ON THE ZEROS OF JACOBI POLYNOMIALS $P_n^{(\alpha_n,\beta_n)}(x)$

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ABSTRACT. If $r_\alpha$ and $s_\alpha$ denote, respectively, the smallest and largest zeros of the Jacobi polynomial $P_n^{(\alpha_n,\beta_n)}$, where $\alpha_n > 1$, $\beta_n = 1$, and if $\lim_{n \to \infty} \frac{\alpha_n}{(2n + \alpha_n + \beta_n + 1)} = a$ and if $\lim_{n \to \infty} \frac{\beta_n}{(2n + \alpha_n + \beta_n + 1)} = b$, then the numbers $r_{\alpha,b}$ and $s_{\alpha,b}$ are determined where

$$
\lim_{n \to \infty} r_n = r_{\alpha,b}, \quad \lim_{n \to \infty} s_n = s_{\alpha,b}.
$$

Furthermore, the zeros of $(P_n^{(\alpha_n,\beta_n)}(x))_{n=0}^{\infty}$ are dense in $[r_{\alpha,b}, s_{\alpha,b}]$.

While a great deal is known (see Szegö [2]) about the asymptotic behavior of the zeros of Jacobi polynomials $(P_n^{(\alpha_n,\beta_n)}(x))_{n=0}^{\infty}$ for a fixed type $(\alpha, \beta)$, there do not appear in the literature results concerning the limiting behavior of zeros of sequences of Jacobi polynomials $(P_n^{(\alpha_n,\beta_n)}(x))_{n=0}^{\infty}$ where $\alpha_n$ or $\beta_n$ (or both) are allowed to grow with $n$. Results on this latter problem have application to the study of incomplete polynomials, as is discussed in The sharpness of Lorentz's theorem on incomplete polynomials [1]. The present note is used in that paper (cf. [1, Lemma 3.4]), and is published separately here because of its independent interest.

Because the polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, $n = 0, 1, 2, \ldots$, are in general not orthogonal on $[-1, 1]$, our results are not as detailed as the known theorems for a fixed type $(\alpha, \beta)$. Of course, we do know that for $\alpha_n > -1$, $\beta_n > -1$ all the zeros of $P_n^{(\alpha_n,\beta_n)}(x)$ lie in the open interval $(-1, 1)$ and, using the Sturm Comparison Theory, we can easily prove

**Theorem 1.** Let $r_n$ and $s_n$ be, respectively, the smallest and largest zeros of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, where $\alpha_n > -1$, $\beta_n > -1$. Suppose that

$$
\lim_{n \to \infty} \frac{\alpha_n}{(2n + \alpha_n + \beta_n)} = a \quad \text{and} \quad \lim_{n \to \infty} \frac{\beta_n}{(2n + \alpha_n + \beta_n)} = b, \quad (1)
$$

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and set

\[ r_{a,b} := b^2 - a^2 - \left[ (a^2 + b^2 - 1)^2 - 4a^2b^2 \right]^{1/2}, \]  
(2)

\[ s_{a,b} := b^2 - a^2 + \left[ (a^2 + b^2 - 1)^2 - 4a^2b^2 \right]^{1/2}. \]  
(3)

Then,

\[ \lim_{n \to \infty} r_n = r_{a,b} \quad \text{and} \quad \lim_{n \to \infty} s_n = s_{a,b}. \]  
(4)

Furthermore, the zeros of the sequence \( \{ P_n^{(\alpha, \beta)}(x) \}_{n=0}^{\infty} \) are dense in the interval \([r_{a,b}, s_{a,b}].\)

The proof requires the following known results related to the Sturm Comparison Theory: (cf. Szegö [2, pp. 19-20]):

**Lemma 2.** Let \( H(\theta) \) be continuous on \((\theta_1, \theta_2)\) and suppose that \( u(\theta) \) satisfies \( u'' + H(\theta)u = 0 \) for \( \theta \in (\theta_1, \theta_2). \) If \( H(\theta) > n > 0 \) on \((\theta_1, \theta_2), \) then \( u(\theta) \) has a zero in every subinterval of \((\theta_1, \theta_2)\) of length \( \geq \pi / \sqrt{n}. \)

**Lemma 3.** Let \( H(\theta) \) be continuous and negative in \((\theta_1, \theta_2).\) Then an arbitrary solution \( u(\theta) \neq 0 \) of \( u'' + H(\theta)u = 0, \) for which \( u(\theta) \to 0 \) if \( \theta \to \theta_2^-, \) cannot vanish in \( \theta_1 < \theta < \theta_2. \)

**Proof of Theorem 1.** As is known [2, p. 67], the function

\[ u_n(\theta) := \left( \frac{\sin \theta}{2} \right)^{\alpha_n + 1/2} \left( \frac{\cos \theta}{2} \right)^{\beta_n + 1/2} P_n^{(\alpha_n, \beta_n)}(\cos \theta) \]  
(5)

satisfies the differential equation

\[ \frac{d^2u}{d\theta^2} + H_n(\theta)u = 0, \quad 0 < \theta < \pi, \]  
(6)

where

\[ H_n(\theta) := \frac{1 - 4\alpha_n^2}{16 \sin^2(\theta/2)} + \frac{1 - 4\beta_n^2}{16 \cos^2(\theta/2)} + \left( n + \frac{\alpha_n + \beta_n + 1}{2} \right)^2. \]  
(7)

It is convenient to rewrite \( H_n(\theta) \) in the form

\[ H_n(\theta) = \frac{- (2n + \alpha_n + \beta_n + 1)^2 \cos^2 \theta + 2(\beta_n^2 - \alpha_n^2) \cos \theta}{4(1 - \cos^2 \theta)} \]  

\[ + \frac{(2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2}{4(1 - \cos^2 \theta)}. \]  
(8)

Notice that the numerator of \( H_n(\theta) \) in (8) is, for \( n > 1, \) a quadratic in \( x = \cos \theta \) having negative leading coefficient. The roots of this quadratic are:
\[ x_n^\pm := \frac{\beta_n^2 - \alpha_n^2}{(2n + \alpha_n + \beta_n + 1)^2} \]
\[ \pm \left[ 1 + \frac{(\beta_n^2 - \alpha_n^2)^2}{(2n + \alpha_n + \beta_n + 1)^4} + \frac{1 - 2\alpha_n^2 - 2\beta_n^2}{(2n + \alpha_n + \beta_n + 1)^2} \right]^{1/2}. \] (9)

Because \( \alpha_n > -1 \) and \( \beta_n > -1 \), then \( 2n + \alpha_n + \beta_n + 1 > 2n - 1 > 0 \) for all \( n > 1 \), so that \( \lim_{n \to \infty} \frac{1}{(2n + \alpha_n + \beta_n + 1)^2} = 0 \). Thus, with (1), the roots of (9) approach
\[ b^2 - a^2 \pm \left[ 1 + (b^2 - a^2)^2 - 2(a^2 + b^2) \right]^{1/2}, \] (10)
which are precisely the numbers \( r_{a,b} \) and \( s_{a,b} \) defined in (2) and (3). Since \( \alpha_n > -1 \) and \( \beta_n > -1 \), it easily follows from (1) that \( a, b \in [0, 1] \). Furthermore, from definitions (2) and (3), it can be verified that
\[ -1 \leq r_{a,b} \leq s_{a,b} < 1, \] (11)
and that
\[ r_{a,b} = -1 \iff b = 0, \quad s_{a,b} = 1 \iff a = 0. \]

Returning to the differential equation (6), it follows from the above discussion that for each \( \epsilon > 0 \) sufficiently small,
\[ H_n(\theta) < 0 \text{ for } \cos \theta \in \begin{cases} (-1, r_{a,b} - \epsilon), & \text{if } b > 0, \\ (s_{a,b} + \epsilon, 1), & \text{if } a > 0, \end{cases} \] (12)
provided that \( n \) is sufficiently large. Hence, by applying Lemma 3 to the function \( u_n(\theta) \) in (5), we have \( P_n^{(\alpha, \beta)}(x) \neq 0 \) in \([ -1, r_{a,b} - \epsilon ) \cup (s_{a,b} + \epsilon, 1] \) for all \( n \) large. In terms of the largest and smallest zeros of \( P_n^{(\alpha, \beta)}(x) \), this means that
\[ r_{a,b} - \epsilon \leq \lim_{n \to \infty} \inf r_n, \quad \lim_{n \to \infty} \sup s_n \leq s_{a,b} + \epsilon, \]
and letting \( \epsilon \to 0^+ \) yields
\[ r_{a,b} \leq \lim_{n \to \infty} \inf r_n, \quad \lim_{n \to \infty} \sup s_n \leq s_{a,b}, \] (13)
the inequalities (13) being valid even if \( a \) and/or \( b \) are zero.

Next, we consider the inequality
\[ H_n(\theta) \geq n, \] (14)
which, using (8), is equivalent to
\[ (-A_n \cos^2 \theta + B_n \cos \theta + C_n)/4(1 - \cos^2 \theta) > 0, \]
where
\[ A_n := (2n + \alpha_n + \beta_n + 1)^2 - 4n, \quad B_n := 2(\beta_n^2 - \alpha_n^2), \]
\[ C_n := (2n + \alpha_n + \beta_n + 1)^2 + 1 - 2\alpha_n^2 - 2\beta_n^2 - 4n. \] (16)
It is immediately verified from (1) that the roots of the quadratic numerator in (15) again approach the number \( r_{a,b} \) and \( s_{a,b} \) as \( n \to \infty \). Consequently, for each \( \varepsilon > 0 \) sufficiently small,

\[
H_n(\theta) > n \quad \text{for} \quad \theta \in \left[ \cos^{-1}(s_{a,b} - \varepsilon), \cos^{-1}(r_{a,b} + \varepsilon) \right],
\]

(17) provided that \( n \) is sufficiently large. Thus, by Lemma 2, the function \( u_n(\theta) \) has zeros within \( \pi/\sqrt{n} \) of each of the endpoints of the interval in (17), and so

\[
\lim_{n \to \infty} \sup r_n < r_{a,b} + \varepsilon, \quad \lim_{n \to \infty} \inf s_n > s_{a,b} - \varepsilon.
\]

Letting \( \varepsilon \to 0^+ \) and using (13) we have proved (4).

The fact that the zeros of the sequence \( \{P_n^{(a,b)}(x)\}_{n=0}^\infty \) are dense in \( [r_{a,b}, s_{a,b}] \) also follows from Lemma 2 and the previous discussion. □

As a special case of Theorem 1, we have

**COROLLARY 1.** If \( \alpha \) and \( \beta \) are finite such that \( \lim_{n \to \infty} \alpha_n/n = \alpha \) and \( \lim_{n \to \infty} \beta_n/n = \beta \), then the conclusions of Theorem 1 are valid with \( a := \alpha/(2 + \alpha + \beta) \) and with \( b := \beta/(2 + \alpha + \beta) \).

We remark that Theorem 1 also includes cases where \( \alpha_n/n \to +\infty \) and/or \( \beta_n/n \to +\infty \).

**REFERENCES**


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