

**SWEEDLER'S TWO-COCYCLES AND GENERALIZATIONS
 OF THEOREMS ON AMITSUR COHOMOLOGY**

BY

DAVE RIFFELMACHER

ABSTRACT. For any (not necessarily commutative) algebra C over a commutative ring k Sweedler defined a cohomology set, denoted here by $\mathcal{H}^2(C/k)$, which generalizes Amitsur's second cohomology group $H^2(C/k)$. In this paper, if I is a nilpotent ideal of C and $\bar{C} \equiv C/I$ is k -projective, a natural bijection $\mathcal{H}^2(C/k) \xrightarrow{\sim} \mathcal{H}^2(\bar{C}/k)$ is established. Also, when $k \subseteq B$ are fields and C is a commutative B -algebra, the sequence $\{1\} \rightarrow \mathcal{H}^2(B/k) \xrightarrow{\iota^*} \mathcal{H}^2(C/k) \xrightarrow{r} \mathcal{H}^2(C/B)$ is shown to be exact if the natural map $C \otimes_k C \rightarrow C \otimes_B C$ induces a surjection on units, ι^* is induced by the inclusion, and r is the "restriction" map.

Introduction. Given a commutative algebra C over a commutative ring k , Amitsur introduced a cochain complex and demonstrated that in certain cases the second cohomology group $H^2(C/k)$ of this complex is isomorphic to the Brauer group of similarity classes of central separable k -algebras split by C [1]. Sweedler generalized the notion of an Amitsur two-cocycle to noncommutative algebras C and has shown that various sets of certain equivalence classes of (generalized) two-cocycles classify various types of k -algebras [8]. In this paper we study several functorial properties of the full set, denoted here by $\mathcal{H}^2(C/k)$, of certain equivalence classes of Sweedler's two-cocycles.

Sweedler's basic definitions are recalled in §1. §2 provides a generalization of a theorem of Rosenberg and Zelinsky [6, Proposition 3.3] which states that under suitable faithfully flatness conditions the Amitsur cohomology of a commutative algebra is not changed by factoring out a nilpotent ideal. We prove that, if I is a nilpotent ideal of the (not necessarily commutative) k -algebra C and $\bar{C} \equiv C/I$ is k -projective, the natural projection $C \rightarrow \bar{C}$ induces a bijection $\mathcal{H}^2(C/k) \xrightarrow{\sim} \mathcal{H}^2(\bar{C}/k)$. In §3 we restrict our attention to commutative algebras. If $k \subseteq B$ are fields, C is a commutative B -algebra, and the natural map $C \otimes_k C \rightarrow C \otimes_B C$ induces a surjection on units, we prove there is an exact sequence

$$\{1\} \rightarrow \mathcal{H}^2(B/k) \xrightarrow{\iota^*} \mathcal{H}^2(C/k) \xrightarrow{r} \mathcal{H}^2(C/B)$$

where ι^* is induced by the inclusion $\iota: B \rightarrow C$ and r is the "restriction" map.

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This sequence is a partial generalization and synthesis of two sequences Yuan used in his study of p -algebras [10, Theorems 3.4, 3.5].

1. Definitions. Let C be an algebra over a commutative ring k and let unadorned \otimes represent \otimes_k . Following Sweedler [8] we call an element $\delta = \sum_i x_i \otimes y_i$ in $C \otimes C$ a C -one-cocycle if $\sum_i x_i \otimes 1 \otimes y_i = \sum_{i,j} x_i \otimes y_j x_j \otimes y_j$. Two C -one-cocycles $\delta = \sum_i x_i \otimes y_i$ and $\eta = \sum_i u_i \otimes v_i$ are said to be equivalent if there is a unit z in C with $\sum_i z^{-1} x_i \otimes y_i z = \sum_i u_i \otimes v_i$. An element $\sum_i x_i \otimes y_i$ in $C \otimes C$ is called vertible if $\sum_i x_i \otimes y_i^0$ is invertible in $C \otimes C^0$. (Here C^0 is the opposite k -algebra.) We denote the set of equivalence classes of vertible C -one-cocycles by $\mathcal{H}^1(C/k)$ or by $\mathcal{H}^1(C)$ if the ground ring is clear from the context.

A C -two-cocycle is an element $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes C \otimes C$ such that

$$\sum_{i,j} a_i a_j \otimes b_j \otimes c_j b_i \otimes c_i = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j \otimes c_j c_i$$

and there is an e_σ in C with $\sum_i a_i e_\sigma b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i e_\sigma c_i$. We shall refer to these equations as the associativity and unitary conditions, respectively, since they arose to guarantee a certain construction C^σ of Sweedler's would be a k -algebra. The C -two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is cohomologous to the C -two-cocycle $\tau = \sum_i r_i \otimes s_i \otimes t_i$ via $\delta = \sum_i x_i \otimes y_i$ in $C \otimes C$ if

$$\begin{aligned} \sum_i x_i e_\sigma y_i &= e_\tau, \\ \sum_{i,j} x_i a_j \otimes b_j \otimes c_j y_i &= \sum_{i,j,l} r_i x_j \otimes y_j s_i x_l \otimes y_l t_i. \end{aligned} \tag{1.1}$$

If σ is cohomologous to τ via a vertible δ , we indicate this by $\sigma \sim^\delta \tau$ and say that σ and τ are equivalent. We denote the set of equivalence classes of C -two-cocycles by $\mathcal{H}^2(C/k)$ or by $\mathcal{H}^2(C)$ if no confusion seems likely. Given a C -two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ and a vertible element $\delta = \sum_i x_i \otimes y_i$ with verse $\bar{\delta} = \sum_i \bar{x}_i \otimes \bar{y}_i$ (i.e. $\sum_i \bar{x}_i \otimes \bar{y}_i^0$ is the inverse of $\sum_i x_i \otimes y_i^0$) we will often wish to consider the C -two-cocycle $\sigma_1 = \sum_{i,j,l,m} \bar{x}_i a_j x_l \otimes y_l b_j x_m \otimes y_m c_j \bar{y}_i$ obtained by "altering" σ by δ . We shall say in this situation that $\sigma \sim^\delta \sigma_1$ defines the C -two-cocycle σ_1 . For example, if C is commutative and $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C -two-cocycle, the unitary condition implies that e_σ is invertible (with inverse $\sum_i a_i b_i c_i$) and $\sigma \sim^{1 \otimes e_\sigma^{-1}} \sigma_1$ defines a C -two-cocycle σ_1 with $e_{\sigma_1} = 1$.

Any k -algebra map $C \xrightarrow{f} D$ induces a map

$$C \otimes \cdot^n \otimes C \xrightarrow{f^n} D \otimes \cdot^n \otimes D$$

given by $x_1 \otimes \cdots \otimes x_n \rightarrow f(x_1) \otimes \cdots \otimes f(x_n)$ for each positive integer n which we will denote by f if no confusion seems likely. For $i = 1, 2$, we have

an induced map on cohomology $\mathcal{H}^2(C) \xrightarrow{f^*} \mathcal{H}^2(D)$.

2. A theorem of Rosenberg and Zelinsky. Throughout this section, $\mathcal{H}^2(D/k)$ will be denoted by $\mathcal{H}^2(D)$ for any k -algebra D . We prove:

THEOREM 2.1. *Let C be an algebra over a commutative ring k . Suppose I is a nilpotent ideal of C with C/I projective as a k -module, and let $p: C \rightarrow C/I$ be the natural projection. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

We begin by noting that Theorem 2.1 is equivalent to:

THEOREM 2.1'. *Let C be an algebra over a commutative ring k . Suppose I is an ideal of C with $I^2 = \{0\}$ and C/I is projective as a k -module. Let $p: C \rightarrow C/I$ be the natural projection. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

PROOF THAT 2.1' IMPLIES 2.1. For any positive integer n we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^2(C) & \xrightarrow{p^*} & \mathcal{H}^2(C/I) \\ \searrow & & \nearrow \\ & \mathcal{H}^2(C/I^n) & \end{array}$$

with each map induced by the appropriate canonical projection. Hence if 2.1' is known, the general result 2.1 may be shown by an induction on the index of nilpotency of I .

Therefore we need only prove Theorem 2.1' and accordingly in the remainder of this section we will assume $I^2 = \{0\}$. With this assumption I is a C/I -bimodule and we may view C as a Hochschild extension of C/I by I [4]. Thus we have $C = C/I \oplus_{\psi} I$ as k -algebras where $\psi: C/I \otimes C/I \rightarrow I$ is a Hochschild two-cocycle. Recall that this means $C = C/I \oplus I$ as k -modules and the multiplication $*_{\psi}$ in $C/I \oplus_{\psi} I$ is given by $(a, x) *_{\psi} (b, y) = (ab, ay + xb + \psi(a \otimes b))$ for a, b in $C/I, x, y$ in I . The next theorem considers the special case of Theorem 2.1' where ψ is a trivial Hochschild two-cocycle.

THEOREM 2.2. *Let k, C, I and p be as in Theorem 2.1'. Also assume $C = C/I \oplus_{\psi} I$ with ψ a trivial Hochschild two-cocycle. Then $\mathcal{H}^2(C) \xrightarrow{p^*} \mathcal{H}^2(C/I)$ is bijective.*

PROOF. Since ψ is trivial there is an algebra map $\iota: C/I \rightarrow C$ with $p\iota$ the identity of C/I . Denote the composite ψ by q . Then $q: C \rightarrow C$ is an algebra endomorphism satisfying $q^2 = q$ and $\ker(q) = I$. It thus follows from Theorem 4.7 of [5] that any C -two-cocycle σ is equivalent to its projection $q(\sigma)$ modulo I , which establishes the theorem.

We are now ready to prove Theorem 2.1'. The idea of the proof is to mimic the argument used to prove Theorem 4.7 of [5] while making allowance for the fact that C/I is not, in general, a subalgebra of $C/I \oplus_{\psi} I$.

p^* is injective. Let σ, τ be C -two-cocycles with $p(\sigma)$ and $p(\tau)$ equivalent C/I -two-cocycles, i.e. there is an element δ in $C \otimes C$ with $p(\sigma)$ equivalent to $p(\tau)$ via $p(\delta)$. Since I is nilpotent, $p(\delta)$ vertible implies that δ is vertible. Thus $\sigma \sim^\delta \sigma_1$ defines a C -two-cocycle σ_1 equivalent to σ with $p(\sigma_1) = p(\tau)$.

Hence we may assume

$$\sigma = \tau + \sum_i f_i \otimes g_i \otimes h_i \tag{2.3}$$

with $\sum_i f_i \otimes g_i \otimes h_i$ in $\mathcal{G} = I \otimes C \otimes C + C \otimes I \otimes C + C \otimes C \otimes I$; that is, $\sigma \equiv \tau$ (modulo \mathcal{G}). We argue by induction on n that if $\sigma \equiv \tau$ (modulo \mathcal{G}^n), there exist C -two-cocycles σ_1, τ_1 with σ_1 equivalent to σ, τ_1 equivalent to τ , and $\sigma_1 \equiv \tau_1$ (modulo \mathcal{G}^{n+1}). This will be sufficient to establish the injectivity of p^* because $\mathcal{G}^4 = \{0\}$.

Let $\tau = \sum_i a_i \otimes b_i \otimes c_i$ in (2.3) and define

$$\delta = \sum_i p(a_i e_\sigma) b_i \otimes c_i + \sum_i p(f_i e_\sigma) g_i \otimes h_i.$$

Since

$$\{p \otimes 1\} \left(\sum_i p(a_i e_\sigma) b_i \otimes c_i \right) = 1 \otimes 1$$

and I is nilpotent, $\sum_i p(a_i e_\sigma) b_i \otimes c_i$ is vertible. Denote its verse by $\sum_i x_i \otimes y_i$. $\sigma_1 \sim^\delta \sigma$ and $\tau_1 \sim^{\sum_i p(a_i e_\sigma) b_i \otimes c_i} \tau$ define C -two-cocycles σ_1, τ_1 with

$$\begin{aligned} \sigma_1 &\equiv \tau_1 - \sum_{i,j} a_j p(f_i e_\sigma) g_i \otimes h_i b_j \otimes c_j + \sum_i f_i \otimes g_i \otimes h_i \\ &\quad + \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i - \sum_{i,j} a_j \otimes b_j p(f_i e_\sigma) g_i \otimes h_i c_j \end{aligned} \tag{modulo \mathcal{G}^{n+1} } \tag{2.4}$$

where we have used

$$\sum_i p(a_i e_\sigma) b_i \otimes c_i \equiv 1 \otimes 1 \equiv \sum_i x_i \otimes y_i \quad (\text{modulo } C \otimes I + I \otimes C).$$

It follows from the associativity relation for σ that

$$\begin{aligned} &\sum_{i,j} p(a_j f_i e_\sigma) g_j \otimes h_j b_i \otimes c_i + \sum_{i,j} p(f_i a_j e_\sigma) b_j \otimes c_j g_i \otimes h_i \\ &\equiv \sum_{i,j} p(a_i e_\sigma) b_j f_j \otimes g_j \otimes h_j c_i + \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i \\ &\equiv \sum_j f_j \otimes g_j \otimes h_j + \sum_{i,j} p(f_i e_\sigma) g_i a_j \otimes b_j \otimes c_j h_i \quad (\text{modulo } \mathcal{G}^{n+1}), \end{aligned}$$

so we may rewrite (2.4) as

$$\begin{aligned} \sigma_1 &\equiv \tau_1 + \sum_{i,j} p(f_i a_j e_\sigma) b_j \otimes c_j g_i \otimes h_i \\ &\quad - \sum_{i,j} a_j \otimes b_j p(f_i e_\sigma) g_i \otimes h_i c_j \quad (\text{modulo } \mathfrak{G}^{n+1}) \end{aligned}$$

since $\sum_i f_i \otimes g_i \otimes h_i$ is in \mathfrak{G} and $p(I) = \{0\}$. That is, we have shown

$$\sigma_1 \equiv \tau_1 + \sum_i f'_i \otimes g'_i \otimes h'_i \quad (\text{modulo } \mathfrak{G}^{n+1})$$

with $\sum_i f'_i \otimes g'_i \otimes h'_i$ in $C \otimes I \otimes C + C \otimes C \otimes I$.

Suppose $\tau_1 = \sum_i a'_i \otimes b'_i \otimes c'_i$. If we now repeat the above procedure but "push down on the right", that is alter by

$$\sum_i a'_i \otimes b'_i p(e_\sigma c'_i) + \sum_i f'_i \otimes g'_i p(e_\sigma h'_i),$$

we obtain C -two-cocycles σ_2, τ_2 with σ_2 equivalent to σ_1 , τ_2 equivalent to τ_1 , and

$$\sigma_2 \equiv \tau_2 + \sum_i u_i \otimes v_i \otimes w_i \quad (\text{modulo } \mathfrak{G}^{n+1}) \tag{2.5}$$

with v_i in I for all i .

If $\tau_2 = \sum_i r_i \otimes s_i \otimes t_i$, the associativity relation for σ_2 yields

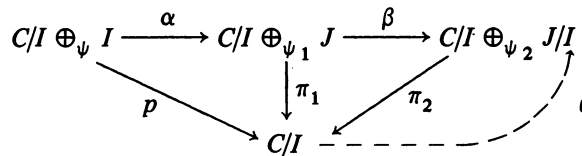
$$\begin{aligned} \sum_{i,j} r_i u_j \otimes v_j \otimes w_j s_i \otimes t_i + \sum_{i,j} u_i r_j \otimes s_j \otimes t_j v_i \otimes w_i \\ \equiv \sum_{i,j} r_i \otimes s_i u_j \otimes v_j \otimes w_j t_i + \sum_{i,j} u_i \otimes v_i r_j \otimes s_j \otimes t_j w_i \quad (\text{modulo } \mathfrak{G}^{n+1}). \end{aligned}$$

Thus it follows that, modulo \mathfrak{G}^{n+1} ,

$$\sum_{i,j} u_i r_j \otimes p(s_j) e_{\sigma_2} t_j v_i \otimes w_i \equiv \sum_{i,j} r_i \otimes p(s_i u_j) e_{\sigma_2} v_j \otimes w_j t_i \equiv 0.$$

Hence (2.5) implies $\sigma_2 \equiv \tau_2$ (modulo \mathfrak{G}^{n+1}) which was to be shown.

p^* is surjective. Let J be the injective hull of I as a C/I -bimodule. Define $\psi_1: C/I \otimes C/I \rightarrow J$ by $\psi_1(a \otimes b) = \psi(a \otimes b)$. Then ψ_1 is a trivial Hochschild two-cocycle since J is injective [3, §IX.6] and, if we set $J^2 = \{0\}$, $C/I \oplus_{\psi_1} J$ is a k -algebra with multiplication $(a, x) *_{\psi_1} (b, y) = (ab, ay + xb + \psi_1(a \otimes b))$. We have a commutative diagram



with p, π_1, π_2 the natural projections and $\alpha(a, x) = (a, x), \beta(a, y) = (a, y + I), \iota(a) = (a, 0)$ for a in $C/I, x$ in I, y in J .

Let σ be a $C/I \oplus_{\psi_1} J$ -two-cocycle. Then $\beta(\sigma)$ is equivalent to $\iota\pi_2\beta(\sigma) = \iota\pi_1(\sigma)$ via δ in

$$1 \otimes 1 + J/I \otimes \{C/I \oplus_{\psi_2} J/I\} + \{C/I \oplus_{\psi_2} J/I\} \otimes J/I$$

by the proof of Theorem 2.2.

Lift δ to δ' in

$$1 \otimes 1 + J \otimes \{C/I \oplus_{\psi_1} J\} + \{C/I \oplus_{\psi_1} J\} \otimes J.$$

δ' is vertible since I is nilpotent. Let $\sigma \sim^{\delta'} \sigma'$ define the $C/I \oplus_{\psi_1} J$ -two-cocycle σ' . Then

$$\iota\pi_2\beta(\sigma) \sim^{\bar{\delta}} \beta(\sigma) \sim^{\beta(\delta')} \beta(\sigma'),$$

where $\bar{\delta}$ is the verse of δ . Thus

$$\iota\pi_2\beta(\sigma) \sim^{1 \otimes 1} \beta(\sigma'),$$

which implies σ' is in $\alpha(C/I \oplus_{\psi} I)$ and α^* is surjective.

We now have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^2(C/I \otimes_{\psi} I) & \xrightarrow{\alpha^*} & \mathcal{H}^2(C/I \otimes_{\psi_1} J) \\ p^* \searrow & & \swarrow \pi_1^* \\ & \mathcal{H}^2(C/I) & \end{array}$$

Since ψ_1 is a trivial Hochschild two-cocycle, π_1^* is bijective by Theorem 2.2 and hence it follows that p^* is surjective. This completes the proof of Theorem 2.1', and hence also of Theorem 2.1.

REMARKS. (1) If we restrict our attention to commutative algebras and invertible (i.e., Amitsur) C -two-cocycles, Theorem 2.1 is included in a theorem of Rosenberg and Zelinsky [6, Proposition 3.3].

(2) One may show that the induced map $\mathcal{H}^1(C) \xrightarrow{p^*} \mathcal{H}^1(C/I)$ is bijective by a similar (and far less tedious) argument.

3. An exact sequence. The object of this section is to establish that if $k \subseteq B$ are fields and C is a commutative B -algebra there is an exact sequence of cohomology sets

$$\{1\} \rightarrow \mathcal{H}^2(B) \xrightarrow{\iota^*} \mathcal{H}^2(C) \xrightarrow{r} \mathcal{H}^2(C/B) \tag{ES}$$

if the natural map $C \otimes C \rightarrow C \otimes_B C$ induces a surjection on the groups of units. Here $\iota: B \rightarrow C$ is the inclusion and r is the "restriction" map arising from the natural projection $C \otimes C \otimes C \rightarrow C \otimes_B C \otimes_B C$. We begin with an easy extension of a theorem of Berkson [2, Theorem 1].

THEOREM 3.1. *Let k be a commutative ring and C be a k -algebra which is free as a k -module. Then $\mathcal{H}^1(C) = \{1\}$.*

PROOF. Since C is free as a k -module, we may write any vertible C -one-cocycle as $\sum_{i=1}^s v_i \otimes w_i$ with $\{v_i\}, \{w_i\}$ sets of linearly independent elements of C . Then the one-cocycle relation

$$\sum_i v_i \otimes 1 \otimes w_i = \sum_{i,j} v_i \otimes w_i v_j \otimes w_j$$

implies

$$\sum_{i,j} v_i \otimes (w_i v_j - \delta_{ij}) \otimes w_j = 0$$

where δ_{ij} is the Kronecker delta. Since $\{v_i\}$ is linearly independent and $\{w_i\}$ is linearly independent we have $w_i v_j = \delta_{ij}$. Suppose $s > 2$. Then

$$(w_1 \otimes v_1^0) \left(\sum_i v_i \otimes w_i^0 \right) = 1 \otimes 1^0 = (w_2 \otimes v_2^0) \left(\sum_i v_i \otimes w_i^0 \right).$$

Hence, since inverses are unique and we are assuming that $\sum_i v_i \otimes w_i^0$ is invertible, $w_1 \otimes v_1 = w_2 \otimes v_2$, a contradiction to the linear independence hypothesis. Thus $\sum_i v_i \otimes w_i = v_1 \otimes w_1$. Since $v_1 \otimes w_1$ is a C -one-cocycle, $v_1 w_1 = 1$. Above we showed that $w_1 v_1 = 1$. Hence $v_1 \otimes w_1 = v_1 \otimes v_1^{-1}$, which is clearly equivalent to $1 \otimes 1$.

REMARK. The proof of Theorem 3.1 is Berkson's proof slightly modified to deal with the noncommutativity of C .

A close look at the proof of Theorem 3.1 yields a fact we will later need.

LEMMA 3.2. *If C is a commutative k -algebra and a free k -module, every C -one-cocycle is invertible.*

PROOF. The vertible hypothesis in Theorem 3.1 is needed since $w_i v_j = \delta_{ij}$.

Notation. If $k \subseteq L \subseteq C$ are commutative rings and C is an L -algebra, we define $m_L: C \otimes C \rightarrow C \otimes_L C$ by

$$m_L \left(\sum_i x_i \otimes y_i \right) = \sum_i x_i \otimes_L y_i$$

and

$$m_{L,L} = \{1 \otimes m_L\} \circ \{m_L \otimes 1\}: C \otimes C \otimes C \rightarrow C \otimes_L C \otimes_L C.$$

THEOREM 3.3. *Let $k \subseteq L$ be fields and C be a commutative L -algebra. If σ, τ are L -two-cocycles and $\sigma \sim^\delta \tau$ with δ a vertible element of $C \otimes C$, then $\delta = (\alpha \otimes \alpha^{-1})\delta'$ with δ' in $L \otimes L$ and α a unit of C .*

PROOF. Since L is commutative we may without loss of generality (cf. §1) assume $e_\sigma = 1 = e_\tau$. $m_{L,L}$ applied to (1.1) shows that $m_L(\delta)$ is a one-cocycle

because σ, τ are in $L \otimes L \otimes L$. By Theorem 3.1 and Lemma 3.2 $m_L(\delta) = \alpha \otimes_L \alpha^{-1}$ for some α in C and hence $\delta = \alpha \otimes \alpha^{-1} + \sum_i d_i \otimes e_i$ with $\{d_i\}$ k -linearly independent, $\{e_i\}$ k -linearly independent, and $\sum_i d_i \otimes_L e_i = 0$. Starting again with (1.1), applying the map $1 \otimes m_L$ and substituting $\alpha \otimes \alpha^{-1} + \sum_i d_i \otimes e_i$ for δ , we find that

$$\sum_i d_i \otimes 1 \otimes_L e_i = \sum_i d_i \otimes e_i \alpha \otimes_L \alpha^{-1}.$$

A straightforward application of linear algebra yields $e_i = m_i \alpha^{-1}$ for all i , with m_i in L . Similarly $d_i = n_i \alpha$ for all i , with n_i in L . Therefore

$$\delta = (\alpha \otimes \alpha^{-1}) \left(1 \otimes 1 + \sum_i n_i \otimes m_i \right),$$

which was to be shown.

An immediate consequence of Theorem 3.3 is

COROLLARY. *Let $k \subseteq L$ be fields and C be a commutative L -algebra. Then the inclusion $\iota: L \rightarrow C$ induces an injection $\mathcal{H}^2(L) \xrightarrow{\iota^*} \mathcal{H}^2(C)$.*

The next theorem is the key result we need to prove exactness at $\mathcal{H}^2(C)$ in the purportedly exact sequence (ES) exhibited at the beginning of this section. First, however, we need a preliminary lemma.

LEMMA 3.4. *Let $k \subseteq B$ be commutative rings and C be a commutative B -algebra. Suppose $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C -two-cocycle with $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$. Then under the map*

$$C \otimes C \otimes_B C \rightarrow (C \otimes C) \otimes_{C \otimes B} (C \otimes C)$$

given by $x \otimes y \otimes_B z \rightarrow (x \otimes y) \otimes_{C \otimes B} (1 \otimes z)$ the element $\{1 \otimes m_B\}(\sigma)$ maps to a one-cocycle.

PROOF. Applying $1 \otimes m_{B,B}$ to the associativity relation for σ yields

$$\sum_{i,j} a_i a_j \otimes b_j \otimes_B c_j b_i \otimes_B c_i = \sum_i a_i \otimes b_i \otimes_B 1 \otimes_B c_i \tag{3.4a}$$

since $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$ by assumption. Under the map

$$C \otimes C \otimes_B C \otimes_B C \xrightarrow{\sim} (C \otimes C) \otimes_{C \otimes B} (C \otimes C) \otimes_{C \otimes B} (C \otimes C)$$

given by

$$w \otimes x \otimes_B y \otimes_B z \rightarrow (w \otimes x) \otimes_{C \otimes B} (1 \otimes y) \otimes_{C \otimes B} (1 \otimes z),$$

(3.4a) transforms to

$$\begin{aligned} & \sum_{i,j} (a_j \otimes b_j) \otimes_{C \otimes B} (a_i \otimes c_j b_i) \otimes_{C \otimes B} (1 \otimes c_i) \\ &= \sum_i (a_i \otimes b_i) \otimes_{C \otimes B} (1 \otimes 1) \otimes_{C \otimes B} (1 \otimes c_i). \end{aligned}$$

This establishes the lemma.

THEOREM 3.5. *Let $k \subseteq B$ be fields and C be a commutative B -algebra. Suppose σ is a C -two-cocycle with $m_{B,B}(\sigma) = 1 \otimes_B 1 \otimes_B 1$. Then σ is equivalent to a B -two-cocycle.*

PROOF. By Lemma 3.4, $\{1 \otimes m_B\}(\sigma)$ is a one-cocycle. Since $C \otimes C$ is a free $C \otimes B$ -module, Theorem 3.1 and Lemma 3.2 imply

$$\{1 \otimes m_B\}(\sigma) = \sum_{i,j} x_i \bar{x}_j \otimes y_i \otimes \bar{y}_j$$

where $\delta = \sum_i x_i \otimes y_i$ is a unit in $C \otimes C$ with inverse $\bar{\delta} = \sum_i \bar{x}_i \otimes \bar{y}_i$ and $m_C(\delta) = 1$. $\sigma_1 \sim^\delta \sigma$ defines a C -two-cocycle σ_1 with $\{1 \otimes m_B\}(\sigma_1) = 1 \otimes 1 \otimes_B 1$ since

$$\{m_B \circ (m_C \otimes 1)\}(\sigma) = 1 \otimes_B 1$$

implies $m_B(\bar{\delta}) = 1 \otimes_B 1$. Since B is a field, an application of $1 \otimes 1 \otimes m_B$ to the associativity relation for σ_1 demonstrates that σ_1 is in $C \otimes C \otimes B$.

By a similar argument using the analog of Lemma 3.4 for the map

$$C \otimes_B C \otimes B \xrightarrow{\sim} (C \otimes B) \otimes_{B \otimes B} (C \otimes B)$$

given by $x \otimes_B y \otimes z \rightarrow (x \otimes 1) \otimes_{B \otimes B} (y \otimes z)$, there exists a unit δ_1 in $C \otimes B$ with $m_C(\delta_1) = 1$ such that $\sigma_2 \sim^{\delta_1} \sigma_1$ defines a C -two-cocycle σ_2 with

$$\{m_B \otimes 1\}(\sigma_2) = 1 \otimes_B 1 \otimes 1.$$

Since δ_1 is in $C \otimes B$, $\{1 \otimes m_B\}(\sigma_2) = 1 \otimes 1 \otimes_B 1$.

By applying $m_B \otimes 1 \otimes 1$ to the associativity relation for σ_2 one may show that σ_2 is in $B \otimes C \otimes B$. That σ_2 is actually in $B \otimes B \otimes B$ then follows from an application of $1 \otimes m_B \otimes 1$ to the associativity relation for σ_2 . Hence we are done.

We are now ready to prove:

THEOREM 3.6. *Let $k \subseteq B$ be fields and let C be a commutative B -algebra. Suppose also that the natural map $m_B: C \otimes C \rightarrow C \otimes_B C$ induces a surjection on the groups of units. Then*

$$\{1\} \rightarrow \mathcal{H}^2(B) \xrightarrow{\iota^*} \mathcal{H}^2(C) \xrightarrow{r} \mathcal{H}^2(C/B)$$

is exact.

PROOF. The injectivity of ι^* follows from the corollary to Theorem 3.3. Hence to finish the proof we must show exactness at $\mathcal{H}^2(C)$.

Let σ be a C -two-cocycle whose cohomology class is mapped to the trivial class under r , that is, $m_{B,B}(\sigma)$ is cohomologous to $1 \otimes_B 1 \otimes_B 1$ via an invertible element δ in $C \otimes_B C$. By hypothesis, we may lift δ to δ_1 in $C \otimes C$.

Then $\sigma_1 \sim \delta_1^{-1} \sigma$ defines a C -two-cocycle σ_1 with $m_{B,B}(\sigma_1) = 1 \otimes_B 1 \otimes_B 1$. Thus by Theorem 3.5 there is a B -two-cocycle σ_2 with σ_2 equivalent to σ_1 , and hence equivalent to σ . This completes the proof.

COROLLARY 3.7. *Let $k \subseteq B$ be fields and C be a commutative B -algebra. Then*

$$\{1\} \rightarrow \mathcal{H}^2(B) \xrightarrow{i^*} \mathcal{H}^2(C) \xrightarrow{r} \mathcal{H}^2(C/B)$$

is exact if either

- (i) B is a purely inseparable field extension of k , or
- (ii) B is a separable field extension of k and C is purely inseparable over B .

PROOF. By the theorem, it suffices to show that either (i) or (ii) implies m_B induces a surjection on units.

(i) Assume B is purely inseparable over k . The kernel of m_B is generated as a C -bimodule by $\{b \otimes 1 - 1 \otimes b \mid b \text{ is in } B\}$. When B is purely inseparable over k , this is clearly an ideal generated by nilpotent elements and hence lies in the Jacobson radical of $C \otimes C$. The surjectivity on units thus follows since units may be lifted modulo the radical.

(ii) Since C is a purely inseparable B -algebra (cf. [9]), the kernel of the multiplication map $C \otimes_B C \rightarrow C$ given by $x \otimes_B y \rightarrow xy$ is contained in the Jacobson radical of $C \otimes_B C$. Thus any unit of $C \otimes_B C$ may be written as $\alpha \otimes_B 1 + \sum_i u_i \otimes_B v_i$ with α a unit in C and $\sum_i u_i \otimes_B v_i$ nilpotent. If $\sum_i x_i \otimes y_i$ is a separability idempotent for B over k , $\alpha \otimes 1 + \sum_{i,j} u_i x_j \otimes y_j v_i$ is a unit of $C \otimes C$ which maps to $\alpha \otimes_B 1 + \sum_i u_i \otimes_B v_i$. Therefore we are done.

REMARK. If we restrict our attention in Corollary 3.7 to the groups of units, we obtain well-known results on Amitsur cohomology. The reader is referred to Yuan [10, Theorems 3.4, 3.5].

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853

WASHINGTON OPERATIONS, GENERAL RESEARCH CORPORATION, MCLEAN, VIRGINIA 22101

Current address: Bell Laboratories, Holmdel, New Jersey 07733