

JORDAN RINGS WITH NONZERO SOCLE

BY

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ABSTRACT. Let \mathcal{J} be a nondegenerate Jordan algebra over a commutative associative ring Φ containing $\frac{1}{2}$. Defining the socle \mathcal{S} of \mathcal{J} to be the sum of all minimal inner ideals of \mathcal{J} , we prove that \mathcal{S} is the direct sum of simple ideals of \mathcal{J} . Our main result is that if \mathcal{J} is prime with nonzero socle, then either (i) \mathcal{J} is simple unital and satisfies DCC on principal inner ideals, (ii) \mathcal{J} is isomorphic to a Jordan subalgebra \mathcal{J}' of the plus algebra A^+ of a primitive associative algebra A with nonzero socle S , and \mathcal{J}' contains S^+ , or (iii) \mathcal{J} is isomorphic to a Jordan subalgebra \mathcal{J}'' of the Jordan algebra of all symmetric elements H of a primitive associative algebra A with nonzero socle S , and \mathcal{J}'' contains $H \cap S$. Conversely, any algebra of type (i), (ii), or (iii) is a prime Jordan algebra with nonzero socle. We also prove that if \mathcal{J} is simple then \mathcal{J} contains a completely primitive idempotent if and only if either \mathcal{J} is unital and satisfies DCC on principal inner ideals or \mathcal{J} is isomorphic to the Jordan algebra of symmetric elements of a \star -simple associative algebra A with involution \star containing a minimal one-sided ideal.

1. Introduction. Our purpose in this paper is to obtain for Jordan rings an analogue of the theory of primitive associative rings with a minimal one-sided ideal ([3], [2]). In [13] the first author considered a class of Jordan rings which he called primitive. However this class appears to be too broad for our purpose. Accordingly our approach will not be "module-theoretical" but rather intrinsic. We first recall some facts and notation in the associative situation which will be used subsequently.

Throughout A will denote an associative ring. If A is semi-prime and contains a minimal left ideal $L \neq 0$ then $L = Ae$ for some $e \in A$, where $e^2 = e \neq 0$ and eAe is a division ring. Moreover eA is a minimal right ideal. Conversely if $e^2 = e \neq 0$ is such that eAe is a division ring then Ae is a minimal left ideal of A and eA is a minimal right ideal.

If V is a left vector space over a division ring Δ and W a right vector space over Δ then V and W are said to be *dual* with respect to an inner product $(,) : V \times W \rightarrow \Delta$ if for $v \in V$, $(v, W) = 0$ implies $v = 0$ and for $w \in W$,

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$(V, w) = 0$ implies $w = 0$. An element $a \in \text{Hom}_\Delta(V, V)$ is said to be *continuous* if there exists an $a^* \in \text{Hom}_\Delta(W, W)$ such that $(va, w) = (v, a^*w)$ for all $v \in V, w \in W$. Denote by $L_W(V)$ the ring of continuous linear transformations on V and by $F_W(V)$ the subring of all elements of finite rank.

Let A be a primitive ring with nonzero socle. Then there exist a division ring Δ and dual vector spaces V and W such that A is isomorphic to a subring of $L_W(V)$ containing $F_W(V)$. Conversely any subring of $L_W(V)$ containing $F_W(V)$ is primitive with nonzero socle. $F_W(V)$ is a simple ring with a minimal left ideal. Moreover it is locally a matrix ring, that is, every finite subset of $F_W(V)$ can be embedded in a subring which is isomorphic to $M_n(\Delta)$. If A is not isomorphic to some matrix ring $M_n(\Delta)$, then, for every integer $m > 0$, A contains a subring isomorphic to $M_n(\Delta)$. If A has an involution then Δ has an involution, V is self-dual with respect to a hermitian or a symplectic inner product $(,)$ (in the latter case Δ is a field), and the involution is $*$, the adjoint with respect to $(,)$.

THEOREM 1. *For an associative ring A the following are equivalent:*

- (1) *A is a primitive ring with nonzero socle;*
- (2) *A is a prime ring with a minimal ideal which, considered as a ring, possesses a minimal right ideal.*

PROOF. (1) \Rightarrow (2). It is well known that any primitive ring is prime [3], and, as was recalled above, A is isomorphic to a subring A' of $L_W(V)$ containing $F_W(V)$. $F_W(V)$ is an ideal of A' and since $F_W(V)$ is simple it must be a minimal ideal of A' . That $F_W(V)$ has a minimal right ideal was also recalled above.

(2) \Rightarrow (1). Let A be a prime ring with minimal ideal B . By Proposition 2 of [3, p. 65], any nonzero ideal of B contains a nonzero ideal of A . So by minimality B is a simple ring and hence is prime. Thus its minimal one-sided ideal is generated by an idempotent e with eBe a division ring. Now $eAe \subset B$. Therefore $eAe = eBe$ and eA is a minimal right ideal of A , hence an irreducible right A -module. It is also faithful since if the right annihilator $\text{Ann}_r(eA) = \{a \in A \mid eAa = 0\}$ is nonzero then by primeness its left annihilator $\text{Ann}_l(\text{Ann}_r(eA))$ must be $\{0\}$, a contradiction. Therefore (2) \Rightarrow (1).

Primeness is an intrinsic notion which has been studied in Jordan rings [14], [1]. We will define the socle \mathfrak{S} of a nondegenerate Jordan ring \mathcal{J} to be the span of the minimal inner ideals of \mathcal{J} . To each minimal inner ideal will be "linked" a completely primitive idempotent. If moreover \mathcal{J} is prime and of characteristic not 2, then the socle will be shown to be a simple ideal or $\{0\}$. A result of McCrimmon will be used to prove that either \mathfrak{S} is unital and has a capacity or \mathfrak{S} is isomorphic to one of the rings $F_W(V)^+$ or $\mathcal{C}(F_W(V), *)$, the symmetric elements of the ring with involution $(F_W(V), *)$. In the first case \mathcal{J}

equals its socle; in the other two cases \mathcal{J} is isomorphic to a Jordan subring of $L_W(V)^+$, respectively $\mathcal{J}(L_W(V), *)$.

2. Simple Jordan rings with a primitive idempotent. From now on \mathcal{J} will denote a Jordan algebra over a commutative associative ring Φ containing $\frac{1}{2}$. We assume as known the structure theory of simple unital Jordan algebras satisfying DCC on principal inner ideals [4], [5]. An idempotent $e \in \mathcal{J}$ is *completely primitive* if $(\mathcal{J}U_e, U, e)$ is a Jordan division algebra.

We recall next a few results from [8]. If e_1, \dots, e_{n-1} are pairwise orthogonal nonsupplementary idempotents, we consider the Peirce decomposition

$$\mathcal{J} = \bigoplus_{0 < i < j < n-1} \mathcal{J}_{ij} \tag{1}$$

and for $x \in \mathcal{J}$ denote the projection of x on \mathcal{J}_{ij} by $[x]_{ij}$ (or $[x]_i$ when $i = j$). The Peirce relations can be found, for example, in [4], [5]. Two orthogonal idempotents $e_1, e_2 \in \mathcal{J}$ are said to be *strongly connected* (*connected*, respectively *interconnected*) if $e_i = [x^2]_i, i = \{1, 2\}$, for some $x \in \mathcal{J}_{12}$ (if \mathcal{J}_{12} contains an element which is invertible in $\mathcal{J}U_{e_1+e_2}$, respectively if $\mathcal{J}_{ii} = [\mathcal{J}_{12}^2]_i, i \in \{1, 2\}$). Strong connectedness implies connectedness which in turn implies interconnectedness. A Peirce decomposition, as in (1), is *n-interconnected* if

$$\mathcal{J}_{ii} = [\mathcal{J}_{ij}^2]_i, \quad i \neq j, \tag{2}$$

and

$$a_{ij}\mathcal{J}_{jk} = 0 \Rightarrow a_{ij} = 0, \quad k \neq i, j, \tag{3}$$

for $0 < i, j, k < n - 1$. McCrimmon [8] has proved:

THEOREM 2. *Any Peirce decomposition of an arbitrary simple Jordan algebra \mathcal{J}/Φ is interconnected.*

THEOREM 3. *Any n-interconnected Jordan algebra \mathcal{J} over a field Φ of characteristic not 2, for $n > 4$, is special.*

McCrimmon [12] has also shown:

THEOREM 4. *If e is a nontrivial idempotent of a simple Jordan algebra \mathcal{J} then the 1 and 0 components of the Peirce decomposition of \mathcal{J} with respect to e are simple.*

We use this result to establish:

LEMMA 5. *If \mathcal{J}/Φ is a simple Jordan algebra and e is a nontrivial idempotent, then $\mathcal{J}_1(e)$ is a special Jordan algebra.*

PROOF. Since e is nontrivial, $e \neq 0$ or 1 if \mathcal{J} is unital. By Theorem 2, $\mathcal{J}_{01}(e) \neq \{0\}$. The map $\nu: \mathcal{J}_1(e) \rightarrow \text{End}_{\Phi}(\mathcal{J}_{01})^+$ given by $\nu(x) = V_x|_{\mathcal{J}_{01}}$, the restriction of V_x to $\mathcal{J}_{1/2}$, is a homomorphism of Jordan algebras [5] whose kernel is $\{0\}$ since \mathcal{J}_{11} is simple and $V_e|_{\mathcal{J}_{01}}$ is the identity.

LEMMA 6. *Let \mathcal{J}/Φ be a simple Jordan algebra and $e \in \mathcal{J}$ a completely primitive idempotent. If $\mathcal{J}_1(e) \neq \mathcal{J}$ then $\mathcal{J}_0(e)$ contains a completely primitive idempotent g connected to e .*

PROOF. If $\mathcal{J}_{11} = \mathcal{J}_1(e) \neq \mathcal{J}$ then by Theorem 2 and equation (2), there exists an $x \in \mathcal{J}_{01}$ such that $[x^2]_1 \neq 0$. Let y be the inverse of $[x^2]_1$ in the Jordan division ring (\mathcal{J}_{11}, U, e) . By the Peirce relations $g = yU_x \in \mathcal{J}_{00}$. Now

$$(yU_x)^2 = x^2U_yU_x = [x^2]_1U_yU_x = yU_x$$

so $g^2 = g$. Moreover,

$$yU_xU_x = yU_{x^2} = yU_{[x^2]_1} = [x^2]_1 \neq 0$$

and $g \neq 0$.

Since

$$x \circ g = x \circ (yU_x) = yU_xV_x = yU_{x,x^2} = \{xy[x^2]_1\} = (x \circ y) \circ [x^2]_1 = x$$

by the Peirce relations, we have $x \in \mathcal{J}_{01}(g) \cap \mathcal{J}_{01}(e)$, the half space of e and g . Now $(y + yU_x)U_x = g + [x^2]_1$ which is invertible in $\mathcal{J}_1(e + g)$, so $e + g \in \mathcal{J}_1(e + g)U_x$, and by the theorem on inverses [5 p. 1.58], x is invertible in $\mathcal{J}_1(e + g)$. Hence e and g are connected by x .

Finally we show that $\mathcal{J}_1(g)$ is a division algebra, so g is completely primitive. Observe that

$$\mathcal{J}_1(g) = \mathcal{J}U_g = \mathcal{J}U_xU_yU_x \subset \mathcal{J}_1(e)U_x$$

and

$$\mathcal{J}_1(e) = \mathcal{J}_1(e)U_{[x^2]_1} = \mathcal{J}_1(e)U_xU_xU_yU_xU_x \subset \mathcal{J}U_gU_x.$$

Therefore any nonzero $b_0 \in \mathcal{J}_1(g)$ can be written as $b_0 = b_1U_x$ with $b_1 \in \mathcal{J}_1(e)$, $b_1 \neq 0$, and hence invertible in $\mathcal{J}_1(e)$. Thus

$$g = yU_x \in \mathcal{J}_1(e)U_x = \mathcal{J}_1(e)U_{b_1}U_x = \mathcal{J}U_gU_xU_{b_1}U_x = \mathcal{J}U_gU_{b_0} = \mathcal{J}_1(g)U_{b_0}$$

and b_0 is invertible in $\mathcal{J}_1(g)$.

COROLLARY 7. *If \mathcal{J}/Φ is a simple Jordan algebra containing a completely primitive idempotent, then either \mathcal{J} has a capacity or \mathcal{J} contains a simple subalgebra of capacity n for any positive integer n .*

PROOF. If $e_1 \in \mathcal{J}$ is a completely primitive idempotent then either $\mathcal{J} = \mathcal{J}_1(e)$ or there exists a completely primitive idempotent $e_2 \in \mathcal{J}_0(e_1)$ connected to e_1 . Assume we have e_1, e_2, \dots, e_{n-1} mutually orthogonal completely primitive idempotents with e_i connected to e_1 , $i > 1$. Then e_i is connected to

$e_j, i \neq j$ [5] and if $\mathcal{J}_1(e_1 + \dots + e_{n-1}) = \mathcal{J}$, \mathcal{J} has a capacity. If not, apply Lemma 6 to the algebra $\mathcal{J}_0(e_2 + \dots + e_{n-1})$ which is simple by Theorem 4, constructing a completely primitive idempotent $e_n \in \mathcal{J}_0(e_1 + \dots + e_{n-1})$ connected to e_1 . The corollary follows by induction and Theorem 4.

Our next lemma is a special case of Lemma 3.1 of [11]. While \mathcal{J} is assumed to be unital in [11], the proof goes through with minor modifications and can be simplified in some parts since we assume the presence of $\frac{1}{2}$. Accordingly we will not give a proof. Denote by $(S(\mathcal{J}), \sigma)$ the special universal envelope of \mathcal{J} and by $*$ the canonical involution of $S(\mathcal{J})$. \mathcal{J} is said to be reflexive if $\mathcal{J}\sigma = \mathcal{K}(S(\mathcal{J}), *)$. Note that reflexivity does not imply speciality since σ need not be injective.

LEMMA 8. *If \mathcal{J}/Φ is an n -interconnected Jordan algebra, $n \geq 3$, then \mathcal{J} is reflexive. If $n \geq 4$, \mathcal{J} is special and reflexive.*

We are ready to prove the main theorem of this section.

THEOREM 9. *\mathcal{J}/Φ is a simple Jordan algebra containing a completely primitive idempotent if and only if either \mathcal{J} is unital and satisfies DCC on principal inner ideals or \mathcal{J} is isomorphic to $\mathcal{K}(A, *)$ the symmetric elements of a $*$ -simple associative Φ algebra A with involution $*$ containing a minimal one-sided ideal.*

PROOF. Let \mathcal{J} be a simple Jordan algebra containing a completely primitive idempotent. By Corollary 7 either \mathcal{J} is unital and has a capacity or it contains simple subalgebras of arbitrary large capacity. In the first case, using the coordinatization theorem in case \mathcal{J} has capacity ≥ 3 or the structure theory of simple algebras of capacity 2 in case \mathcal{J} has capacity 2, one obtains that \mathcal{J} is isomorphic to one of the algebras in the classification of simple unital Jordan algebras satisfying DCC on principal inner ideals.

In the second case, by Theorem 3, \mathcal{J} is special (\mathcal{J} can always be considered as an algebra over its centroid which is a field of characteristic not 2 [9]). By Lemma 8, $\mathcal{J}\sigma = \mathcal{K}(S(\mathcal{J}), *)$. Let \mathcal{I} be a $*$ -ideal of $S(\mathcal{J})$. Then $\mathcal{I} \cap \mathcal{J}\sigma$ is an ideal of $\mathcal{J}\sigma$. So $\mathcal{I} \cap \mathcal{J}\sigma = \{0\}$ or $\mathcal{J}\sigma$. Since $\mathcal{J}\sigma$ generates $S(\mathcal{J})$ (associatively), if $\mathcal{I} \cap \mathcal{J}\sigma = \mathcal{J}\sigma$ then $\mathcal{I} = S(\mathcal{J})$. If $\mathcal{I} \cap \mathcal{J} = \{0\}$ then for any $b \in \mathcal{I}, b + b^*$ and $bb^* \in \mathcal{I} \cap \mathcal{J}\sigma$ so $b^* = -b$ and $b^2 = 0$. If $a \in \mathcal{J}\sigma$ and $b \in \mathcal{I}$ then $ab \in \mathcal{I}$ and $-ab = (ab)^* = -ba$ so $[\mathcal{J}\sigma, \mathcal{I}] = \{0\}$. Since $\mathcal{J}\sigma$ generates $S(\mathcal{J})$, \mathcal{I} is contained in the centre of $S(\mathcal{J})$. Let \mathcal{I} be a maximal proper $*$ -ideal. Clearly \mathcal{I} is unique. Let $A = S(\mathcal{J})/\mathcal{I}; *$ induces an involution on A which we will also denote $*$. The map $\mathcal{J} \rightarrow S(\mathcal{J}) \rightarrow A$ is injective since \mathcal{I} consists entirely of skew-symmetric elements. Moreover since $\frac{1}{2} \in \Phi$, any element of $S(\mathcal{J})$ is the sum of a symmetric and a skew-symmetric element and if $x \in S(\mathcal{J})$ is skew then $(x + \mathcal{I})^* = -x + \mathcal{I} \neq x + \mathcal{I}$ unless $x \in \mathcal{I}$. Therefore $\mathcal{J} \cong \mathcal{K}(A, *)$ and A is $*$ -simple by the maximality of \mathcal{I} .

Identify \mathcal{J} with $\mathcal{K}(A, *)$. Let e_1 be a completely primitive idempotent of \mathcal{J} . As in Corollary 7 choose $e_2, e_3, e_4 \in \mathcal{J}_0(e_1)$ mutually orthogonal completely primitive idempotents connected to e_1 . Coordinatizing, we obtain that e_1 is either primitive in A in which case $e_1 A e_1$ is a division algebra, or $e_1 = e' + e'^*$ and $e_1 A e_1 = M_2(F)$ or $\Delta \oplus \Delta^0$, where F is a field and the restriction of $*$ to $M_2(F)$ is the symplectic involution, and Δ is a division algebra, Δ^0 the opposite algebra. In this last case A is not simple, $A = B \oplus B^0$, B a simple associative algebra and $*$ is the exchange involution. Since $e' B e' = \Delta$, B has a minimal one-sided ideal. In the other cases, A is simple and contains a minimal one-sided ideal.

The converse follows from structure theory and [6].

The classification of simple unital Jordan algebras satisfying DCC on principal inner ideals and Theorem 9 imply:

COROLLARY 10. *If \mathcal{J}/Φ is a simple exceptional Jordan algebra which contains a completely primitive idempotent then \mathcal{J} is either a Jordan division algebra or is isomorphic to $\mathcal{K}(\mathcal{O}_3, J_\gamma)$.*

3. The socle of a Jordan ring. An element $z \in \mathcal{J}$ is an *absolute zero divisor* if $U_z \equiv 0$. \mathcal{J} is said to be *nondegenerate* if $U_z \equiv 0$ implies $z = 0$. First we recall the theorem on minimal inner ideals [5]:

Any minimal inner ideal \mathfrak{B} of a unital Jordan algebra \mathcal{J}/Φ is one of the following types:

- (I) $\mathfrak{B} = \Phi z, z \neq 0$, an absolute zero divisor;
- (II) $\mathfrak{B} = \mathcal{J}U_b$ for every $b \neq 0$ in \mathfrak{B} but $\mathfrak{B}U_b = \{0\}$ and $b^2 = 0$ for every $b \in \mathfrak{B}$;
- (III) $\mathfrak{B} = \mathcal{J}U_e, e$ a completely primitive idempotent.

We finally recall:

AUXILIARY LEMMA ([5, p. 3.11]). *Let $a, b \in \mathcal{J}$. If $aU_b = b$, let $d = bU_a$. Then $U_a U_b$ and $U_b U_a$ are idempotent elements of $\text{End } \mathcal{J}$ and $dU_b = b, bU_a = d$.*

In the structure theory of simple Jordan ring with DCC on principal inner ideals, the inner ideals play a role analogous to that played by one-sided ideals in associative theory. In view of the minimal inner ideal theorem (MII Theorem) and especially since it is not known whether a simple radical Jordan ring can exist in the presence of DCC on inner ideals, it seems reasonable to assume that \mathcal{J} is nondegenerate which we do from now on. We define the *socle* \mathfrak{S} of a nondegenerate Jordan ring \mathcal{J} to be span of the minimal inner ideals of \mathcal{J} . Since inner ideals are preserved by the structure group [10], \mathfrak{S} is left stable by the structure group. We wish to show that the socle is an ideal.

LEMMA 11. *Let \mathcal{J} be a nondegenerate Jordan ring. If $T \in \text{End } \mathcal{J}$ satisfies $U_{xT} = T'U_xT, \forall x \in \mathcal{J}$, and some $T' \in \text{End } \mathcal{J}$, then for any minimal inner ideal \mathfrak{B} either $\mathfrak{B}T = 0$ or $\mathfrak{B}T$ is again a minimal inner ideal.*

PROOF. If $bT \neq 0$ for some $b \in \mathfrak{B}$ then $\mathcal{J}U_{bT} = \mathcal{J}T'U_bT \neq \{0\}$ by nondegeneracy, so $\mathfrak{B} \supset \mathcal{J}T'U_b \neq \{0\}$ implies $\mathcal{J}T'U_b = \mathfrak{B}$ by minimality. Hence $\mathcal{J}U_{bT} = \mathfrak{B}T$ and $\mathfrak{B}T$ is a minimal inner ideal.

PROPOSITION 12. *The socle \mathfrak{S} of a nondegenerate Jordan ring is an ideal of \mathcal{J} .*

PROOF. If \mathfrak{B} is a minimal inner ideal of \mathcal{J} then, by Lemma 11, $\mathfrak{B}U_x$ is either $\{0\}$ or a minimal inner ideal. Therefore \mathfrak{S} is an outer ideal. To prove that \mathfrak{S} is also inner it suffices to show that $\mathcal{J}U_s \subset \mathfrak{S}$ for all $s \in S$ a spanning set of \mathfrak{S} . Let $S = \cup \alpha \mathfrak{B}_\alpha$ where $\{\mathfrak{B}_\alpha\}$ is the set of all minimal inner ideals. Then $\mathcal{J}U_b \subset \mathfrak{B} \in S$ and $\mathcal{J}U_{b,b'} = \{b\mathcal{J}b'\} = bV_{\mathcal{J},b'} \subset \mathfrak{S}$ since \mathfrak{S} is outer.

LEMMA 13. *Let \mathcal{J} be a nondegenerate Jordan algebra over Φ . If \mathcal{J} has a minimal inner ideal \mathfrak{B} then \mathcal{J} contains a completely primitive idempotent.*

PROOF. The MII theorem is proved in [5] under the assumption that \mathcal{J} is unital. Under this assumption any inner ideal is closed under squaring. However, in the proof of the MII theorem this is used only for minimal inner ideals of type (II) or (III) in which case, as we shall see, the results also hold even if \mathcal{J} is not unital. Let \mathfrak{B} be a minimal ideal containing $b \neq 0$ which is not an absolute zero divisor. Then $\mathcal{J}U_b = \mathfrak{B}$ by minimality and $(xU_b)^2 = b^2U_xU_b \in \mathfrak{B}$ for all $x \in \mathcal{J}$ and so $\mathfrak{B}^2 \subset \mathfrak{B}$. Thus the MII theorem holds even if \mathcal{J} is not unital.

Let \mathfrak{B} be a minimal inner ideal of \mathcal{J} . By nondegeneracy we must be in case (II) or (III) of the MII theorem. In case (III), $\mathfrak{B} = \mathcal{J}U_e, e$ a completely primitive idempotent, and we are done. It remains to show that having a minimal inner ideal \mathfrak{B} of type II implies the presence of a completely primitive idempotent in \mathcal{J} . The methods used are those of [5, p. 3.13]. We have $\mathfrak{B}U_{\mathfrak{B}} = \mathfrak{B}^2 = \{0\}$. There are two cases. Either there exist a $b \in \mathfrak{B}$ and an $a \in \mathcal{J}$ such that $aU_b = b$ and $a^2U_b \neq 0$, or for all $a \in \mathcal{J}$ such that $aU_b = b$ then $a^2U_b = 0$.

In the first case let $c = a^2U_b$. Since $c \in \mathfrak{B}$, there exists $b_0, c_0 \in \mathcal{J}$ such that $b_0U_c = b$ and $c_0U_c = c$. Let $e = c_0U_bU_a$. Then

$$e^2 = (c_0U_bU_a)^2 = a^2U_bU_{c_0}U_bU_a = cU_{c_0}U_bU_a = cU_{c_0}U_cU_{b_0}U_cU_a$$

which by the auxiliary lemma equals

$$cU_{b_0}U_cU_a = c_0U_cU_{b_0}U_cU_a = c_0U_{b_0c}U_a = c_0U_bU_a = e.$$

Now

$$\begin{aligned} eU_c &= c_0U_bU_aU_c = c_0U_bU_aU_{a^2U_b} = c_0U_bU_aU_bU_{a^2U_b} \\ &= c_0U_{aU_b}U_{a^2U_b} = c_0U_bU_{a^2U_b} = c_0U_{a^2U_b} = c_0U_c = c \neq 0. \end{aligned}$$

Therefore $e^2 = e \neq 0$ and $eU_c = c$. Also

$$\begin{aligned} cU_e &= cU_{c_0U_bU_a} = cU_aU_bU_{c_0}U_bU_a = a^2U_bU_aU_bU_{c_0}U_bU_a \\ &= a^2U_{aU_b}U_{c_0}U_bU_a = a^2U_bU_{c_0}U_bU_a = a^2U_{c_0U_b}U_a = (c_0U_bU_a)^2 = e^2 = e. \end{aligned}$$

Therefore $cU_e = e$, $U_e = U_eU_cU_e$ and $\mathcal{J}U_e = \mathcal{J}U_eU_cU_e \subset \mathfrak{B}U_e$. Thus $\mathcal{J}U_e = \mathfrak{B}U_e$. To show that $(\mathcal{J}U_e, U, e)$ is a Jordan division algebra (and hence that e is completely primitive) it suffices to show that for all $x \in \mathcal{J}U_e$, $x \neq 0$, $\mathcal{J}U_eU_x = \mathcal{J}U_e$. If $x \in \mathcal{J}U_e$, $x \neq 0$, then $x = dU_e$ for some $d \in \mathfrak{B}$, so $\mathcal{J}U_eU_x = \mathcal{J}U_eU_{dU_e} = \mathcal{J}U_eU_dU_e = \mathcal{J}U_{dU_e} = \mathcal{J}U_x \neq \{0\}$ by nondegeneracy. Now $\mathcal{J}U_eU_d \subset \mathfrak{B}$ and is a nonzero inner ideal. By minimality $\mathcal{J}U_eU_d = \mathfrak{B}$ and $\mathcal{J}U_eU_dU_e = \mathfrak{B}U_e = \mathcal{J}U_e$.

Consider now the second possibility, namely that given any $b \in \mathfrak{B}$, $b \neq 0$, then for any $a \in \mathcal{J}$, $aU_b = b$ implies $a^2U_b = 0$. Fix $b \in \mathfrak{B}$, $b \neq 0$, and choose $a \in \mathcal{J}$ such that $aU_b = b$. Let $d = bU_a$. By the auxiliary lemma $bU_d = d$ and $dU_b = b$. So $d^2U_b = 0$ and $d^2 = (bU_d)^2 = d^2U_bU_d = 0$. Since $b^2 = 0$, by QJ 30 of [5],

$$(b \circ d)^2 = b^2U_d + d^2U_b + bU_d \circ b = b \circ d$$

and by QJ 17 of [5],

$$\begin{aligned} bU_{b \circ d} &= -bU_{d,b} + bU_dU_b + bU_bU_d + bV_dU_bV_d \\ &= -b^2 \circ d + b + 0 + dV_bU_bV_d = 0 + b + dU_bV_bV_d \\ &= b + bV_bV_d = b. \end{aligned}$$

Therefore $b \circ d$ is a nonzero idempotent. By symmetry $dU_{b \circ d} = d$. Finally,

$$(b \circ d)U_b = dV_bU_b = dU_bV_b = bV_b = 2b^2 = 0,$$

so $(d + b \circ d)U_b = b$ and hence

$$\begin{aligned} 0 &= (d + b \circ d)^2U_b = (d^2 + bV_d^2 + (b \circ d)^2)U_b = (bV_d^2 + b \circ d)U_b \\ &= bV_d^2U_b = b(2U_d - V_{d^2})U_b = 2bU_dU_b = 2dU_b = 2b. \end{aligned}$$

Therefore $b = 0$, a contradiction. This completes the proof of Lemma 13.

Note that the idempotent e constructed above lies in the ideal of \mathcal{J} generated by \mathfrak{B} . Combining Lemma 13 and Theorem 9 we obtain:

THEOREM 14. *If \mathcal{J}/Φ is a simple nondegenerate Jordan algebra having a minimal inner ideal then \mathcal{J} is either unital and satisfies the DCC on principal inner ideals or is isomorphic to $\mathcal{K}(A, *)$, A a $*$ -simple associative algebra with involution having a minimal one-sided ideal.*

LEMMA 15. *If e is a completely primitive idempotent of \mathcal{J} let $\mathfrak{B} = \mathcal{J}_{11} + \mathcal{J}_{01} + [\mathcal{J}_{01}^2]_0$. Then \mathfrak{B} is the ideal of \mathcal{J} generated by e . Moreover \mathfrak{B} is simple.*

PROOF. Since \mathfrak{B} is contained in the ideal generated by e , it suffices to show that \mathfrak{B} is an ideal to prove the first part of the lemma. We need only consider the products of Peirce components landing in \mathcal{J}_{00} . By the Peirce relations, only $\mathcal{J}_{00}(\mathcal{J}_{01}^2)$ need be checked. By PD 5 of [4]

$$(x_{01}y_{01})z_{00} = [(x_{01}z_{00})y_{01}]_0 + [(y_{01}z_{00})x_{01}]_0 \in [\mathcal{J}_{01}^2]_0.$$

So \mathfrak{B} is an ideal.

We wish to show that \mathfrak{B} is simple. Note that the nondegeneracy of \mathcal{J} implies that of \mathfrak{B} . Indeed if $\mathfrak{B}U_b = \{0\}$ for some $b \in \mathfrak{B}$ then $\mathcal{J}U_aU_b = \mathcal{J}U_bU_aU_b \subset \mathfrak{B}U_b = \{0\}$ so $aU_b = 0$ for all $a \in \mathcal{J}$ and $b = 0$. Thus \mathfrak{B} is nondegenerate. Let \mathfrak{C} be a proper ideal of \mathfrak{B} . If $c \in \mathfrak{C}$ then its Peirce components $c_{11}, c_{01}, c_{00} \in \mathfrak{C}$ and $c_{11} = 0$ for otherwise $e \in \mathcal{J}_{11} \subset \mathfrak{C}$ and $\mathfrak{C} = \mathfrak{B}$. Therefore $\mathfrak{C} = \mathfrak{C}_{01} + \mathfrak{C}_{00}$ and $\mathfrak{C}_{01}\mathcal{J}_{01} \subset \mathfrak{C}_{00}$. In particular if $c \in \mathfrak{C}_{01}$ then $c^2 \in \mathfrak{C}_{00}$. For $c \in \mathfrak{C}_{01}$, the Peirce relations imply $\mathcal{J}_{11}U_c = \{0\}$ and $\mathcal{J}_{01}U_c = \{0\}$. Also $\mathfrak{B}_{00}U_c \subset \mathfrak{C}_{11} = \{0\}$, so $\mathfrak{B}_{00}U_c = \{0\}$, c^2 is an absolute zero-divisor of \mathfrak{B} . Hence $c^2 = \{0\}$ and $\mathfrak{C}_{01} = \{0\}$. We claim that $c = 0$. We already have $\mathfrak{B}_{00}U_c = \{0\}$. In general

$$xU_c = 2(xc)c - xc^2 = 2(xc)c, \quad x \in \mathcal{J}.$$

If $x \in \mathfrak{B}_{00} + \mathfrak{B}_{11}$, $xc \in \mathfrak{C}_{01}$ and $(xc)c \in \mathfrak{C}_{01}^2 = \{0\}$. If $x \in \mathcal{J}_{01}$, by PD 1 of [4, Lemma 3, p. 121],

$$2[xc]_{00}c = 2[xc]_{11}c + [c^2]_{11}x - [c^2]_{00}x = 0$$

since $[xc]_{11} = 0$. Thus $2(xc)c = 0$ for $x \in \mathfrak{B}$ and $\mathfrak{B}U_c = 0$. By nondegeneracy $c = 0$ and $\mathfrak{C}_{01} = \{0\}$.

So $\mathfrak{C} = \mathfrak{C}_{00}$. If $c \in \mathfrak{C}$, $c\mathcal{J}_{01} \subset \mathfrak{C}_{01} = \{0\}$ so $\mathfrak{C}\mathcal{J}_{01} = \{0\}$. By the Peirce relations, if $x \in \mathcal{J}_{01}$, $cU_x \in \mathfrak{C}_{11} = \{0\}$, and $cx^2 = 2(cx)x = 0$ since $cx = 0$. So $\mathfrak{C}\mathcal{J}_{01}^2 = \{0\}$ and by the Peirce relations $\mathfrak{C}\mathcal{J}_{11} = \{0\}$. Thus $\mathfrak{C}\mathfrak{B} = \{0\}$ and $c = 0$ by the nondegeneracy of \mathfrak{B} . We have shown that $\mathfrak{C} = \{0\}$ and therefore that \mathfrak{B} is a simple ideal of \mathcal{J} .

Clearly \mathfrak{B} must then be a minimal ideal of \mathcal{J} . We would like to show that \mathfrak{B} is spanned by minimal inner ideals of \mathcal{J} and ultimately that the socle of \mathcal{J} is a direct sum of simple components.

LEMMA 16. *Let e and \mathfrak{B} be as in the previous lemma. Then \mathfrak{B} is spanned by minimal inner ideals of \mathcal{J} .*

PROOF. $\mathfrak{B} = \mathcal{J}_{11} + \mathcal{J}_{01} + [\mathcal{J}_{01}^2]_0$ so \mathfrak{B} is spanned by $\mathcal{J}_{11}, \mathcal{J}_{11}U_{x_{01}}, \mathcal{J}_{11}U_{e+x_{01}}, x_{01} \in \mathcal{J}_{01}$. Since \mathcal{J}_{11} is a minimal inner ideal so are $\mathcal{J}_{11}U_{x_{01}}$ and $\mathcal{J}_{11}U_{e+x_{01}}$ by Lemma 11.

THEOREM 17. *If \mathcal{J}/Φ is a nondegenerate Jordan algebra then its socle \mathfrak{S} is an ideal which is the direct sum of simple ideals.*

PROOF. If e is a completely primitive idempotent of \mathcal{J} and \mathfrak{C} the ideal of \mathcal{J} it generates, since \mathfrak{C} is simple any completely primitive idempotent $g \in \mathfrak{C}$ also generates \mathfrak{C} .

If \mathfrak{B} is a minimal inner ideal of type (II), a completely primitive idempotent e was constructed in Lemma 13, and for $c \in \mathfrak{B}$, $c \neq 0$ chosen as in the proof of Lemma 13, we showed that $eU_c = c$. Let \mathfrak{C} be the ideal of \mathcal{J} generated by e . Then $c \in \mathfrak{C} \cap \mathfrak{B}$. So, by the minimality of \mathfrak{B} , $\mathfrak{B} \subset \mathfrak{C}$. Denote by \mathfrak{C}_e the ideal generated by a completely primitive idempotent e and let \mathfrak{S}_0 be the sum of all ideals \mathfrak{C}_e for completely primitive idempotents e . Since all \mathfrak{C}_e 's are simple by Lemma 15, \mathfrak{S}_0 is a sum of simple ideals. By Lemma 15 \mathfrak{C}_e is a sum of minimal ideals, hence $\mathfrak{C}_e \subset \mathfrak{S}$ and $\mathfrak{S}_0 \subset \mathfrak{S}$. If \mathfrak{B} is a minimal inner ideal then \mathfrak{B} is contained in some \mathfrak{C}_e and $\mathfrak{S} \subset \mathfrak{S}_0$. So $\mathfrak{S} = \mathfrak{S}_0$ and is thus a sum of simple ideals. It must therefore be a direct sum of simple ideals (since we are in the linear case).

4. Prime Jordan rings with nonzero socle. Recall that a Jordan ring \mathcal{J} is said to be *prime* if $\mathfrak{I}_1 U_{\mathfrak{I}_2} = 0$, \mathfrak{I}_i ideals of \mathcal{J} , implies \mathfrak{I}_1 or $\mathfrak{I}_2 = 0$ [14].

THEOREM 18. *If \mathcal{J}/Φ is a nondegenerate prime Jordan ring with nonzero socle \mathfrak{S} , then either \mathcal{J} is simple unital and satisfies DCC on principal inner ideals or \mathcal{J} is isomorphic to a Jordan subalgebra of $\mathcal{K}(A, *)$ containing $\mathcal{K}(F, *)$ or a subalgebra of A^+ containing F^+ , where A/Φ is a primitive associative algebra with nonzero socle, F the socle of A and in the first case $*$ is an involution. Conversely if \mathcal{J} is a Jordan subalgebra of $L_W(V)^+$ containing $F_W(V)$, or of $\mathcal{K}(L_V(V), *)$ containing $F_V(V)$, then \mathcal{J} is a nondegenerate prime Jordan algebra with nonzero socle.*

PROOF. Let \mathcal{J}/Φ be a nondegenerate prime Jordan algebra with nonzero socle \mathfrak{S} . \mathfrak{S} is a direct sum of simple ideals. By primeness this sum must have only one component and \mathfrak{S} is a simple ideal. By Lemma 13 \mathfrak{S} contains a completely primitive idempotent.

If \mathfrak{S} is unital, say with unit f , then $\mathcal{J} = \mathfrak{J}_{00} \oplus \mathfrak{J}_{01} \oplus \mathfrak{J}_{11}$ with respect to f , and $\mathfrak{J}_{01} \oplus \mathfrak{J}_{11} \subset \mathfrak{S}$. Hence $\mathfrak{S} = \mathfrak{J}_{11}$ and by primeness $\mathfrak{J}_{00} = \{0\}$. Therefore $\mathcal{J} = \mathfrak{S}$. In all cases when $\mathcal{J} = \mathfrak{S}$ we are done by Theorem 9.

Assume $\mathcal{J} \neq \mathfrak{S}$. Then \mathfrak{S} is not unital and $\mathfrak{S} \cong \mathcal{K}(F, *)$ or F^+ where $F = F_W(V)$ in the notation of §1. ($W = V$ in case $\mathfrak{S} = \mathcal{K}(F, *)$.) We will use the fact that if X is a finite dimensional subspace of V then there exists an idempotent $e \in \mathfrak{S}$ such that $we = w$ for all $w \in X$. This is clear if $\mathfrak{S} = F^+$ and is proved in [6] if $\mathfrak{S} = \mathcal{K}(F, *)$. Let $x \in \mathcal{J}$ and $v \in V$. Pick an idempotent $e \in \mathfrak{S}$ with $ve = v$ and define vx to be $v(x_1 + x_{01})$, x_1, x_{01} the Peirce

components of x with respect to e . This makes sense since $x_1, x_{01} \in \mathfrak{S}$. We must show that this definition is independent of the choice of e . Choose an idempotent $f \in \mathfrak{S}$ which acts as the identity on the subspace of V spanned by Ve and vx . Taking the Peirce decomposition of \mathcal{J} with respect to $e_1 = e$ and $e_2 = f - e$ one sees that $vx = v(x_{11} + x_{12}) = v(xU_f)$. Since f can be any idempotent of \mathfrak{S} as the identity on Ve and vx , if e' is an idempotent of \mathfrak{S} such that $ve' = v$ then f can be chosen to act as the identity on Ve, Ve', vx and $v(x_1(e') + x_{01}(e'))$. In this case $v(x_1(e') + x_{01}(e')) = v(xU_f) = vx$. It is easy to see that x acts linearly on V .

We must show that Jordan products are preserved under this action of \mathcal{J} on V . Let $x, y \in \mathcal{J}, v \in V$. Choose f an idempotent of \mathfrak{S} fixing $v, vx, vy, (vx)y, (vy)x$ and $v(xy)$. By the choice of f , we have $vx = v(x_1 + x_{01}) = vx_1$ and so on. Then

$$\frac{1}{2}((vx)y + (vy)x) = \frac{1}{2}((vx_1)y_1 + (vy_1)x_1) = v(x_1y_1)$$

and

$$v(xy) = v([xy]_1) = v(x_1y_1 + [x_{01}y_{01}]_1).$$

But

$$v[x_{01}y_{01}]_1 = v(x_{01}y_{01}) = \frac{1}{2}((vx_{01})y_{01} + (vy_{01})x_{01}) = 0$$

since $vx_{01} = 0 = vy_{01}$. Therefore $v(xy) = \frac{1}{2}((vx)y + (vy)x)$ and we have a homomorphism of \mathcal{J} into $(\text{End}_\Delta V)^+$.

To see that this homomorphism is actually into $L_W(V)^+$ we must define x^* . Let $w \in W, x \in \mathcal{J}$. Choose an idempotent $e^* \in F^*$ such that $e^*w = w$. Then define x^*w to be $(x_1 + x_{01})^*w$ where x_1 and x_{01} are the Peirce components of x in the Peirce decomposition of \mathcal{J} with respect to $e = (e^*)^*$. An argument similar to that given above shows that x^* is well defined and that $(vx, w) = (v, x^*w)$ for all $v \in V$ and $w \in W$.

Finally since the map $\mathcal{J} \rightarrow L_W(V)^+$ is a Jordan homomorphism its kernel \mathcal{K} is an ideal of \mathcal{J} which lies in the 0-Peirce component of \mathcal{J} with respect to any idempotent of \mathfrak{S} . Since any element of \mathfrak{S} lies in the 1-space of a suitable chosen idempotent of \mathfrak{S} we have $\mathcal{K}U_{\mathfrak{S}} = 0$ and $\mathcal{K} = 0$ by primeness.

To prove the converse, by Theorem 10, we need only consider $\mathcal{J} \supseteq F^+$ or $\mathcal{J} \supseteq \mathcal{K}(F, *)$ and only the primeness of \mathcal{J} need be shown. Let $\mathfrak{B} = F^+$ or $\mathcal{K}(F, *)$ as the case may be. If \mathcal{I} is an ideal of \mathcal{J} then $\mathcal{I} \cap \mathfrak{B} = \{0\}$ or \mathfrak{B} by the simplicity of \mathfrak{B} . If $\mathcal{I} \neq \{0\}$ then for $x \in \mathcal{I}, x \neq 0, vx \neq 0$ for some $v \in V$ and choosing $e \in \mathfrak{B}$ with $ve = v, vx = v(x_1 + x_{01})$ and hence $x_1 + x_{01} \neq 0$ and $x_1 + x_{01} \in \mathfrak{B} \cap \mathcal{I}$. Therefore, all nonzero ideals of \mathcal{J} contain \mathfrak{B} and \mathcal{J} is prime since $\mathfrak{B}U_{\mathfrak{B}} \neq \{0\}$. This completes the proof of Theorem 18.

We call a Jordan subalgebra of $\mathcal{K}(A, *)$ or A^+ *dense* if it contains $\mathcal{K}(F, *)$ or F^* , respectively. Using this term, we can restate the most important half of

Theorem 18 without using the notation developed in the first section of this paper.

COROLLARY 19. *If \mathcal{J}/Φ is a nondegenerate prime Jordan ring with nonzero socle \mathfrak{S} , then either \mathcal{J} is simple unital and satisfies DCC on principal inner ideals or \mathcal{J} is isomorphic to a dense Jordan subalgebra either of the plus algebra of a primitive associative algebra with nonzero socle or of the Jordan algebra of symmetric elements of such an algebra under some involution.*

We end with an example which shows that there do exist Jordan rings satisfying the hypotheses of Theorem 18 or Corollary 19 which are not of the form $\mathcal{K}(\mathfrak{B}, *)$ for some associative algebra \mathfrak{B} . Let \mathfrak{C} be a Jordan algebra determined by a nondegenerate quadratic form with base point. Assuming that the characteristic of the underlying field Φ is not 2 and that $\dim_{\Phi} \mathfrak{C}$ is odd > 3 then the special universal envelope of \mathfrak{C} is a Clifford algebra \mathcal{C} which is central simple. So $\mathcal{C} \cong M_n(\Delta)$, Δ a division algebra. \mathfrak{C} generates \mathcal{C} associatively. Let V be a countable vector space over Δ ; pick a basis and let $*$ be the $-$ transpose involution on A the row-finite matrices with respect to this basis, where $-$ is the involution of Δ induced by the canonical involution of \mathcal{C} . Embed \mathcal{C} in A by mapping each $n \times n$ matrix C of \mathcal{C} into that countable matrix in A which has C repeated down the diagonal ad infinitum, and which has zeros except in these diagonal blocks. Identify \mathcal{C} with this isomorphic subalgebra and \mathfrak{C} with the corresponding Jordan subalgebra of \mathcal{C}^+ . Let $\mathcal{J} = \mathfrak{C} + \mathfrak{S}$ where $\mathfrak{S} = \mathcal{K}(F, *)$, F the transformations of finite rank. Then \mathcal{J} generates $\mathcal{C} + F$ associatively but is properly contained in $\mathcal{K}(\mathcal{C} + F, *)$ provided $\dim_{\Phi} \mathfrak{C} > 5$.

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