AN ALGEBRAIC CHARACTERIZATION OF CONNECTED SUM FACTORS OF CLOSED 3-MANIFOLDS

BY

W. H. ROW

Abstract. Let $M$ and $N$ be closed connected 3-manifolds. A knot group of $M$ is the fundamental group of the complement of a tame simple closed curve in $M$. Denote the set of knot groups of $M$ by $K(M)$. A knot group $G$ of $M$ is realized in $N$ if $G$ is the fundamental group of a compact submanifold of $N$ with connected boundary.

Theorem. Every knot group of $N$ is realized in $M$ iff $N$ is a connected sum factor of $M$.

Corollary 1. $K(M) = K(N)$ iff $M$ is homeomorphic to $N$.

Given $M$, there exists a knot group $G_M$ of $M$ that serves to characterize $M$ in the following sense.

Corollary 2. $G_M$ is realized in $N$ and $G_N$ is realized in $M$ iff $M$ is homeomorphic to $N$.

Our proof depends heavily on the work of Bing, Feustal, Haken, and Waldhausen in the 1960s and early 1970s. A. C. Conner announced Corollary 1 for orientable 3-manifolds in 1969 which Jaco and Myers have recently obtained using different techniques.

1. Preliminaries. We will work exclusively in the PL category. [Hem] is an excellent reference for definitions, notation and techniques. Manifolds are usually connected but not necessarily compact, orientable, or without boundary. $\partial M$ denotes the boundary of a manifold $M$. A 2-manifold $S$ properly embedded in a 3-manifold $M$ or contained in $\partial M$ is compressible provided (1) there exists a 2-cell $D$ in $M$ such that $D \cap S = \partial D$ and $\partial D$ does not bound a 2-cell in $S$, or (2) $S$ bounds a 3-cell in $M$. We call a 2-cell $D$ as in (1) a compressing 2-cell for $S$ in $M$. If $S$ is not compressible we say $S$ is incompressible.

A 3-manifold $M$ is $P^2$-irreducible if every 2-sphere in $M$ bounds a 3-cell in $M$ and $M$ contains no 2-sided projective planes. $M$ is $3$-irreducible if every component of $\partial M$ is incompressible. We usually follow Waldhausen [W, p. 57] in using $U(\cdot)$ for nice regular neighborhoods. (One exception is when $U(J_{i+1})$ is defined.) If $N$ and $M$ are compact manifolds with connected

Received by the editors May 3, 1978.


Key words and phrases. Connected sum, knot group, submanifold group, cube-with-a-knotted-hole, $P^2$-irreducible.

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0002-9947/79/0000-0267/$03.50

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boundary and $N \subseteq \text{Int } M$, we use the notation $[N, M] = M - \text{Int } N$.

Projective planes, 2-spheres, and 2-cells share the property that any 2-sided simple closed curve they contain bounds a 2-cell, which is the key to proving Lemma A. (Proof omitted.)

**Lemma A.** Suppose $N_1$ and $N_2$ are $P^2$-irreducible, $\partial$-irreducible 3-manifolds such that $N_1 \cup N_2$ is a 3-manifold, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$ is a collection of 2-manifolds, $N_1 \cap N_2$ is incompressible in both $N_1$ and $N_2$, and no component of $N_1 \cap N_2$ is a 2-cell. Then $N_1 \cup N_2$ is $P^2$-irreducible and $\partial$-irreducible.

The next two lemmas are more general than we need but have other applications. First we give some definitions. Let $B$ be a 3-cell and $A$ an annulus on $\partial B$. Suppose $\alpha$ is an arc properly embedded in $B$ with an endpoint in each component of $\partial B - A$ and such that $C$, the closure of $B - U(\alpha)$, is not a solid torus. We call $C$ a cube-with-a-knotted-hole. If $M$ is a 3-manifold such that $C \cap M = \partial C \cap \partial M = A$. We say the 3-manifold $M \cup C$ is obtained from $M$ by attaching a cube-with-a-knotted-hole to $M$ along $A$.

**Lemma B.** Suppose $T$ is a torus or Klein bottle boundary component of a 3-manifold $M$. Let $A$ be an annulus in $T$ that is not contractible in $M$. If $N$ is obtained from $M$ by attaching a cube-with-a-knotted-hole to $M$ along $A$, then the boundary component of $N$ that intersects $T$ is incompressible in $N$.

**Proof.** Let $L$ be a cube-with-a-knotted-hole such that $L \cup M = N$ and $L \cap M = (\partial L) \cap (\partial M) = A$. Note $A$ is an incompressible surface in $N$. Let $S$ be the boundary component of $N$ that intersects $T$. Let $S \cap L = A'$, an annulus with $\partial A' = \partial A$.

**Case 1.** $T - \text{Int } A$ is an annulus. Suppose $S$ is compressible. Then there exists a properly embedded 2-cell $E$ in $N$ such that $\partial E \subseteq S$, $\partial E$ does not bound a 2-cell in $S$, and $E$ is in general position with respect to $A$. If $(\partial E) \cap A = \emptyset$, there exists a properly embedded 2-cell $E'$ such that $\partial E = \partial E'$ and $E' \cap A = \emptyset$. But then $A$ is contractible in either $M$ or $L$, a contradiction. So assume $(\partial E) \cap A \neq \emptyset$. Since the closures of both components of $S - A$ are annuli, we can adjust $E$ by an ambient isotopy of $M$ so that the closure of each component of $\partial E - A$ is an arc with endpoints in different components of $\partial A$, and $E$ is in general position with respect to $A$. Note that if $\alpha$ is an arc that is a component of $E \cap A$, then the endpoints of $\alpha$ must also lie in different components of $\partial A$. Using the incompressibility of $A$ we can remove the simple closed curve components of $E \cap A$. Hence we can assume that $E \cap A$ consists of a finite number of arcs. Let $F$ be a component of $E \cap L$. $F$ is a 2-cell properly embedded in $L$ such that the algebraic intersection number of $\partial F$ with a component $J$ of $\partial A$ is a nonzero integer. Since $L$ is $\partial$-irreducible, $\partial F$ must bound a 2-cell in $\partial L$. Hence $\partial F$ must have zero...
algebraic intersection number with $J$. So we must conclude that $S$ is incompressible.

*Case 2.* $T - \text{Int } A$ consists of two Möbius bands $Q_1$ and $Q_2$. Suppose $S$ is compressible. As in Case 1, there exists a properly embedded 2-cell $E$ in $N$ such that $\partial E \subseteq S$, $\partial E$ does not bound a 2-cell in $S$, $E$ is in general position with respect to $A$, and $\partial E \cap A \neq \emptyset$. We can assume each component of $(\partial E) \cap A'$ is an arc with an endpoint in each component of $\partial A'$, and each component of $\partial E \cap Q_i$ is an arc that does not separate $Q_i$. Since $A$ is incompressible in $N$ we can further assume that all the components of $E \cap A$ are arcs. Let $D$ be a 2-cell in $E$ such that $D \cap A = \alpha$ is an arc, $D \cap \partial E = \alpha'$ is an arc, and $\partial D = \alpha \cup \alpha'$.

If $D \subseteq L$, then $\alpha' \subseteq A'$. Recall that $\alpha'$ must have an endpoint in each component of $\partial A'$. Hence the existence of $D$ violates the fact that $L$ is not a solid torus.

If $D \subseteq M$, then $\partial D \subseteq T$. Suppose $\alpha' \subseteq Q_1$. Since $\alpha'$ does not separate $Q_1$, $\alpha \cup \alpha'$ must be 1-sided in $T$. But $\partial D = \alpha \cup \alpha'$ must be 2-sided in $T$.

We are forced to conclude again that $S$ is incompressible.

**Lemma C.** Suppose $M$ is a compact 3-manifold with no 2-sphere or projective plane boundary components. Then there exists a simple closed curve $J$ contained in $\text{Int } M$ such that $M - \text{Int } U(J)$ is $P^2$-irreducible and $\partial$-irreducible. Furthermore we may choose $J$ so that $U(J)$ is orientable.

**Proof.** Let $C$ be a compact collar on $\partial M$ in $M$ and let $N$ be the closure of $M - C$. Suppose $L$ is a triangulation of $M$ with subcomplex $K$ that triangulates $N$. Let $G$ be the 1-skeleton of $K''$, the second derived barycentric subdivision of $K$. Since $M - \text{Int } U(G)$ is homeomorphic to $C$ with a finite number of 1-handles attached, $M - G$ is $P^2$-irreducible and $\partial$-irreducible. We want to use one of Bing's techniques to find a simple closed curve $J_1$ that approximates $G$. Then we can tie a knot in $J_1$ to obtain $J$ to insure that $\partial U(J)$ is incompressible in $M - \text{Int } U(J)$.

Note $G$ has the following properties:

1. $G$ is a connected finite graph with no points of order one;
2. for all vertices $v$ of $G$, $G - \text{st}(v, G)$ is connected ($\text{st}(v, G)$ denotes the open star of $v$ in $G$);
3. for all vertices $v$ of $G$, $G - \text{st}(v, G)$ contains either $K^1$, the 1-skeleton of $K$, or $K_2$, the dual 1-skeleton of $K$.

1 and (3) clearly hold. (2) holds since $\text{st}(v, G)$ is one component of $G - \text{lk}(v, K'')$ and $\text{lk}(v, K'')$ is connected ($\text{lk}(v, K'')$ denotes the link of $v$ in $K''$). Recall that $\text{lk}(v, K'')$ is the 1-skeleton of $\text{lk}(v, K'')$, a 2-sphere or 2-cell.) Note any subdivision of a graph with properties (1)--(3) also has properties (1)--(3).
There are an even number of vertices of $G$ with odd order. Hence we can pair these vertices and connect them by polygonal arcs that intersect $G$ only in their endpoints. So we obtain a finite graph $G_1$ that contains $G$, has properties (1)-(3), and such that each point of $G_1$ has even order. Note that $M - G_1$ is $P^2$-irreducible and $\partial$-irreducible.

We now wish to modify $G_1$ to obtain a graph $G_2$ that satisfies (1) and (2) such that each point of $G_2$ has order 2 or 4, and $M - G_2$ is homeomorphic to $M - G_1$.

Let $w_1, \ldots, w_k$ be the points of $G_1$ that have order greater than 4. Let $D_1, \ldots, D_k$ be small regular neighborhoods of $w_1, \ldots, w_k$, respectively, in $M$ such that $D_i$ collapses to $G_1 \cap D_i$, $i = 1, \ldots, k$. There exist 2-cells $B_i$ properly embedded in $D_i$ such that $G_1 \cap D_i \subseteq B_i$. There exist trees $F_i \subseteq D_i$ such that $F_i$ collapses to $F_i$, $F_i \cap \partial D_i = G_1 \cap \partial D_i$ consists of the endpoints of $F_i$, and each point of $F_i$ in $\text{Int} B_i$ has order 2 or 4. Let $D = \bigcup_{i=1}^{k} B_i$ and $G_2 = (G_1 - D) \cup \bigcup_{i=1}^{k} F_i$.

Let $z_1, \ldots, z_s$ be the points of $G_2$ that have order 4, and let $H_1, \ldots, H_s$ be small regular neighborhoods of $z_1, \ldots, z_s$, respectively, in $\text{Int} D$ such that $G_2 \cap \partial H_j$ consists of 4 points. Let $H = \bigcup_{j=1}^{s} H_j$. Replace $G_2 \cap H_j = 1, \ldots, s$, by a pair of arcs $a_j, a_j'$ properly embedded in $H_j$ such that $(a_j \cup a_j') \cap \partial H_j = G_2 \cap \partial H_j$, $a_j$ and $a_j'$ cannot be separated in $H_j$ by a properly embedded 2-cell, and such that

$$J_1 = (G_2 - H) \cup \left( \bigcup_{j=1}^{s} (a_j \cup a_j') \right)$$

is a simple closed curve. See [B, Figure 3 and Lemma 6] for one way to choose $a_j$ and $a_j'$. In the next four paragraphs we will show $M - J_1$ is $P^2$-irreducible and $\partial$-irreducible.

$H_j - J_1$ is clearly $P^2$-irreducible. Using linking arguments it is easy to verify that $H_j - J_1$ is $\partial$-irreducible.

We claim $D_i - (J_1 \cup \text{Int} H)$ is $P^2$-irreducible and $\partial$-irreducible. $D_i - (J_1 \cup H)$, $D_i - G_2, D_i - G_1$ and $(\partial D_i - J_1) \times [0, 1)$ are homeomorphic. So it is sufficient to show that if $E$ is a properly embedded 2-cell in $D_i - (J_1 \cup \text{Int} H)$ with $\partial E \subseteq \partial H_j - J_1$, then $\partial E$ bounds a 2-cell in $\partial H_j - J_1$. By property (2), $G_2 - \text{Int} H_j$ is connected. Note $E$ separates $M - \text{Int} H_j$. Hence $J_1 \cap \partial H_j = G_2 \cap \partial H_j$ must be contained in one component of $\partial H_j - \partial E$. Therefore $\partial E$ bounds a 2-cell in $\partial H_j - J_1$.

We now show $M - (J_1 \cup \text{Int} D)$ is $P^2$-irreducible and $\partial$-irreducible. Recall $N - G_1$, which is homeomorphic to $M - (J_1 \cup D)$, is $P^2$-irreducible and $\partial$-irreducible. So we just need to show $\partial D_i - J_1$ is incompressible in $M - (J_1 \cup \text{Int} D)$. Let $E$ be a properly embedded 2-cell in $M - (J_1 \cup \text{Int} D)$ with $\partial E \subseteq \partial D_i - J_1$. We claim $E$ separates $M - (J_1 \cup \text{Int} D)$. Let $E'$ be a 2-cell
in $\partial D_i$ with $\partial E' = \partial E$. The 2-sphere $E \cup E'$ is contained in either $M - K_1$ or $M - K'$ by property (3) for $G_1$. Since both $M - K_1$ and $M - K'$ are irreducible, $E \cup E'$ separates $M$. Hence $E$ separates $M - (J_1 \cup \text{Int } D)$. Using property (2) for $G_1$, we see $G_1 - \text{Int } D_i$ is contained in one component of $M - (J_1 \cup \text{Int } D \cup E)$. Hence $G_1 \cap \partial D_i = J_1 \cap \partial D_i$ cannot meet both components of $\partial D_i - \partial E$. Therefore $\partial E$ bounds a 2-cell in $\partial D_i - J_1$.

The results of the preceding three paragraphs imply that each component of $(\partial H \cup \partial D) - J_1$ is a separating incompressible surface in $M - J_1$, and the closures of the components of $(M - J_1) - ((\partial H \cap \partial D) - J_1)$ are $P^2$-irreducible and $\partial$-irreducible. By Lemma A, $M - J_1$ is $P^2$-irreducible and $\partial$-irreducible.

We claim $J_1$ does not pierce any 2-sphere in $M$. Suppose $S$ is a 2-sphere in $M$ with $S \cap J_1 = \{p\}$. We can assume $p \notin D$ and that $S \cap \partial D = \emptyset$. So $S \cap (J_1 \cup D) = \{p\} = S \cap G_1$. Hence $S$ is contained in an irreducible submanifold of $M$, either $M - K_1$ or $M - K'$. We conclude $J_1$ does not pierce $S$.

$\partial U(J_1)$ may not be incompressible in $M - \text{Int } U(J_1)$. So we need to “tie a knot” in $J_1$. More precisely, let $P$ be a 3-cell in $U(J_1)$ such that $P \cap \partial U(J_1) = A$ is an annulus, $\text{cl}(U(J_1) - P) = P'$ is a 3-cell, and $P \cap J_1 = \alpha$ is an arc with an endpoint in each component of $\partial P - A$. Let $\alpha'$ be the arc $P' \cap J_1$. Let $\beta$ be a properly embedded arc in $P$ such that $\partial \beta = \partial \alpha$ and $L = \text{cl}(P - U(\beta))$ is a cube-with-a-knotted-hole. ($U(\beta)$ is a standard regular neighborhood of $\beta$ in $P$.) Let $J_2 = \beta \cup \alpha'$. We say $J_2$ is obtained from $J_1$ by tying a knot in $J_1$. Note $U(\beta) \cup P'$ is a regular neighborhood of $J_2$ in $M$. Hence $M - \text{Int } U(J_2)$ is homeomorphic to $M - \text{Int } U(J_1)$ with the cube-with-a-knotted-hole $L$ attached along $A$. Since $J_1$ does not pierce any 2-sphere in $M$, $A$ must not be contractible in $M - \text{Int } U(J_1)$. Applying Lemma B we see $M - \text{Int } U(J_2)$ is $\partial$-irreducible. $M - \text{Int } U(J_2)$ is clearly $P^2$-irreducible. Hence $J_2$ is our required $J$.

Suppose $U(J_2)$ is nonorientable. We wish to find a simple closed curve $J$ such that $U(J)$ is orientable and $M - \text{Int } U(J)$ is $P^2$-irreducible and $\partial$-irreducible. Let $Q$ be a Möbius band, with center line $J_2$, that is properly embedded as a 2-sided subset of $U(J_2)$. Let $J_3 = \partial Q$. Note $U(J_3)$ is orientable and $J_3$ is not contractible in $M - J_2$.

We claim Int $Q$ is incompressible in $M - J_3$. Let $E$ be a 2-cell in $M - J_3$ such that $E \cap Q = \partial E$. $\partial E$ must be 2-sided in $Q$. So we can assume $\partial E \cap J_2 = \emptyset$. Now $\partial E$ is not parallel to $J_3$ in $Q$ since $\partial E$ is contractible in $M - J_2$. Hence $\partial E$ must bound a 2-cell in $Q$.

Since $M - Q$ and $M - J_2$ are homeomorphic, $M - Q$ is $P^2$-irreducible and $\partial$-irreducible. It follows that $M - J_3$ is $P^2$-irreducible and $\partial$-irreducible. $J_3$ cannot pierce a 2-sphere in $M$ since $J_3$ bounds $Q$. Let $J$ be obtained from
$J_3$ by tying a knot in $J_3$. $U(J)$ is orientable. As before, $M - \text{Int } U(J)$ is $P^2$-irreducible and 3-irreducible.

This completes the proof of Lemma C.

2. Proof of the main theorem and corollaries. Let $M$ be a closed 3-manifold. Recall that $K(M)$ is the set of fundamental groups of complements of PL simple closed curves in $M$. $S(M)$ is the set of fundamental groups of compact sub-3-manifolds of $M$ that have connected boundary.

Main Theorem. Let $M, N$ be closed 3-manifolds. $S(M)$ contains $K(N)$ if and only if $N$ is a connected sum factor of $M$.

Proof. Suppose $N$ is a connected sum factor of $M$. Let $\alpha$ be a simple closed curve in $N$, $U(\alpha)$ a regular neighborhood of $\alpha$ in $N$, and $B$ a 3-cell in $\text{Int } U(\alpha)$. Since $N - \text{Int } B$ is homeomorphic to a subset of $M$, $\pi_1(N - \text{Int } U(\alpha))$ belongs to $S(M)$.

Now suppose $S(M)$ contains $K(N)$. By Lemma C there exists an orientable simple closed curve $J_0$ contained in $N$ such that $L_0 = N - \text{Int } U(J_0)$ is $P^2$-irreducible and 3-irreducible. Let $h$ be a positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori contained in $M$ [Ha]. We wish to add structure to $L_0$ by doubling $J_0$ and then tying a knot in the result, repeating the process $h + 1$ times. Note that $h$ is the only information about $M$ used in this construction. More precisely assume $J_i$, $U(J_i)$, and $L_i$ have been defined.

Let $J_i^+$ be a simple closed curve in $\text{Int } U(J_i)$ with winding number two. Let $A_i$ be an annulus on $\partial U(J_i^+)$ that is contractible in $U(J_i^+)$ and does not separate $\partial U(J_i^+)$. Let $L_i^+ = N - \text{Int } U(J_i^+)$. Let $C_i$ be a cube-with-a-knotted-hole in $U(J_i^+)$ such that $C_i \cap \partial U(J_i^+) = A_i$ and $L_{i+1} = L_i^+ \cup C_i$ is obtained from $L_i^+$ by attaching $C_i$ along $A_i$. Now $N - \text{Int } L_{i+1}$, being a solid torus, is a regular neighborhood of a simple closed curve $J_{i+1}$. We depart from our $U(\cdot)$ convention and define $U(J_{i+1}) = N - \text{Int } L_{i+1}$. Recall $[L_i, L_i^+] = L_i^+ - \text{Int } L_i$. We collect some useful facts in the following lemma. Figure 1 should help picture the construction.

Lemma 1. $[L_i, L_i^+]$ is 3-irreducible. $[L_i^+, L_i^{+1}]$ is not a parallelity component. $L_i$ is $P^2$-irreducible and 3-irreducible. No nontrivial loop on $A_i$ is freely homotopic in $[L_i, L_i^+]$ to a loop on $\partial L_i^+$.

Proof. Suppose $D$ is a compressing 2-cell for $[L_i, L_i^+]$. Let $U(D)$ be a regular neighborhood of $D$ in $[L_i, L_i^+]$. If $\partial D \subseteq \partial L_i$, the closure of $U(J_i) - U(D)$ is a 3-cell in $U(J_i)$ that contains $J_i^+$. If $\partial D \subseteq \partial L_i^+$, then $U(J_i^+) \cup U(D)$ is a 3-cell in $U(J_i)$ containing $J_i^+$. In either case the winding number of $J_i^+$ in $U(J_i)$ would be zero, contrary to construction.
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Figure 1

\[ \text{Int} [L_i^+, L_{i+1}^+] = U(J_i) - \text{Int} U(J_i^+) \]

\[ U(J_{i+1}) = \text{cl}(U(J_i^+) - C_i) \]

Int \([L_i^+, L_{i+1}^+]\) contains an incompressible torus (a parallel copy of \(\partial C_i\) in Int \(C_i\)) that does not separate \(\partial L_i^+\) from \(\partial L_{i+1}^+\) in \([L_i^+, L_{i+1}^+]\). So by the appendix to [Ha], \([L_i^+, L_{i+1}^+]\) cannot be a parallelity component.

Using Lemmas A and B with the above facts we see \(L_i\) is \(P^2\)-irreducible and \(\partial\)-irreducible.

Suppose \(l^+\) is a nontrivial loop on \(A_i\) that is freely homotopic in \([L_i, L_i^+]\) to a loop \(l\) on \(\partial L_i\). Since \(A_i\) is incompressible \(l\) must be nontrivial on \(\partial L_i\). By [W2] there exists an annulus \(A\) in \([L_i, L_i^+]\) with one boundary component \(\alpha^+\) on \(A_i\), the other \(\alpha\) on \(\partial L_i\), and both are nontrivial. Note \(\alpha\) bounds a 2-cell in \(U(J_i)\). So if we consider \(U(J_i)\) as embedded in \(E^3\), \(\alpha\) does not link \(J_{i+1}\) mod 2 but \(\alpha^+\) does link \(J_{i+1}\) mod 2. This contradiction completes the proof of Lemma 1.

Lemma 2. \(M\) contains a compact \(P^2\)-irreducible, \(\partial\)-irreducible 3-manifold \(K\) such that \(\partial K\) is a torus or a Klein bottle and \(K\) is homotopy equivalent to \(L_{n+1}\).
Proof. By hypothesis there is a compact 3-manifold $K'$ in $M$ such that
\[ \pi_1(K') \cong \pi_1(L_{n+1}) \] and $\partial K'$ is connected. Note $\pi_1(L_{n+1})$ has no elements of
finite order [E, Theorem 3.2], is not a nontrivial free product [Hem, Theorem
7.1], and is not free abelian. $\partial K'$ is incompressible since otherwise $\pi_1(K')$
would be free abelian or a nontrivial free product. If $K'$ contained a 2-sided
projective plane, $\pi_1(K')$ would have elements of order two. Hence $K' = K \# H$
where $H$ is a homotopy 3-sphere and $K$ is $P^2$-irreducible with $\partial K$
connected and incompressible. We can assume $K$ is contained in $M$. Since $K$
and $L_{n+1}$ are $K(\pi, 1)$'s, $K$ is homotopy equivalent to $L_{n+1}$. Hence
\[ 0 = X(\partial L_{n+1}) = 2X(L_{n+1}) = 2X(K) = X(\partial K) \]
which implies $\partial K$ is a torus on a Klein bottle. ($X(\cdot)$ denotes the Euler
characteristic.) Lemma 2 is completed.

Suppose $f : K \to \text{Int} L_{n+1}$ is a homotopy equivalence. We can assume
$f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$ are collections of 2-sided properly embedded
incompressible surfaces in $K$ [Hel] with a minimum number of components.
Both collections are nonempty since $f_* : \pi_1(K) \to \pi_1(L_{n+1})$ is an isomorphism
and $\pi_1(L_{n+1})$ splits as nontrivial free products with amalgamation along both
$\pi_1(A_h)$ and $\pi_1(\partial L_h)$. Suppose $S$ is a component of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$
that has boundary.

Each component of $\partial S$ is a nontrivial simple closed curve on $\partial K$. In order
to see this note $(f|_S)_* : \pi_1(S) \to G$ is a monomorphism where $G$ is either
$\pi_1(A_h)$ or $\pi_1(\partial L_h)$. Hence $\pi_1(S)$ is free abelian. If $S$ is an annulus the
boundary components must be nontrivial. If $S$ were a Möbius band with
trivial boundary we could find a 2-sided projective plane in $K$. If $S$ were a
2-cell with trivial boundary, one component of $K \setminus \text{Int} U(S)$ would be a
3-cell and we could reduce the number of components of either $f^{-1}(A_h)$ or
$f^{-1}(\partial L_h)$. Our assertion follows.

Note $f^{-1}(A_h) \cap \partial K$ is nonempty.

Lemma 3. $f^{-1}(\partial L_h) \cap \partial K$ is empty.

Proof. Suppose not. Then there exist nontrivial 2-sided simple closed
curves $\alpha$ and $\beta$ on $\partial K$ such that $f(\alpha) \subseteq A_h$ and $f(\beta) \subseteq \partial L_h$. Let $U(\alpha \cup \beta)$ be
a regular neighborhood of $\alpha \cup \beta$ in $\partial K$. Each component of $\partial K \setminus \text{Int} U(\alpha \cup \beta)$
must have Euler characteristic zero. Since at least one such component
must have two boundary simple closed curves, $\alpha \cup \beta$ must bound an annulus
$A'$ in $\partial K$. We can assume $\text{Int} A'$ misses $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$. Hence $f(\alpha)$ is a
nontrivial loop in $A_h$ that is freely homotopic in $[L_h, L_h^+]$ to a loop in $\partial L_h$,
which violates Lemma 1. So Lemma 3 holds.

Hence $f(\partial K) \subseteq \text{Int}[L_h, L_{n+1}]$. We can modify $f$ so that $f(\partial K) \subseteq [L_h, L_{n+1}]$
but $f(\partial K) \cap \partial L_h$ is nonempty. Let $x$ be a point in $\partial K$ such that $f(x)$ belongs
to $\partial L_h$. We now wish to apply the main result of [F]. Consider the geometric
splitting of $L_{h+1} = L_h \cup_{d_L} [L_h, L_{h+1}]$ and $f_{\circlearrowright x} : \pi_1(K, x) \to \pi_1(L_{h+1}, f(x))$. Under $(f_{\circlearrowright x})^{-1}$ we obtain an algebraic splitting of $\pi_1(K, x)$ that respects the peripheral structure of $\pi_1(K, x)$, i.e., $\pi_1(\partial K, x)$ under inclusion is a subgroup of $(f_{\circlearrowright x})^{-1}(\pi_1([L_h, L_{h+1}], f(x)))$. So applying [F] there exists a separating incompressible torus $T \subseteq \text{Int } K$ (denote the closure of the component of $K - T$ that has boundary $T$ as $K_h$; then the closure of the other component is $K - \text{Int } K_h = [K_h, K]$; let $y$ belong to $T$) and an isomorphism $d: \pi_1(K, y) \to \pi_1(K, x)$ such that

$$f_{\circlearrowright x}d(\pi_1(T, y)) = \pi_1(\partial L_h, f(x))$$

and

$$f_{\circlearrowright x}d(\pi_1(K_h, y)) = \pi_1(L_h, f(x)) \text{ or } \pi_1([L_h, L_{h+1}], f(x)).$$

Since $\pi_1(\partial L_h)$ is a peripheral subgroup of both $\pi_1(L_h)$ and $\pi_1([L_h, L_{h+1}])$ and neither $L_h$ nor $[L_h, L_{h+1}]$ is a product $I$-bundle, [W] and [He] apply to conclude $K_h$ is homeomorphic to either $L_h$ or $[L_h, L_{h+1}]$. The only possibility is $L_h$.

Let $g: L_h \to K_h$ be a homeomorphism. First some notation is needed. Let $K_j = g(L_j)$ and $K_j^+ = g(L_j^+)$. Now $\partial K_0^+, \ldots, \partial K_{h-1}$ are nonparallel disjoint 2-sided tori in $M$. At least one must be compressible, say $\partial K_j^+$, $0 < j < h - 1$. Let $J$ be a boundary component of the annulus $A_j$.

**Lemma 4.** $g(J)$ bounds a 2-cell in $M - \text{Int } K_{j+1}$.

**Proof.** We adapt the Bing-Martin proof that composite knots have property P [B, M]. Recall $L_{j+1} = C_j \cup L_j^+$. Let $K_j'$ be a concentric copy of $K_j^+$ in $\text{Int } K_j^+$. Since $\partial K_j^+$ is compressible in $M$, there exists a compressing 2-cell $D$ for $\partial K_j'$ in $M - \text{Int } K_j'$. We can assume $D$ misses $g(C_j)$ (put $D$ in general position with respect to $g(C_j)$). If all the simple closed curves in $D \cap g(\partial C_j)$ are trivial on $g(\partial C_j)$, we can find the desired compressing 2-cell. Otherwise we can find a compact 3-manifold $Q'$ that contains $g(C_j)$, is contained in $M - K_j'$ and has a 2-sphere boundary. Once again we can find the desired 2-cell.) Now put $D$ in general position with respect to $\partial K_j^+$. We can assume all simple closed curves in $D \cap \partial K_j^+$ are nontrivial on $\partial K_j^+$. $D \cap \partial K_j^+$ is nonempty since $K_j^+ - \text{Int } K_j'$ has incompressible boundary. Let $E \subseteq D$ be a 2-cell such that $E \cap \partial K_j^+ = \partial E$. $E$ is contained in $M - \text{Int } K_{j+1}$ since $D$ misses $g(C_j)$ and $K_j^+ - K_j'$ has incompressible boundary. Since $\partial E$ misses $g(A_j)$ and is nontrivial in $\partial K_j^+$, $\partial E$ is parallel to $g(J)$ on $\partial K_{j+1}$. The required 2-cell exists and the proof of Lemma 4 is complete.

Let $D_1$ be a 2-cell in $N$ with $D_1 \cap L_{j+1} = \partial D_1 = J$. Let $D_2$ be a 2-cell in $M$ with $D_2 \cap K_j^+ = \partial D_2 = g(J)$. Extend $g|_{L_{j+1}}: L_{j+1} \to K_{j+1}$ first to $g': L_{j+1} \cup D_1 \to K_{j+1} \cup D_2$ and then to regular neighborhoods $g''$: $U(L_{j+1} \cup D_1) \to U(K_{j+1} \cup D_2)$. Since $N - \text{Int } U(L_{j+1} \cup D_1)$ is a 3-cell, there exists a closed 3-manifold $Q$ such that $M$ is a connected sum of $N$ and $Q$. The proof of the main theorem is complete.
Corollary 1. Suppose $M$ and $N$ are closed 3-manifolds. $K(M) = K(N)$ if and only if $M$ is homeomorphic to $N$.

Proof. By the main theorem, $M$ is a connected sum of $N$ and a closed 3-manifold $Q_1$. Again $N$ is a connected sum of $M$ and a closed 3-manifold $Q_2$. So $M$ is a connected sum of $M$, $Q_2$, and $Q_1$. [Hem, Theorem 3.21] applies to allow us to conclude $Q_1$ and $Q_2$ are trivial. Hence $M$ and $N$ are homeomorphic.

Suppose $N$ is a closed 3-manifold. Let $h_N = h$ be the positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori in $N$ [Ha]. Choose $L_{h+1} \subseteq N$ as in the proof of the main theorem. Denote $\pi_1(L_{h+1})$ by $G_N$. If $M$ is a closed 3-manifold and $G_N$ belongs to $S(M)$, we say $G_N$ is realized in $M$.

Corollary 2. Let $M$ and $N$ be closed 3-manifolds. $G_M$ is realized in $N$ and $G_N$ is realized in $M$ if and only if $M$ is homeomorphic to $N$.

Proof. Consider the positive integers $h_N$ and $h_M$ as defined above. Suppose $h_M < h_N$. By the proof of the main theorem, $N$ is a connected sum factor of $M$. So $h_N < h_M$. Apply the proof of the main theorem again to conclude $M$ is a connected sum factor of $N$. As in Corollary 1, $M$ is homeomorphic to $N$.

We conclude with the following question.

Question. Does the main theorem hold for compact 3-manifolds with no 2-sphere boundary components?

References

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916

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