

ON THE ZEROS OF JACOBI POLYNOMIALS $P_n^{(\alpha_n, \beta_n)}(x)$

BY

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ABSTRACT. If r_n and s_n denote, respectively, the smallest and largest zeros of the Jacobi polynomial $P_n^{(\alpha_n, \beta_n)}$, where $\alpha_n > -1$, $\beta_n > -1$, and if $\lim_{n \rightarrow \infty} \alpha_n / (2n + \alpha_n + \beta_n + 1) = a$ and if $\lim_{n \rightarrow \infty} \beta_n / (2n + \alpha_n + \beta_n + 1) = b$, then the numbers $r_{a,b}$ and $s_{a,b}$ are determined where

$$\lim_{n \rightarrow \infty} r_n = r_{a,b}, \quad \lim_{n \rightarrow \infty} s_n = s_{a,b}.$$

Furthermore, the zeros of $\{P_n^{(\alpha_n, \beta_n)}(x)\}_{n=0}^{\infty}$ are dense in $[r_{a,b}, s_{a,b}]$.

While a great deal is known (see Szegő [2]) about the asymptotic behavior of the zeros of Jacobi polynomials $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$ for a fixed type (α, β) , there do not appear in the literature results concerning the limiting behavior of zeros of sequences of Jacobi polynomials $\{P_n^{(\alpha_n, \beta_n)}(x)\}_{n=0}^{\infty}$ where α_n or β_n (or both) are allowed to grow with n . Results on this latter problem have application to the study of incomplete polynomials, as is discussed in *The sharpness of Lorentz's theorem on incomplete polynomials* [1]. The present note is used in that paper (cf. [1, Lemma 3.4]), and is published separately here because of its independent interest.

Because the polynomials $P_n^{(\alpha_n, \beta_n)}(x)$, $n = 0, 1, 2, \dots$, are in general *not* orthogonal on $[-1, 1]$, our results are not as detailed as the known theorems for a fixed type (α, β) . Of course, we do know that for $\alpha_n > -1$, $\beta_n > -1$ all the zeros of $P_n^{(\alpha_n, \beta_n)}(x)$ lie in the open interval $(-1, 1)$ and, using the Sturm Comparison Theory, we can easily prove

THEOREM 1. *Let r_n and s_n be, respectively, the smallest and largest zeros of the Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}(x)$, where $\alpha_n > -1$, $\beta_n > -1$. Suppose that*

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{(2n + \alpha_n + \beta_n)} = a \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\beta_n}{(2n + \alpha_n + \beta_n)} = b, \quad (1)$$

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and set

$$r_{a,b} := b^2 - a^2 - [(a^2 + b^2 - 1)^2 - 4a^2b^2]^{1/2}, \quad (2)$$

$$s_{a,b} := b^2 - a^2 + [(a^2 + b^2 - 1)^2 - 4a^2b^2]^{1/2}. \quad (3)$$

Then,

$$\lim_{n \rightarrow \infty} r_n = r_{a,b} \quad \text{and} \quad \lim_{n \rightarrow \infty} s_n = s_{a,b}. \quad (4)$$

Furthermore, the zeros of the sequence $\{P_n^{(\alpha_n, \beta_n)}(x)\}_{n=0}^\infty$ are dense in the interval $[r_{a,b}, s_{a,b}]$.

The proof requires the following known results related to the Sturm Comparison Theory: (cf. Szegő [2, pp. 19–20]):

LEMMA 2. Let $H(\theta)$ be continuous on (θ_1, θ_2) and suppose that $u(\theta)$ satisfies $u'' + H(\theta)u = 0$ for $\theta \in (\theta_1, \theta_2)$. If $H(\theta) \geq n > 0$ on (θ_1, θ_2) , then $u(\theta)$ has a zero in every subinterval of (θ_1, θ_2) of length $\geq \pi/\sqrt{n}$.

LEMMA 3. Let $H(\theta)$ be continuous and negative in (θ_1, θ_2) . Then an arbitrary solution $u(\theta)$ ($\not\equiv 0$) of $u'' + H(\theta)u = 0$, for which $u(\theta) \rightarrow 0$ if $\theta \rightarrow \theta_2^-$, cannot vanish in $\theta_1 \leq \theta < \theta_2$.

PROOF OF THEOREM 1. As is known [2, p. 67], the function

$$u_n(\theta) := \left(\sin \frac{\theta}{2}\right)^{\alpha_n+1/2} \left(\cos \frac{\theta}{2}\right)^{\beta_n+1/2} P_n^{(\alpha_n, \beta_n)}(\cos \theta) \quad (5)$$

satisfies the differential equation

$$d^2u/d\theta^2 + H_n(\theta)u = 0, \quad \text{for } 0 < \theta < \pi, \quad (6)$$

where

$$H_n(\theta) := \frac{1 - 4\alpha_n^2}{16 \sin^2(\theta/2)} + \frac{1 - 4\beta_n^2}{16 \cos^2(\theta/2)} + \left(n + \frac{\alpha_n + \beta_n + 1}{2}\right)^2. \quad (7)$$

It is convenient to rewrite $H_n(\theta)$ in the form

$$H_n(\theta) = \frac{-(2n + \alpha_n + \beta_n + 1)^2 \cos^2 \theta + 2(\beta_n^2 - \alpha_n^2) \cos \theta}{4(1 - \cos^2 \theta)} + \frac{(2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2}{4(1 - \cos^2 \theta)}. \quad (8)$$

Notice that the numerator of $H_n(\theta)$ in (8) is, for $n \geq 1$, a quadratic in $x = \cos \theta$ having negative leading coefficient. The roots of this quadratic are:

$$x_n^\pm := \frac{\beta_n^2 - \alpha_n^2}{(2n + \alpha_n + \beta_n + 1)^2} \pm \left[1 + \frac{(\beta_n^2 - \alpha_n^2)^2}{(2n + \alpha_n + \beta_n + 1)^4} + \frac{1 - 2\alpha_n^2 - 2\beta_n^2}{(2n + \alpha_n + \beta_n + 1)^2} \right]^{1/2}. \tag{9}$$

Because $\alpha_n > -1$ and $\beta_n > -1$, then $2n + \alpha_n + \beta_n + 1 > 2n - 1 > 0$ for all $n \geq 1$, so that $\lim_{n \rightarrow \infty} 1/(2n + \alpha_n + \beta_n + 1)^2 = 0$. Thus, with (1), the roots of (9) approach

$$b^2 - a^2 \pm [1 + (b^2 - a^2)^2 - 2(a^2 + b^2)]^{1/2}, \tag{10}$$

which are precisely the numbers $r_{a,b}$ and $s_{a,b}$ defined in (2) and (3). Since $\alpha_n > -1$ and $\beta_n > -1$, it easily follows from (1) that $a, b \in [0, 1]$. Furthermore, from definitions (2) and (3), it can be verified that

$$-1 \leq r_{a,b} \leq s_{a,b} \leq 1, \tag{11}$$

and that

$$r_{a,b} = -1 \text{ iff } b = 0, \quad s_{a,b} = 1 \text{ iff } a = 0.$$

Returning to the differential equation (6), it follows from the above discussion that for each $\epsilon > 0$ sufficiently small,

$$H_n(\theta) < 0 \text{ for } \cos \theta \in \begin{cases} (-1, r_{a,b} - \epsilon), & \text{if } b > 0, \\ (s_{a,b} + \epsilon, 1), & \text{if } a > 0, \end{cases} \tag{12}$$

provided that n is sufficiently large. Hence, by applying Lemma 3 to the function $u_n(\theta)$ in (5), we have $P_n^{(\alpha_n, \beta_n)}(x) \neq 0$ in $[-1, r_{a,b} - \epsilon) \cup (s_{a,b} + \epsilon, 1]$ for all n large. In terms of the largest and smallest zeros of $P_n^{(\alpha_n, \beta_n)}(x)$, this means that

$$r_{a,b} - \epsilon \leq \liminf_{n \rightarrow \infty} r_n, \quad \limsup_{n \rightarrow \infty} s_n \leq s_{a,b} + \epsilon,$$

and letting $\epsilon \rightarrow 0^+$ yields

$$r_{a,b} \leq \liminf_{n \rightarrow \infty} r_n, \quad \limsup_{n \rightarrow \infty} s_n \leq s_{a,b}, \tag{13}$$

the inequalities (13) being valid even if a and/or b are zero.

Next, we consider the inequality

$$H_n(\theta) \geq n, \tag{14}$$

which, using (8), is equivalent to

$$(-A_n \cos^2 \theta + B_n \cos \theta + C_n)/4(1 - \cos^2 \theta) \geq 0, \tag{15}$$

where

$$\begin{aligned} A_n &:= (2n + \alpha_n + \beta_n + 1)^2 - 4n, & B_n &:= 2(\beta_n^2 - \alpha_n^2), \\ C_n &:= (2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2 - 4n. \end{aligned} \tag{16}$$

It is immediately verified from (1) that the roots of the quadratic numerator in (15) again approach the number $r_{a,b}$ and $s_{a,b}$ as $n \rightarrow \infty$. Consequently, for each $\varepsilon > 0$ sufficiently small,

$$H_n(\theta) \geq n \quad \text{for } \theta \in [\cos^{-1}(s_{a,b} - \varepsilon), \cos^{-1}(r_{a,b} + \varepsilon)], \quad (17)$$

provided that n is sufficiently large. Thus, by Lemma 2, the function $u_n(\theta)$ has zeros within π/\sqrt{n} of each of the endpoints of the interval in (17), and so

$$\limsup_{n \rightarrow \infty} r_n \leq r_{a,b} + \varepsilon, \quad \liminf_{n \rightarrow \infty} s_n \geq s_{a,b} - \varepsilon.$$

Letting $\varepsilon \rightarrow 0^+$ and using (13) we have proved (4).

The fact that the zeros of the sequence $\{P_n^{(\alpha_n, \beta_n)}(x)\}_{n=0}^\infty$ are dense in $[r_{a,b}, s_{a,b}]$ also follows from Lemma 2 and the previous discussion. \square

As a special case of Theorem 1, we have

COROLLARY 1. *If α and β are finite such that $\lim_{n \rightarrow \infty} \alpha_n/n = \alpha$ and $\lim_{n \rightarrow \infty} \beta_n/n = \beta$, then the conclusions of Theorem 1 are valid with $a := \alpha/(2 + \alpha + \beta)$ and with $b := \beta/(2 + \alpha + \beta)$.*

We remark that Theorem 1 also includes cases where $\alpha_n/n \rightarrow +\infty$ and/or $\beta_n/n \rightarrow +\infty$.

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