TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 250, June 1979

AN ALGEBRAIC CHARACTERIZATION OF CONNECTED SUM FACTORS OF CLOSED 3-MANIFOLDS

BY

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ABSTRACT. Let M and N be closed connected 3-manifolds. A *knot group* of M is the fundamental group of the complement of a tame simple closed curve in M. Denote the set of knot groups of M by $K(M)$. A knot group G of M is *realized* in N if G is the fundamental group of a compact submanifold of N with connected boundary.

THEOREM. *Every knot group of N is realized in M iff N is a connected sum factor of M.*

COROLLARY 1. *$K(M) = K(N)$ iff M is homeomorphic to N.*

Given M, there exists a knot group G_M of M that serves to characterize M in the following sense.

COROLLARY 2. *G_M is realized in N and G_N is realized in M iff M is homeomorphic to N.*

Our proof depends heavily on the work of Bing, Feustal, Haken, and Waldhausen in the 1960s and early 1970s. A. C. Conner announced Corollary 1 for orientable 3-manifolds in 1969 which Jaco and Myers have recently obtained using different techniques.

1. Preliminaries. We will work exclusively in the PL category. **[Hem]** is an excellent reference for definitions, notation and techniques. Manifolds are usually connected but not necessarily compact, orientable, or without boundary. ∂M denotes the boundary of a manifold M. A 2-manifold S properly embedded in a 3-manifold M or contained in ∂M is *compressible* provided (1) there exists a 2-cell D in M such that $D \cap S = \partial D$ and ∂D does not bound a 2-cell in S, or (2) S bounds a 3-cell in M. We call a 2-cell D as in (1) a *compressing 2-cell for S in M*. If S is not compressible we say S is *incompressible*.

A 3-manifold M is *P^2-irreducible* if every 2-sphere in M bounds a 3-cell in M and M contains no 2-sided projective planes. M is *∂-irreducible* if every component of ∂M is incompressible. We usually follow Waldhausen [$\mathbf{W}_1$, p. 57] in using $U(\cdot)$ for nice regular neighborhoods. (One exception is when $U(J_{i+1})$ is defined.) If N and M are compact manifolds with connected

Received by the editors May 3, 1978.

AMS (MOS) subject classifications (1970). Primary 55A25, 57A10, 55A40.

Key words and phrases. Connected sum, knot group, submanifold group, cube-with-a-knotted-hole, P^2-irreducible.

boundary and $N \subseteq \operatorname{Int} M$, we use the notation $[N, M] = M - \operatorname{Int} N$.

Projective planes, 2-spheres, and 2-cells share the property that any 2-sided simple closed curve they contain bounds a 2-cell, which is the key to proving Lemma A. (Proof omitted.)

LEMMA A. *Suppose N_1 and N_2 are P^2-irreducible, ∂-irreducible 3-manifolds such that $N_1 \cup N_2$ is a 3-manifold, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$ is a collection of 2-manifolds, $N_1 \cap N_2$ is incompressible in both N_1 and N_2, and no component of $N_1 \cap N_2$ is a 2-cell. Then $N_1 \cup N_2$ is P^2-irreducible and ∂-irreducible.*

The next two lemmas are more general than we need but have other applications. First we give some definitions. Let B be a 3-cell and A an annulus on ∂B. Suppose α is an arc properly embedded in B with an endpoint in each component of $\partial B - A$ and such that C, the closure of $B - U(\alpha)$, is not a solid torus. We call C a *cube-with-a-knotted-hole*. If M is a 3-manifold such that $C \cap M = \partial C \cap \partial M = A$. We say the 3-manifold $M \cup C$ *is obtained from M by attaching a cube-with-a-knotted-hole to M along A.*

LEMMA B. *Suppose T is a torus or Klein bottle boundary component of a 3-manifold M. Let A be an annulus in T that is not contractible in M.If N is obtained from M by attaching a cube-with-a-knotted-hole to M along A, then the boundary component of N that intersects T is incompressible in N.*

PROOF. Let L be a cube-with-a-knotted-hole such that $L \cup M = N$ and $L \cap M = (\partial L) \cap (\partial M) = A$. Note A is an incompressible surface in N. Let S be the boundary component of N that intersects T. Let $S \cap L = A'$, an annulus with $\partial A' = \partial A$.

Case 1. $T - \operatorname{Int} A$ is an annulus. Suppose S is compressible. Then there exists a properly embedded 2-cell E in N such that $\partial E \subseteq S$, ∂E does not bound a 2-cell in S, and E is in general position with respect to A. If $(\partial E) \cap A = \varnothing$, there exists a properly embedded 2-cell E' such that $\partial E = \partial E'$ and $E' \cap A = \varnothing$. But then A is contractible in either M or L, a contradiction. So assume $(\partial E) \cap A \neq \varnothing$. Since the closures of both components of $S - A$ are annuli, we can adjust E by an ambient isotopy of M so that the closure of each component of $\partial E - A$ is an arc with endpoints in different components of ∂A, and E is in general position with respect to A. Note that if α is an arc that is a component of $E \cap A$, then the endpoints of α must also lie in different components of ∂A. Using the incompressibility of A we can remove the simple closed curve components of $E \cap A$. Hence we can assume that $E \cap A$ consists of a finite number of arcs. Let F be a component of $E \cap L$. F is a 2-cell properly embedded in L such that the algebraic intersection number of ∂F with a component J of ∂A is a nonzero integer. Since L is ∂-irreducible, ∂F must bound a 2-cell in ∂L. Hence ∂F must have zero

algebraic intersection number with J. So we must conclude that S is incompressible.

Case 2. $T - \operatorname{Int} A$ consists of two Möbius bands Q_1 and Q_2. Suppose S is compressible. As in Case 1, there exists a properly embedded 2-cell E in N such that $\partial E \subseteq S$, ∂E does not bound a 2-cell in S, E is in general position with respect to A, and $\partial E \cap A \neq \varnothing$. We can assume each component of $(\partial E) \cap A'$ is an arc with an endpoint in each component of $\partial A'$, and each component of $\partial E \cap Q_i$ is an arc that does not separate Q_i. Since A is incompressible in N we can further assume that all the components of $E \cap A$ are arcs. Let D be a 2-cell in E such that $D \cap A = \alpha$ is an arc, $D \cap \partial E = \alpha'$ is an arc, and $\partial D = \alpha \cup \alpha'$.

If $D \subseteq L$, then $\alpha' \subseteq A'$. Recall that α' must have an endpoint in each component of $\partial A'$. Hence the existence of D violates the fact that L is not a solid torus.

If $D \subseteq M$, then $\partial D \subseteq T$. Suppose $\alpha' \subseteq Q_1$. Since α' does not separate Q_1, $\alpha \cup \alpha'$ must be 1-sided in T. But $\partial D = \alpha \cup \alpha'$ must be 2-sided in T.

We are forced to conclude again that S is incompressible.

LEMMA C. *Suppose M is a compact 3-manifold with no 2-sphere or projective plane boundary components. Then there exists a simple closed curve J contained in* $\operatorname{Int} M$ *such that* $M - \operatorname{Int} U(J)$ *is P^2-irreducible and ∂-irreducible. Furthermore we may choose J so that $U(J)$ is orientable.*

PROOF. Let C be a compact collar on ∂M in M and let N be the closure of $M - C$. Suppose L is a triangulation of M with subcomplex K that triangulates N. Let G be the 1-skeleton of K'', the second derived barycentric subdivision of K. Since $M - \operatorname{Int} U(G)$ is homeomorphic to C with a finite number of 1-handles attached, $M - G$ is P^2-irreducible and ∂-irreducible. We want to use one of Bing's techniques to find a simple closed curve J_1 that approximates G. Then we can tie a knot in J_1 to obtain J to insure that $\partial U(J)$ is incompressible in $M - \operatorname{Int} U(J)$.

Note G has the following properties:

(1) G is a connected finite graph with no points of order one;

(2) for all vertices v of G, $G - \operatorname{st}(v, G)$ is connected ($\operatorname{st}(v, G)$ denotes the open star of v in G);

(3) for all vertices v of G, $G - \operatorname{st}(v, G)$ contains either K^1, the 1-skeleton of K, or K_1, the dual 1-skeleton of K.

(1) and (3) clearly hold. (2) holds since $\operatorname{st}(v, G)$ is one component of $G - \operatorname{lk}(v, K'')^1$ and $\operatorname{lk}(v, K'')^1$ is connected ($\operatorname{lk}(v, K'')$ denotes the link of v in K''. Recall that $\operatorname{lk}(v, K'')^1$ is the 1-skeleton of $\operatorname{lk}(v, K'')$, a 2-sphere or 2-cell.) Note any subdivision of a graph with properties (1)–(3) also has properties (1)–(3).

There are an even number of vertices of G with odd order. Hence we can pair these vertices and connect them by polygonal arcs that intersect G only in their endpoints. So we obtain a finite graph G_1 that contains G, has properties (1)–(3), and such that each point of G_1 has even order. Note that $M - G_1$ is P^2-irreducible and ∂-irreducible.

We now wish to modify G_1 to obtain a graph G_2 that satisfies (1) and (2) such that each point of G_2 has order 2 or 4, and $M - G_2$ is homeomorphic to $M - G_1$.

Let $w_1, \ldots, w_k$ be the points of G_1 that have order greater than 4. Let $D_1, \ldots, D_k$ be small regular neighborhoods of $w_1, \ldots, w_k$, respectively, in M such that D_i collapses to $G_1 \cap D_i$, $i = 1, \ldots, k$. There exist 2-cells B_i, properly embedded in D_i, such that $G_1 \cap D_i \subseteq B_i$. There exist trees $F_i \subseteq B_i$ such that B_i collapses to F_i, $F_i \cap \partial D_i = G_1 \cap \partial D_i$ consists of the endpoints of F_i, and each point of F_i in Int B_i has order 2 or 4. Let $D = \bigcup_{i=1}^k D_i$ and $G_2 = (G_1 - D) \cup (\bigcup_{i=1}^k F_i)$.

Let $z_1, \ldots, z_s$ be the points of G_2 that have order 4, and let $H_1, \ldots, H_s$ be small regular neighborhoods of $z_1, \ldots, z_s$, respectively, in Int D such that $G_2 \cap \partial H_j$ consists of 4 points. Let $H = \bigcup_{j=1}^s H_j$. Replace $G_2 \cap H_j$, $j = 1, \ldots, s$, by a pair of arcs α_j, α_j' properly embedded in H_j such that $(\alpha_j \cup \alpha_j') \cap \partial H_j = G_2 \cap \partial H_j$, α_j and α_j' cannot be separated in H_j by a properly embedded 2-cell, and such that

$$J_1 = (G_2 - H) \cup \left(\bigcup_{j=1}^{s} (\alpha_j \cup \alpha_j') \right)$$

is a simple closed curve. See [**B**, Figure 3 and Lemma 6] for one way to choose α_j and α_j'. In the next four paragraphs we will show $M - J_1$ is P^2-irreducible and ∂-irreducible.

$H_j - J_1$ is clearly P^2-irreducible. Using linking arguments it is easy to verify that $H_j - J_1$ is ∂-irreducible.

We claim $D_i - (J_1 \cup \text{Int } H)$ is P^2-irreducible and ∂-irreducible. $D_i - (J_1 \cup H)$, $D_i - G_2$, $D_i - G_1$ and $(\partial D_i - J_1) \times [0, 1)$ are homeomorphic. So it is sufficient to show that if E is a properly embedded 2-cell in $D_i - (J_1 \cup \text{Int } H)$ with $\partial E \subseteq \partial H_j - J_1$, then ∂E bounds a 2-cell in $\partial H_j - J_1$. By property (2), $G_2 - \text{Int } H_j$ is connected. Note E separates $M - \text{Int } H_j$. Hence $J_1 \cap \partial H_j = G_2 \cap \partial H_j$ must be contained in one component of $\partial H_j - \partial E$. Therefore ∂E bounds a 2-cell in $\partial H_j - J_1$.

We now show $M - (J_1 \cup \text{Int } D)$ is P^2-irreducible and ∂-irreducible. Recall $N - G_1$, which is homeomorphic to $M - (J_1 \cup D)$, is P^2-irreducible and ∂-irreducible. So we just need to show $\partial D_i - J_1$ is incompressible in $M - (J_1 \cup \text{Int } D)$. Let E be a properly embedded 2-cell in $M - (J_1 \cup \text{Int } D)$ with $\partial E \subseteq \partial D_i - J_1$. We claim E separates $M - (J_1 \cup \text{Int } D)$. Let E' be a 2-cell

in ∂D_i with $\partial E' = \partial E$. The 2-sphere $E \cup E'$ is contained in either $M - K_1$ or $M - K^1$ by property (3) for G_1. Since both $M - K_1$ and $M - K^1$ are irreducible, $E \cup E'$ separates M. Hence E separates $M - (J_1 \cup \operatorname{Int} D)$. Using property (2) for G_1, we see $G_1 - \operatorname{Int} D_i$ is contained in one component of $M - (J_1 \cup \operatorname{Int} D \cup E)$. Hence $G_1 \cap \partial D_i = J_1 \cap \partial D_i$ cannot meet both components of $\partial D_i - \partial E$. Therefore ∂E bounds a 2-cell in $\partial D_i - J_1$.

The results of the preceding three paragraphs imply that each component of $(\partial H \cup \partial D) - J_1$ is a separating incompressible surface in $M - J_1$, and the closures of the components of $(M - J_1) - ((\partial H \cap \partial D) - J_1)$ are P^2-irreducible and ∂-irreducible. By Lemma A, $M - J_1$ is P^2-irreducible and ∂-irreducible.

We claim J_1 does not pierce any 2-sphere in M. Suppose S is a 2-sphere in M with $S \cap J_1 = \{p\}$. We can assume $p \notin D$ and that $S \cap \partial D = \varnothing$. So $S \cap (J_1 \cup D) = \{p\} = S \cap G_1$. Hence S is contained in an irreducible submanifold of M, either $M - K_1$ or $M - K^1$. We conclude J_1 does not pierce S.

$\partial U(J_1)$ may not be incompressible in $M - \operatorname{Int} U(J_1)$. So we need to "tie a knot" in J_1. More precisely, let P be a 3-cell in $U(J_1)$ such that $P \cap \partial U(J_1) = A$ is an annulus, $\operatorname{cl}(U(J_1) - P) = P'$ is a 3-cell, and $P \cap J_1 = \alpha$ is an arc with an endpoint in each component of $\partial P - A$. Let α' be the arc $P' \cap J_1$. Let β be a properly embedded arc in P such that $\partial\beta = \partial\alpha$ and $L = \operatorname{cl}(P - U(\beta))$ is a cube-with-a-knotted-hole. ($U(\beta)$ is a standard regular neighborhood of β in P.) Let $J_2 = \beta \cup \alpha'$. We say J_2 is obtained from J_1 by tieing a knot in J_1. Note $U(\beta) \cup P'$ is a regular neighborhood of J_2 in M. Hence $M - \operatorname{Int} U(J_2)$ is homeomorphic to $M - \operatorname{Int} U(J_1)$ with the cube-with-a-knotted-hole L attached along A. Since J_1 does not pierce any 2-sphere in M, A must not be contractible in $M - \operatorname{Int} U(J_1)$. Applying Lemma B we see $M - \operatorname{Int} U(J_2)$ is ∂-irreducible. $M - \operatorname{Int} U(J_2)$ is clearly P^2-irreducible. Hence J_2 is our required J.

Suppose $U(J_2)$ is nonorientable. We wish to find a simple closed curve J such that $U(J)$ is orientable and $M - \operatorname{Int} U(J)$ is P^2-irreducible and ∂-irreducible. Let Q be a Möbius band, with center line J_2, that is properly embedded as a 2-sided subset of $U(J_2)$. Let $J_3 = \partial Q$. Note $U(J_3)$ is orientable and J_3 is not contractible in $M - J_2$.

We claim $\operatorname{Int} Q$ is incompressible in $M - J_3$. Let E be a 2-cell in $M - J_3$ such that $E \cap Q = \partial E$. ∂E must be 2-sided in Q. So we can assume $\partial E \cap J_2 = \varnothing$. Now ∂E is not parallel to J_3 in Q since ∂E is contractible in $M - J_2$. Hence ∂E must bound a 2-cell in Q.

Since $M - Q$ and $M - J_2$ are homeomorphic, $M - Q$ is P^2-irreducible and ∂-irreducible. It follows that $M - J_3$ is P^2-irreducible and ∂-irreducible. J_3 cannot pierce a 2-sphere in M since J_3 bounds Q. Let J be obtained from

J_3 by tieing a knot in J_3. $U(J)$ is orientable. As before, $M - \operatorname{Int} U(J)$ is P^2-irreducible and ∂-irreducible.

This completes the proof of Lemma C.

2. Proof of the main theorem and corollaries. Let M be a closed 3-manifold. Recall that $K(M)$ is the set of fundamental groups of complements of PL simple closed curves in M. $S(M)$ is the set of fundamental groups of compact sub-3-manifolds of M that have connected boundary.

MAIN THEOREM. *Let M, N be closed 3-manifolds. $S(M)$ contains $K(N)$ if and only if N is a connected sum factor of M.*

PROOF. Suppose N is a connected sum factor of M. Let α be a simple closed curve in N, $U(\alpha)$ a regular neighborhood of α in N, and B a 3-cell in $\operatorname{Int} U(\alpha)$. Since $N - \operatorname{Int} B$ is homeomorphic to a subset of M, $\pi_1(N - \operatorname{Int} U(\alpha))$ belongs to $S(M)$.

Now suppose $S(M)$ contains $K(N)$. By Lemma C there exists an orientable simple closed curve J_0 contained in N such that $L_0 = N - \operatorname{Int} U(J_0)$ is P^2-irreducible and ∂-irreducible. Let h be a positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori contained in M **[Ha]**. We wish to add structure to L_0 by doubling J_0 and then tieing a knot in the result, repeating the process $h + 1$ times. Note that h is the only information about M used in this construction. More precisely assume J_i, $U(J_i)$, and L_i have been defined.

Let J_i^+ be a simple closed curve in $\operatorname{Int} U(J_i)$ with winding number two. Let A_i be an annulus on $\partial U(J_i^+)$ that is contractible in $U(J_i^+)$ and does not separate $\partial U(J_i^+)$. Let $L_i^+ = N - \operatorname{Int} U(J_i^+)$. Let C_i be a cube-with-a-knotted-hole in $U(J_i^+)$ such that $C_i \cap \partial U(J_i^+) = A_i$ and $L_{i+1} = L_i^+ \cup C_i$ is obtained from L_i^+ by attaching C_i along A_i. Now $N - \operatorname{Int} L_{i+1}$, being a solid torus, is a regular neighborhood of a simple closed curve J_{i+1}. We depart from our $U(\cdot)$ convention and define $U(J_{i+1}) = N - \operatorname{Int} L_{i+1}$. Recall $[L_i, L_i^+] = L_i^+ - \operatorname{Int} L_i$. We collect some useful facts in the following lemma. Figure 1 should help picture the construction.

LEMMA 1. *$[L_i, L_i^+]$ is ∂-irreducible. $[L_i^+, L_{i+1}^+]$ is not a parallelity component. L_i is P^2-irreducible and ∂-irreducible. No nontrivial loop on A_i is freely homotopic in $[L_i, L_i^+]$ to a loop on ∂L_i.*

PROOF. Suppose D is a compressing 2-cell for $[L_i, L_i^+]$. Let $U(D)$ be a regular neighborhood of D in $[L_i, L_i^+]$. If $\partial D \subseteq \partial L_i$, the closure of $U(J_i) - U(D)$ is a 3-cell in $U(J_i)$ that contains J_i^+. If $\partial D \subseteq \partial L_i^+$, then $U(J_i^+) \cup U(D)$ is a 3-cell in $U(J_i)$ containing J_i^+. In either case the winding number of J_i^+ in $U(J_i)$ would be zero, contrary to construction.

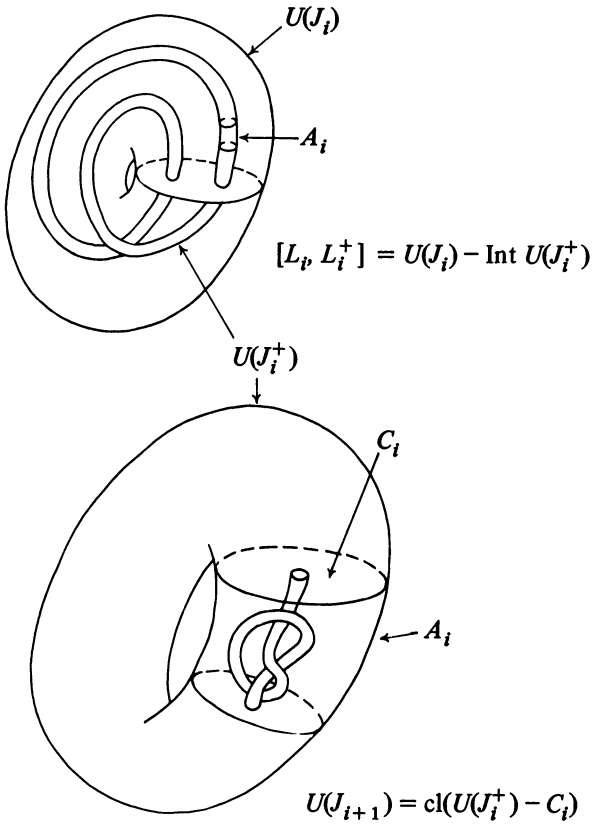

FIGURE 1

Int $[L_i^+, L_{i+1}^+]$ contains an incompressible torus (a parallel copy of ∂C_i in Int C_i) that does not separate ∂L_i^+ from ∂L_{i+1}^+ in $[L_i^+, L_{i+1}^+]$. So by the appendix to **[Ha]**, $[L_i^+, L_{i+1}^+]$ cannot be a parallelity component.

Using Lemmas A and B with the above facts we see L_i is P^2-irreducible and ∂-irreducible.

Suppose l^+ is a nontrivial loop on A_i that is freely homotopic in $[L_i, L_i^+]$ to a loop l on ∂L_i. Since A_i is incompressible l must be nontrivial on ∂L_i. By **[W$_2$]** there exists an annulus A in $[L_i, L_i^+]$ with one boundary component α^+ on A_i, the other α on ∂L_i, and both are nontrivial. Note α bounds a 2-cell in $U(J_i)$. So if we consider $U(J_i)$ as embedded in E^3, α does not link J_{i+1} mod 2 but α^+ does link J_{i+1} mod 2. This contradiction completes the proof of Lemma 1.

LEMMA 2. *M contains a compact P^2-irreducible, ∂-irreducible 3-manifold K such that ∂K is a torus or a Klein bottle and K is homotopy equivalent to L_{h+1}.*

PROOF. By hypothesis there is a compact 3-manifold K' in M such that $\pi_1(K') \cong \pi_1(L_{h+1})$ and $\partial K'$ is connected. Note $\pi_1(L_{h+1})$ has no elements of finite order [**E**, Theorem 3.2], is not a nontrivial free product [**Hem**, Theorem 7.1], and is not free abelian. $\partial K'$ is incompressible since otherwise $\pi_1(K')$ would be free abelian or a nontrivial free product. If K' contained a 2-sided projective plane, $\pi_1(K')$ would have elements of order two. Hence $K' = K \# H$ where H is a homotopy 3-sphere and K is P^2-irreducible with ∂K connected and incompressible. We can assume K is contained in M. Since K and L_{h+1} are $K(\pi, 1)$'s, K is homotopy equivalent to L_{h+1}. Hence

$$0 = X(\partial L_{h+1}) = 2X(L_{h+1}) = 2X(K) = X(\partial K)$$

which implies ∂K is a torus on a Klein bottle. ($X(\cdot)$ denotes the Euler characteristic.) Lemma 2 is completed.

Suppose $f\colon K \to \operatorname{Int} L_{h+1}$ is a homotopy equivalence. We can assume $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$ are collections of 2-sided properly embedded incompressible surfaces in K [**Hei**] with a minimum number of components. Both collections are nonempty since $f_*\colon \pi_1(K) \to \pi_1(L_{h+1})$ is an isomorphism and $\pi_1(L_{h+1})$ splits as nontrivial free products with amalgamation along both $\pi_1(A_h)$ and $\pi_1(\partial L_h)$. Suppose S is a component of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$ that has boundary.

Each component of ∂S is a nontrivial simple closed curve on ∂K. In order to see this note $(f|_S)_*\colon \pi_1(S) \to G$ is a monomorphism where G is either $\pi_1(A_h)$ or $\pi_1(\partial L_h)$. Hence $\pi_1(S)$ is free abelian. If S is an annulus the boundary components must be nontrivial. If S were a Möbius band with trivial boundary we could find a 2-sided projective plane in K. If S were a 2-cell with trivial boundary, one component of $K - \operatorname{Int} U(S)$ would be a 3-cell and we could reduce the number of components of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$. Our assertion follows.

Note $f^{-1}(A_h) \cap \partial K$ is nonempty.

LEMMA 3. *$f^{-1}(\partial L_h) \cap \partial K$ is empty.*

PROOF. Suppose not. Then there exist nontrivial 2-sided simple closed curves α and β on ∂K such that $f(\alpha) \subseteq A_h$ and $f(\beta) \subseteq \partial L_h$. Let $U(\alpha \cup \beta)$ be a regular neighborhood of $\alpha \cup \beta$ in ∂K. Each component of $\partial K - \operatorname{Int} U(\alpha \cup \beta)$ must have Euler characteristic zero. Since at least one such component must have two boundary simple closed curves, $\alpha \cup \beta$ must bound an annulus A' in ∂K. We can assume $\operatorname{Int} A'$ misses $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$. Hence $f(\alpha)$ is a nontrivial loop in A_h that is freely homotopic in $[L_h, L_h^+]$ to a loop in ∂L_h, which violates Lemma 1. So Lemma 3 holds.

Hence $f(\partial K) \subseteq \operatorname{Int}[L_h, L_{h+1}]$. We can modify f so that $f(\partial K) \subseteq [L_h, L_{h+1}]$ but $f(\partial K) \cap \partial L_h$ is nonempty. Let x be a point in ∂K such that $f(x)$ belongs to ∂L_h. We now wish to apply the main result of [**F**]. Consider the geometric

splitting of $L_{h+1} = L_h \cup_{\partial L_h} [L_h, L_{h+1}]$ and f_*: $\pi_1(K, x) \to \pi_1(L_{h+1}, f(x))$. Under $(f_*)^{-1}$ we obtain an algebraic splitting of $\pi_1(K, x)$ that respects the peripheral structure of $\pi_1(K, x)$, i.e., $\pi_1(\partial K, x)$ under inclusion is a subgroup of $f_*^{-1}(\pi_1([L_h, L_{h+1}], f(x)))$. So applying [F] there exists a separating incompressible torus $T \subseteq \operatorname{Int} K$ (denote the closure of the component of $K - T$ that has boundary T as K_h; then the closure of the other component is $K - \operatorname{Int} K_h = [K_h, K]$; let y belong to T) and an isomorphism d: $\pi_1(K, y) \to \pi_1(K, x)$ such that

$$f_* d(\pi_1(T, y)) = \pi_1(\partial L_h, f(x))$$

and

$$f_* d(\pi_1(K_h, y)) = \pi_1(L_h, f(x)) \quad \text{or} \quad \pi_1([L_h, L_{h+1}], f(x)).$$

Since $\pi_1(\partial L_h)$ is a peripheral subgroup of both $\pi_1(L_h)$ and $\pi_1([L_h, L_{h+1}])$ and neither L_h nor $[L_h, L_{h+1}]$ is a product I-bundle, [W] and [Hei] apply to conclude K_h is homeomorphic to either L_h or $[L_h, L_{h+1}]$. The only possibility is L_h.

Let g: $L_h \to K_h$ be a homeomorphism. First some notation is needed. Let $K_i = g(L_i)$ and $K_i^+ = g(L_i^+)$. Now $\partial K_0^+, \ldots, \partial K_{h-1}^+$ are nonparallel disjoint 2-sided tori in M. At least one must be compressible, say ∂K_j^+, $0 < j < h - 1$. Let J be a boundary component of the annulus A_j.

LEMMA 4. *$g(J)$ bounds a 2-cell in $M - \operatorname{Int} K_{j+1}$.*

PROOF. We adapt the Bing-Martin proof that composite knots have property P [B, M]. Recall $L_{j+1} = C_j \cup L_j^+$. Let K_j' be a concentric copy of K_j^+ in $\operatorname{Int} K_j^+$. Since ∂K_j^+ is compressible in M, there exists a compressing 2-cell D for $\partial K_j'$ in $M - \operatorname{Int} K_j'$. We can assume D misses $g(C_j)$ (put D in general position with respect to $g(\partial C_j)$. If all the simple closed curves in $D \cap g(\partial C_j)$ are trivial on $g(\partial C_j)$ we can find the desired compressing 2-cell. Otherwise we can find a compact 3-manifold Q' that contains $g(C_j)$, is contained in $M - K_j'$ and has a 2-sphere boundary. Once again we can find the desired 2-cell.) Now put D in general position with respect to ∂K_j^+. We can assume all simple closed curves in $D \cap \partial K_j^+$ are nontrivial on ∂K_j^+ ($D \cap \partial K_j^+$ is nonempty since $K_j^+ - \operatorname{Int} K_j'$ has incompressible boundary). Let $E \subseteq D$ be a 2-cell such that $E \cap \partial K_j^+ = \partial E$. E is contained in $M - \operatorname{Int} K_{j+1}$ since D misses $g(C_j)$ and $K_j^+ - K_j'$ has incompressible boundary. Since ∂E misses $g(A_j)$ and is nontrivial in ∂K_j^+, ∂E is parallel to $g(J)$ on ∂K_{j+1}. The required 2-cell exists and the proof of Lemma 4 is complete.

Let D_1 be a 2-cell in N with $D_1 \cap L_{j+1} = \partial D_1 = J$. Let D_2 be a 2-cell in M with $D_2 \cap K_{j+1} = \partial D_2 = g(J)$. Extend $g|_{L_{j+1}}$: $L_{j+1} \to K_{j+1}$ first to g': $L_{j+1} \cup D_1 \to K_{j+1} \cup D_2$ and then to regular neighborhoods g'': $U(L_{j+1} \cup D_1) \to U(K_{j+1} \cup D_2)$. Since $N - \operatorname{Int} U(L_{j+1} \cup D_1)$ is a 3-cell, there exists a closed 3-manifold Q such that M is a connected sum of N and Q. The proof of the main theorem is complete.

COROLLARY 1. *Suppose M and N are closed 3-manifolds. $K(M) = K(N)$ if and only if M is homeomorphic to N.*

PROOF. By the main theorem, M is a connected sum of N and a closed 3-manifold Q_1. Again N is a connected sum of M and a closed 3-manifold Q_2. So M is a connected sum of M, Q_2, and Q_1. [**Hem**, Theorem 3.21] applies to allow us to conclude Q_1 and Q_2 are trivial. Hence M and N are homeomorphic.

Suppose N is a closed 3-manifold. Let $h_N = h$ be the positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori in N [**Ha**]. Choose $L_{h+1} \subseteq N$ as in the proof of the main theorem. Denote $\pi_1(L_{h+1})$ by G_N. If M is a closed 3-manifold and G_N belongs to $S(M)$, we say G_N is *realized in M*.

COROLLARY 2. *Let M and N be closed 3-manifolds. G_M is realized in N and G_N is realized in M if and only if M is homeomorphic to N.*

PROOF. Consider the positive integers h_N and h_M as defined above. Suppose $h_M < h_N$. By the proof of the main theorem, N is a connected sum factor of M. So $h_N < h_M$. Apply the proof of the main theorem again to conclude M is a connected sum factor of N. As in Corollary 1, M is homeomorphic to N.

We conclude with the following question.

QUESTION. Does the main theorem hold for compact 3-manifolds with no 2-sphere boundary components?

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