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RAMSEY'S THEOREM FOR SPACES

BY

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ABSTRACT. A short proof is given of the following known result. For all k, r, t there exists n so that if the t-spaces of an n-space are r-colored there exists a k-space all of whose t-spaces are the same color. Here t-space refers initially to a t-dimensional affine space over a fixed finite field. The result is also shown for a more general notion of t-space.

In §1 we give a short proof of the Affine Ramsey Theorem. §2 is an appendix to the proof, covering some technical points. §3 gives extensions to Vector and Projective Ramsey Theorems. In §4 we define parameter systems and modify the proof to show parameter systems are Ramsey.

Notation. $[r] = \{1, 2, \ldots, r\}$.

1. Let F be an arbitrary, but fixed, finite field. Functions defined in this section are dependent on F in addition to the written variables. In this section "space" shall refer to an affine space (i.e. a translate of a vector space) over F. Let $\dim(T)$ be the dimension of T and call T a t-space if $\dim(T) = t$. Let V be an n-space. We define, for $t \geqslant 0$,

$$\begin{bmatrix} V \\ t \end{bmatrix} = \{T \subset V: \dim(T) = t\}.$$

The points of V are its 0-spaces.

Our object is to give a short proof of the following result.

THEOREM 1 (AFFINE RAMSEY THEOREM). *For all $t \geqslant 0$, r, k there exists $n = N^{(t)}(k: r)$ with the following property. Let V be an n-space and let χ be an r-coloring of the t-spaces of V. Then there exists a k-space W such that all t-spaces $T \subset W$ are the same color.*

We restate in formal terms. Let $\dim(V) = n$ and $\chi: \left[{V \atop t}\right] \to [r]$. Then there exists $W \in \left[{V \atop k}\right]$ such that χ is constant on $\left[{W \atop t}\right]$.

Given a coloring χ of t-spaces we call a k-space W monochromatic (resp.: Red) if all t-spaces $T \subset W$ are the same color (resp.: Red).

The Projective Ramsey Theorem and the Vector Space Ramsey Theorem are given in §3 and derived as immediate corollaries of Theorem 1. The

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original proof of these theorems is given in [1]. Our proof follows the same basic lines as [1], though with strong differences in notation and emphasis. We are indebted to Klaus Leeb for a number of fruitful discussions on this topic.

Our proof of Theorem 1 is self-contained with one large exception: Hales-Jewett Theorem. The statement of that theorem (though not its proof) and some technical corollaries involving applications of it are given in §2.

Let χ be a coloring of $\left[{V \atop t}\right]$. Let $B \in \left[{V \atop u+1}\right]$ and let $p\colon B \to F^u$ be a surjective projection. If $T \in \left[{B \atop t}\right]$ then $p|_T\colon T \to F^u$ is either bijective or it is not. In the former case, $p(T) \in \left[{F^u \atop t}\right]$ and T is called transversal. In the latter case, $p(T) \in \left[{F^u \atop t-1}\right]$, $T = p^{-1}(p(T))$ and T is called vertical. (Intuitively, p defines a vertical direction.) The next definition is essential to the entire proof. We call a space $B \in \left[{V \atop u+1}\right]$ *special* (with respect to a coloring χ and a projection p) if whenever T_1, $T_2 \in \left[{B \atop t}\right]$, transversals, and $p(T_1) = p(T_2)$ then $\chi(T_1) = \chi(T_2)$. That is, B is special if the color of a transversal t-space is determined by its projection.

The following result is central to the proof of Theorem 1.

LEMMA 1. *For all $t > 0$, u, r there exists $m = M^{(t)}(u\colon r)$ such that for any r-coloring of the t-spaces of F^{u+m} there exists a special $(u+1)$-space B.*

In fact, we show somewhat more. Let $p\colon F^{u+m} \to F^u$ be the natural projection given by taking the first u coordinates. Then there exists B which is special with respect to p (technically: $p|_B$).

PROOF. Let $v = v(t, u)$ be the number of t-spaces of a u-space. We prove Lemma 1 for $m = \mathrm{HJ}(|F|^{u+1}\colon c)$ where $c = r^v$ and HJ is the Hales-Jewett function (see §2).

Fix the coloring

$$\chi\colon \left[{F^{u+m} \atop t}\right] \to [r]$$

and $p\colon F^{u+m} \to F^u$ as before. Define $\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_u \in F^u$ by $\vec{e}_0 = \vec{0}$ and $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$, the ith basis vector. Set $A_i = p^{-1}(e_i) \subset F^{u+m}$, $0 < i < u$. That is, A_i is a copy of F^m consisting of all vectors "starting" with $\vec{e}_i$. Figure 1 illustrates the case $F = Z_3$, $u = 2$.

Let $(\vec{x}_0, \ldots, \vec{x}_u)$ be a $(u+1)$-tuple, $\vec{x}_i \in F^m$. Set $\vec{y}_i = \vec{e}_i\vec{x}_i \in A_i$. (That is, place $\vec{e}_i$ to the left of $\vec{x}_i$. Technically, $\vec{y}_i$ is defined by $\vec{y}_i \in A_i$ and $p(\vec{y}_i) = \vec{x}_i$.) The $\vec{y}_i$ generate a unique u-space $X \subset F^{u+m}$. The projection $p|_X\colon X \to F^u$ is bijective (i.e. X "transverses" F^u). Let $T_1, \ldots, T_v$ denote the t-spaces of F^u in some preassigned order. We induce a c-coloring χ' of $(F^m)^{u+1}$ by

$$\chi'[(\vec{x}_0, \ldots, \vec{x}_u)] = (\chi(T_1'), \ldots, \chi(T_v'))$$

where T_i' is that unique t-space in X such that $p(T_i') = T_i$. That is, two

$(u + 1)$-tuples are colored the same iff the u-spaces generated by them are colored identically–identifying them under the projection p.

By the Hales-Jewett Theorem (§2, Corollary 2) there exists parallel lines $L_0, \ldots, L_u$, $L_i \in A_i$, such that all $(\vec{x}_0, \ldots, \vec{x}_u)$, $\vec{x}_i \in L_i$, have the same χ' color. Then $\{L_i\}$ generate a $(u + 1)$-space B which we show is special with respect to χ and the projection $p|_B$ which we shall denote by p.

Let $T \in [{}^B_t]$, transverse. For some j, $1 \leqslant j \leqslant v$, $p(T) = T_j$. By linear algebra we may extend T to a transverse u-space $X \subset B$. Set $X \cap A_i = \{\vec{y}_i\}$ so that T is contained in the u-space X generated by the $\vec{y}_i$. Hence $\chi(T)$ is the jth coordinate of $\chi'[(\vec{x}_0, \ldots, \vec{x}_u)]$, where $\vec{y}_i = \vec{e}_i\vec{x}_i$. As χ' is constant, $\chi(T)$ depends only on j, as desired.

To prove Theorem 1 we first restate it in an equivalent form more suitable for an inductive proof.

THEOREM 1′. *For all $t \geqslant 0$, r, $k_1, \ldots, k_r$ there exists $n = N^{(t)}(k_1, \ldots, k_r)$ with the following property. Let V be an n-space and χ: $[{}^V_t] \to [r]$. Then for some i, $1 \leqslant i \leqslant r$, there exists $W \in [{}^V_{k_i}]$ such that for all $T \in [{}^W_t]$, $\chi(T) = i$. (That is, there is a k_i-space colored i.)*

The proof is a double induction–first on t (for all $(k_1, \ldots, k_r)$) and then on $(k_1, \ldots, k_r)$. For $t = 0$, Theorem 1′ is a simply consequence of the Hales-Jewett Theorem (§2, Corollary 1). Assume the existence of n for $t' < t$ (all $(k_1, \ldots, k_r)$) and for t and all $(k'_1, \ldots, k'_r) < (k_1, \ldots, k_r)$. We set

$$s = \max_{1 \leqslant i \leqslant r} N^{(t)}(k_1, \ldots, k_i - 1, \ldots, k_r),$$

$$u = N^{(t-1)}(s: r), \qquad m = M^{(t)}(u: r), \qquad n = u + m.$$

We shall show that n has the desired property. Let χ:$[{}^{F^u}_{\ t}{}^{+m}] \to [r]$ be arbitrary. Be definition of m (that is, by Lemma 1) there exists a special $(u + 1)$-space B which we fix. Henceforth p refers to the projection with domain B. We induce a coloring χ': $[{}^{F^u}_{t-1}] \to [r]$ by $\chi'(T) = \chi(p^{-1}(T))$. (That is, a $(t - 1)$-space $T \subset F^u$ is colored by the color of the vertical t-space $p^{-1}(T) \subset B$ which projects onto it.) By definition of u (i.e.: induction on t) there exists $X \in [{}^{F^u}_s]$ which is monochromatic–say, by symmetry, color 1–under χ'. Then $p^{-1}(X)$ is a special $(s + 1)$-space all of whose vertical t-spaces are colored 1.

We induce a coloring χ'': $[{}^X_t] \to [r]$ by $\chi''(T) = \chi(T_1)$ where $T_1 \in [{}^{p^{-1}(X)}_t]$, $p(T_1) = T$. Of course, there are many such T_1 but they all have the same color as $p^{-1}(X)$ is special. (That is, we project the coloring of the transverse $T \subset p^{-1}(X)$ down onto X.) As $s \geqslant N^{(t)}(k_1 - 1, k_2, \ldots, k_r)$ (i.e.: induction on $(k_1, \ldots, k_r)$) there exists $W_1 \subset X$ so that either

(i) $\dim(W_1) = k_1 - 1$ and $[{}^{W_1}_t]$ is colored 1 under χ'', or

(ii) $2 \leqslant j \leqslant r$, $\dim(W_1) = k_j$ and $[{}^{W_1}_t]$ is colored j under χ''.

In case (ii), by linear algebra we find a transverse k_j-space $W \subset p^{-1}(X)$ so

that $p(W) = W_1$. (That is, we may lift W_1 to W.) Then $[{W \atop t}]$ is colored j under χ. In case (i) (the moment of induction) we set $W = p^{-1}(W_1)$. W is a (vertical) k_1-space of B. Let T be a t-space of W. If T is transversal, $\chi(T) = \chi''(p(T)) = 1$ as $p(T) \in [{W_1 \atop t}]$. If T is vertical it is a vertical t-space of $p^{-1}(X)$; hence $\chi(T) = 1$.

This completes the proof of Theorem 1. To review: inside an arbitrary n-space V we find a special $(u + 1)$-space B. Inside B we find an $(s + 1)$-space $p^{-1}(X)$ which is special and such that all of its vertical t-spaces are colored 1. Inside $p^{-1}(X)$ we find (in case (ii)) a $((k_1 - 1) + 1)$-space W all of whose transverse t-spaces are colored 1. Then W is colored 1.

2. Let A be an arbitrary finite set and p_i: $A^n \to A$ denote the ith coordinate projection, $1 \leqslant i \leqslant n$. A set $L \subset A^n$, $|L| = |A|$, is called a (combinatorial) line if for some nonempty $I \subset [n]$ all p_i, $i \in I$, are bijective and identical and all p_i, $i \notin I$, are (possibly different) constant maps. (E.g.: $A = \{a, b, c\}$, $n = 4$, $\{(a, b, a, c), (b, b, b, c), (c, b, c, c)\}$ is a line.) Note if A is a field the combinatorial lines of A^n are affine lines but not conversely. If $A = Z_3$, $n = 2$, $\{(0, 2), (1, 1), (2, 0)\}$ is an affine line but not a combinatorial line. We use the following celebrated result.

THE HALES-JEWETT THEOREM [3]. *For all k, c there exists $n = \mathrm{HJ}(k\colon c)$ with the following property. Let $|A| = k$ and χ: $A^n \to [c]$. Then there exists a monochromatic line $L \subset A^n$.*

The two corollaries following are used in the proof of Theorem 1.

COROLLARY 1. *Theorem* 1 (*and hence Theorem* 1′) *holds for $t = 0$.*

PROOF. Set $n = km$ where $m = \mathrm{HJ}(|F|^k, r)$. Let Ψ: $F^n \to (F^k)^m$ denote the natural bijection given by grouping the coordinates of $\vec{x} \in F^n$ into m disjoint sets of k coordinates each. A coloring χ: $F^n \to [r]$ induces the coloring $\chi\Psi^{-1}$: $(F^k)^m \to [r]$ so, by definition of m (i.e. the Hales-Jewett Theorem) there is a monochromatic combinatorial line $L \subset (F^k)^m$ so that $\Psi^{-1}(L) \subset F^n$ is monochromatic.

We claim $\Psi^{-1}(L)$ is an affine k-space in F^n. The proof is conceptually easy but notationally difficult so we only illustrate by an example. In $(F^2)^3$ the set

$$L = \{((x,y), (2, 5), (x,y)): (x,y) \in F^2\}$$

is a combinatorial line which corresponds to the affine plane

$$\Psi^{-1}(L) = \{(x, y, 2, 5, x, y): x, y \in F\} \subset F^6.$$

COROLLARY 2. *Let* $m = \mathrm{HJ}(|F|^{u+1}: c)$. *Let* χ *be a c-coloring of the ordered* $(u+1)$*-tuples* $(\vec{x}_0, \ldots, \vec{x}_u)$, $\vec{x}_i \in F^m$. *Then there exist parallel affine lines* $L_0, \ldots, L_u \subset F^m$ *so that* $\{(\vec{x}_0, \ldots, \vec{x}_u): \vec{x}_i \in L_i\}$ *is monochromatic.*

PROOF. Consider the natural bijection

$$\Phi: (F^m)^{u+1} \to (F^{u+1})^m$$

that sends $(\vec{x}_0, \ldots, \vec{x}_u)$, $\vec{x}_i \in F^m$, into $(\vec{y}_1, \ldots, \vec{y}_m)$, $\vec{y}_j \in F^{u+1}$, where $\vec{x}_i = (x_{i1}, \ldots, x_{im})$, $0 \leq i \leq u$, and $\vec{y}_j = (x_{1j}, \ldots, x_{uj})$, $1 \leq j \leq m$. As before, χ induces $\chi\Phi^{-1}$: $(F^{u+1})^m \to [c]$ under which some combinatorial line $L \subset (F^{u+1})^m$ is monochromatic, so that $\Phi^{-1}(L)$ is also monochromatic. Thus it suffices to show

$$\Phi^{-1}(L) = L_0 \times \cdots \times L_u$$

where the $L_i \subset F^m$ are parallel affine lines. Once again, we give only an example. Let $u = 1$, $m = 3$ and

$$L = \{((x,y), (2,5), (x,y)); (x,y) \in F^2\}.$$

Then

$$\Phi^{-1}(L) = \{((x,2,x), (y,5,y)): x,y \in F\} = L_0 \times L_1$$

where $L_0 = \{(x, 2, x): x \in F\}$ and $L_1 = \{(y, 5, y): y \in F\}$ are parallel lines.

3. Other formulations.

PROJECTIVE RAMSEY THEOREM. *Theorem* 1 *holds where "space" refers to projective space.*

VECTOR SPACE RAMSEY THEOREM. *Theorem* 1 *holds* (*for* $t > 0$) *where "space" refers to vector space.*

The above theorems are equivalent by the canonical association between vector $(t+1)$-spaces and projective t-spaces. In addition, the Vector Space Ramsey Theorem follows immediately from Theorem 1. For let $\dim(V) = n$ and let χ be an r-coloring of the vector subspaces $W \subset V$. We induce a coloring χ' of the affine t-spaces $A \subset V$. Any such A may be written $A = \vec{c} + W$ where W is a vector space determined by A. Set $\chi'(A) = \chi(W)$. Then (selecting n by Theorem 1) there is an affine k-space B which is, say, Red under χ'. Set $B = \vec{d} + X$ where X is a vector space. If T is a vector subspace of X of dimension t then $\chi(T) = \chi'(\vec{d} + T) = \mathrm{Red}$.

4. Parameter systems. Let A be a finite set and $F = \bigcup_{i=1}^{\infty} F_i$, where F_i is a family of functions f: $A^i \to A$. (In §1, A is a finite field and F_i is the family of affine linear functions $f(x_1, \ldots, x_i) = c + \Sigma a_j x_j$.) We fix A, F throughout. A set $S \subset A^n$ is called a t-space if there exists $J = \{j_1, \ldots, j_t\} \subset [n]$ and, for

$i \notin J$, functions $f_i \in F_t$ so that

$$S = \{(x_1, \ldots, x_n): x_i = f_i(x_{j_1}, \ldots, x_{j_t}), i \notin J\}.$$

We call J a basis for S. Generally J is not determined by S, but as $|S| = |A|^{|J|}$, $t = |J|$ is determined. We call t the dimension of S and write $\dim(S) = t$. The singleton subsets of A^n are called 0-spaces.

For any distinct $j_1, \ldots, j_t \in [n]$ we define $p_{j_1 j_2 \ldots j_t}: A^n \to A^t$ by

$$p_{j_1 j_2 \ldots j_t}(x_1, \ldots, x_n) = (x_{j_1}, \ldots, x_{j_t}).$$

As order shall be unimportant, we write p_J for $p_{j_1 j_2 \ldots j_t}$, $J = \{j_1, \ldots, j_t\}$. We call $p: A^n \to A^t$ a projection if $p = p_J$ for some J. If $S \subset A^n$ we write p_J: $S \to A^{|J|}$ for the restriction $p_J|_S$.

DEFINITION. We say (A, F) is *Ramsey* if for all $t \geqslant 0$, r, k there exists n with the following property. If V is an n-space and the t-spaces of V are r-colored there exists a k-space W all of whose t-spaces are the same color.

We call (A, F) a *parameter system* if it satisfies (A1)–(A6) below.

(A1) *Constants*: For all $a \in A$, m, the constant function $f(x_1, \ldots, x_m) = a$ is in F_m. (A generalization (A1′) is given at the end of this section.)

(A2) *Identity*: F_1 contains the identity function $f(x) = x$.

(A3) *Extension*: If $f \in F_u$ and $p: A^n \to A^u$ is a projection then $f' = fp \in F_n$. (E.g.: If $f \in F_2$ then $f' \in F_3$ where $f'(x, y, z) = f(x, z)$.)

(A4) *Composition*: If $f_1, \ldots, f_s \in F_u, f \in F_s$, then $f' \in F_u$ where

$$f'(x_1, \ldots, x_u) = f(f_1(x_1, \ldots, x_u), \ldots, f_s(x_1, \ldots, x_u)).$$

(A5) *Basis*: If $S \subset A^n$, $\dim(S) = t$, $J \subset [n]$ and p_J: $S \to A^{|J|}$ is bijective then J is a basis for S.

(A6) *Projection*: If $S \subset A^n$, a subspace, and $J \subset [n]$ then $p_J(S)$ is a subspace of $A^{|J|}$.

THEOREM 2. *Parameter systems are Ramsey.*

Of course, axioms (A1)–(A6) were chosen so that Theorem 2 would hold. The n-parameter sets of Graham-Rothschild [2] are generalized by parameter systems. Also, affine spaces are an example of parameter systems. On the other hand, the categorical approach of [1] generalizes parameter systems. To this author, the above axiom schema provides an aesthetically pleasing balance between generality and clarity.

There is nothing unique in the choice of (A1)–(A6). We note (without proof) that (A5) may be replaced by

(A5′) *Inverse*: If $f_1, \ldots, f_t \in F_t$ and $(f_1, \ldots, f_t)$: $A^t \to A^t$ is bijective then there exist $g_1, \ldots, g_t \in F_t$ so that $(g_1, \ldots, g_t)$: $A^t \to A^t$ is the inverse mapping of $(f_1, \ldots, f_t)$.

We first derive some elementary corollaries of (A1)–(A6). Let $S \subset A^n$, S a

space, $I \subset [n]$. We say I is a spanning set (for S) if the projection p_I: $S \to A^{|I|}$ is injective. We say I is independent (for S) if the projection p_I: $S \to A^{|I|}$ is surjective. From (A5), I is a basis iff it is independent and a spanning set.

(A7) If I is a spanning set for S there exists a basis $J \subset I$.

PROOF. By (A6), $p_I(S)$ is a subspace of $A^{|I|}$ so there is a bijective projection p: $p_I(S) \to A^t$ onto its basis. Then pp_I: $S \to A^t$ is bijective and $pp_I = p_J$ for some $J \subset I$. By (A5), J is a basis for S.

(A8) If I is independent for S there exists a basis $J \supseteq I$.

PROOF. Let J be a maximal independent set, $J \supseteq I$. If p_J: $S \to A^{|J|}$ is not bijective there exist $x, y \in S$, $p_j(x) = p_j(y)$ for $j \in J$ but $p_u(x) \neq p_u(y)$ for some $u \notin J$. Then $p_{J \cup \{u\}}$: $S \to A^{|J|+1}$ has an image with more than $|A|^{|J|}$ elements. As the image is a space (A6) it must be all of $A^{|J|+1}$, contradicting the maximality of J.

(A9) If I is a basis for S, the map p_I: $S \to A^{|I|}$ is a combinatorial isomorphism. That is, p_I is a bijection such that it and its inverse preserve subspaces and their dimensions.

PROOF. p_I is bijective by definition and sends spaces into spaces by (A6). If U is a subspace of $A^{|I|}$ we may, by (A4), express $p_I^{-1}(U)$ as a subspace of S. In both cases dimension is preserved as cardinality is.

(A9) says, essentially, that all n-spaces (A, F fixed) are isomorphic.

Now we follow the lines of the proof of Theorem 1. Suppose $\dim(B) = u + 1$ and p: $B \to A^u$ is a surjective projection. Let $T \subset B$, $\dim(T) = t$. Then p: $T \to A^u$ is either bijective or it is not. If it is we call T transverse (relative to p) and T has a basis $I \subset [u]$. If it is not, we call T vertical (relative to p). As $p(T)$ is a space and the inverse image of $x \in A^u$ under $p|_T$ has at most $|A|$ points, $p(T)$ must be a $(t-1)$-space, $p|_T$ is an $|A|$-to-1 function, and $T = p^{-1}(p(T))$. Now let χ be a coloring of the t-spaces of B. We call B special (relative to χ, p) if whenever $T_1, T_2 \subset B$, both transverse t-spaces with $p(T_1) = p(T_2)$, then $\chi(T_1) = \chi(T_2)$.

LEMMA 2. *For all $t > 0$, u, r there exists $m = M^{(t)}(u\colon r)$ (dependent on (A, F) of course) so that for any r-coloring χ of the t-spaces of A^{u+m} there exists a $(u+1)$-space B special relative to χ and the projection p: $A^{u+m} \to A^u$ onto the first u coordinates.*

PROOF. Let v be the number of t-spaces in a u-space. We prove Lemma 2 for $m = \mathrm{HJ}(|F_u|, r^v)$. Let $T \subset A^u$, $\dim(T) = t$, $f_1, \ldots, f_m \in F_u$. We define $(T, f_1, \ldots, f_m) \subset A^{u+m}$ by

$$(T, f_1, \ldots, f_m) = \{(x_1, \ldots, x_u, y_1, \ldots, y_m)\colon (x_1, \ldots, x_u) \in T, y_i = f_i(x_1, \ldots, x_u)\}.$$

The basis for T provides (A4) a basis for $(T, f_1, \ldots, f_m)$, so it is a transverse

t-space in A^{u+m}. Conversely, let T' be a transverse t-space in A^{u+m}. By (A7), T' has a basis $J \subset [u]$. All y_i may be written as functions of the basis variables, so $T' = (T, f_1, \ldots, f_m)$ where f_i is the extension of that function for y_i to u variables. (In general, T' will not have a unique expression in this form.)

Now, the critical step, we induce a coloring

$$\chi^*: (F_u)^m \to [r^v]$$

setting $\chi^*(f_1, \ldots, f_m) = \chi^*(f_1', \ldots, f_m')$ iff for all $T \subset A^u$, $\dim(T) = t$,

$$\chi((T, f_1, \ldots, f_m)) = \chi((T, f_1', \ldots, f_m')).$$

(We may consider $(f_1, \ldots, f_m)$ as representing a lifting from A^u to A^{u+m}, inverse to the projection p. Two liftings are colored the same iff the images of A^u are colored identically–identifying the images under p.)

For convenience, let us renumber so the monochromatic "line" in $(F_u)^m$ varies in the first r (> 0) coordinates with constants f_i, $r < i \leq m$. That is, for all $T \subset A^u$, $\dim(T) = t$,

$$\chi((T, f, f, \ldots, f, f_{r+1}, \ldots, f_m))$$

is independent of $f \in F_u$. Now set

$$B = \{(x_1, \ldots, x_u, y_1, \ldots, y_m): y_i = y_1, 2 \leq i \leq r, \\ y_i = f_i(x_1, \ldots, x_u), r < i \leq m\}.$$

B is the desired space. Its basis is the first $u + 1$ coordinates. Let T' be a transverse t-space of B, with $p(T') = T$. Then we may express $T' = (T, g_1, \ldots, g_m)$ for some $g_i \in F_u$. Since $T' \subset B$ we can also express $T' = (T, g_1, \ldots, g_1, f_{r+1}, \ldots, f_m)$, so $\chi(T')$ depends only on T.

At this point the proof of Theorem 2, follows, practically word for word (changing F to A) from that of Theorem 1′. (Note that in Corollary 1 the set $\Psi^{-1}(L)$ is a k-space in A^n. In case (ii) the lifting from W_1 to W is accomplished by setting $W = (W_1, f)$ for arbitrary f.) We omit the details.

The following result, given in [2], somewhat strengthens Theorem 2. Let $\varnothing \neq C \subset A$ and replace (A1) by

(A1′) CONSTANTS: The constant function $f(x_1, \ldots, x_m) = c$ is in F_m iff $c \in C$.

We call (A, F, C) satisfying (A1′), (A2)–(A6) a parameter system with restricted coefficients. The 0-spaces of A^n are the points $x \in C^n$.

THEOREM 3. *Parameter sets with restricted coefficients are Ramsey.*

OUTLINE OF PROOF. Let an element $0 \in C$ be specified. In A^n we call spaces in (A, F) *spaces* and spaces in (A, F, C) *restricted spaces*. If S is a t-space we define Rest(S), the restriction of S, to be that restricted t-space

given by changing all coordinates equal constant $a \notin C$ to the constant 0. Now assume $t \geqslant 0$, r, k, n satisfy the Ramsey property for (A, F) and consider an r-coloring of the restricted t-spaces of A^n. We induce a coloring of all t-spaces, giving S the color of Rest(S). Then there exists a k-space W all of whose t-spaces have the same induced color. $W' = \text{Rest}(W)$ is a restricted k-space all of whose restricted t-spaces have the same color.

When A is a finite field, F the set of all linear functions (without constant term) and $C = \{0\}$, Theorem 3 gives Ramsey's Theorem for vector spaces.

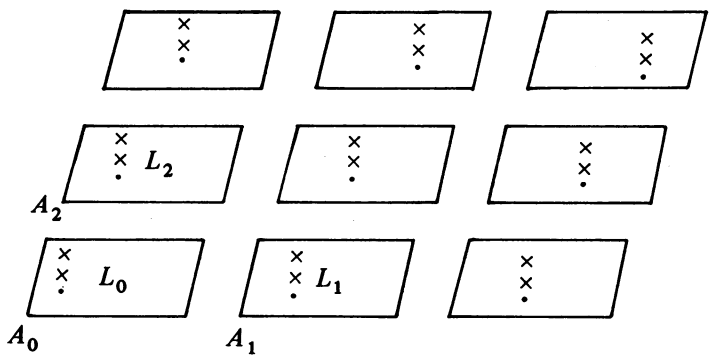

FIGURE 1. $\cdot$ = points of X; $\cdot$, $\times$ = point of B

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