpha,\beta} = \{x \in J | \alpha(I)(x) \neq \beta(I)(x)\},$$

where α, β are unary H'-operations.

In all our examples, ξ and η of (S1) may in fact be chosen independently of q, so that

$$\forall_{x \in J^4}\big((\xi(I)(x) = \eta(I)(x)) \Leftrightarrow x_1 = x_2 \text{ or } x_3 = x_4\big).$$

For example, when I is an integral domain in the category of rings we may take

$$\xi(x) = (x_1 - x_2)(x_3 - x_4), \qquad \eta(x) = 0.$$

THEOREM 4.2. *Suppose* (U, F) *satisfies* (D1) *and* (D2) *and* (I, J) *satisfies* (S1) *and* (S2) *with regard to some* $\mathcal{Q}'$ *containing the image of* F. *Then every object of* $\mathcal{Q}'$ *is Stonean, hence* UF *and* FU *are idempotent.*

PROOF. Suppose first that $\mathcal{Q}' = \mathcal{Q}$. Let A be an object of $\mathcal{Q}$. Then $U(A)$ has a basis of open sets consisting of all finite intersections of sets

$$W_{a,\alpha,\beta} = \{s \in U(A) | H(s)(a) \in V_{\alpha,\beta}\},$$

where a is any element of $H(A)$ and $V_{\alpha,\beta}$ was defined in (S2). Thus

$$\begin{aligned} s \in W_{a,\alpha,\beta} &\Leftrightarrow H(s)(a) \in V_{\alpha,\beta} \\ &\Leftrightarrow \alpha(I)(H(s)(a)) \neq \beta(I)(H(s)(a)) \\ &\Leftrightarrow H(s)(\alpha(A)(a)) \neq H(s)(\beta(A)(a)), \end{aligned}$$

since $H(s)$ is a homomorphism. Thus, in the notation of Proposition 4.1,

$$W_{a,\alpha,\beta} = V(\alpha(A)(a), \beta(A)(a)) \in \mathcal{V}(A).$$

To show that $\mathcal{V}(A)$ is a basis of open sets, we need only show that, if $s_0 \in E = V(a_1, a_2) \cap V(a_3, a_4)$, then $s_0 \in V(c_1, c_2) \subseteq E$ for some $c_1, c_2 \in H(A)$.

Let

$$q = (H(s_0)(a_1), \ldots, H(s_0)(a_4)),$$

then $q \in H(I)^4$, but $q \notin P$ since $s_0 \in E$. Let $\xi, \eta\colon H^4 \to H$ be the operations of (S_1), then

$$\xi(I)(H(s_0)(a_1), \ldots) \neq \eta(I)(H(s_0)(a_1), \ldots),$$

hence

$$H(s_0)(c_1) \neq H(s_0)(c_2),$$

where

$$c_1 = \xi(A)(a_1, \ldots), \qquad c_2 = \eta(A)(a_1, \ldots).$$

Thus $s_0 \in V(c_1, c_2)$. It remains to show that $V(c_1, c_2) \subseteq E$.

Suppose $s \notin V(a_1, a_2) \cap V(a_3, a_4)$, then $H(s)(a_1) = H(s)(a_2)$ or $H(s)(a_3) = H(s)(a_4)$, so $(H(s)(a_1), \ldots, H(s)(a_4)) \in P$. Therefore

$$\xi(I)(H(s)(a_1), \ldots) = \eta(I)(H(s)(a_1), \ldots),$$

that is,

$$H(s)\xi(A)(a_1, \ldots) = H(s)\eta(A)(a_1, \ldots),$$

that is,

$$H(s)(c_1) = H(s)(c_2),$$

hence $s \notin V(c_1, c_2)$. Thus $V(c_1, c_2) \subseteq E$, and our proof is complete.

If $\mathfrak{A}' \neq \mathfrak{A}$ we apply the previous result to the functors

$$\mathfrak{A}'^{\mathrm{op}} \underset{F'}{\overset{U'}{\rightleftarrows}} \mathrm{Top}$$

obtained from U, F in the obvious way.

We now ask: are the conditions of Proposition 4.1 and Theorem 4.2 necessary for idempotence? The answer is that they are, provided we assume a weak form of the condition (C4) of §3 on J.

J is called *n-quasi-injective* if, for every closed subspace C of J^n, every continuous map $C \to J$ can be extended to a continuous map $J^n \to J$. (I, J) is called *n-functionally complete* with regard to $(\mathfrak{A}, H)$ if every continuous map $J^n \to J$ is $\omega(I)$ for some H-operation $\omega\colon H^n \to H$.

LEMMA 4.3. *Suppose* $\mathfrak{A}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ *satisfies* (D1) *and* (D2), *J has at least two points and, for $n = 1$ and $n = 4$, J is n-quasi-injective and (I, J) is n-functionally complete. Then (I, J) satisfies* (S1) *and* (S2) *with regard to* $(\mathfrak{A}, H)$.

PROOF. To verify (S1), we note that P is a closed set in J^4 and so is $P \cup \{q\}$. Since $q \notin \overline{P} = P$, we can find continuous functions $f, g\colon P \cup \{q\} \to J$ which are equal on P and unequal at q. Clearly these can be extended to continuous functions $J^4 \to J$, and the extensions are $\xi(I)$, $\eta(I)$ for some $\xi, \eta\colon H^4 \to H$.

To verify (S2), suppose C is any closed subset of J not containing the point

$j \in J$. As above, we can find H-operations $\alpha, \beta\colon H \to H$ such that $\alpha(I)$ and $\beta(I)$ are equal on C and unequal at j. (S2) now follows.

PROPOSITION 4.4. *Suppose* $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ *satisfies* (D1) *and* (D2), *J has at least two points and J is n-quasi-injective for $n = 1$ and $n = 4$. Then the following are equivalent*:

(a) (I, J) *satisfies* (S1) *and* (S2) *with regard to* Image F;

(b) *all objects of* Image F *are Stonean*;

(c) *UF and FU are idempotent*;

(d) (I, J) *is functionally complete with regard to* Image F;

(e) (I, J) *is n-functionally complete with regard to* Image F, *for $n = 1$ and $n = 4$.*

PROOF. (a) ⇒ (b) This is Theorem 4.2, with $\mathcal{C}$ replaced by Image F.

(b) ⇒ (c) This is Proposition 4.1.

(c) ⇒ (d) This was proved in Theorem 3.4, and inspection of the proof shows that it is valid in the present case.

(d) is clearly stronger than (e).

(e) ⇒ (a) This is Lemma 4.3.

We observe that we now have a strengthened form of the part (2) ⇒ (1) of Proposition 3.4, with a different proof.

Examples 2 and 3 of §3 can also be treated by the method of this section, in fact more easily (we do not need to use the Weierstrass theorem).

EXAMPLE 1. As in Example 2 of §3, let $\mathcal{C}$ be the category of real Banach algebras, $I = R$, $J = [-1, 1]$ and $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ the associated adjoint pair. Condition (S1) is satisfied with regard to $\mathcal{C}$, e.g. by the operations ξ, η: $H^4 \to H$, $\xi(x_1, x_2, x_3, x_4) = (\frac{1}{2}x_1 - \frac{1}{2}x_2)(\frac{1}{2}x_3 - \frac{1}{2}x_4)$, $\eta(x_1, x_2, x_3, x_4) = 0$.

Hence (S1) is satisfied with regard to Image F. (S2) is also satisfied with regard to Image F. Recall that it was shown in §3 that the binary sup and inf are H_1-operations, where $H_1 = H|\mathrm{Image}\ F$. Since there is a nullary H_1-operation whose value at R is a, $\sup(x, a)$ and $\inf(x, a)$ are unary H_1-operations for any fixed a in J. Now the set of intervals $[-1, a)$ and $(a, 1]$ is a subbasis of open sets of J and

$$[-1, a) = \{x \in J | \sup(x, a) \neq x\}, \qquad (a, 1] = \{x \in J | \inf(x, a) \neq x\}.$$

Thus (S2) is satisfied.

By Theorem 4.2, UF and FU are idempotent.

EXAMPLE 2. As in Example 3 of §3, let $\mathcal{C}$ be the category of complex Banach algebras, $I = K$, J the unit disc in K, and $\mathcal{C}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$ the associated adjoint pair. (S1) is satisfied with regard to $\mathcal{C}$ exactly as in Example 1. To verify (S2), recall from Example 3 of §3 that conjugation is an H_1-operation, where $H_1 = H|\text{Image } F$. It follows, as in Example 2 of §3, that there is a unary H_1-operation which assigns to every f in Top(B, K) the positive part of its real part, call it f_{r+}. Now J has a basis of open sets consisting of the open balls

$$\{z \in J \mid |z - c|^2 < \varepsilon^2\} = \{z \in J \mid \varepsilon^2 - (z - c)(\bar{z} - \bar{c}) > 0\}$$
$$= \left\{z \in J \,\middle|\, \left(\tfrac{1}{2}(\varepsilon/2)^2 - \tfrac{1}{2}\left(\tfrac{1}{2}z - \tfrac{1}{2}c\right)\left(\tfrac{1}{2}\bar{z} - \tfrac{1}{2}\bar{c}\right)\right)_{r+} \neq 0\right\}.$$

Thus (S2) is satisfied with regard to Image F.

By Theorem 4.2, UF and FU are idempotent.

EXAMPLE 3. This is not of the type considered in §3. Let $\mathcal{Q}$ be the category of rings (with unit), H its underlying set functor. As discussed in §2, any topological ring determines an adjoint pair $\mathcal{Q}^{\mathrm{op}} \underset{F}{\overset{U}{\rightleftarrows}} \mathrm{Top}$. We consider the topological ring R of real numbers, and the associated (F, U). Since R is an integral domain, (S1) holds with regard to $\mathcal{Q}$, hence with regard to Image F. We shall show that (S2) holds with regard to Image F. We observe that $J = R$ has a subbasis of open sets consisting of the open intervals $(m/n, \infty)$, $(-\infty, m/n)$ for all integers m and positive integers n. Now

$$(m/n, \infty) = \{x \in R \mid (nx - m)_+ \neq 0\},$$
$$(-\infty, m/n) = \{x \in R \mid (m - nx)_+ \neq 0\}.$$

Clearly the function f, where $f(x) = m - nx$, is a unary $\mathcal{Q}$-operation on R. One proves, as in Example 2 of §3, that the "positive part" function which takes x to x_+ is also an Image F-operation on R. Thus (S2) is satisfied.

By Theorem 4.2, UF and FU are idempotent.

By Proposition 2.3, Image U consists of all homeomorphs of subspaces of powers R^X of R which are equalizers of some pair of continuous maps $R^X \rightrightarrows R^Y$. In fact we show (by a standard argument) that any closed subspace B of a power R^X is in Image U. Clearly, for such a B, the map $\eta(B)$: $B \to UF(B)$ is injective, since the continuous maps $B \to R$ distinguish points of B. Since $\eta(B)$ is a reflection map, the inclusion $B \to R^X$ can be factored as $g\eta(B)$. It is easily shown that $g^{-1}(B) = \text{image } \eta(B)$, whence image $\eta(B)$ is closed in $UF(B)$. But this image is dense in $UF(B)$, as shown in the proof of Proposition 4.1, hence equal to $UF(B)$.

Thus Image U consists of homeomorphs of closed subspaces of powers of R. It is the well-known category of realcompact spaces, and UF is the realcompactification functor.

The category Image F does not seem to have been described in any way really independent of the duality.

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