

THE ALBANESE MAPPING FOR A PUNCTUAL HILBERT SCHEME: I. IRREDUCIBILITY OF THE FIBERS

BY

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ABSTRACT. Let $f: X \rightarrow A$ be the canonical mapping from an algebraic surface X to its Albanese variety A , $X(n)$ the n -fold symmetric product of X , and H_X^n the punctual Hilbert scheme parameterizing 0-dimensional closed subschemes of length n on X . The latter is a nonsingular and irreducible variety of dimension $2n$, and the "Hilbert-Chow" morphism $\sigma_n: H_X^n \rightarrow X(n)$ is a birational map which desingularizes $X(n)$.

This paper studies the composite morphism

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A,$$

where f_n is obtained from f by addition on A . The main result (Part 1 of the paper) is that for $n \gg 0$, all the fibers of φ_n are irreducible and of dimension $2n - q$, where $q = \dim A$. An interesting special case (Part 2 of the paper) arises when $X = A$ is an abelian surface; in this case we show (for example) that the fibers of φ_n are nonsingular, provided n is prime to the characteristic.

Introduction. Let X be an irreducible and nonsingular projective variety of dimension $d > 0$, defined over an algebraically closed ground field. Let $f: X \rightarrow A$ be the canonical map from X to its Albanese variety A ; over the complex numbers this is the complex torus C^q/L , where L is the lattice generated by the $2q$ periods of a basis of holomorphic 1-forms on X . If $X(n)$ is the n -fold symmetric product of X , whose points represent the 0-cycles on X of degree n , then the map $f: X \rightarrow A$ induces by addition on A a map,

$$f_n: X(n) \rightarrow A,$$

which is in fact the Albanese mapping of $X(n)$.

In case X is a curve C , the Albanese variety is the Jacobian J of C , and the fibers of $f_n: C(n) \rightarrow J$ are projective spaces which represent the complete linear systems of degree n on C (the Abel-Jacobi theorem). The classical Riemann-Roch theorem computes the dimensions of these fibers for all $n > 0$. In particular, Riemann's theorem is the assertion that for $n \gg 0$, all of the fibers of $f_n: C(n) \rightarrow J$ have the same dimension $n - g$, where g is the genus of C .

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It is known that the natural generalization of Riemann's theorem to the mapping $f_n: X(n) \rightarrow A$ is true where $\dim X = d > 1$. That is, for $n \gg 0$, all of the fibers of f_n are irreducible and of the same dimension $dn - q$, where $q = \dim A$ [10]. In seeking further generalizations (for example, analogues of the full Riemann-Roch theorem) one confronts the fact that $X(n)$ has singularities when $\dim X > 1$, a "defect" not shared by the nonsingular variety $C(n)$. Consequently the fibers of f_n can, and do, acquire singularities which can seriously interfere.

If X is a surface, however, there is available a natural desingularization of $X(n)$. This is the Hilbert scheme H_X^n , an irreducible and nonsingular projective variety of dimension $2n$ whose points represent the closed subschemes of X having Hilbert polynomial n . With each such subscheme Z there is naturally associated a 0-cycle of degree n (a weighted sum of the points in the support of Z); this gives a birational map

$$\sigma_n: H_X^n \rightarrow X(n)$$

which is an isomorphism over the smooth locus of $X(n)$, whose points represent the 0-cycles $x_1 + \cdots + x_n$ with n distinct summands. (All these results may be found in [2]. Morphisms analogous to σ_n exist for X of any dimension $d > 0$, but H_X^n is not in general irreducible for $d > 2$ [6]. If $d = 1$, $H_C^n = C(n)$ [7].)

The main purpose of this paper is to prove that when X is a surface and $n \gg 0$, all of the fibers of the composite mapping

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

are irreducible and of the same dimension (Corollary 1 of Theorem 3). The proof of this result is the focus of Part 1 of the paper. The necessary preliminaries include a detailed study of the map f_n restricted to the singular locus of $X(n)$, which is carried out for X of any dimension.

Part 2 of the paper treats the special case where X is taken to be an abelian variety A , that is, X equals its own Albanese variety. The principal results of Part 1 (those that follow Theorem 3) are shown to hold for all $n > 0$, rather than just for $n \gg 0$. Moreover, additional information is obtained, notably Corollary 1 of Theorem 6, which states that if A is an abelian surface and n is prime to the characteristic, then all fibers of the mapping $\varphi_n: H_A^n \rightarrow A$ are nonsingular varieties which desingularize the fibers of $f_n: A(n) \rightarrow A$. The arguments in Part 2 are based on the group structure of A , and are essentially independent of Part 1.

A forthcoming paper will study the question of smoothness for the mapping $\varphi_n: H_X^n \rightarrow A$, when X is a surface; it can be viewed as a step in the direction of finding a "Roch" theorem to complement the "Riemann" theorem proved here. It should be noted, however, that unlike the case of curves,

certain fibers of φ_n can be singular, even in characteristic 0 and for arbitrarily large n ; examples will be given.

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PART I

Remark on terminology. In addition to those established in the introduction, we make the following conventions. A *variety* is a reduced, separated, not necessarily irreducible scheme of finite type over a field k , the latter usually taken to be algebraically closed. By a *point* of V we mean a geometric point, i.e., a map $\text{Spec } K \rightarrow V$ where K is an algebraically closed extension of the ground field. If V is irreducible, a *generic point* of V is a point localized at the scheme-theoretic generic point of V , and a set $\{v_1, \dots, v_n\}$ of points of V is a set of *independent generic points* if (v_1, \dots, v_n) is a generic point of the product V^n .

The fibers of a morphism of varieties are taken to be varieties, i.e., they are always given their reduced structure. The generic fiber refers to the fiber over a generic point as understood above. (In Part 2 we do not need the language of generic points; there, a point is a rational point over the (algebraically closed) ground field.)

Finally, if $v_1 + \dots + v_n$ is a 0-cycle on V , the associated point of $V(n)$ is written $(v_1 + \dots + v_n)$.

1. The varieties $X(\pi)$. Until further notice, let X be a projective variety of dimension $d > 0$, irreducible and nonsingular, and let $f: X \rightarrow A$ be its Albanese mapping. We assume throughout that $\dim A = q > 0$. Of particular interest for us is the nonsmooth locus of $X(n)$ and the behavior of the map $f_n: X(n) \rightarrow A$ restricted to the nonsmooth locus. If $(x_1 + \dots + x_n) \in X(n)$, let $r < n$ be the number of distinct x_i . Then, as noted above, $(x_1 + \dots + x_n)$ is a smooth point of $X(n)$ if and only if $r = n$. We call r the *size* of $(x_1 + \dots + x_n)$.

Let $\pi = (n_1, \dots, n_r)$ be a partition of n into r parts, which means each $n_i > 0$ and $\sum_{i=1}^r n_i = n$. We call r the *size* of π and n , following standard usage, the *weight* of π .

To every partition $\pi = (n_1, \dots, n_r)$ of weight n and size r there corresponds a closed irreducible subvariety $X(\pi)$ of $X(n)$. By definition, $X(\pi)$ is the image of the composite morphism

$$\theta_\pi: X^r \xrightarrow{\Delta n_1 \times \dots \times \Delta n_r} X^{n_1} \times \dots \times X^{n_r} = X^n \xrightarrow{\theta_n} X(n), \quad (1)$$

where $\Delta n_i: X \rightarrow X^{n_i}$ is the diagonal map and θ_n is the quotient map (which sends (x_1, \dots, x_n) to $(x_1 + \dots + x_n)$). Thus $\theta_\pi(x_1, \dots, x_r) = (\sum_{i=1}^r n_i x_i)$. Since X^r is complete and irreducible, it follows that its image $X(\pi)$ is a closed irreducible subvariety of $X(n)$.

The following are immediate consequences of the definition:

(2) If $\{x_1, \dots, x_r\}$ is a set of independent generic points of X , then $(\sum_{i=1}^r n_i x_i)$ is a generic point of $X(\pi)$.

(3) The points of $X(\pi)$ represent the 0-cycles of the form $\sum_{i=1}^r n_i x_i$, where not all the x_i need be distinct.

(4) $\dim X(\pi) = dr$, where $d = \dim X$ and r is the size of π .

We write $f_\pi: X(\pi) \rightarrow A$ for the restriction to $X(\pi)$ of the morphism $f_n: X(n) \rightarrow A$. (If, as is standard, we let n represent the partition $(1, \dots, 1)$ of weight and size n , then $X(n)$, θ_n and f_n are exactly $X(\pi)$, θ_π , and f_π for $\pi = n$, so the notation is consistent.)

The next two sections study the map f_π . In §2 we show that f_π is surjective if the size of π is sufficiently large, and in §3 we consider the fibers of f_π (dimension and irreducibility).

2. Surjectivity of f_π . Let B be an abelian variety and Z, Z' subvarieties of B . We write $Z + Z'$ for the image of the map

$$Z \times Z' \rightarrow B \times B \xrightarrow{+} B;$$

the sum $Z + \dots + Z$ (m summands) is abbreviated mZ . We say Z generates B if the inclusion $mZ \hookrightarrow B$ dominates B for $m \gg 0$. As usual we write $m\delta: B \rightarrow B$ for the isogeny of multiplication by m on B . Then $m\delta(Z)$ is the image of Z under $m\delta$ and should not be confused with mZ .

It is clear that

(5) if Z and Z' are closed (resp. irreducible) subvarieties of B , then so are $Z + Z'$, mZ , and $m\delta(Z)$, since each of these is the image of a complete (resp. irreducible) variety.

PROPOSITION 1. *Let B be an abelian variety of dimension $p > 0$. Let Z_1, \dots, Z_s be closed irreducible subvarieties of B each of which generates B , with, say, $\dim Z_1 \geq \dim Z_i$ for $1 < i \leq s$. Then for $s > p - \dim Z_1$, and all $n_i > 0$,*

$$n_1\delta(Z_1) + \dots + n_s\delta(Z_s) = B.$$

To prove this, we use two lemmas.

LEMMA 1. *If Z is an irreducible subvariety of B that generates B , then so is $n\delta(Z)$ for any $n > 0$.*

PROOF. We have that $n\delta(Z)$ is irreducible by (5). To see $n\delta(Z)$ generates B consider the commutative diagram (m factors)

$$\begin{array}{ccccc} Z \times \dots \times Z & \hookrightarrow & B \times \dots \times B & \xrightarrow{+} & B \\ \downarrow & & \downarrow & & \downarrow n\delta \\ n\delta(Z) \times \dots \times n\delta(Z) & \hookrightarrow & B \times \dots \times B & \xrightarrow{+} & B \end{array}$$

$\longleftarrow n\delta \times \dots \times n\delta \longrightarrow$

where m is chosen so that the top row is dominating. Since the rightmost vertical arrow is also dominating, it follows that the bottom row is dominating, whence the lemma. \square

LEMMA 2. *Let W be a proper irreducible subvariety of B , and Z an irreducible subvariety of B that generates B . Then $\dim W < \dim(Z + W)$.*

PROOF. Clearly we may assume Z contains the identity element of B , so $W \subseteq Z + W$ and $\dim(W) < \dim(Z + W)$. In addition, since $\dim(W) = \dim(\overline{W})$ and (as is easily checked)

$$\dim(Z + W) = \dim(\overline{Z} + \overline{W}),$$

we may assume Z and W are closed. Thus $Z + W$ is closed and irreducible (5), and the equality $\dim(W) = \dim(Z + W)$ forces $W = Z + W$, hence

$$W = Z + W = 2Z + W = \dots = mZ + W.$$

But $mZ = B$ for $m \gg 0$, since Z generates B , so $W = B + W = B$, a contradiction. \square

PROOF OF PROPOSITION 1. Since $\dim Z_i = \dim n_i \delta(Z_i)$ for $1 < i \leq s$, Lemma 1 reduces us to proving $Z_1 + \dots + Z_s = B$. If $Z_1 + \dots + Z_s \neq B$, by using (5) and Lemma 2 we see

$\dim B > \dim(Z_1 + \dots + Z_s) > \dim(Z_1 + \dots + Z_{s-1}) > \dots > \dim Z_1$. Since the dimension falls by at least one in each of the s steps to Z_1 , it follows that

$$\dim B = p > \dim Z_1 + s,$$

contradicting the choice of s . \square

PROPOSITION 2. *Let $\pi = (n_1, \dots, n_r)$ be any partition of size $r > q - \dim f(X)$, where $q = \dim A$. Then $f_\pi: X(\pi) \rightarrow A$ is surjective.*

PROOF. Since X is an irreducible projective variety, $f(X)$ is a closed irreducible subvariety of A that generates A . Keeping in mind (3), it is easy to check that

$$\begin{aligned} f_\pi(X(\pi)) &= n_1 \delta(f(X)) + \dots + n_r \delta(f(X)) \\ &= A, \text{ by Proposition 1.} \end{aligned}$$

3. Dimension and irreducibility of the fibers of f_π . The following results are proved in [10].

(6) If $n \gg 0$, the generic fiber of $f_n: X(n) \rightarrow A$ is (nonempty and) irreducible. (It suffices to take $n > g$, where g is the genus of a generic 1-dimensional linear section on X .)

(7) Let n_0 be the least $n > 0$ for which the conclusion of (6) holds. If $n > n_0 + q$, then every fiber of f_n is irreducible and of dimension $dn - q$.

Theorems 1 and 2 below extend these results to the mapping $f_\pi: X(\pi) \rightarrow A$.

Theorem 1 (in case X is a surface) plays an important role in the proof of Theorem 3, our main goal. Theorem 2 has no application in this paper, but is included for completeness.

For the proofs of Theorems 1 and 2 we need

LEMMA 3. *Let Y_1, Y_2 and Z be three irreducible varieties of dimensions d_1, d_2 , and s , respectively, and let $G_1: Y_1 \rightarrow Z, G_2: Y_2 \rightarrow Z$ be morphisms, where G_1 is assumed dominating. Form the cartesian diagram:*

$$\begin{array}{ccc}
 & Y_1 \times_Z Y_2 & \\
 & \downarrow & \\
 & Z & \\
 pr_1 \swarrow & & \searrow pr_2 \\
 Y_1 & & Y_2 \\
 G_1 \searrow & & \swarrow G_2 \\
 & Z &
 \end{array}$$

(a) *Let V be an irreducible subvariety of $Y_1 \times_Z Y_2$, which dominates Y_1 by the first projection, and satisfies $\dim V \geq d_1 + d_2 - s$. Then*

- (i) $\dim V = d_1 + d_2 - s$,
- (ii) V dominates Y_2 by the second projection, and
- (iii) \bar{V} (the closure of V) is an irreducible component of $Y_1 \times_Z Y_2$.

(b) *If G_2 is assumed dominating and G_1 has irreducible generic fiber, then there exists a unique irreducible component W of $Y_1 \times_Z Y_2$ that dominates Y_2 by the second projection, and $\dim W = d_1 + d_2 - s$.*

PROOF. (a) Let $v = (v_1, v_2)$ be a generic point of V , where $v_i \in Y_i, i = 1, 2$; by hypothesis v_1 is a generic point of Y_1 and $G_1(v_1) = z = G_2(v_2)$ is a generic point of Z . Note that both residue fields $k(v_1)$ and $k(v_2)$ contain $k(z)$, and that $\text{dtr}_k k(v_1) = d_1, \text{dtr}_k k(z) = s$, and $\text{dtr}_k k(v_2) \leq d_2$ ($\text{dtr}_K L$ denotes degree of transcendence of L/K). It follows easily that $\text{dtr}_k k(v_1, v_2) \leq d_1 + d_2 - s$, and in view of our hypothesis on $\dim V$, (i) holds. In turn this implies $\text{dtr}_k k(v_2) = d_2$, so that v_2 is a generic point of Y_2 , and (ii) holds. Finally, let V' be an irreducible component of $Y_1 \times_Z Y_2$ containing V ; replacing V by V' above shows that $\dim V' = \dim V = d_1 + d_2 - s$, whence $\bar{V} = V'$, and (iii) holds.

(b) Since G_1 and G_2 are dominating, so are the maps pr_1 and pr_2 and thus at least one irreducible component of $Y_1 \times_Z Y_2$ must dominate (the irreducible) Y_2 . Let W_1, \dots, W_k be the set of all such components. Now the generic (geometric) fiber F of pr_2 is irreducible, it being isomorphic to the corresponding irreducible generic fiber of G_1 . Since $F \subseteq \cup_{i=1}^k W_i$, we have $F \subseteq W_c = W$ for some index c . On the other hand, a generic point of each W_i lies in $F \subseteq W$, hence $W_i \subseteq W$, which proves the uniqueness of W . To see $\dim W = d_1 + d_2 - s$, note that W dominates Y_2 (of dimension d_2) with generic fiber F of dimension $d_1 - s$. \square

Theorems 1 and 2 describe the fibers of the mapping $f_\pi: X(\pi) \rightarrow A$, given hypotheses on the partition π . A key idea in the proof of both theorems is to interpret the fiber $f_\pi^{-1}(a)$ ($a \in A$) as a fiber product of two smaller varieties (and then to apply Lemma 3) as follows: Break up $\pi = (n_1, \dots, n_r)$ into subpartitions $\pi_1 = (n_1, \dots, n_s)$, $\pi_2 = (n_{s+1}, \dots, n_r)$, and form the diagram with cartesian square:

$$\begin{array}{ccc}
 X(\pi_1) \times X(\pi_2) & \xrightarrow{\alpha} & X(\pi) \\
 \text{\scriptsize } pr_1 \swarrow & & \searrow \text{\scriptsize } pr_2 \\
 X(\pi_1) & & X(\pi_2) \\
 \text{\scriptsize } f_{\pi_1} \searrow & & \swarrow \text{\scriptsize } a - f_{\pi_2} \\
 & A &
 \end{array} \tag{8}$$

where $a - f_{\pi_2}$ is the map given by composing f_{π_2} with inversion and then translation by a , and α is the map that “adds 0-cycles.” One checks easily that the image of α in $X(\pi)$ is the fiber $f_\pi^{-1}(a)$.

THEOREM 1. *Let m_0 be the least $m > 0$ such that $f_\pi: X(\pi) \rightarrow A$ is surjective whenever size $(\pi) \geq m$. Then for any partition π of size $r \geq m_0 + q$, every fiber of f_π is equidimensional of dimension $dr - q$, where $d = \dim X$, $q = \dim A$.*

REMARK. According to Proposition 2, $m_0 \leq q - \dim f(X) + 1$.

PROOF. Let a be a point of A and let U be an irreducible component of the fiber $f_\pi^{-1}(a)$. By extending the ground field, we may assume that X, A and U are all defined over the algebraically closed ground field K , and that a is K -rational. We have that $\dim U \geq dr - q$ by the theorem on the dimension of fibers of a morphism [13, p. 60]. We must prove this inequality is an equality.

Let $u = (\sum_{i=1}^r n_i x_i)$ be a generic point of U , where $\pi = (n_1, \dots, n_r)$, as in (3). We first claim that

(9) after renumbering the x_i (and n_i accordingly), $\{x_1, \dots, x_{r-q}\}$ is a set of independent generic points of X over K .

To prove this, note that the map $\theta_\pi: X^r \rightarrow X(\pi)$ which defined $X(\pi)$ above (1) induces an inclusion of residue fields

$$K(u) \rightarrow K(x_1, \dots, x_r),$$

since $\theta_\pi((x_1, \dots, x_r)) = (\sum_{i=1}^r n_i x_i) = u$. Since $K(u)$ is the function field of U , we see that

$$\text{dtr}_K K(x_1, \dots, x_r) \geq \text{dtr}_K K(u) = \dim U \geq dr - q.$$

But $K(x_1, \dots, x_r)$ is the top of the tower

$$K \subseteq K(x_1) \subseteq K(x_1, x_2) \subseteq \dots \subseteq K(x_1, \dots, x_r),$$

and since $\text{dtr}_K K(x_i) \leq d$ for all i , at least $r - q$ of the steps in the tower must

raise the dtr by the amount d , otherwise $\text{dtr}_K K(x_1, \dots, x_r) < dr - q$. Renumbering the x_i , we may suppose these to be the first $r - q$ steps, and we obtain $\text{dtr}_K K(x_1, \dots, x_{r-q}) = d(r - q)$, which is equivalent to (9).

Assuming this renumbering accomplished, break up π as follows: $\pi = (\pi_1, \pi_2)$, with $\pi_1 = (n_1, \dots, n_{r-q})$ and $\pi_2 = (n_{r-q+1}, \dots, n_r)$, and form the diagram (8). Let $v_1 = (\sum_{i=1}^{r-q} n_i x_i)$ and $v_2 = (\sum_{i=r-q+1}^r n_i x_i)$, and let V be the closed irreducible subvariety of $X(\pi_1) \times_A X(\pi_2)$ having generic point (v_1, v_2) . Clearly V dominates U under the mapping α (since $\alpha(v_1, v_2) = u$), so $\dim V > \dim U > dr - q$. We now show, using (a) of Lemma 3, that $\dim V = dr - q$, which completes the proof. The last inequality already checks the hypothesis required on $\dim V$; it remains to check that V dominates $X(\pi_1)$ and that $X(\pi_1)$ dominates A . But by construction $v_1 = (\sum_{i=1}^{r-q} n_i x_i)$ is a generic point (2) of $X(\pi_1)$, so V dominates $X(\pi_1)$, and $X(\pi_1)$ dominates A since π_1 has size $r - q \geq m_0$. Now Lemma 3, part (a), (i) shows that $\dim V = dr - q$, and Theorem 1 is proved. \square

Let r_0 be the least $r > 0$ such that for any partition π of size $\geq r$, the map f_π is surjective and has all fibers equidimensional of dimension $dr - q$. Then by Theorem 1,

$$(10) \quad r_0 \text{ exists and satisfies } r_0 \leq 2q - \dim f(X) + 1.$$

PROPOSITION 3. *If $n > r_0$, then every irreducible component of every fiber of $f_n: X(n) \rightarrow A$ meets the smooth locus of $X(n)$.*

PROOF. Let U be a component of $f_n^{-1}(a)$. If U contains no smooth points of $X(n)$ (points of size n), then clearly $U \subseteq X(\pi) \cap f_n^{-1}(a) = f_\pi^{-1}(a)$, where $\pi = (1, 1, \dots, 1, 2)$ is the partition of weight n and size $n - 1$. Since $n > r_0$, $\dim U = dn - q$. On the other hand, since $n - 1 \geq r_0$, $f_\pi^{-1}(a)$ has dimension $d(n - 1) - q$, contradicting $U \subseteq f_\pi^{-1}(a)$. \square

THEOREM 2. *Suppose that $f_m: X(m) \rightarrow A$ is surjective and has irreducible generic fiber. Let π be a partition of size r for which at least $m + q$ of the parts equal 1. Then every fiber of $f_\pi: X(\pi) \rightarrow A$ is irreducible and of dimension $dr - q$.*

REMARKS. For the partition $(1, \dots, 1)$ of weight and size $n \geq m + q$, Theorem 2 is just (7). According to (6), the hypothesis on f_m is verified for $m \geq g$, where g is the genus of a generic curve of X .

PROOF. Let a be a point of A and let U be an irreducible component of the fiber $f_\pi^{-1}(a)$. We may assume that X, A , and U are all defined over the same algebraically closed ground field K and that a is K -rational. We have that $\dim U \geq dr - q$ by the theorem on the dimension of fibers.

Write $\pi = (n_1, \dots, n_r)$ and let $u = (\sum_{i=1}^r n_i x_i)$ be a generic point of U . Call n_i the associated part of x_i . We claim that we can choose a subset of m of the

x_i , all having associated part 1, to form a set of independent generic points of X over K .

First note that, according to (9), we can choose from among the x_i a set $S = \{x_{i_1}, \dots, x_{i_{m+q}}\}$ of independent generic points of X over K . Now at least m of the $x_i \in S$ must have associated part 1, since by hypothesis at least $m + q$ of the x_i have associated part 1, and only q of the x_i are not in S . By renumbering the x_i and n_i , therefore, we may assume that (x_1, \dots, x_m) is a set of independent generic points with associated parts $n_1 = n_2 = \dots = n_m = 1$.

Now break up $\pi = (\pi_1, \pi_2)$ with $\pi_1 = (1, 1, \dots, 1)$ (m parts), $\pi_2 = (n_{m+1}, \dots, n_r)$, and form the diagram (8). Let $v_1 = (\sum_{i=1}^m n_i x_i)$ and $v_2 = (\sum_{i=m+1}^r n_i x_i)$, and let V be the closed irreducible subvariety of $X(\pi_1) \times_A X(\pi_2)$ having generic point (v_1, v_2) . Arguing as in the proof of Theorem 1, we see that V covers U under α and that the hypotheses of Lemma 3, part (a), are satisfied; consequently:

- (i) $\dim V = dr - q \ (\Rightarrow \dim U = dr - q)$,
- (ii) V dominates $X(\pi_2)$, and
- (iii) V is an irreducible component of $X(\pi_1) \times_A X(\pi_2)$.

On the other hand, the hypotheses of Lemma 3, part (b) are also satisfied for (8), namely: $f_{\pi_1} = f_m$ is surjective and has irreducible generic fiber by hypothesis, and since f_{π_2} is surjective (size $(\pi_2) = r - m > q$), so also is $a - f_{\pi_2}$. Thus there is a unique irreducible component W of $X(\pi_1) \times_A X(\pi_2)$ dominating $X(\pi_2)$. By (ii) and (iii) above, $V = W$; thus W covers U under α . If U' were another component of $f_{\pi}^{-1}(a)$, we could repeat the above argument for U' and thereby conclude that W covers U' , whence $U = U'$. Thus $f_{\pi}^{-1}(a)$ is irreducible (and of dimension $dr - q$), and Theorem 2 is proved. \square

EXAMPLE. Let $X = C$, a curve of genus $g > 1$, and let J be its Jacobian. Let $n = cr$ with $r > g$, $c > 1$, and let $\pi = (c, \dots, c)$ (r parts), a partition of weight n and size r . (Thus π does not satisfy the hypothesis of Theorem 2.) Let $i: C(r) \rightarrow C(\pi)$ be the map sending $(x_1 + \dots + x_r)$ to $(cx_1 + \dots + cx_r)$, evidently a bijective morphism. Consider the commutative diagram:

$$\begin{array}{ccc}
 C(r) & \xrightarrow{i} & C(\pi) \\
 f_r \downarrow & & \downarrow f_{\pi} \\
 J & \xrightarrow{c\delta} & J
 \end{array}$$

Now the fibers of f_r are projective spaces, and the map $c\delta$ is finite to one. Thus the fibers of f_{π} are (set theoretically) finite disjoint unions of projective spaces, and in particular are not irreducible.

The end of [10] consists of two examples relevant to statements (6) and (7).

4. Irreducibility of the fibers of φ_n . We now restrict X to be a (projective, irreducible, and nonsingular) surface, and let H_X^n be the Hilbert scheme representing 0-dimensional closed subschemes of length n on X . As discussed in the introduction, there is a birational map $\sigma_n: H_X^n \rightarrow X(n)$, which is an isomorphism over the smooth locus of $X(n)$, and which desingularizes $X(n)$. We are out to show that for $n \gg 0$, all fibers of the composite mapping

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

(which is the Albanese mapping of H_X^n) are irreducible and of dimension $2n - q$; this is a corollary of Theorem 3. For the proof of Theorem 3 we need, in addition to the work above on the morphisms $f_\pi: X(\pi) \rightarrow A$, the following fact concerning the behavior of σ_n over the singular locus:

(11) If $(x_1 + \dots + x_n) \in X(n)$ has size r , then $\dim \sigma_n^{-1}(x_1 + \dots + x_n) = n - r$ [5, p. 820].

THEOREM 3. *Let X be an irreducible, nonsingular, and projective algebraic surface, defined over an algebraically closed field of any characteristic. If $n > r_0 + q$ (where r_0 is as in (10)), then*

(a) *every fiber of the Albanese mapping*

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

is equidimensional of dimension $2n - q$, and

(b) *for all $a \in A$, the mapping $V \mapsto \sigma_n(V)$ is a bijection from the set of irreducible components of $\varphi_n^{-1}(a)$ to the set of irreducible components of $f_n^{-1}(a)$. Moreover, V and $\sigma_n(V)$ are birationally equivalent under σ_n .*

PROOF. We remark that the hypothesis $n > r_0 + q$ implies, in particular, that f_n and (therefore) φ_n are surjective.

Let a be any point of A , and V an irreducible component of $\varphi_n^{-1}(a)$. By the theorem on the dimension of fibers, $\dim V > 2n - q$.

Since V is irreducible, its image $\sigma_n(V)$ is an irreducible subvariety of some component U of $f_n^{-1}(a)$. We shall prove that $\sigma_n(V) = U$.

Let v be a generic point of V , and suppose that $\sigma_n(v) = (\sum_{i=1}^r n_i x_i)$ is a point of size r , where $r < n$. Then clearly

$$\sigma_n(V) \subseteq X(\pi) \cap f_n^{-1}(a) = f_n^{-1}(a),$$

where π is the partition (n_1, \dots, n_r) . We claim that $r \geq n - q$.

To prove the claim, first note that the theorem on the dimension of fibers applied to $\sigma_n|V$ shows

$$\begin{aligned} 2n - q &< \dim V = \dim \sigma_n(V) + \dim(\sigma_n|V)^{-1}(\sigma_n(v)) \\ &< \dim \sigma_n(V) + n - r, \end{aligned} \tag{12}$$

where the second line follows from (11). Now using in (12) the crude estimate

$$\dim \sigma_n(V) < \dim f_n^{-1}(a) < \dim X(\pi) = 2r,$$

we see that

$$2n - q < \dim V < 2r + n - r,$$

thus $n - q < r$, as claimed.

Since π has size $r > n - q > r_0$, we have that $f_n^{-1}(a)$ is equidimensional of dimension $2r - q$. This yields the refined estimate

$$\dim \sigma_n(V) < \dim f_n^{-1}(a) = 2r - q;$$

using this in (12) instead of $2r$ gives

$$2n - q < n + r - q,$$

which shows that $n < r$, therefore $n = r$, and $\dim \sigma_n(v) < 2n - q$. Consequently (again using (12)),

$$\dim V = \dim \sigma_n(V) = 2n - q,$$

which proves (a).

Moreover, every fiber of $f_n: X(n) \rightarrow A$ is equidimensional of dimension $2n - q$ since $n > r_0$. Thus $\dim U = 2n - q$, and since $\sigma_n(V)$ is an irreducible subvariety of U of dimension $2n - q$, we conclude that $\sigma_n(V) = U$, i.e., $\sigma_n(V)$ is an irreducible component of the fiber $f_n^{-1}(a)$.

It remains to prove (b), which asserts the bijectivity of the map $V \mapsto \sigma_n(V) = U$, and the birationality of V and U under σ_n . But by Proposition 3 (which applies since $n > r_0$) every component of $f_n^{-1}(a)$ meets the open set where σ_n is an isomorphism; from this (b) follows immediately and Theorem 3 is proved. \square

COROLLARY 1. *If $n \gg 0$ (more precisely: if n is such that the conclusions of Theorem 3 and (7) hold simultaneously), then every fiber of the Albanese mapping*

$$\varphi_n: H_X^n \xrightarrow{\sigma_n} X(n) \xrightarrow{f_n} A$$

is irreducible of dimension $2n - q$, and birationally equivalent to the corresponding fiber of f_n .

PROOF. An immediate deduction from (7) and Theorem 3. \square

We recall that a variety is called regular if it has trivial Albanese variety. In [8] it is proved that if V is a projective irreducible variety of any dimension, then for $n \gg 0$ every fiber of the map

$$f_n: V(n) \rightarrow \text{Alb}(V)$$

is a regular variety. Applying this when $V = X$, we obtain

COROLLARY 2. *If $n \gg 0$, every fiber of $\varphi_n: H_X^n \rightarrow A$ is a regular variety.*

PROOF. Since being a regular variety is a birational invariant, the corollary follows from Corollary 1. \square

Let $g: V \rightarrow W$ be a dominating morphism of irreducible varieties. We say that a property P holds for the *general fiber* of g if there is a nonempty open subset of W over which all the fibers have the property P . We recall that

(13) the general fiber of g is irreducible if and only if the field extension $k(V)/k(W)$ is primary (*i.e.*, the algebraic closure of $k(W)$ in $k(V)$ is purely inseparable over $k(W)$) [9, p. 90].

Since H_X^n and $X(n)$ are birationally equivalent, the field extensions $k(H_X^n)/k(A)$ and $k(X(n))/k(A)$ are the same. Thus the general fiber of φ_n is irreducible (and birationally equivalent to the corresponding general fiber of f_n) as soon as the general fiber of f_n is irreducible. This is possibly useful in seeking to sharpen the lower bounds on n in the above results, at least in certain cases (*e.g.* Theorem 4(b) of Part 2). The following example shows what can go wrong at the special fiber.

EXAMPLE. Let A be an abelian surface and let X be the surface obtained by blowing up A at its identity e ; the general fiber of the Albanese mapping $f: X \rightarrow A$ is thus a point, and $f_n^{-1}(e) = D$, a projective line. Since the general fiber of f_n is irreducible when $n = 1$, all of the fibers of f_n are irreducible when $n = 1 + 2 = 3$, by (7). However, when $n = 2$, $f_2^{-1}(e)$ consists of two irreducible components of dimension 2: one component U_1 is essentially the fiber of the addition map $A(2) \rightarrow A$ over e , and the other component U_2 is $D(2)$. (The general fiber of $f_2: X(2) \rightarrow A$ is irreducible.)

There are irreducible components V_1, V_2 of $\varphi_2^{-1}(e)$ dominating U_1, U_2 respectively. Since the generic points of U_i have size $n = 2$, it follows that $\sigma_2: V_i \rightarrow U_i$ is generically 1-1, hence $\dim V_i = 2$ for $i = 1, 2$. But let L be the locus in $U_2 = D(2)$ of points of size 1; clearly $\dim L = 1$ and, by (11), $\dim \sigma_2^{-1}(x) = 1$ for all $x \in L$. Thus there is a two-dimensional locus V_3 in $\varphi_2^{-1}(e)$ dominating L , therefore $\varphi_2^{-1}(e)$ has at least three components. This shows that (b) of Theorem 3 need not be true for low values of n .

PART 2

In Part 2 we study the special case of the Albanese mapping where X is taken to be an abelian variety A of dimension $q > 0$. Then the map $f_n: A(n) \rightarrow A$ is given by addition (in A) of the points of the 0-cycles, *i.e.*, $f_n((a_1 + \dots + a_n)) = \sum_{i=1}^n a_i$. We study the fibers of f_n and of the composite mapping

$$\varphi_n: H_A^n \xrightarrow{\sigma_n} A(n) \xrightarrow{f_n} A,$$

the latter especially (but not exclusively) in case A is an abelian surface. We obtain strengthened versions of some of the results of Part 1 (with less effort), and new information by exploiting the group structure of A .

Henceforth, if we say v is a point of a variety V , or write $v \in V$, we mean v is a rational point of V (over the algebraically closed ground field), unless otherwise stated. We write $\mu: A \times A \rightarrow A$ for the group law on A , and $\mu_a: A \rightarrow A$ for translation by $a \in A$ (i.e., $\mu_a(b) = a + b$). The Hilbert scheme H_A^n is abbreviated to H^n ; if Z is a 0-dimensional closed subscheme of length n on A , the associated point of H^n is written $[Z]$.

5. Dimension, irreducibility and regularity of the fibers. We begin by describing natural actions of A on H^n and on $A(n)$ for which the morphism $\sigma_n: H^n \rightarrow A(n)$ is an A -morphism. These actions are key tools for all that follows.

If Z is a 0-dimensional closed subscheme of length n on A , we define its translate Z_a by $a \in A$ to be the pullback

$$\begin{array}{ccc} Z_a & \xrightarrow{\mu_a} & Z \\ \downarrow & & \downarrow \\ A & \xrightarrow{\mu_a} & A \end{array}$$

Similarly, if $(a_1 + \cdots + a_n) \in A(n)$, we define its translate by a to be the point $(\mu_a(a_1) + \cdots + \mu_a(a_n))$. The actions we seek are given by the morphisms:

$$\begin{aligned} T'_n: H^n \times A &\rightarrow H^n, \\ &([Z], a) \mapsto [Z_a], \\ T_n: A(n) \times A &\rightarrow A(n), \\ &((a_1 + \cdots + a_n), a) \mapsto (\mu_a(a_1) + \cdots + \mu_a(a_n)); \end{aligned} \quad (14)$$

which are obtained as follows.

T_n is the morphism induced (by the universal property of a quotient variety [12, pp. 57–60]) by the morphism

$$\begin{aligned} A^n \times A &\rightarrow A(n), \\ ((a_1, \dots, a_n), a) &\mapsto (\mu_a(a_1) + \cdots + \mu_a(a_n)), \end{aligned}$$

which is invariant under the usual action of the symmetric group Σ_n on the first factor.

To obtain T'_n we must (by the universal property of the Hilbert Scheme [3]) give a closed subscheme W'_n of $A \times H^n \times A = A \times V$, which is finite and flat over V , and has as fiber over $([Z], a) \in V$ the subscheme Z_a . If $W_n \subseteq A \times H^n$ is the universal closed subscheme, one checks that the pullback

$$\begin{array}{ccc}
 W'_n & \xrightarrow{\quad} & W \\
 \downarrow & \square & \downarrow \\
 A \times V & \xrightarrow{\quad} & A \times H^n \\
 (b, [Z], a) & \longmapsto & (b - a, [Z])
 \end{array}$$

does the job.

REMARKS-NOTATION. Let H_a^n (resp. F_a^n) denote the fibers over $a \in A$ of the mappings φ_n (resp. f_n), and for all $c \in A$ let $T'_{n,c}: H^n \rightarrow H^n$ (resp. $T_{n,c}: A(n) \rightarrow A(n)$) denote the morphisms of translation by c induced by the actions (14). We remark that translation takes fibers to fibers; more precisely, if $a, b \in A$ and c satisfies $nc = b - a$, then $T'_{n,c}|H_a^n: H_a^n \rightarrow H_b^n$ and $T_{n,c}|F_a^n: F_a^n \rightarrow F_b^n$ are isomorphisms. In this way A acts transitively on the set of fibers of φ_n (resp. f_n) and the stabilizer of any fiber is the n -torsion group of A . In particular, all the fibers of φ_n (resp. f_n) are isomorphic to one another, so anything true for the general fiber or any special fiber is true for all fibers "by translation."

Theorems 4 and 5 below show (among other things) that the conclusions of the corollaries of Theorem 3 hold for all $n > 0$, not just for $n \gg 0$, when $X = A$ is an abelian surface.

THEOREM 4. (a) For all $n > 0$, each fiber F_a^n of the mapping $f_n: A(n) \rightarrow A$ is irreducible and of dimension $nq - q$, where $q = \dim A$.

(b) If A is an abelian surface, then for all $n > 0$, each fiber H_a^n of the mapping

$$\varphi_n: H^n \xrightarrow{\sigma_n} A(n) \xrightarrow{f_n} A$$

is irreducible and of dimension $2n - 2$. Furthermore, H_a^n is birationally equivalent (by restriction of σ_n) to the corresponding fiber F_a^n of f_n .

PROOF. (a) It is enough to prove that the fiber F_e^n over the identity $e \in A$ is irreducible; the rest follows by translation. But F_e^n is the image of the composite map

$$j: A^{n-1} \xrightarrow{i} A^n \xrightarrow{\theta_n} A(n), \tag{15}$$

where i is the imbedding sending (a_1, \dots, a_{n-1}) to $(a_1, \dots, a_{n-1}, -\sum_{i=1}^{n-1} a_i)$, and θ_n is the quotient map. Since A^{n-1} is irreducible, so is its image.

(b) Since A is a surface, H^n is irreducible (and nonsingular), and σ_n is a birational map. The argument sketched following (13) shows that (b) holds for the general fiber of φ_n , therefore (b) holds for all fibers by translation. \square

THEOREM 5. *Let A be an abelian variety of any dimension. Then for all $n > 0$, every fiber of the mapping $f_n: A(n) \rightarrow A$ is a regular variety (i.e. has trivial Albanese variety).*

PROOF. By translation, it is enough to prove this for F_e^n , the fiber over the identity. Let $g: F_e^n \rightarrow B$ be a rational mapping to an abelian variety B ; we must prove that g is constant.

The surjective morphism $j: A^{n-1} \rightarrow F_e^n$ given by (15) yields a diagram

$$\begin{array}{ccc} A^{n-1} & \xrightarrow{j} & F_e^n \\ & H \searrow & \swarrow g \\ & & B \end{array}$$

where H is defined by composition, and it is enough to prove that H is constant.

By the general theory of rational maps of abelian varieties (see, e.g. [1]), after a translation of B , H will be a homomorphism of abelian varieties. Since $H((a_1, \dots, a_{n-1}))$ does not depend on the order of the arguments, all of the homomorphisms

$$h_\nu: A \xrightarrow{i_\nu} A^{n-1} \xrightarrow{H} B,$$

where i_ν ($1 \leq \nu \leq n-1$) is the standard injection to the ν th factor, are equal. Writing h for any of the h_ν , we have

$$H((a_1, \dots, a_{n-1})) = \sum_{i=1}^{n-1} h(a_i)$$

for all $(a_1, \dots, a_{n-1}) \in A^{n-1}$.

But from (15) it is clear that (a_1, \dots, a_{n-1}) and $(a_1, \dots, a_{n-2}, -\sum_{i=1}^{n-1} a_i)$ have the same image in F_e^n under j , thus

$$\begin{aligned} H((a_1, \dots, a_{n-1})) &= H\left(\left(a_1, \dots, a_{n-2}, -\sum_{i=1}^{n-1} a_i\right)\right), \text{ i.e.,} \\ \sum_{i=1}^{n-1} h(a_i) &= \sum_{i=1}^{n-2} h(a_i) - \sum_{i=1}^{n-1} h(a_i) \end{aligned}$$

which becomes

$$\sum_{i=1}^{n-2} h(a_i) + 2h(a_{n-1}) = e.$$

Fixing a_2, \dots, a_{n-1} and letting a_1 vary, we conclude that h and (thus) H are constant, which proves the theorem. \square

COROLLARY. *Let A be an abelian surface. Then for any $n > 0$, every fiber of the map $\varphi_n: H^n \rightarrow A$ is a regular variety.*

PROOF. Being a regular variety is a birational invariant, so the corollary follows from Theorems 4 and 5. \square

6. Normality and nonsingularity of the fibers. As remarked earlier, A acts by translation on the fibers of φ_n and f_n transitively but not simply; the stabilizer of any fiber is the n -torsion group of A . Apparently the fibrations $\varphi_n: H^n \rightarrow A$ and $f_n: A(n) \rightarrow A$ are “twisted products” of the base and a fiber, which one wants to “untwist” into simple products. To do this, form the cartesian diagrams

$$\begin{array}{ccc}
 A(n) & \xleftarrow{(n\delta)^*} & A(n)_1 \\
 f_n \downarrow & \square & \downarrow \\
 A & \xleftarrow{n\delta} & A_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 H^n & \xleftarrow{(n\delta)^*} & H_1^n \\
 \varphi_n \downarrow & \square & \downarrow \\
 A & \xleftarrow{n\delta} & A_1
 \end{array}
 \tag{16}$$

where $A_1 = A$ and $n\delta$ is the isogeny given by multiplication by n . We now show that the fibrations of $A(n)_1$ and H_1^n over A_1 are actually products.

THEOREM 6. *There exist A_1 -isomorphisms*

$$\lambda'_n: H_1^n \rightarrow H_e^n \times A_1, \quad \lambda_n: A(n)_1 \rightarrow F_e^n \times A_1,$$

where e is the identity of $A = A_1$.

PROOF. We write down λ'_n and its inverse using the action T'_n (14); one obtains λ_n and its inverse analogously. We note that as point sets,

$$H_1^n = \{([Z], a) | \varphi_n([Z]) = na\},$$

and

$$H_e^n \times A_1 = \{([Z], a) | \varphi_n([Z]) = e\}.$$

Let λ'_n be the composite morphism

$$\lambda'_n: H_1^n \hookrightarrow H^n \times A_1 \xrightarrow{1 \times -\delta} H^n \times A_1 \xrightarrow{(T'_n, pr_2)} H^n \times A_1 \xrightarrow{1 \times -\delta} H^n \times A_1$$

which on points satisfies

$$\lambda'_n([Z], a) = ([Z_{-a}], a) \in H_e^n \times A_1.$$

The natural candidate for the inverse is the composite morphism

$$H_e^n \times A_1 \hookrightarrow H^n \times A_1 \xrightarrow{(T'_n, pr_2)} H^n \times A_1,$$

which on points satisfies $([Z], a) \mapsto ([Z_a], a) \in H_1^n$.

One checks easily that the composition of these morphisms in either order is the identity, so the theorem is proved. \square

In order to exploit Theorem 6, we need the following two facts. Let V and W be irreducible varieties defined over an algebraically closed field.

(17) The product $V \times W$ is normal (resp. nonsingular) if and only if V and W are each normal (resp. nonsingular).

(18) If $g: V \rightarrow W$ is an étale map, then W normal (resp. nonsingular) implies V normal (resp. nonsingular). (The converse holds if g is surjective.)

PROOF. (17) If V and W are normal, then so is $V \times W$, by [9, p. 148]. Conversely, let V' and W' be the normalizations of V and W , and consider the finite and birational map $V' \times W' \rightarrow V \times W$ defined in the obvious way. Since $V \times W$ is normal, Zariski's Main Theorem implies this map is bijective, and hence (ZMT again) an isomorphism, whence $V = V'$, $W = W'$. Statement (17) for "nonsingular" is proved in [9, p. 199] (and is not difficult).

(18) These assertions are proved in [4, Exp. 1, 9.2, 9.10].

Let p be the characteristic of the ground field. It is well known that the isogeny $n\delta$ is étale if and only if $p \nmid n$ [11, pp. 64, 74]. Since the pullback of an étale morphism is étale, we have that the morphisms $(n\delta)^*$ of (16) are étale if $p \nmid n$. This leads to Corollaries 1 and 2.

COROLLARY 1. *Let A be an abelian surface, and let $p \nmid n > 0$. Then every fiber H_a^n of the map $\varphi_n: H^n \rightarrow A$ is a nonsingular variety. Moreover, the map $\sigma_n|_{H_a^n}: H_a^n \rightarrow F_a^n$ is a resolution of the singularities of F_a^n , for all $a \in A$.*

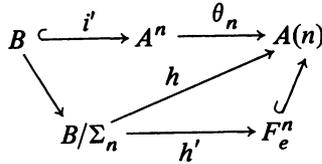
PROOF. Since H^n is nonsingular, the same is true of H_1^n by (18). But by Theorem 6, H_1^n is isomorphic to the product variety $H_e^n \times A_1$; thus H_e^n is nonsingular by (17). By translation all of the fibers of φ_n are nonsingular, and the last statement follows from Theorem 4. \square

COROLLARY 2. *Let A be an abelian variety of any dimension $q > 0$, and let $n > 0$ be such that $p \nmid n$. Then every fiber F_a^n of the map $f_n: A(n) \rightarrow A$ is a normal variety.*

PROOF. Since A^n is normal (being nonsingular), the symmetric product is also normal [12, p. 58]. Arguing as in the proof of Corollary 1, we conclude that $A(n)_1$ and (hence) F_e^n are normal; thus all fibers are normal by translation.

REMARK. The normality of the fibers of $f_n: A(n) \rightarrow A$ (assured when $p \nmid n$) has the consequence that these fibers are themselves quotient varieties for the action of a finite group.

Indeed, let B be the image of the embedding $i: A^{n-1} \rightarrow A^n$ used to define the surjective morphism $j: A^{n-1} \rightarrow F_e^n$ (15). Clearly B is (as set) the kernel of the homomorphism $A^n \rightarrow A$ given by addition, and is stable under the action of the symmetric group Σ_n on A^n . By restriction Σ_n acts on B , and we may form the quotient variety B/Σ_n . There is a commutative diagram



where h is the map resulting from the universal property of the quotient variety, since $\theta_n \circ i'$ is invariant under the action of Σ_n on B . The map h factors through the image F_e^n of $\theta_n \circ i'$ as shown; clearly h' is a bijection. We want to prove that h' is an isomorphism; this will be so if and only if h is an embedding (closed immersion).

As is pointed out in [12, p. 60], h need not be an embedding in general, unless the characteristic is zero or the group acts freely. However, since Σ_n does act freely on the open set of off-diagonal points of A^n (which B intersects nontrivially), it follows that h is an embedding on an open non-empty set. Consequently h' is a bijective birational morphism, and hence an isomorphism (by Zariski's Main Theorem) when F_e^n is normal. Thus if $p \nmid n$, the fibers of $f_n: A(n) \rightarrow A$ are quotient varieties, and the singularities being resolved in Corollary 1 are quotient singularities.

EXAMPLE. Let A be an abelian surface in characteristic 0, and let $n = 2$. Then the map $i: A \rightarrow A^2$ sends $a \in A$ to the pair $(a, -a)$, and B is the set of all such pairs (notation as above). Thus $B/\Sigma_2 = F_e^2$ is the classical Kummer surface, created by identifying each point $a \in A$ with its inverse $-a$. The singular points of this surface correspond to the points of order 2 in A , and are resolved by being blown up into projective lines [14]. This is then a special case of Corollary 1, since the map $\sigma_2|_{H_e^2}: H_e^2 \rightarrow F_e^2$ is an isomorphism except over the singular points of F_e^2 , where the fiber is a projective line.

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