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THE BERGMAN NORM AND THE SZEGÖ NORM

BY

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Dedicated to Professor Yûsaku Komatu on his 65th birthday

ABSTRACT. Let G denote an arbitrary bounded regular region in the plane and $H_2(G)$ the analytic Hardy class on G with index 2. We show that *the generalized isoperimetric inequality*

$$\frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy \leq \frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\,|dz|\,\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\,|dz| \qquad (z = x + iy)$$

holds for any φ and $\psi \in H_2(G)$. We also determine necessary and sufficient conditions for equality.

1. Introduction. Let G denote an arbitrary bounded regular region with boundary contours $\{C_\nu\}_{\nu=1}^N$ in the plane and $H_2(G)$ the analytic Hardy class on G with index 2. Then, in the case of simply-connected regions G, Aronszajn [1] obtained the inequality

$$\frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy \leq \frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\,|dz|\,\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\,|dz| \qquad (z = x + iy)$$

for any φ and $\psi \in H_2(G)$. His proof is based on his general theory of reproducing kernels. In this paper, we show that the same inequality is valid for arbitrary bounded regular regions G by combining Aronszajn's techniques with a profound result of D. A. Hejhal [5].

The main inequality is given in §2; the equality statement is established in §3. In §4, we give a generalization of the inequality for the case of simply-connected regions G. In §5, using the main inequality, we derive several miscellaneous relations among the magnitudes of some extremal quantities in the Bergman and Szegö norms. These relations lead to a general and sharp solution for a problem of Sario-Oikawa. In §6, we discuss the existence of a

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doubly-orthogonal system for the Bergman and Szegö spaces. Finally, in §§7 and 8, we construct examples which show that certain generalized versions of the fundamental inequality are not true.

2. The main inequality. Let F denote the class (the Bergman space) of analytic functions on G such that

$$\|f\| = (f, f)^{1/2} = \left(\iint_G |f(z)|^2 \, dx \, dy \right)^{1/2} < \infty.$$

Then we obtain the following inequalities:

THEOREM 2.1. *Any $f \in F$ can be represented by a series*

$$f(z) = \sum_j \varphi_j(z)\psi_j(z) \qquad (\varphi_j, \psi_j \in H_2(G)) \tag{2.1}$$

and the inequality

$$\frac{1}{\pi} \iint_G |f(z)|^2 \, dx \, dy$$

$$\leqslant \min \sum_j \sum_k \frac{1}{4\pi^2} \int_{\partial G} \varphi_j(z_1) \overline{\varphi_k(z_1)} \, |dz_1| \int_{\partial G} \psi_j(z_2) \overline{\psi_k(z_2)} \, |dz_2| \tag{2.2}$$

is valid. The minimum is taken here over all analytic functions $\sum_j \varphi_j(z_1)\psi_j(z_2)$ on $G \times G$ satisfying (2.1).

Conversely, if the jk sum in (2.2) *is finite, then the function f defined by the series* (2.1) *belongs to the class F.*

In particular, for any φ and $\psi \in H_2(G)$, we obtain the inequality

$$\frac{1}{\pi} \iint_G |\varphi(z)\psi(z)|^2 \, dx \, dy$$

$$\leqslant \frac{1}{2\pi} \int_{\partial G} |\varphi(z)|^2 \, |dz| \, \frac{1}{2\pi} \int_{\partial G} |\psi(z)|^2 \, |dz|. \tag{2.3}$$

PROOF. Let $K(z, \bar{u})$ and $\hat{K}(z, \bar{u})$ denote the Bergman kernel and the Szegö kernel of G, respectively. Let $\{Z_\nu(z)dz\}_{\nu=1}^{N-1}$ denote the basis of analytic differentials which are real along ∂G such that

$$Z_\nu(z) = -\int_{C_\nu} \overline{K(\zeta, \bar{z}) d\zeta} = \int_{C_\nu} L(\zeta, z) \, d\zeta, \tag{2.4}$$

where $L(\zeta, z)$ is the adjoint L-kernel of $K(\zeta, \bar{z})$ (cf. [**2**, pp. 59–62] and [**14**, Chapters 3 and 4]). Then the identity

$$K(z, \bar{u}) = 4\pi \hat{K}(z, \bar{u})^2 + \sum_\nu \sum_\mu C_{\nu\mu} \overline{Z_\nu(u)} \, Z_\mu(z) \tag{2.5}$$

is well known for some uniquely determined constants $\{C_{\nu\mu}\}$. D. A. Hejhal [**5**] established the positive definiteness of the real symmetric matrix $\|C_{\nu\mu}\|$ by means of the theta function. Hence

$$k(z, \bar{u}) \equiv \sum_{\nu} \sum_{\mu} C_{\nu\mu} \overline{Z_\nu(u)}\ Z_\mu(z)$$

is a reproducing kernel in the *finite-dimensional class* F_2 which is generated by $\{Z_\nu\}_{\nu=1}^{N-1}$. See [**1**, pp. 346–347]. The scalar product is given by

$$(f, h)_2 = \sum_{\nu} \sum_{\mu} \alpha_{\nu\mu} \zeta_\nu \bar{\eta}_\mu$$

for $f(z) = \Sigma_\nu \zeta_\nu Z_\nu(z)$ and $h(z) = \Sigma_\nu \eta_\nu Z_\nu(z)$, where the constants $\alpha_{\nu\mu}$ are determined by the $(N-1)(N-1)$ equations

$$\sum_j \alpha_{\nu j} C_{j\mu} = \delta_{\nu\mu}.$$

By the definition in [**1**, p. 354], we have:

$$K(z, \bar{u}) \gg \left(2\sqrt{\pi}\ \hat{K}(z, \bar{u})\right)^2. \tag{2.6}$$

Let F_1 denote the Hilbert space such that $(2\sqrt{\pi}\ \hat{K}(z, \bar{u}))^2$ is the reproducing kernel of the class F_1 with norm $\|\ \|_1$ and inner product $(\ ,\ )_1$. Then from Theorem I in [**1**, p. 354] and (2.6), we have

$$F_1 \subset F \tag{2.7}$$

and

$$\|f_1\|_1 \geqslant \|f_1\| \quad \text{for any } f_1 \in F_1. \tag{2.8}$$

For clarity, we recall the structure of F_1. At first, $2\sqrt{\pi}\ \hat{K}(z, \bar{u})$ is the reproducing kernel of class $H_2(G)$ with the scalar product

$$(f, h)_0 = \frac{1}{2\sqrt{\pi}} \int_{\partial G} f(z)\overline{h(z)}\ |dz| \quad \text{for } f, h \in H_2(G).$$

If this Hilbert space is denoted by F_0, then $(2\sqrt{\pi}\ \hat{K}(z, \bar{u}))^2$ is the reproducing kernel of the space F_1 which is formed by restricting the functions in the direct product $\hat{F}_1 = F_0 \otimes F_0$ to the diagonal set D formed by all the elements $\{(z, z)\colon z \in G\}$. Furthermore, for any such restriction $f \in F_1$, the norm $\|f\|_1$ is defined by $\min\|h\|_{\hat{1}}$ for all $h \in \hat{F}_1$, the restriction of which to D is f [**1**, p. 361, Theorem II]. Here $\|\ \|_{\hat{1}}$ and $(\ ,\ )_{\hat{1}}$ denote the norm and inner product in the direct product $\hat{F}_1$, respectively. On the other hand, the space F must coincide with the class corresponding to the kernel $K(z, \bar{u})$ when it is considered as the sum of the kernels $(2\sqrt{\pi}\ \hat{K}(z, \bar{u}))^2$ and $k(z, \bar{u})$; see [**1**, pp. 352–357]. It follows that any $f \in F$ is representable in the form

$$f(z) = \sum_j \tilde{\varphi}_j(z)\tilde{\psi}_j(z) + \sum_\nu d_\nu Z_\nu(z)$$

for some $\tilde{\varphi}_j$, $\tilde{\psi}_j \in H_2(G)$ and for some constants $\{d_\nu\}$. Thus, f can be represented in the form (2.1). At first, the limit $\Sigma_j \tilde{\varphi}_j(z)\tilde{\psi}_j(z)$ is taken here in the sense of norm convergence:

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} \tilde{\varphi}_j(z_1)\tilde{\psi}_j(z_2) - f_0(z_1, z_2) \right\|_1^{\wedge} = 0 \quad \text{for some } f_0 \in \hat{F}_1.$$

But this convergence implies also that the sequence $\{\sum_{j=1}^{n}\tilde{\varphi}_j(z_1)\tilde{\psi}_j(z_2)\}$ converges uniformly on every compact subset of $G \times G$ in the ordinary sense, $\lim_{n\to\infty}\sum_{j=1}^{n}\tilde{\varphi}_j(z_1)\tilde{\psi}_j(z_2) = f_0(z_1, z_2)$. See **[1**, pp. 391–393 and p. 344].

Using (2.1), (2.8) and the definition of the norm in $\hat{F}_1$ **[1**, pp. 357–361], we now obtain the desired inequality

$$\begin{aligned} \|f(z)\|^2 &\leq (\|f(z)\|_1)^2 \\ &= \min \sum_j \sum_k \frac{1}{4\pi} \int_{\partial G} \varphi_j(z)\overline{\varphi_k(z)}\, |dz| \int_{\partial G} \psi_j(z)\overline{\psi_k(z)}\, |dz|. \end{aligned} \tag{2.9}$$

3. **Equality statement.** In this section, we shall establish the following equality statement:

THEOREM 3.1. *In the fundamental inequality* (2.2), *equality holds for* $f \in F$ *if and only if*

$$(f, f_2)_1 = 0 \quad \textit{for all } f_2 \in F_2.$$

In the inequality (2.3), *equality holds for* φ *and* $\psi \in H_2(G)$ $(\varphi\psi \not\equiv 0)$, *if and only if* G *is simply-connected and* φ *and* ψ *are expressible in the form* $\varphi(z) = \alpha\hat{K}(z, \bar{u})$ *and* $\psi(z) = \beta\hat{K}(z, \bar{u})$ *for some point* $u \in G$ *and for some constants* α *and* β.

PROOF. The space F is the class corresponding to the kernel $K(z, \bar{u})$ which is considered as the sum of the kernels $(2\sqrt{\pi}\, \hat{K}(z, \bar{u}))^2$ and $k(z, \bar{u})$. Hence the norm $\|f\|$ is equal to $[\min((\|f_1\|_1)^2 + (\|f_2\|_2)^2)]^{1/2}$, where the minimum is taken over all decompositions $f = f_1 + f_2$ with $f_1 \in F_1$ and $f_2 \in F_2$ **[1**, pp. 354–355]. Therefore we see that equality holds in (2.2) if and only if $f = f + 0$; $f \in F_1$ and $0 \in F_2$ is the extremal decomposition of f. Hence, from the fact $F_2 \subset F_1$, we obtain

$$\|f\|^2 \leq (\|f - \lambda f_2\|_1)^2 + (\|\lambda f_2\|_2)^2$$

for any λ and $f_2 \in F_2$. From the equality $\|f\| = \|f\|_1$ and arbitrariness of λ, we have the first part of Theorem 3.1, as usual.

Next, in order to establish the equality statement for (2.3), we observe that

$$\frac{1}{\pi}\iint_G |f(z)|^2\,dx\,dy$$
$$\leqslant \min \sum_j \sum_k \frac{1}{4\pi^2} \int_{\partial G} \varphi_j(z_1)\overline{\varphi_k(z_1)}\,|dz_1| \int_{\partial G} \psi_j(z_2)\overline{\psi_k(z_2)}\,|dz_2|; \quad (3.1\mathrm{a})$$

$$\min \sum_j \sum_k \frac{1}{4\pi^2} \int_{\partial G} \varphi_j(z_1)\overline{\varphi_k(z_1)}\,|dz_1| \int_{\partial G} \psi_j(z_2)\overline{\psi_k(z_2)}\,|dz_2|$$
$$\leqslant \frac{1}{4\pi^2}\int_{\partial G} |\varphi(z_1)|^2\,|dz_1| \int_{\partial G} |\psi(z_2)|^2\,|dz_2| \qquad (3.1\mathrm{b})$$

for $f = \varphi\psi$. The minimum is taken over all analytic functions $\Sigma_j \varphi_j(z_1)\psi_j(z_2)$ on $G \times G$ which reduce to $\varphi(z)\psi(z)$ along the diagonal. Of course, φ_j and $\psi_j \in H_2(G)$.

If equality holds in (3.1b), then $\varphi(z_1)\psi(z_2)$ is the extremal function which minimizes $\|f(z_1, z_2)\|_1^{\hat{}}$ among all the functions $f(z_1, z_2) = \Sigma_j \varphi_j(z_1)\psi_j(z_2) \in \hat{F}_1$ such that $f(z, z) = \varphi(z)\psi(z)$ on D. Let $\hat{F}_{1,D(0)}$ denote the closed subspace formed by all the functions of $\hat{F}_1$ which vanish on D. Then we see easily that $\varphi(z_1)\psi(z_2)$ is characterized by the following orthogonality condition:

$$\left(\varphi(z_1)\psi(z_2), f(z_1, z_2)\right)_1^{\hat{}} = 0 \quad \text{for all } f \in \hat{F}_{1,D(0)}. \qquad (3.2)$$

At first, by setting $f(z_1, z_2) = \hat{K}(z_1, \bar{u})\hat{K}(z_2, \bar{v}) - \hat{K}(z_1, \bar{v})\hat{K}(z_2, \bar{u})$ ($u, v \in G$) in (3.2), we find that φ and ψ are linearly dependent. Hence we can assume that $\varphi(z_2) \equiv \psi(z_2)$ in (3.2).

Next, let $\hat{L}(z, u)$ denote the adjoint L-kernel of $\hat{K}(z, \bar{u})$. The $\hat{L}(z, u)$ is meromorphic on $\overline{G}$, has one simple pole at $z = u$ with residue $1/2\pi$ and has no zeros on $\overline{G}$. Furthermore $\hat{L}(z, u) = -\hat{L}(u, z)$ and $\hat{L}(z, u)$ has the following important property on ∂G:

$$\overline{\hat{K}(z, \bar{u})}\,|dz| = i^{-1}\hat{L}(z, u)\,dz \quad \text{along } \partial G. \qquad (3.3)$$

Cf. [2], [5]–[8].

Hence, for any u and $v \in G$ and for fixed $a \in G$, from (3.2), we have

$$\left[\varphi(z_1)\varphi(z_2), \frac{\hat{K}(z_1, \bar{a})\hat{K}(z_1, \bar{u})\hat{K}(z_2, \bar{v})}{\hat{L}(z_1, a)} - \frac{\hat{K}(z_1, \bar{a})\hat{K}(z_2, \bar{u})\hat{K}(z_2, \bar{v})}{\hat{L}(z_2, a)}\right]_1^{\hat{}} = 0; \qquad (3.4)$$

that is,

$$\varphi(v)\int_{\partial G}\varphi(z_1)\overline{\left[\frac{\hat{K}(z_1,\bar{a})\hat{K}(z_1,\bar{u})}{\hat{L}(z_1,a)}\right]}|dz_1|$$

$$-\varphi(a)\int_{\partial G}\varphi(z_2)\overline{\left[\frac{\hat{K}(z_2,\bar{u})\hat{K}(z_2,\bar{v})}{\hat{L}(z_2,a)}\right]}|dz_2| = 0 \quad \text{for all } u, v \in G. \quad (3.5)$$

Using the relation (3.3), we now obtain

$$\varphi(v)\left\{\frac{1}{i}\int_{\partial G}\frac{\varphi(z_1)\hat{L}(z_1,a)\hat{L}(z_1,u)}{\hat{K}(z_1,\bar{a})}\,dz_1\right\}$$

$$-\varphi(a)\left\{\frac{1}{i}\int_{\partial G}\frac{\varphi(z_2)\hat{L}(z_2,u)\hat{L}(z_2,v)}{\hat{K}(z_2,\bar{a})}\,dz_2\right\} = 0. \quad (3.6)$$

We let $\{a_\nu\}_{\nu=1}^{N-1}$ denote the zeros of $\hat{K}(z,\bar{a})$ on G and *assume* that they are simple. The other cases require only a slight modification. From (3.6), by the residue theorem, we obtain

$$\varphi(v)\left\{2\pi\sum_\nu\frac{\varphi(a_\nu)\hat{L}(a_\nu,a)\hat{L}(a_\nu,u)}{\hat{K}'(a_\nu,\bar{a})}+\frac{\varphi(a)\hat{L}(a,u)}{\hat{K}(a,\bar{a})}+\frac{\varphi(u)\hat{L}(u,a)}{\hat{K}(u,\bar{a})}\right\}$$

$$-\varphi(a)\left\{2\pi\sum_\nu\frac{\varphi(a_\nu)\hat{L}(a_\nu,u)\hat{L}(a_\nu,v)}{\hat{K}'(a_\nu,\bar{a})}+\frac{\varphi(u)\hat{L}(u,v)}{\hat{K}(u,\bar{a})}+\frac{\varphi(v)\hat{L}(v,u)}{\hat{K}(v,\bar{a})}\right\}$$

$$= 0 \quad \text{for all } u, v \ (\neq a, a_\nu) \in \bar{G}. \quad (3.7)$$

Choose any $\varphi_0 \in H_2(G)$ such that

$$\int_{\partial G}\varphi(v)\overline{\varphi_0(v)}\,|dv| = 0, \qquad \varphi_0 \not\equiv 0.$$

We may assume that $\varphi_0(v)$ is a rational function. Then from (3.7), we have

$$\int_{\partial G}\frac{\varphi(v)\hat{L}(v,u)}{\hat{K}(v,\bar{a})}\,\overline{\varphi_0(v)}\,|dv| = 0 \quad \text{for all } u \in G. \quad (3.8)$$

We assume here that $\varphi(a) \neq 0$ and recall that

$$\int_{\partial G}\hat{L}(v,u)\,\overline{f(v)}\,|dv| = 0 \quad \text{for all } f \in H_2(G) \text{ and } u \in G.$$

From (3.8) and (3.3), we have

$$\int_{\partial G} \frac{\varphi(v)\overline{\varphi_0(v)}\ \overline{\hat{K}(v,\bar{u})}}{\hat{K}(v,\bar{a})}\ \overline{dv} = 0 \quad \text{for all } u \in G.$$

Since the family $\{\hat{K}(v, \bar{u}) | u \in G\}$ is complete in $H_2(G)$, we obtain

$$\int_{\partial G} \frac{\overline{\varphi(v)}\ \varphi_0(v)}{\overline{\hat{K}(v,\bar{a})}} f(v)\, dv = 0 \quad \text{for all } f \in H_2(G).$$

Hence from the theorem of Cauchy-Read, there exists an $F \in H_2(G)$ such that

$$\frac{\overline{\varphi(v)}\ \varphi_0(v)}{\overline{\hat{K}(v,\bar{a})}} = F(v) \quad \text{a.e. on } \partial G,$$

or

$$\overline{\left(\frac{\varphi(v)}{\hat{K}(v,\bar{a})}\right)} = \frac{F(v)}{\varphi_0(v)} \quad \text{a.e. on } \partial G. \tag{3.9}$$

Since $\varphi \in H_2(G)$ and $\hat{K}(v, \bar{a}) \neq 0$ near ∂G, a modified form of the Schwarz reflection principle shows that both $\varphi(v)$ and $F(v)/\varphi_0(v)$ can be extended analytically across ∂G. See [6, p. 145] and [9, Chapter 4].

Furthermore, from (3.9) and (3.3), we have the relation

$$\varphi_0(v)\overline{\varphi(v)}\ |dv| = i^{-1} F(v)\hat{L}(v, a)\, dv \quad \text{a.e. along } \partial G. \tag{3.10}$$

Note that $F(v)\hat{L}(v, a)/\varphi_0(v)$ has a *finite* number of poles in G. Hence

$$\int_{\partial G} f(v)\overline{\varphi(v)}\ |dv| = \sum_{\nu}\sum_{\mu} \overline{X_{\nu\mu}}\, f^{(\nu)}(p_\mu)$$

$$\text{for all } f \in H_2(G) \text{ and for some points } p_\mu \in G. \tag{3.11}$$

The $X_{\nu\mu}$ are constants which are independent of f. Thus we have the following expression for φ:

$$\varphi(v) = \sum_{\nu}\sum_{\mu} X_{\nu\mu} \frac{\partial^{\nu}\hat{K}(v, \overline{p_\mu})}{\partial\bar{p}_\mu^{\nu}}. \tag{3.12}$$

Using (3.12) and the identity

$$\left(\varphi(z_1)\varphi(z_2), \hat{K}(z_1,\bar{z})\hat{K}(z_2,\bar{z}) - \frac{\hat{K}(z_1,\bar{z})^2 + \hat{K}(z_2,\bar{z})^2}{2}\right)_1^{\wedge} = 0$$

$$\text{for all } z \in G, \tag{3.13}$$

we obtain

$$\left[\sum_{\nu}\sum_{\mu} X_{\nu\mu}\frac{\partial^{\nu}\hat{K}(z,\bar{p}_{\mu})}{\partial\bar{p}_{\mu}^{\nu}}\right]^2$$

$$=\left(\sum_{\mu'} X_{0\mu'}\right)\left[\sum_{\nu''}\sum_{\mu''} X_{\nu''\mu''}\frac{\partial^{\nu''}\left(\hat{K}(z,\bar{p}_{\mu''})\right)^2}{\partial\bar{p}_{\mu''}^{\nu''}}\right]. \tag{3.14}$$

Hence from (3.3), we have

$$\left(\sum_{\nu}\sum_{\mu}\overline{X_{\nu\mu}}\frac{\partial^{\nu}\hat{L}(z,p_{\mu})}{\partial p_{\mu}^{\nu}}\right)^2$$

$$=\left(\sum_{\mu'}\overline{X_{0\mu'}}\right)\left[\sum_{\nu''}\sum_{\mu''}\overline{X_{\nu''\mu''}}\frac{\partial^{\nu''}\left(\hat{L}(z,p_{\mu''})\right)^2}{\partial p_{\mu''}^{\nu''}}\right]. \tag{3.15}$$

At first, by comparing the orders of the poles at each p_μ, we see that $X_{\nu\mu}=0$ for $\nu\neq 0$. Hence we have

$$\left(\sum_{\mu}\overline{X_{0\mu}}\,\hat{L}(z,p_\mu)\right)^2=\left(\sum_{\mu'}\overline{X_{0\mu'}}\right)\left(\sum_{\mu''}\overline{X_{0\mu''}}\left(\hat{L}(z,p_{\mu''})\right)^2\right). \tag{3.16}$$

Therefore we have

$$X_{0\mu}^2=\left(\sum_{\mu'}X_{0\mu'}\right)X_{0\mu}\quad\text{for all }\mu. \tag{3.17}$$

Hence we have the result that $\varphi(z)=C\hat{K}(z,\bar{u})$ for some constant C and for some point u of G.

We have thus proved that equality holds in (3.1b) *only if* φ and ψ are expressible in the form $\varphi(z)=\alpha\hat{K}(z,\bar{u})$, $\psi(z)=\beta\hat{K}(z,\bar{u})$ for some constants α and β, and for some point u of G.

Since $4\pi\hat{K}(z,\bar{\xi})^2$ is the reproducing kernel for F_1, we know that

$$\left(Z_\nu(z),4\pi\hat{K}(z,\bar{\xi})^2\right)_1=Z_\nu(\xi)\quad\text{for } 1<\nu<N-1.$$

We also know that equality holds in (3.1a) if and only if $(\varphi\psi,f_2)_1=0$ for all $f_2\in F_2$. Equality in *both* (3.1a) and (3.1b) will therefore imply that $Z_\nu(u)=0$ for all ν. But, there is no point of G for which all the $Z_\nu(z)$ vanish simultaneously; see [5, p. 75]. Hence: equality in (2.3) is possible *only if* G is simply-connected.

For simply-connected G, our assertion is already clear. We have thus completed the proof of Theorem 3.1.

In the above proof of Theorem 3.1, we are able to see some interesting results and miscellaneous problems for the products of the kernels. Further discussion and results for the products of reproducing kernels will appear elsewhere.

In addition, by setting $P_{\nu\mu} = \int_{C_\mu} Z_\nu(z)\,dz$, we obtain the following interesting theorem. Compare [5, Theorem 39]:

THEOREM 3.2. *The matrix* $\|\alpha_{\nu\mu} + P_{\nu\mu}\|$ *is positive definite.*

PROOF. Indeed from the fact $K(z, \bar{u}) \gg k(z, \bar{u})$, we have

$$\|f_2\| < \|f_2\|_2 \quad \text{for all } f_2 \in F_2, \tag{3.18}$$

and hence, for any $\{d_\nu\}$

$$\iint_G \left|\sum_\nu d_\nu Z_\nu(z)\right|^2 dx\,dy < \sum_\nu \sum_\mu \alpha_{\nu\mu} d_\nu \overline{d_\mu}. \tag{3.19}$$

Further we see that equality holds in (3.19) for f_2 if and only if

$$(f_2, f)_2 = 0 \quad \text{for all } f \in F_2. \tag{3.20}$$

Therefore we see that for $f_2 \not\equiv 0$, equality in (3.19) does not happen.

We note that by virtue of the conformal invariance of (2.5), all our propositions are conformally invariant. In particular, we note that our propositions are also valid in the case of unbounded regular regions G, if we consider the restricted class of $H_2(G)$ such that $\{f \in H_2(G) | f(z) = O(|z|^{-1})$ for $z \to \infty\}$.

4. Generalizations in simply-connected regions. For any positive continuous function ρ on ∂G, we consider the weighted Szegö kernel $\hat{K}_\rho(z, \bar{u})$ of G which is characterized by the following reproducing property

$$f(u) = \int_{\partial G} f(z)\,\overline{\hat{K}_\rho(z, \bar{u})}\,\rho(z)\,|dz| \quad \text{for all } f \in H_2(G).$$

The kernel $\hat{K}_\rho(z, \bar{u})$ was investigated by Z. Nehari [7], [8] (cf. [6, pp. 121–126]). We now obtain

THEOREM 4.1. *Let G be a simply-connected regular region. Then: for any positive continuous function ρ on ∂G, we have*

$$\frac{1}{\pi} \iint_G |\varphi(z)\psi(z)|^2\,dx\,dy$$
$$< \frac{1}{2\pi} \int_{\partial G} |\varphi(z)|^2 \rho(z)|dz| \,\frac{1}{2\pi} \int_{\partial G} |\psi(z)|^2 (\rho(z))^{-1}\,|dz| \qquad \textit{for any } \varphi, \psi \in H_2(G). \tag{4.1}$$

Equality holds if and only if φ and ψ are expressible in the form $\varphi(z) = \alpha\hat{K}_\rho(z, \bar{u})$ and $\psi(z) = \beta\hat{K}_{\rho^{-1}}(z, \bar{u})$ for some constants α and β, and for some point u of G.

PROOF. Let H be an analytic function on G such that $|H(z)|^2 = \rho(z)$ on ∂G and $H(z)$ has no zeros and poles on G. We note that inequality (4.1) is clear from the proof of Theorem 2.1 and the identity $K(z, \bar{u}) = 4\pi\hat{K}_\rho(z, \bar{u})\hat{K}_{\rho^{-1}}(z, \bar{u})$. From the inequality

$$\frac{1}{\pi}\iint_G \left|\varphi(z)H(z)\frac{\psi(z)}{H(z)}\right|^2 dx\,dy$$

$$\leq \frac{1}{2\pi}\int_{\partial G}|\varphi(z)H(z)|^2\,|dz|\,\frac{1}{2\pi}\int_{\partial G}\left|\frac{\psi(z)}{H(z)}\right|^2|dz|$$

and the identities

$$\hat{K}_\rho(z, \bar{u}) = \hat{K}(z, \bar{u})/\left(H(z)\overline{H(u)}\right) \quad \text{and} \quad \hat{K}_{\rho^{-1}}(z, \bar{u}) = \hat{K}(z, \bar{u})H(z)\overline{H(u)},$$

we immediately obtain the desired result.

COROLLARY 4.1. *For any regular region G and for any positive continuous function ρ on ∂G, we take a meromorphic function M on G such that $|M(z)|^2 = \rho(z)$ on ∂G and $M(z)$ has the zeros and poles at $\{a_1, a_2, \ldots, a_P\}$ and $\{b_1, b_2, \ldots, b_Q\}$, respectively. Here each zero or pole is repeated as many times as its order indicates. Then we have*

$$\frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy$$

$$\leq \frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\rho(z)|dz|\,\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\left(\rho(z)\right)^{-1}|dz|$$

$$\textit{for any } \varphi, \psi \in H_2(G) \textit{ such that } \varphi M, \psi M^{-1} \in H_2(G). \tag{4.2}$$

Equality holds for $\varphi\psi \not\equiv 0$ if and only if G is simply-connected and φ and ψ are expressible in the form $\varphi(z) = \alpha\hat{K}_\rho(z, \bar{u})$ and $\psi(z) = \beta\hat{K}_{\rho^{-1}}(z, \bar{u})$ for some point u of G, and for some constants α and β.

Concerning the existence of the functions M in Corollary 4.1, Nehari **[8]** established that for any ρ, such functions M always exist and, in fact, we can assume that $P + Q \leq N - 1$, when G is N-ply connected.

5. Comparison of extremal quantities. As an application of the fundamental inequality (2.3), we shall examine some relations among the magnitudes of extremal quantities in the Bergman space and the Szegö space. Let $\{\sigma_j\}_j$ and $\{\tau_j\}_j$ denote the complete orthonormal systems which are obtained by the Gram-Schmidt process from the sets $\{K(z, \bar{u}_j)\}_j$ and $\{\hat{K}(z, \bar{u}_j)\}_j$, respectively. Here $\{u_j\}_{j=1}^\infty$ is any point set of G such that $\lim_{j\to\infty} u_j = u_0$ (for some u_0 of G) and $u_j \neq u_k$ for $j \neq k$. See **[6**, pp. 56–62**]**. For any φ and $\psi \in H_2(G)$, we set

$$\varphi(z) = \sum_j a_j\tau_j(z), \qquad \psi(z) = \sum_j b_j\tau_j(z)$$

and

$$\varphi(z)\psi(z) = \sum_j \sum_k a_j b_k \tau_j(z)\tau_k(z) = \sum_j c_j\sigma_j(z).$$

By setting $z = u_n$, we have the relations

$$\sum_{j<n} c_j\sigma_j(u_n) = \Big(\sum_{j<n} a_j\tau_j(u_n)\Big)\Big(\sum_{j<n} b_j\tau_j(u_n)\Big),$$

or

$$\sum_{j<n} c_j\sigma_j(u_k) = \Big(\sum_{j<n} a_j\tau_j(u_k)\Big)\Big(\sum_{j<n} b_j\tau_j(u_k)\Big)$$

$$\text{for } k = 1, 2, \ldots, n \text{ and any fixed } n > 1. \tag{5.1}$$

We note that $\sigma_j(u_k) = \tau_j(u_k) = 0$ for any pair (j, k) such that $j > k > 1$. Since the function $\sum_{j<n} c_j\sigma_j(z)$ is the uniquely determined extremal function which minimizes the Bergman norm in the class $\{f \in F | f(u_k) = c_k;\ k = 1, 2, \ldots, n\}$ [6, p. 115], equation (5.1) implies that

$$\begin{aligned}\frac{1}{\pi} \sum_{j<n} |c_j|^2 &= \frac{1}{\pi} \iint_G \Big|\sum_{j<n} c_j\sigma_j(z)\Big|^2 dx\,dy \\ &< \frac{1}{\pi} \iint_G \Big|\Big(\sum_{j<n} a_j\tau_j(z)\Big)\Big(\sum_{j<n} b_j\tau_j(z)\Big)\Big|^2 dx\,dy \\ &< \frac{1}{2\pi} \int_{\partial G} \Big|\sum_{j<n} a_j\tau_j(z)\Big|^2 |dz|\ \frac{1}{2\pi} \int_{\partial G} \Big|\sum_{j<n} b_j\tau_j(z)\Big|^2 |dz| \\ &= \frac{1}{4\pi^2} \Big(\sum_{j<n} |a_j|^2\Big)\Big(\sum_{j<n} |b_j|^2\Big).\end{aligned} \tag{5.2}$$

We thus obtain

THEOREM 5.1. *For any n, the values $\{\sigma_j(u_k)\}$ and $\{\tau_j(u_k)\}$ $(j, k < n)$ satisfy the following property: for any constants $\{a_j\}$, $\{b_j\}$, and $\{c_j\}$ satisfying the condition* (5.1), *we have*

$$4\pi \sum_{j<k} |c_j|^2 < \Big(\sum_{j<k} |a_j|^2\Big)\Big(\sum_{j<k} |b_j|^2\Big) \quad \textit{for } k = 1, 2, \ldots, n. \tag{5.3}$$

By considering the function

$$f(z) = \tau_k(z)\tau_n(z)/\tau_k(u_n)\tau_n(u_n)$$

and using the extremal property of $\sigma_n(z)$ [6, p. 61], we deduce:

COROLLARY 5.1. *For any $k \leqslant n$, we have*

$$|\sigma_n(u_n)|^2 \geqslant 4\pi|\tau_n(u_n)|^2\,|\tau_k(u_n)|^2.$$

Equality holds if and only if G is simply-connected and $n = 1$.

We shall examine another type of complete orthonormal system. For any fixed $u \in G$, let $\{\tilde{\sigma}_j\}_j$ and $\{\tilde{\tau}_j\}_j$ denote the complete orthonormal systems which are obtained by the Gram-Schmidt process from the sets

$$\left\{ \frac{\partial^{j-1}K(z,\bar{u})}{\partial \bar{u}^{j-1}} \right\}_{j=1}^{\infty} \quad \text{and} \quad \left\{ \frac{\partial^{j-1}\hat{K}(z,\bar{u})}{\partial \bar{u}^{j-1}} \right\}_{j=1}^{\infty},$$

respectively. Then from a modification of Theorem 5.1, we obtain

THEOREM 5.2. *For any n, the values $\{\tilde{\sigma}_j^{(k)}(u)\}$ and $\{\tilde{\tau}_j^{(k)}(u)\}$ $(j, k \leqslant n)$ satisfy the following property:*

$$\sum_{j\leqslant n} c_j \tilde{\sigma}_j^{(k)}(u) = \frac{d^k}{du^k}\left[\left(\sum_{j\leqslant n} a_j\tilde{\tau}_j(u)\right)\left(\sum_{j\leqslant n} b_j\tilde{\tau}_j(u)\right)\right] \quad \textit{for } 0 \leqslant k \leqslant n-1 \tag{5.4}$$

implies

$$4\pi \sum_{j\leqslant k} |c_j|^2 \leqslant \left(\sum_{j\leqslant k} |a_j|^2\right)\left(\sum_{j\leqslant k} |b_j|^2\right) \quad \textit{for } 1 \leqslant k \leqslant n-1. \tag{5.5}$$

COROLLARY 5.2. *For any $k \leqslant n$, we have*

$$\left|\frac{d^{n-1}}{du^{n-1}}\,\tilde{\sigma}_n(u)\right|^2 \geqslant 4\pi|\tilde{\tau}_k(u)|^2\left|\frac{d^{n-1}}{du^{n-1}}\,\tilde{\tau}_n(u)\right|^2.$$

Equality holds if and only if G is simply-connected and $n = 1$.

Taking $n = k = 1$ in Corollaries 5.1 and 5.2, we have

$$\frac{4\pi}{K(u,\bar{u})} \leqslant 4\pi \iint_G \left|\frac{\hat{K}(z,\bar{u})}{\hat{K}(u,\bar{u})}\right|^4 dx\,dy \leqslant \frac{1}{\hat{K}(u,\bar{u})^2}, \tag{5.6}$$

which can be considered as a sharp form of the Sario-Oikawa problem **[13**, p. 342]; cf. also **[5]** and **[15]**. We remark that (5.6) implies the following estimate of the constant $C_{1,1}$ in (2.5) in the case of a doubly connected region G:

$$C_{1,1} < 1 \Big/ \iint_G |Z_1(z)|^2\,dx\,dy. \tag{5.7}$$

6. Existence of doubly orthogonal systems. As typical complete orthonormal systems in the Szegö space, we shall consider the systems $\{\tau_j\}$ and $\{\tilde{\tau}_j\}$ defined in §5. In connection with Theorems 5.1 and 5.2, it is natural to ask whether these systems have the following double-orthogonality property:

$$I_{(j,j')(k,k')} = \iint_G \tau_j(z)\tau_{j'}(z)\overline{\tau_k(z)\tau_{k'}(z)}\, dx\, dy = 0 \tag{6.1}$$

and

$$\tilde{I}_{(j,j')(k,k')} = \iint_G \tilde{\tau}_j(z)\tilde{\tau}_{j'}(z)\overline{\tilde{\tau}_k(z)\tilde{\tau}_{k'}(z)}\, dx\, dy = 0$$
$$\text{for } (j,j') \neq (k,k') \text{ and } (k',k), \tag{6.2}$$

respectively.

In the case of simply-connected regions G, we shall show that (6.2) *is valid.*

We assume that $j + j' \geqslant k + k'$. From the construction of $\{\tilde{\tau}_j\}$, we can set as follows:

$$\tilde{\tau}_j(z) = \sum_{n=0}^{j-1} \tilde{A}_{j,n} \frac{\partial^n \hat{K}(z,\bar{u})}{\partial \bar{u}^n} \qquad (j \geqslant 1), \tag{6.3}$$

for some constants $\{\tilde{A}_{j,n}\}$. For simply-connected regions G, we have the following identities:

$$\frac{\partial^n \hat{K}(z,\bar{u})}{\partial \bar{u}^n} = \frac{1}{4\pi(n+1)\hat{K}(z,\bar{u})} \frac{\partial^n K(z,\bar{u})}{\partial \bar{u}^n} \qquad (n \geqslant 0) \tag{6.4}$$

and

$$\frac{1}{\hat{K}(z,\bar{u})^2} \frac{\partial^n K(z,\bar{u})}{\partial \bar{u}^n} \frac{\partial^m K(z,\bar{u})}{\partial \bar{u}^m}$$
$$= \frac{4\pi(n+1)!\,(m+1)!}{(n+m+1)!} \frac{\partial^{n+m} K(z,\bar{u})}{\partial \bar{u}^{n+m}} \qquad (n, m \geqslant 0), \tag{6.5}$$

as is easily seen. Hence from (6.2), (6.4) and (6.5), we have

$$\tilde{I}_{(j,j')(k,k')} = \iint_G \tilde{\tau}_j(z)\tilde{\tau}_{j'}(z)$$
$$\times \overline{\left[\sum_{0 \leqslant n \leqslant k-1} \sum_{0 \leqslant m \leqslant k'-1} \frac{\tilde{A}_{k,n}\tilde{A}_{k',m'}\, n!\, m!}{4\pi(n+m+1)!} \frac{\partial^{n+m} K(z,\bar{u})}{\partial \bar{u}^{n+m}} \right]}\, dx\, dy.$$

Since $\tilde{\tau}_j(z)\tilde{\tau}_{j'}(z)$ has at least a zero of order $(j-1)+(j'-1)$ at $z = u$, we have the desired result.

In the other cases, we find that equations (6.1) and (6.2) are *not*, in general, valid, by means of explicit computation. This fact seems to imply a serious difficulty in any proof of Theorem 2.1 which does not use the general theory of reproducing kernels and the positive definiteness of $\|C_{\nu\mu}\|$.

7. Rudin kernels on doubly connected regions. In this section we shall show that Theorem 4.1 is not valid, in general, for the case of multiply connected regions G. In particular, we show that Theorem 4.1 is not valid in a typical case of $\rho(z) = \partial g(z, t)/\partial \nu$ in a doubly connected region G. Here $g(z, t)$ is the Green function of G with pole at t $(\in G)$ and $\partial/\partial\nu$ the inner normal derivative with respect to G.

Without loss of generality, we asssume that $G = \{\sqrt{q} < |z| < 1/\sqrt{q}\}$ $(0 < q < 1)$ and we set

$$K_{0,t}(z, \bar{u}) = 2\pi \hat{K}_{\rho}(z, \bar{u}) \quad \text{and} \quad K_{1,t}(z, \bar{u}) = 2\pi \hat{K}_{\rho^{-1}}(z, \bar{u}),$$

respectively. Then, after setting $\tilde{Z}_1(z) = 1/iz$, we have the following identities:

$$K_{0,t}(z, \bar{u}) K_{1,t}(z, \bar{u}) = \pi K(z, \bar{u}) + D_{1,1}(t) \overline{\tilde{Z}_1(u)}\, \tilde{Z}_1(z) \tag{7.1}$$

and

$$L_{0,t}(z, u) L_{1,t}(z, u) = \pi L(z, u) - D_{1,1}(t) \tilde{Z}_1(u) \tilde{Z}_1(z), \tag{7.2}$$

for a uniquely determined real $D_{1,1}(t)$ which is independent of the variables z and u. Here $L_{0,t}(z, u)$ and $L_{1,t}(z, u)$ denote the adjoint L-kernels of $K_{0,t}(z, \bar{u})$ and $K_{1,t}(z, \bar{u})$, respectively. Taking $u = t$ in (7.1) and (7.2), we have

$$K_{1,t}(z, \bar{t}) = \pi K(z, t) + D_{1,1}(t) \overline{\tilde{Z}_1(t)}\, \tilde{Z}_1(z) \tag{7.3}$$

and

$$- L_{1,t}(z, t) W'(z, t) = \pi L(z, t) - D_{1,1}(t) \tilde{Z}_1(t) \tilde{Z}_1(z). \tag{7.4}$$

Here we have used the following facts:

$$K_{0,t}(z, \bar{t}) \equiv 1 \quad \text{and} \quad L_{0,t}(z, t) \equiv -2 \frac{\partial g(z, t)}{\partial z} \equiv -W'(z, t) \quad [\mathbf{11}].$$

Let t_1 denote the unique critical point of $g(z, t)$. By setting $z = t_1$ in (7.4), we obtain

$$D_{1,1}(t) = \pi L(t_1, t)/\tilde{Z}_1(t) \tilde{Z}_1(t_1). \tag{7.5}$$

We claim that $L(t_1, t) \neq 0$ for all $t \in G$. Indeed from the identities $W'(t_1(t), t) \equiv 0$ and

$$K(z, \bar{u}) = -\frac{2}{\pi} \frac{\partial^2 g(z, u)}{\partial z \partial \bar{u}}, \qquad L(z, u) = -\frac{2}{\pi} \frac{\partial^2 g(z, u)}{\partial z \partial u}$$

(cf. [**14**], [**2**]), we have

$$W''(t_1, t) \frac{\partial t_1}{\partial t} - \frac{\pi}{4} L(t_1, t) = 0 \tag{7.6}$$

and

$$W''(t_1, t)\frac{\partial t_1}{\partial \bar{t}} - \frac{\pi}{4} K\left(t_1, \bar{t}\right) = 0. \tag{7.7}$$

From the symmetry of the Green function, we see that if $t \in (\sqrt{q}, 1/\sqrt{q})$, then $t_1 \in (-1/\sqrt{q}, -\sqrt{q})$ (cf. **[10]**). Hence we have $L(t_1, t) = K(t_1, \bar{t})$ for real t and so $L(t_1, t) = \exp(-2i \arg t)K(t_1, \bar{t})$, in general. Suppose that $L(t_1, t) = 0$ and hence at the same time $K(t_1, \bar{t}) = 0$. Then from (7.3) and (7.5), we have

$$K_{1,t}\left(t, \bar{t}_1\right) = \overline{K_{1,t}\left(t_1, \bar{t}\right)} = 0.$$

Further we have

$$L_{1,t}(t, t_1) = -L_{0,t}(t_1, t) = W'(t_1, t) = 0 \quad [\mathbf{11}].$$

But, in general, from the relation

$$i^{-1}L_{1,t}(z, u)K_{1,t}(z, \bar{u})dz = |L_{1,t}(z, u)|^2\, idW(z, t) \quad \text{along } \partial G,$$

we see that the function $L_{1,t}(z, u)K_{1,t}(z, \bar{u})$ has precisely one zero on $\bar{G}$. Here any zero on the boundary is to be counted with half its multiplicity in this enumeration. We thus have a contradiction.

Here we should incidentally remark that $L(z, t)$ has at least one zero in G always, as we see by the application of the maximum principle to the function

$$F(z, t) = K(z, \bar{t})/L(z, t), \qquad |F(z, t)| = 1 \quad \text{on } \partial G.$$

This fact leads us to the conjecture that the number of the zeros of the Bergman kernel $K(z, \bar{u})$ is at most $N - 1$, when G is N-ply connected (cf. **[16]**).

Now we shall show that $D_{1,1}(t)$ is positive in G. Since $L(t_1, t) \neq 0$ for all $t \in G$, it is sufficient to prove this fact for a special t, say $t = 1$. In that case, $t_1 = -1$. We shall prove that $L(-1, 1) > 0$ directly.

Recall the representation

$$\begin{aligned} g(z, c) = -\mathrm{Re}\Bigg[&\frac{1}{4} \log q - \frac{\log c}{\log q} \log z + \log\left(\sqrt{\frac{z}{c}} - \sqrt{\frac{c}{z}}\right) \\ &+ \sum_{\nu=1}^{\infty} \log\left\{\left(1 - q^{2\nu} \frac{z}{c}\right)\left(1 - q^{2\nu} \frac{c}{z}\right)\right\} \\ &- \sum_{\nu=1}^{\infty} \log\left\{(1 - q^{2\nu-1} cz)\left(1 - q^{2\nu-1} \frac{1}{cz}\right)\right\}\Bigg] \end{aligned} \tag{7.8}$$

[3, pp. 386–388]. By direct computation, we get

$$L(z,c) = \frac{2}{\pi}\left[-\frac{1}{\log q}\,\frac{1}{2zc} + \frac{1}{2(z-c)^2}\right.$$

$$+\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{q^{2\nu}/c^2 + q^{2\nu}/z^2 - 4q^{4\nu}/zc + q^{6\nu}/c^2 + q^{6\nu}/z^2}{(1-q^{2\nu}z/c)^2(1-q^{2\nu}c/z)^2}$$

$$\left.-\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{-q^{2\nu-1} - q^{2\nu-1}/c^2z^2 + 4q^{4\nu-2}/cz - q^{6\nu-3} - q^{6\nu-3}/c^2z^2}{(1-q^{2\nu-1}zc)^2\left(1-q^{2\nu-1}(zc)^{-1}\right)^2}\right] \tag{7.9}$$

and

$$L(-1,1) = \frac{1}{\pi}\left[\frac{1}{\log q} + \frac{1}{4} + \sum_{\nu=1}^{\infty}\frac{2q^{2\nu} + 4q^{4\nu} + 2q^{6\nu}}{(1+q^{2\nu})^4}\right.$$

$$\left.+\sum_{\nu=1}^{\infty}\frac{2q^{2\nu-1} + 4q^{4\nu-2} + 2q^{6\nu-3}}{(1+q^{2\nu-1})^4}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{\log q} + \frac{1}{4} + \sum_{n=1}^{\infty}\frac{2q^n}{(1+q^n)^2}\right]. \tag{7.10}$$

The desired result is now obvious for $q \to 0$, and hence for all $q \in (0, 1)$ since the domain function $L(z, u)$ is continuous with respect to q. Cf. [2, Chapter 3].

Finally from (7.1), we obtain

$$K_{0,t}(z,\bar{u})K_{1,t}(z,\bar{u}) \gg \pi K(z,\bar{u}) \tag{7.11}$$

and

$$K_{0,t}(u,\bar{u})K_{1,t}(u,\bar{u}) > \pi K(u,\bar{u}). \tag{7.12}$$

Suppose that the inequality

$$\frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy$$

$$< \frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\frac{|dz|}{\partial g(z,t)/\partial\nu}\;\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\frac{\partial g(z,t)}{\partial\nu}|dz| \tag{7.13}$$

were valid for all φ and $\psi \in H_2(G)$. Then, for $\varphi(z) = K_{1,t}(z,\bar{u})$ and $\psi(z) = K_{0,t}(z,\bar{u})$, we have

$$\frac{1}{\pi}\iint_G |K_{0,t}(z,\bar{u})K_{1,t}(z,\bar{u})|^2\,dx\,dy < K_{0,t}(u,\bar{u})K_{1,t}(u,\bar{u}).$$

From the extremal property of $K(z, \bar{u})$ [2, pp. 21–23], we have

$$\frac{1}{\pi K(u, \bar{u})} < \frac{1}{\pi} \iint_G \left| \frac{K_{0,t}(z, \bar{u}) K_{1,t}(z, \bar{u})}{K_{0,t}(u, \bar{u}) K_{1,t}(u, \bar{u})} \right|^2 dx\, dy$$

$$< \frac{1}{K_{0,t}(u, \bar{u}) K_{1,t}(u, \bar{u})},$$

which implies a contradiction with (7.12).

Establishing (7.11) for an arbitrary compact bordered Riemann surface seems to be an interesting problem. The inequality (7.11) implies a phenomenon more delicate than that in the case of the exact Bergman kernel. Cf. [12].

For example, (7.12) implies the following inequalities (for doubly-connected regions) which we list without proof.

Let $K^E(z, \bar{u})$ and $L^E(z, u)$ denote the exact Bergman kernel and its adjoint L-kernel, respectively.

(I) Inequality

$$L^E(t_1, t)/\tilde{Z}_1(t)\tilde{Z}_1(t_1) > 0$$

was known, but (7.12) implies that

$$\frac{L^E(t_1, t)}{\tilde{Z}_1(t)\tilde{Z}_1(t_1)} > \frac{1}{\iint_G |\tilde{Z}_1(z)|^2\, dx\, dy}.$$

(II) $$K_{1,t}(t_1, \bar{t}_1)\big(K_{1,t}(t, \bar{t}) - \pi K(t, \bar{t})\big)$$

$$= K_{1,t}(t, \bar{t}_1)\big(K_{1,t}(t_1, \bar{t}) - \pi K(t_1, \bar{t})\big) > 0.$$

(III) $$L(t_1, t) K_{1,t}(t_1, \bar{t}) / W''(t_1, t) > 0.$$

(IV) $$1/\pi > K(t, \bar{t}_1)/K_{1,t}(t, \bar{t}_1).$$

(V) $$\frac{K^E(t_1, \bar{t})}{\tilde{Z}_1(t_1)\overline{\tilde{Z}_1(t)}} < 0, \quad \text{but} \quad \frac{K(t_1, \bar{t})}{\tilde{Z}_1(t_1)\overline{\tilde{Z}_1(t)}} > 0.$$

(VI) If B_1 is a canonical cut connecting the two boundary components of G, then

$$\int_{B_1}\left(\int_{B_1} \hat{K}(z, \bar{u})^2\, dz\right)\overline{du} + \int_{B_1}\left(\int_{B_1} \hat{L}(z, u)^2\, dz\right) du > 0,$$

but

$$\int_{B_1}\left(\int_{B_1} K_{0,t}(z, \bar{u}) K_{1,t}(z, \bar{u})\, dz\right)\overline{du} + \int_{B_1}\left(\int_{B_1} L_{0,t}(z, u) L_{1,t}(z, u)\, dz\right) du < 0.$$

There are also some connections here with the identity (39) in [4].

Finally, we remark that in contrast with the inequality (7.13), the inequality

$$\left(\frac{1}{\pi}\iint_G |f'(z)|^2\,dx\,dy\right)^2 \leqslant \frac{1}{2\pi}\int_{\partial G}|f(z)|^2\,idW(z,t)\,\frac{1}{2\pi}\int_{\partial G}\frac{|f'(z)dz|^2}{idW(z,t)}$$

is valid on any compact bordered Riemann surface G, as we see easily from the Green-Stokes formula. Cf. [2, p. 59].

8. Multiplicative functions. The integrals in the fundamental inequality (2.3) can still be considered in the case of multiplicative functions φ and ψ. We shall show that (2.3) is not, in general, valid, in this wider class. We assume that $G = \{q < |z| < 1\}$, $\varphi(z) = \exp(\Gamma_1 W_1(z))$ and $\psi(z) = \exp(\Gamma_2 W_1(z))$, where W_1 denotes $2\,\partial\omega_1(z)/\partial z$ and ω_1 is the harmonic measure of $\{|z| = q\}$ on G. Further Γ_1 and Γ_2 denote real numbers. Then, we have

$$\frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy = \left(\frac{\Gamma_1+\Gamma_2}{\log q}+1\right)^{-1}\left[1 - q^{2((\Gamma_1+\Gamma_2)/\log q+1)}\right] \tag{8.1}$$

and

$$\frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\,|dz|\,\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\,|dz| = \left[1 + q\exp(2\Gamma_1)\right]\left[1+q\exp(2\Gamma_2)\right]. \tag{8.2}$$

Take $\Gamma_1 = \Gamma_2 = -(1/2)\log q$ and q such that $-2\log q > 4$. This yields:

$$\begin{aligned} -2\log q &= \frac{1}{\pi}\iint_G |\varphi(z)\psi(z)|^2\,dx\,dy \\ &> \frac{1}{2\pi}\int_{\partial G}|\varphi(z)|^2\,|dz|\,\frac{1}{2\pi}\int_{\partial G}|\psi(z)|^2\,|dz| = 4, \end{aligned} \tag{8.3}$$

which implies the desired result.

We note that $\varphi(z) = \psi(z) = z^{-1/2}$ in this case and that this function belongs to the L_e-section for $e = \frac{1}{2}\{{}^1_0\}$. Cf. [4, p. 125] and [5, pp. 94–95]. For the Szegö kernel $\hat{K}_e(z,\bar{u})$ of this section, the identity

$$K(z,\bar{u}) = 4\pi\hat{K}_e(z,\bar{u})^2 + C_{1,1}^{(e)}\overline{Z_1(u)}\;Z_1(z)$$

is still valid, as in (2.5) (cf. [12]), but from the fact (8.3) and Theorem 2.1, we see that

$$K(z,\bar{u}) \ll 4\pi\hat{K}_e(z,\bar{u})^2, \quad \text{i.e. } C_{1,1}^{(e)} < 0, \tag{8.4}$$

and

$$K(u,\bar{u}) < 4\pi\hat{K}_e(u,\bar{u})^2. \tag{8.5}$$

Again this fact seems to imply a phenomenon more delicate than that in the case of the exact Bergman kernel. Cf. [12].

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