THE MODULE OF INDECOMPOSABLES FOR
mod 2 FINITE H-SPACES

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ABSTRACT. The module of indecomposables obtained from the mod 2
cohomology of a finite H-space is studied. It is shown that this module is
trivial in dimensions \equiv 0 \pmod{4}.

1. Introduction. For the purposes of this paper an H-space \((X, \mu)\) will be a
pointed topological space \(X\) which has the homotopy type of a connected CW
complex of finite type together with a basepoint preserving map \(\mu: X \times X \to X\)
with two-sided homotopy unit. An H-space is (mod 2) finite if \(H^* = H^*(X ; \mathbb{Z}/2)\) is a finite dimensional \(\mathbb{Z}/2\) module (\(\mathbb{Z}/2\) are the integers
reduced mod 2). In this paper we will study the module of indecomposables
\(Q = Q(H^*(X ; \mathbb{Z}/2))\) for (mod 2) finite H-spaces.

**Theorem 1.1.** Let \((X, \mu)\) be a (mod 2) finite H-space. Then \(Q^{2n} = 0\) unless
\(n \equiv 1 \pmod{2}\).

Theorem 1.1 can be viewed as a partial result towards proving the loop
space hypothesis for 1-connected (mod 2) finite H-spaces. This hypothesis
asserts that \(H^*(\Omega X ; \mathbb{Z})\) has no 2 torsion. It is equivalent to asserting that \(Q^{2n}\)
is trivial for all \(n\) (see [4]). Theorem 1.1 is an extension of the main result of
[3] and seems to be the ultimate fact towards which the arguments of [3] were
directed.

In proving 1.1 we will assume that \(X\) is 1-connected. For consider the
fibration \(\widetilde{X} \to X \to G\) where \(G = K(\pi_1(X), 1)\) and \(\widetilde{X}\) is the 1-connected
universal covering space of \(X\). Then \(\widetilde{X}\) is (mod 2) finite and the induced map
\(f^*: Q^{4n} \to Q^{4n}(H^*(\widetilde{X} ; \mathbb{Z}/2))\) is injective. This follows from the spectral
sequence argument in [1]. Thus it suffices to prove 1.1 for \(\widetilde{X}\).

The main tool in proving 1.1 is the theory of secondary operations as
outlined in [13], [9] and [10]. Let \(A(2)\) be the (mod 2) Steenrod algebra. The
ring \(H^*\) has a natural structure as a Hopf algebra over \(A(2)\). Letting \(\mu^*: H^* \to H^* \otimes H^*\) be the comultiplication we will use \(\nu\) to denote the reduced

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comultiplication \( \nu(x) = \mu^*(x) - x \otimes 1 - 1 \otimes x \). There is a duality between \( H^* \) and \( H_* = H_*(X; \mathbb{Z}/2) \) as Hopf algebras over \( A(2) \). This induces a duality of \( A(2) \) modules between \( Q \) and the module of primitives \( P = P(H_*) \). The action of \( A(2) \) on \( H_* \) and on \( P \) is the right action obtained by duality from the left action on \( H^* \) and on \( Q \). We will use the same symbol to denote an element in a module and its image in a quotient module. Similarly for maps between modules. Let \( B(m) \) be the sub Hopf algebra of \( H^* \) generated by \( \sum_{i < m} H^i \). The main technical result we need is

**Theorem 1.2.** Let \( 0 \neq \omega \in P_{2n} \). Pick \( x \in H^{2n} \) and \( B(m) \) such that \( \langle \omega, x \rangle \neq 0, \langle \omega, B(m) \rangle = 0, \) and \( \nu(x) \in B(m) \otimes B(m) \). Suppose that in dimension \( 2n - |\theta| + 1 \) we have the relation \( \text{Sq}^{2n+1}\theta = \sum_{i=1}^{l} a_i b_i \) where \( x = \theta(y) \) and

1. \( b_i(y) \in B(m) \cdot B(m) \) for \( 1 \leq i \leq l \);
2. \( \omega \otimes \omega \in \text{kernel } a_i \) for \( 1 < i < l \). Then \( \omega^2 \neq 0 \).

(See 3.1.1 of [10].) We will prove 1.1 by using 1.2 to show

**Proposition 1.3.** Let \( (X, \mu) \) be a 1-connected (mod 2) finite \( H \)-space. If \( 0 \neq \alpha \in P_{4n} \) and \( P_{4i} = 0 \) for \( 1 < n \) then \( \alpha^{2^q} \neq 0 \) for \( q > 1 \).

Since \( \alpha^{2^q} \neq 0 \) for \( q > 1 \) obviously contradicts the finiteness of \( X \) we conclude that there is no \( n \) such that \( P_{4n} \neq 0 \) while \( P_{4i} = 0 \) for \( i < n \). Thus \( P_{4n} = 0 \) for all \( n \) and, by duality \( Q^{4n} = 0 \) for all \( n \).

The main innovation in our arguments lies in the use of unstable relations in our applications of 1.2. This is made possible by combining Milnor’s description of \( A(2) \) (see [11]) with the results of Kraines in [8].

The outline of this paper is as follows. In §2 we discuss relations among the elements of \( A(2) \). In §3 we discuss the action of \( A(2) \) on \( H^* \) and \( H_* \). In §4 and §5 we prove Proposition 1.3.

**2. The algebra \( A^*(2) \).** In this section we outline some results about \( A(2) \). They follow from Milnor’s description of \( A(2) \) in [11]. The algebra \( A(2) \) has a \( \mathbb{Z}/2 \) basis \( \{ \text{Sq}^R \} \) where \( R \) ranges through all sequences \( R = (r_1, r_2, \ldots) \) of nonnegative integers with only finitely many nonzero terms. The element \( \text{Sq}^R \) is of dimension \( |R| = \sum r_i(p^i - 1) \). In particular \( \text{Sq}^{(n,0,0,\ldots)} = \text{Sq}^n \). Let \( \Delta_s \) denote the sequence \( (0, \ldots, 0, 1, 0, \ldots) \) where 1 occurs in the \( s \)th position (by convention \( \Delta_0 = (0, 0, 0, \ldots) \)). Let \( 2^s R \) denote the sequence where each term in \( R \) is multiplied by \( 2^s \).

Milnor provides a rule by which any two elements in his basis can be multiplied together. Using this rule, we will derive some relations among the elements of \( A(2) \). Any future reference simply to [11] should be taken as a
reference to this rule. First of all, for any $R$ and $s > 1$, there is the relation

$$\text{Sq}^A \cdot \text{Sq}^R = \text{Sq}^R \text{Sq}^A + \sum_{i > 0} \text{Sq}^A \cdot \text{Sq}^{R-2\Delta_i}. \quad (2.1)$$

We can deduce from 2.1 that, for each $R$ and $s > 1$, there exists a relation of the form

$$\text{Sq}^A \cdot \text{Sq}^R = \text{Sq}^R \text{Sq}^A + \sum \text{Sq}^R \text{Sq}^A, \quad (2.2)$$

for a set of $R_i$ and $\Delta_i$ where $i > s$.

For the rest of this section we will deduce unstable relations. These relations will be used in applications of Theorem 1.2 (see §5). With regards to Theorem 1.2 it should be emphasized that unstable relations of dimension $k$ can only be applied in dimensions $< k - 1$. Consequently we must show that a given unstable relation holds in dimension $k$ if we intend to apply it in dimension $k - 1$.

By Kraines' definition of excess in [8] we can eliminate $\text{Sq}^R$ from an unstable relation in dimension $k$, provided $\Sigma r_i > k$. Any future reference to [8] will refer to this fact.

In dimension $4n - 2^s + 2$ ($t > 2$) we have the unstable relation

$$\text{Sq}^{4n+1} \cdot \text{Sq}^A = \text{Sq}^{4n}(\text{Sq}^I \text{Sq}^A) + \text{Sq}^A \cdot (\text{Sq}^2 \text{Sq}^{4n-2^s+1}). \quad (2.3)$$

**Proof.**

$$\begin{align*}
\text{Sq}^{4n+1} \cdot \text{Sq}^A &= \text{Sq}^I \text{Sq}^{4n} \text{Sq}^A, \quad \text{(by [11])} \\
&= \text{Sq}^{4n} \text{Sq}^I \text{Sq}^A + \text{Sq}^{01} \text{Sq}^{4n-2} \text{Sq}^A, \quad \text{(by 2.1)} \\
&= \text{Sq}^{4n} \text{Sq}^I \text{Sq}^A + \text{Sq}^{01} \text{Sq}^A \cdot \text{Sq}^{4n-2^s-2}, \quad \text{(by [11] and [8])} \\
&= \text{Sq}^{4n} \text{Sq}^I \text{Sq}^A + \text{Sq}^A \cdot \text{Sq}^{01} \text{Sq}^{4n-2^s-2}, \quad \text{(by 2.1)} \\
&= \text{Sq}^{4n} \text{Sq}^I \text{Sq}^A + \text{Sq}^A \cdot \text{Sq}^{01} \text{Sq}^A \cdot \text{Sq}^{4n-2^s-2}, \quad \text{(by 2.1)} \\
&= \text{Sq}^{4n+1} \cdot \text{Sq}^A + \text{Sq}^A \cdot \text{Sq}^{01} \text{Sq}^{4n-2^s-1} \quad \text{(by [11]).}
\end{align*}$$

In dimension $4n - 2^s + 3$ ($s > 2$) we have the relation

$$\begin{align*}
\text{Sq}^{4n+1} \cdot \text{Sq}^2 \Delta_i &= \text{Sq}^{4n}(\text{Sq}^I \text{Sq}^{2\Delta_i}) + \text{Sq}^{2\Delta_i} \cdot (\text{Sq}^{01} \text{Sq}^{4n-2^{s+1}-2}) \\
&+ \text{Sq}^{A+1} \cdot \text{Sq}^{01} \text{Sq}^{4n-2^{s+1}-3}. \quad (2.4)
\end{align*}$$
PROOF.

$$
\begin{align*}
\text{Sq}^{4n+1}\text{Sq}^{2\Delta} &= \text{Sq}^1\text{Sq}^{4n}\text{Sq}^{2\Delta} \\
&= \text{Sq}^{4n}\text{Sq}^1\text{Sq}^{2\Delta} + \text{Sq}^{01}\text{Sq}^{4n-2}\text{Sq}^{2\Delta} \\
&= \text{Sq}^{4n}\text{Sq}^1\text{Sq}^{2\Delta} \\
&+ \text{Sq}^{01}\left(\text{Sq}^{2\Delta+1}\text{Sq}^{4n-2^{s+1}-2} + \text{Sq}^{4n-2^{s+1}-3}\right) \\
&= \text{Sq}^{4n}\text{Sq}^1\text{Sq}^{2\Delta} \\
&+ \text{Sq}^{2\Delta+1}\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2} \\
&+ \text{Sq}^{4n-2^{s+1}-3}\text{Sq}^{2\Delta}. 
\end{align*}
$$

(by [11] and [8])

In dimension $4n - 2^{s+1} - 2^{t+1} + 4$ ($s, t > 2$) we have the relation

$$
\begin{align*}
\text{Sq}^{4n+1}\text{Sq}^{2\Delta}\text{Sq}^{\Delta} &= \text{Sq}^{4n}\left(\text{Sq}^1\text{Sq}^{2\Delta}\text{Sq}^{\Delta}\right) \\
&+ \text{Sq}^{2\Delta+1}\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2} \\
&+ \text{Sq}^{2\Delta+1}\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2} \\
&+ \text{Sq}^{4n-2^{s+1}-3}\text{Sq}^{2\Delta}. 
\end{align*}
$$

(2.5)

PROOF. This follows from 2.4 plus the identity

$$
\begin{align*}
\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2}\text{Sq}^{\Delta} &= \text{Sq}^{01}\text{Sq}^{2\Delta+1}\text{Sq}^{4n-2^{s+1}-2} \\
&= \text{Sq}^{2\Delta+1}\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2} \\
&= \left(\text{Sq}^1\text{Sq}^{2\Delta} + \text{Sq}^{2\Delta}\text{Sq}^1\right)\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2} \\
&= \text{Sq}^{2\Delta}\text{Sq}^{01}\text{Sq}^{4n-2^{s+1}-2}. 
\end{align*}
$$

(2.1)

We will use 2.3, 2.4, and 2.5 in applying 1.2. The elements in brackets will play the role of the $b_i$ operations.

3. Action of $A(2)$ on $H^*$ and $H_*$. In this section we describe how $A(2)$ acts on $H^*$ and on $H_*$ when $(X, \mu)$ is a 1-connected (mod 2) finite $H$-space. The results obtained are used in applications of 1.2. They will enable us to pick $x, y$, and $B(m)$ such that the hypotheses of 1.2 are satisfied. We remark that the arguments used in this section depend heavily on the work in [3].

As a preliminary remark we note that $A(2)$ acts on $H^*$ and $H_*$ so as to satisfy the Cartan formula. The action of $\text{Sq}^R$ on $H^*$ satisfies the formula $\text{Sq}^R(x) = \Sigma_{R_r, R_{r-1}} x^{R} \text{Sq}^{R_r}(x) \text{Sq}^{R}(y)$ while the action of $\text{Sq}^R$ on $H_*$ satisfies the formula $(\beta y)\text{Sq}^R = \Sigma_{R_r, R_{r-1}} \beta y \text{Sq}^{R_r} y \text{Sq}^{R_z}$.

We begin by considering the action of $A(2)$ on $H^*$. Besides the map $\nu$:...
\[ H^* \to H^* \otimes H^* \text{ consider the extended map } v: H^* \to H^* \otimes H^* \to Q \otimes Q. \]

Then

1. any \( x \in Q^{2n} \) is represented by \( x \in H^{2n} \) such that \( v(x) \in Q^{\text{even}} \otimes Q^{\text{even}} \).

2. given \( x \in H^{2n} \) such that \( v(x) \in B(m) \otimes B(m) \) then \( \text{Sq}^R(x) \in B(m) \cdot B(m) \) if \( R \neq 2S \) for some \( S \).

3. given \( x \in H^{4n} \) such that \( v(x) \in B(m) \otimes B(m) \) and \( v(x) \in Q^{\text{even}} \otimes Q^{\text{even}} \) then \( \text{Sq}^{2R}(x) \in B(m) \cdot B(m) \) if \( R \neq 2S \) for some \( S \).

For statement 3.1 see 2.8 of [3]. Statements 3.2 and 3.3 are extensions of 2.4 and 2.5 of [3]. The proofs of 2.4 and 2.5 of [3] can be modified to prove 3.2 and 3.3 in the cases \( \text{Sq}^i(x) \) and \( \text{Sq}^{i+1}(x) \) respectively. (In particular, in proving 3.3, we use 3.1 plus the fact (see [3]) that \( \text{Sq}^{2n-2} : Q^{4n-2} \to Q^{4n} \) is surjective for \( n > 0 \).) The general case then follows from the fact that \( \text{Sq}^R \) belongs to the ideal \( \langle \text{Sq}^1 \rangle \) if \( R \neq 2S \) while \( \text{Sq}^{2R} \) belongs to \( \langle \text{Sq}^1, \text{Sq}^2 \rangle \) if \( 2R \neq 4S \).

Next we consider the action of \( A(2) \) on \( P \).

4. Given \( \beta \in P_{2l+1} \) then \( \beta \text{Sq}^R = 0 \) if \( R \neq 2S \) for some \( S \).

For a proof of 3.4 see [2]. It is equivalent to the fact that elements of the ideal \( \langle \text{Sq}^1 \rangle \) act trivially on \( P_{\text{odd}} \).

5. Given \( \beta \in P_{4l+2} \) then \( \beta \text{Sq}^{2R} = 0 \) if \( R \neq 2S \) for some \( S \).

To prove 3.5 observe, first of all, that, by 3.4, all even dimensional elements of \( \langle \text{Sq}^1 \rangle \) act trivially on \( P_{\text{even}} \). Hence there is a well-defined action of \( A(2)/\langle \text{Sq}^1 \rangle \) on \( P_{\text{even}} \). Furthermore there is an isomorphism \( A(2)/\langle \text{Sq}^1 \rangle \cong A(2) \) of algebra where \( \text{Sq}^R \) corresponds to \( \text{Sq}^R \). Then 3.5 is equivalent to asserting that the ideal \( \langle \text{Sq}^2 \rangle \subset A(2)/\langle \text{Sq}^1 \rangle \) acts trivially on \( P_{4l+2} \). This follows from the relation \( (\text{Sq}^2)^2 = 0 \) in \( A(2)/\langle \text{Sq}^1 \rangle \) plus the fact that \( \text{Sq}^2: P_{4k} \to P_{4k-2} \) is injective (see [3]).

Finally we consider the Lie algebra structure of \( P \). We define the Lie bracket product \( [ , ] \) by \( [\beta, \gamma] = \beta \gamma + \gamma \beta \)

6. given \( \beta \in P_{2l+1}, \gamma \in P_{2j+1} \) then \( [\beta, \gamma] = \beta^2 = \gamma^2 = 0 \),

7. given \( \beta \in P_{4l+2}, \gamma \in P_{4j+2} \) then \( [\beta, \gamma] = \beta^2 = \gamma^2 = 0 \).

For a proof of 3.6 see [5]. Regarding 3.7, \( \beta^2 \text{Sq}^2 = \gamma^2 \text{Sq}^2 = [\beta, \gamma] \text{Sq}^2 = 0 \) by 3.4, 3.5, and 3.6. But \( \text{Sq}^2: P_{4k} \to P_{4k-2} \) is injective for all \( k > 0 \).

4. Proof of Proposition 1.3, Part A. In this section we study the action of \( A(2) \) on any \( \alpha \) satisfying the hypothesis of 1.3. So suppose \( 0 \neq \alpha \in P_{4n} \) and \( P_{4i} = 0 \) for \( i \neq n \). Pick \( q > 1 \). Let \( \omega = \alpha^{2^q} \).

(4.1) \( \omega \text{Sq}^R = 0 \) if \( |R| > 0 \) and \( |R| \equiv 0 \mod 4 \).

Proof. By induction on \( q \). The case \( q = 1 \) follows from the fact that \( P_{4i} = 0 \) for \( i < n \). Suppose that the result is true for \( \omega = \alpha^{2^q} \). Pick \( R \) such that \( |R| > 0 \) and \( |R| \equiv 0 \mod 4 \). By the Cartan formula \( \omega^2 \text{Sq}^R \) can be written as a sum of the elements
\[ [\omega \text{Sq}^R, \omega \text{Sq}^R] \quad \text{where } R_1 + R_2 = R, \]
\[(\omega \text{Sq}^R)^2 \quad \text{when } R = 2S.\]

By the induction hypothesis \(|R_1| \equiv 0 \pmod{4}, |R_2| \equiv 0 \pmod{4}, |S| \equiv 0 \pmod{4} \).
Then, by 3.6 and 3.7, \([\omega \text{Sq}^R, \omega \text{Sq}^R] = 0 \) and \((\omega \text{Sq}^R)^2 = 0. \) Q.E.D.

We now prove the main result which we still need about the action of \(A(2)\) on \(\omega\).

(4.2) Any element of \(A(2)\) which acts nontrivially on \(\omega\) can be written as a sum of the elements \(\{\text{Sq}^{2k}\} \). \(\forall k \geq 0.\)

**Proof.** First of all we show that any element which acts nontrivially on \(P_{\text{even}}\) can be written as a sum of the elements \(\{\text{Sq}^R\}\). It suffices to consider the basis elements \(\{\text{Sq}^R\}\). We will argue by induction on the degree of \(R\). Suppose \(|R| = d\) and the statement is true for elements of degree \(< d.\) We can assume \(R \neq 2S\) since, otherwise, we are done. Suppose \(r_i \equiv 1 \pmod{2}\). Then, using 2.2,
\[ \text{Sq}^R = \text{Sq}^A \text{Sq}^{R-A} = \sum \text{Sq}^R \text{Sq}^A \]
where \(|R_j| < |R|\) and \(j > i\). Suppose \(\text{Sq}^R \text{Sq}^A \neq 0 \) on \(P_{\text{even}}\). By 3.4, \(|R_j|\) is even. By the induction hypothesis \(\text{Sq}^R\) can be written as a sum \(\sum \text{Sq}^{R_k}\).

Secondly, we show that if \(\omega \text{Sq}^R \neq 0\) then \(R = \Delta\) for some \(s > 0.\) By 4.1, \(R = 2S.\) Suppose \(r_i \equiv 1 \pmod{2}\). Then
\[ \omega \text{Sq}^R = \omega \text{Sq}^A \text{Sq}^{R-2\Delta} = \sum \omega \text{Sq}^{2R_k} \text{Sq}^{2\Delta} \]
where \(j > i.\) To prove these identities we pass to the action of \(A(2)/\langle \text{Sq}^1 \rangle\) on \(P_{\text{even}}\) considered in the proof of 3.5 and use the isomorphism \(A(2)/\langle \text{Sq}^1 \rangle \cong A(2)\). The first identity is trivial and the second identity is equivalent to 2.2. Now suppose \(\omega \text{Sq}^R \text{Sq}^{2\Delta} \neq 0.\) By 3.5, \(|R_j|\) is even. By 4.1, \(|R_j| = 0.\) Thus \(\text{Sq}^{2R} = 1\) and \(\text{Sq}^{2R} \text{Sq}^{2\Delta} = \text{Sq}^{2\Delta}\).

5. Proof of Proposition 1.3, Part B. We prove Proposition 1.3 by induction on \(q.\) Assume \(X, \alpha, \) and \(n\) are as in 1.3. Suppose \(\omega = \alpha^{2q} \neq 0.\) We will show \(\omega^2 \neq 0.\) Define the integers \(k\) and \(l\) by:

\[ \omega \text{Sq}^{2\Delta_i} \neq 0 \quad \text{while } \omega \text{Sq}^{2\Delta_i} = 0 \text{ for } i > k. \]
\[ \omega \text{Sq}^{\Delta_i} \neq 0 \quad \text{while } \omega \text{Sq}^{\Delta_i} = 0 \text{ for } i > l. \]

Pick an element \(\text{Sq}^{2\Delta_i} - \text{Sq}^{\Delta_i}\) of maximal degree such that \(\omega \text{Sq}^{2\Delta_i} \text{Sq}^{\Delta_i} \neq 0.\) Also pick \(t\) to be the maximal possible integer. By 2.2 we have the relation \(\omega \text{Sq}^{2\Delta_i} \text{Sq}^{\Delta_i} = \omega \text{Sq}^{\Delta_i} \text{Sq}^{2\Delta_i}.\) Also \(s < k\) and \(t < l.\) By 4.2 any element from \(A(2)\) of degree greater than that of \(\text{Sq}^{2\Delta_i} \text{Sq}^{\Delta_i}\) acts trivially on \(\omega.\)

We divide our proof that \(\omega^2 \neq 0\) into three parts. In Part I we show that we can assume that \(t < l - 1,\) that is \(\omega^2 \neq 0\) if \(t = l.\) In Part II we show that we can make the stronger assumption that \(2 < t < l - 1 = s = l.\) In Part III we show \(\omega^2 \neq 0\) in this case as well.
Part I. We show that we can reduce to the case \( t < l - 1 \). First of all we can assume

\[(5.1) \ t > 2.\]

Proof. We will show that \( \omega \in \text{kernel } \text{Sq}^{01} \) implies \( \omega^2 \neq 0 \). We will apply 1.2 to the relation \( \text{Sq}'\text{Sq}^4 = \text{Sq}^4\text{Sq}' + \text{Sq}^{01}\text{Sq}^{4n-2} \). Pick \( z \in H^{2n-2^r+1} \) such that \( \langle \omega \text{Sq}^2 \text{Sq}^4, z \rangle \neq 0 \). Let \( x = \text{Sq}^2 \text{Sq}^4 z \). Thus \( \langle \omega, x \rangle = \langle \omega, \text{Sq}^2 \text{Sq}^4 z \rangle = \langle \omega \text{Sq}^2 \text{Sq}^4, z \rangle \neq 0 \). Let \( m = 4n - 2^r + 1 - 2s + 2 \). By definition \( \nu(z) \in B(m) \otimes B(m) \) (for \( z \) lies in dimension \( m + 1 \)). Since \( B(m) \) is invariant under the action of \( A(2) \) it follows that \( \nu(z) \in B(m) \otimes B(m) \). Also \( \langle \omega, B(m) \rangle = 0 \). For pick \( u \in B(m) \). We can assume that either \( u \) is decomposable or \( u = \theta(v) \) where \( v \) is of dimension \( < m \). But if \( u \) is decomposable then \( \langle \omega, \theta(v) \rangle = 0 \) since \( \omega \) is primitive and annihilates decomposables (see [12]). If \( u = \theta(v) \) then \( \langle \omega, u \rangle = \langle \omega, \theta(v) \rangle = \langle \omega, v \rangle = 0 \). The last equality follows from the fact that all elements of \( A(2) \) of degree greater than that of \( \text{Sq}^2 \text{Sq}^4 \) must act trivially on \( \omega \).

Regarding hypothesis (a) of 1.2 we have \( y = x \) in our case. Thus we must show

\[ \text{Sq}'(x) \in B(m) \cdot B(m), \quad \text{Sq}^{4n-2}(x) \in B(m) \cdot B(m). \]

But this follows from 3.2 and 3.3. For, by 3.1, we can always pick \( z \) such that \( \nu(z) \in \text{Q}^{\text{even}} \otimes \text{Q}^{\text{even}} \). Since \( \text{Q}^{\text{even}} \) is invariant under the action of \( A(2) \) it follows that \( \nu(x) \in \text{Q}^{\text{even}} \otimes \text{Q}^{\text{even}} \) as well.

Regarding hypothesis (b), \( \omega \otimes \omega \in \text{kernel } \text{Sq}^{4n} \) for dimension reasons while \( \omega \otimes \omega \in \text{kernel } \text{Sq}^{01} \) by assumption. We conclude from 1.2 that \( \omega^2 \neq 0 \).

Next we can assume

\[(5.2) \ t > 2, \ t < l.\]

Proof. Suppose \( 2 < t = l \). We will show that \( \omega^2 \neq 0 \). We apply 1.2 to the relation 2.3. Let \( m = 4n - 2^r + 1 - 2s + 2 \). Pick \( z \in H^{m+1} \) such that \( \langle \omega \text{Sq}^2 \text{Sq}^4, z \rangle \neq 0 \). Let \( y = \text{Sq}^2 \text{Sq}^4(y) \) and let \( x = \text{Sq}^2 \text{Sq}^4(y) \). By arguments analogous to 5.1 we can show \( \langle \omega, x \rangle \neq 0, \langle \omega, B(m) \rangle = 0 \) and \( \nu(x) \in B(m) \cdot B(m) \). Regarding hypothesis (a), we must show that

\[ \text{Sq}'\text{Sq}^4(y) \in B(m) \cdot B(m), \quad \text{Sq}^{4n-1}(y) \in B(m) \cdot B(m). \]

Again this follows from 3.2 and 3.3. Regarding hypothesis (b), \( \omega \otimes \omega \in \text{kernel } \text{Sq}^{4n} \) for dimension reasons while \( \omega \otimes \omega \in \text{kernel } \text{Sq}^{01} \) since \( t + 1 > l \). We conclude from 1.2 that \( \omega^2 \neq 0 \).

Part II. We will show that the inequality in Part I extends to

\[ 2 < t < l = s = k, \quad l - 1 < s = k. \]

Proof. First of all \( l - 1 < s \). For \( s < l - 2 \) (plus \( t < l - 1 \) from Part I) implies that \( \text{Sq}^4 \) is of larger degree than \( \text{Sq}^2 \). This contradicts the fact
that all elements of degree larger than \(\text{Sq}^{2^k} \text{Sq}^\Delta\) acts trivially on \(\omega\). Secondly, 
\(s = k\). For \(s < k\) (plus the now established fact that \(t < s\)) implies that \(\text{Sq}^{2^k}\) is of larger degree than \(\text{Sq}^{2^k} \text{Sq}^\Delta\). This again contradicts the maximality of 
\(\text{Sq}^{2^k} \text{Sq}^\Delta\).

Next we show that we can assume

\((5.4)\) \(l - 1 = s\).

**Proof.** Suppose \(l - 1 < s\). We will show \(\omega^2 \neq 0\). We will apply 1.2 to relation 2.4. Let \(m = 4n - 2^{s+1} - 2^{s'} + 2\). Pick \(z \in H^{m+1}\) such that 
\(\langle \omega \text{Sq}^{2^k} \text{Sq}^\Delta, z \rangle \neq 0\). Let \(y = \text{Sq}^\Delta(z)\) and let \(x = \text{Sq}^{2^k}(y)\). By arguments analogous to those in 5.1 we verify that \(x, \omega,\) and \(B(m)\) are related in the correct way. For hypothesis (a) we must verify that

\[
\text{Sq}^{1} \text{Sq}^{2^k}(y) \in B(m) \cdot B(m),
\]
\[
\text{Sq}^{1} \text{Sq}^4 - 2^{s+1} - 2^a \in B(m) \cdot B(m),
\]
\[
\text{Sq}^{4^n - 2^{s+1} - 3} \in B(m) \cdot B(m).
\]

This follows from 3.2. Regarding hypothesis (b), \((\omega \otimes \omega) \text{Sq}^{4n} = 0\) for dimension reasons. Also

\[(\omega \otimes \omega) \text{Sq}^{2^k + \omega} = \omega \text{Sq}^{2^k} \otimes \omega + \omega \text{Sq}^{2^k} \otimes \omega \text{Sq}^{2^k + \omega} + \omega \otimes \omega \text{Sq}^{2^k + \omega} = 0\]

and

\[(\omega \otimes \omega) \text{Sq}^{2^k + 2} = \omega \text{Sq}^{2^k + 2} \otimes \omega + \omega \otimes \omega \text{Sq}^{2^k + 2} = 0\]

since \(s + 1 > k, l\). We conclude that \(\omega^2 \neq 0\). It follows from 5.4 that we can also assume

\((5.5)\) \(t > 2\).

**Proof.** If \(t = 1\) then \(\text{Sq}^{2^k} \text{Sq}^\Delta\) and \(\text{Sq}^\Delta\) have the same degree, while \(t < l\) (for \(t < l - 1 - s\) by 5.1 and 5.4). This situation contradicts our choice of \(t\).

**Part III.** We eliminate the final case \(2 < t < l - 1 = s = k\). We will show \(\omega^2 \neq 0\) by applying 1.2 to relation 2.5. Let \(m = 4n - 2^{s+1} - 2^{s'} + 2\). Pick \(z \in H^{m+1}\) such that 
\(\langle \omega \text{Sq}^{2^k} \text{Sq}^\Delta, z \rangle \neq 0\). Let \(y = \text{Sq}^\Delta(z)\) and let \(x = \text{Sq}^{2^k}(y)\). By the same arguments as in 5.1, \(x, \omega,\) and \(B(m)\) are related in the correct way and hypothesis (a) is satisfied because of 3.2 and 3.3. Regarding hypothesis (b), \((\omega \otimes \omega) \text{Sq}^{4n} = 0\) for dimension reasons. Also

\[(\omega \otimes \omega) \text{Sq}^{2^k + \omega} \otimes \text{Sq}^{2^k} = (\omega \text{Sq}^{2^k + \omega} \otimes \omega \text{Sq}^{2^k}) \text{Sq}^{2^k} = (\omega \text{Sq}^{2^k + \omega} \otimes \omega \text{Sq}^{2^k + 1}) \text{Sq}^{2^k}\]

(since \(s + 1 > k\))

\[
= \omega \text{Sq}^{2^k + 1} \otimes \text{Sq}^{2^k} \otimes \omega \text{Sq}^{2^k + 1}
\]

\[
+ \omega \text{Sq}^{2^k + 1} \otimes \omega \text{Sq}^{2^k + 1} \text{Sq}^{2^k}
\]

(by 3.4)

\[
= 0.
\]

The last equality follows from the fact that \(\text{Sq}^{2^k + 1} \text{Sq}^{2^k}\) has larger degree than \(\text{Sq}^{2^k} \text{Sq}^\Delta\) and, hence, acts trivially on \(\omega\). Finally
\[(\omega \otimes \omega)\text{Sq}^{2s+1} = (\omega\text{Sq}^{2s+1} \otimes \omega + \omega \otimes \omega\text{Sq}^{2s+1})\text{Sq}^1 \quad \text{(by 3.4)}\]

\[= 0 \quad \text{(since } s + 1 > k)\]

and

\[(\omega \otimes \omega)\text{Sq}^{b+2}\text{Sq}^0 = (\omega\text{Sq}^{b+2} \otimes \omega + \omega \otimes \omega\text{Sq}^{b+2})\text{Sq}^0 \]

\[= 0 \quad \text{(since } s + 2 > l).\]

We conclude that \(\omega^2 \neq 0\).

REFERENCES