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ON INVARIANT OPERATOR RANGES

BY

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ABSTRACT. A matricial representation is given for the algebra of operators leaving a given dense operator range invariant. It is shown that every operator on an infinite-dimensional Hilbert space has an uncountable family of invariant operator ranges, any two of which intersect only in $\{0\}$.

1. Introduction. By an *operator range* we mean a linear manifold in a Hilbert space $\mathcal{H}$ which is the range of some bounded linear operator on $\mathcal{H}$. Operator ranges have been studied in several contexts: the paper [6] of Fillmore and Williams contains an excellent account of the known results. Foiąs [7] proposed the study of the operator ranges invariant under given collections of operators. One of the many interesting results of [7] is a version of Burnside's theorem: if $\mathcal{Q}$ is an algebra of operators on $\mathcal{H}$ and the only operator ranges invariant under $\mathcal{Q}$ are $\{0\}$ and $\mathcal{H}$, then $\mathcal{Q}$ is strongly dense in $\mathfrak{B}(\mathcal{H})$ (this theorem is also discussed in [14]). Other results on invariant operator ranges can be found in [5], [9], [12] and [13].

There are two general questions about which little is known: given an algebra of operators, what can be said about its lattice of invariant operator ranges, and given a lattice of operator ranges what can be said about the operators which leave them invariant? We make a beginning on these questions by considering the cases of singly generated algebras and singly generated lattices.

Our first main result (Theorem 3) is a structure theorem for the algebra $\mathcal{Q}(P)$ of all operators leaving the range of an operator P invariant. We show that $\mathcal{Q}(P)$ is the sum of a certain algebra of upper triangular matrices and an algebra of lower triangular matrices relative to a decomposition of the space corresponding to certain spectral subspaces of P. We mention some consequences of this theorem below; another application can be found in [9].

In §3 of this paper we prove that every operator has a large number of invariant operator ranges.

2. The algebra of operators leaving a dense range invariant. In this section we consider the algebra of all operators which leave $P\mathcal{H}$ invariant, where P is

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any given operator whose range is dense in but not equal to $\mathcal{H}$. We can assume, with no loss of generality, that P is positive (by the polar decomposition). For a positive operator P with dense range we use the notation $\mathcal{A}(P)$ for the collection of all operators which take $P\mathcal{H}$ into itself. Our main result (Theorem 3) gives a representation of $\mathcal{A}(P)$. We begin with some easy properties of this algebra.

THEOREM 1. *The uniform closure of $\mathcal{A}(P)$ contains all compact operators.*

PROOF. If F is an operator of rank 1, then the rank-one operator PF is in $\mathcal{A}(P)$. Clearly $\mathcal{A}(P)$ is transitive (in the sense that there is no closed subspace invariant under $\mathcal{A}(P)$). Hence the result follows from the well-known result of Barnes [2] (also discussed in [15]).

The algebra $\mathcal{A}(P)$ has few invariant operator ranges in the following sense.

THEOREM 2. *If $B\mathcal{H}$ is invariant under $\mathcal{A}(P)$ and $B \neq 0$, then $B\mathcal{H} \supset P\mathcal{H}$.*

PROOF. Assume $Bx_0 \neq 0$. Then $PBx_0 \neq 0$ and $PBx_0 \in P\mathcal{H}$. For each Py there is an $A \in \mathcal{A}(P)$ such that $APBx_0 = Py$. Thus $P\mathcal{H} \subset \mathcal{A}(P)PBx_0 \subset \mathcal{A}(P)B\mathcal{H} \subset B\mathcal{H}$. In fact, as we see below (Corollary 2), $\mathcal{A}(P)$ does have invariant operator ranges other than $P\mathcal{H}$.

The next lemma is required for our main theorem; it was also found by D. O'Donovan and is probably known to others as well.

LEMMA 1. *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be a sequence of Hilbert spaces and $A = ((A_{ij}))$ be a matrix of operators (where $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i$ for each i, j). Suppose that there exists an infinite matrix of positive numbers $B = ((b_{ij}))$ such that $\|A_{ij}\| \leq b_{ij}$ for all i, j and such that B defines a bounded operator on l^2. Then A defines a bounded operator on $\mathcal{H} = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$.*

PROOF. Suppose that $x_i \in \mathcal{H}_i$ and $\sum_{i=1}^{\infty}\|x_i\|^2 = 1$. We claim that $\sum_i\|\sum_j A_{ij}x_j\|^2 \leq \|B\|^2$, which would imply that $\|A\| \leq \|B\|$. The vector $u = (\|x_1\|, \|x_2\|, \|x_3\|, \ldots)$ in l^2 has norm 1, so $\|Bu\|^2 \leq \|B\|^2$. Therefore

$$\sum_i \left\| \sum_j A_{ij}x_j \right\|^2 \leq \sum_i \left(\sum_j \|A_{ij}\| \, \|x_j\| \right)^2 \leq \sum_i \left(\sum_j b_{ij}\|x_j\| \right)^2 = \|Bu\|^2 \leq \|B\|^2.$$

The next lemma follows immediately from the result of Halmos and Douglas [4].

LEMMA 2. *If $A \in \mathcal{A}(P)$ then $P^{-1}AP$ is bounded.*

In order to state our structure theorem simply it is convenient to assume P has norm at most 1; since P can be divided by $\|P\|$, this involves no loss of generality. For P any positive noninvertible operator with dense range and norm at most 1, and λ any positive number less than 1, we form the algebra

$\mathfrak{T}(P, \lambda)$ as follows. Let the spectral measure of P be $E(\cdot)$. For $j = 1, 2, 3, \ldots$ let $\mathcal{H}_j = E((\lambda^j, \lambda^{j-1}])$ (some of the $\{\mathcal{H}_j\}$ may be $\{0\}$). Then $\mathfrak{T}(P, \lambda)$ is the algebra of all operators which are upper-triangular with respect to the decomposition $\sum_{j=1}^{\infty} \oplus \mathcal{H}_j$ of $\mathcal{H}$. That is, $\mathfrak{T}(P, \lambda)$ consists of those operators which leave the subspaces $\mathcal{H}_1, \mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \ldots$ invariant.

THEOREM 3. *If P is a noninvertible positive operator with dense range and norm at most* 1, *if $\lambda \in (0, 1)$, and if $\mathcal{Q}(P)$ and $\mathfrak{T}(P, \lambda)$ are formed as above, then $\mathcal{Q}(P) = \mathfrak{T}(P, \lambda) + (P^{-1}\mathfrak{T}(P, \lambda)P)^*$.*

(The theorem includes the fact that $B \in \mathfrak{T}(P, \lambda)$ implies that $P^{-1}BP$ is bounded.)

PROOF. Let $\mathcal{H} = \sum_{j=1}^{\infty} \oplus \mathcal{H}_j$ be the decomposition defining $\mathfrak{T}(P, \lambda)$ as above, and let $P = \sum_{j=1}^{\infty} \oplus P_j$ be the corresponding decomposition of P. Let $J = \{j\colon \mathcal{H}_j \neq \{0\}\}$; then $\lambda^j \leq P_j \leq \lambda^{j-1}$ for $j \in J$.

We begin by proving that $\mathfrak{T}(P, \lambda) \subset \mathcal{Q}(P)$. Let $B = ((B_{ij}))$ be any operator in $\mathfrak{T}(P, \lambda)$. The above gives $\|P_i^{-1}B_{ij}P_j\| \leq \lambda^{j-1}(1/\lambda^i)\|B_{ij}\|$, so $\|P_i^{-1}B_{ij}P_j\| \leq \lambda^{j-i-1}\|B\|$ $(i, j \in J)$.Now the operator C with matrix $((c_{ij}))$ where

$$c_{ij} = \begin{cases} \lambda^{j-i-1} & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases}$$

represents the adjoint of an analytic Toeplitz operator, and thus is bounded. Therefore so is its compression $((c_{ij}))_{i,j \in J}$, and so is $((c_{ij}\|B\|))_{i,j \in J}$. The above inequality and Lemma 1 show that $P^{-1}BP$ is a bounded operator D, and $BP = PD$ implies $B \in \mathcal{Q}(P)$.

We next show that $(P^{-1}\mathfrak{T}(P, \lambda)P)^* \subset \mathcal{Q}(P)$. Note that we showed above that $P^{-1}\mathfrak{T}(P, \lambda)P$ consists of bounded operators. Thus $P(\mathfrak{T}(P, \lambda))^*P^{-1}$ consists of bounded operators, and obviously $P^{-1}(P(\mathfrak{T}(P, \lambda))^*P^{-1})P \subset B(\mathcal{H})$, so, as above, we conclude that $P(\mathfrak{T}(P, \lambda))^*P^{-1} \subset \mathcal{Q}(P)$.

Hence $\mathfrak{T}(P, \lambda) + (P^{-1}\mathfrak{T}(P, \lambda)P)^* \subset \mathcal{Q}(P)$. For the other inclusion, let $A \in \mathcal{Q}(P)$. Write A as a matrix $((A_{ij}))$ with respect to the decomposition $\mathcal{H} = \sum \oplus \mathcal{H}_j$. For each $i, j \in J$ with $i > j$,

$$\|A_{ij}\| = \|P_i P_i^{-1} A_{ij} P_j P_j^{-1}\| \leq \lambda^{i-j-1}\|P_i^{-1}A_{ij}P_j\| \leq \lambda^{i-j-1}\|P^{-1}AP\|$$

(note that $A \in \mathcal{Q}(P)$ implies $P^{-1}AP$ is bounded, by Lemma 2). Thus the "lower triangular part" of A, say A', whose matrix $((A'_{ij}))$ is defined by

$$A'_{ij} = \begin{cases} A_{ij} & \text{if } i > j, \\ 0 & \text{if } i \leq j \end{cases}$$

is bounded (by Lemma 1). Clearly $A - A' \in \mathfrak{T}(P, \lambda)$, and we already know that $\mathfrak{T}(P, \lambda) \subset \mathcal{Q}(P)$. Thus $A' = A - (A - A')$ is in $\mathcal{Q}(P)$, $P^{-1}A'P$ is bounded and $(P^{-1}A'P)^* \in \mathfrak{T}(P, \lambda)$. Thus $A \in \mathfrak{T}(P, \lambda) + (P^{-1}\mathfrak{T}(P, \lambda)P)^*$.

Theorem 1 above discussed the uniform closure of $\mathcal{Q}(P)$.

COROLLARY 1. *The algebra $\mathcal{Q}(P)$ is neither uniformly dense in $B(\mathcal{H})$ nor uniformly closed.*

PROOF. The proof of Theorem 3 shows that $A \in \mathcal{Q}(P)$ implies $\|A_{2n,n}\| \leq \lambda^{n-1}\|P^{-1}AP\|$ (where $((A_{ij}))$ is the decomposition of A relative to $\Sigma_{i=1}^{\infty} \oplus \mathcal{H}_i$). Let $i_1, i_2, \ldots$ be the enumeration of J such that $i_1 < i_2 < \ldots$. Define B by any matrix $((B_{ij}))$, where $\|B_{i_{2n}, i_n}\| = 1$ for all n and $\|B_{ij}\| = 0$ otherwise. Then $\|B - A\| \geq 1$ for all $A \in \mathcal{Q}(P)$ since $\lim_{n\to\infty}\|A_{i_{2n}, i_n}\| = 0$. Thus $\mathcal{Q}(P)$ is not dense. Also $\mathcal{Q}(P)$ is never uniformly closed: if F is an operator of rank 1 whose range is not in $P\mathcal{H}$, then $F \notin \mathcal{Q}(P)$ (but F is in the closure of $\mathcal{Q}(P)$ by Theorem 1 above).

COROLLARY 2. *If $0 < r < s$ then $\mathcal{Q}(P^r) \supset \mathcal{Q}(P^s)$. In particular, for each $t \in (0, 1)$ the range of P^t is invariant under $\mathcal{Q}(P)$.*

PROOF. For each $\lambda \in (0, 1)$, the decomposition of $\mathcal{H}$ obtained in forming $\mathcal{T}(P, \lambda)$ is the same as the decomposition obtained in forming $\mathcal{T}(P^t, \lambda^t)$ for any $t > 0$; hence $\mathcal{T}(P^t, \lambda^t) = \mathcal{T}(P, \lambda)$ for all $t > 0$. By Theorem 3, then, it suffices to show that $P^{-r}\mathcal{T}(P, \lambda)P^r \supset P^{-s}\mathcal{T}(P, \lambda)P^s$. But $(P^{-(s-r)}\mathcal{T}(P, \lambda)P^{s-r})^* \subset \mathcal{Q}(P^{s-r})$ so we know that $P^{(s-r)}(\mathcal{T}(P, \lambda))^*P^{-(s-r)}$ consists of bounded operators. Since P^{s-r} is diagonal with respect to $\Sigma_{i=1}^{\infty} \oplus \mathcal{H}_i$, we have that $P^{(s-r)}(\mathcal{T}(P, \lambda))^*P^{-(s-r)} \subset (\mathcal{T}(P, \lambda))^*$, or $P^{(r-s)}\mathcal{T}(P, \lambda)P^{(s-r)} \subset \mathcal{T}(P, \lambda)$. Hence $P^{-s}\mathcal{T}(P, \lambda)P^s \subset P^{-r}\mathcal{T}(P, \lambda)P^r$.

REMARK. There is a more general result than Corollary 2. It was shown by Foiaş [7, Chapter II, Proposition 5] that if the range of the positive operator P is invariant under a closed subalgebra $\mathfrak{S}$ of $\mathfrak{B}(\mathcal{H})$, and if ϕ is a continuous, concave, nondecreasing function on $[0, \|P\|^2]$, then the range of $(\phi(P^2))^{1/2}$ is invariant under $\mathfrak{S}$. Foiaş' proof can be modified to apply in the case where $\mathfrak{S}$ is not closed. (Let $\|B\|_p = \sup\{\|PBx\|: x \in \mathcal{H}, \|Px\| \leq 1\}$ for every B. Then [4] implies that $\|B^*\|_p < \infty$ if and only if the range of P is invariant under B. Thus the lemma of J. Peetre, as quoted in [7, p. 895], immediately yields the above.) Given this, the proof of Foiaş' Proposition 7 goes through even when $\mathfrak{S}$ is not closed. Hence if $\mathfrak{S}$ is any operator algebra whose lattice of invariant operator ranges is totally ordered it follows that every invariant operator range is closed and Lat $\mathfrak{S}$ is well-ordered from above. Consequently $\mathcal{Q}(P)$ includes no subalgebra with a totally ordered lattice of invariant operator ranges.

3. Existence of invariant operator ranges for single operators. We show that every operator has an uncountable set of invariant operator ranges, any pair of which intersect only in $\{0\}$. This is very different from the situation for invariant subspaces!

The proof will be given after establishing the existence of certain $\mathcal{H}^{\infty}$

functions with prescribed boundary behavior. For $|z| < 1$, let $H_z(e^{i\theta})$ be the Herglotz kernel for evaluation at z:

$$H_z(e^{i\theta}) = (e^{i\theta} + z)/(e^{i\theta} - z).$$

Let $P_z(e^{i\theta})$ be the Poisson kernel,

$$P_z(e^{i\theta}) = \operatorname{Re} H_z(e^{i\theta}).$$

LEMMA 3. *If $\rho(x) = 1/x^p + 1/(2\pi - x)^p$ for $0 < x < 2\pi$ and $\frac{1}{2} < p < 1$, and if ϕ in $\mathcal{H}^\infty$ is such that $|\phi(e^{i\theta})| \leq 1/\rho(\theta)$ a.e., then, for every f in $\mathcal{H}^2$,*

$$\lim_{r\to 1-} \phi(r)f(r) = 0.$$

PROOF. Let ϕ_0 and f_0 be the outer parts of ϕ and f, respectively. For $|z| < 1$, $|\phi(z)f(z)| \leq |\phi_0(z)f_0(z)|$. By the arithmetic-geometric mean inequality, applied to the measure $(1/2\pi)P_z(e^{i\theta})\,d\theta$ and the function $|\phi(e^{i\theta})f(e^{i\theta})|$,

$$|\phi_0(z)f_0(z)| = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} P_z(e^{i\theta})\log|\phi(e^{i\theta})f(e^{i\theta})|\,d\theta\right]$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} P_z(e^{i\theta})|\phi(e^{i\theta})f(e^{i\theta})|\,d\theta.$$

Thus the hypothesis yields

$$|\phi(z)f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} P_z(e^{i\theta})\,\frac{|f(e^{i\theta})|}{\rho(\theta)}\,d\theta.$$

Denoting the function on the right-hand side of the preceding inequality by $\psi(z)$, we see that it suffices to prove $\lim_{r\to 1-}\psi(r) = 0$.

Let ρ be extended periodically to the entire real line, and let

$$F(\theta) = \int_{-\pi}^{\theta} \frac{|f(e^{it})|}{\rho(t)}\,dt.$$

Then,

$$\psi(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_z(e^{i\theta})\,dF(\theta),$$

and, by Fatou's theorem, $\lim_{r\to 1-}\psi(re^{i\theta}) = F'(\theta)$ wherever $F'(\theta)$ exists. We need only show that $F'(0) = 0$. For $t > 0$ Hölder's inequality implies

$$\frac{F(t) - F(0)}{t} = \frac{1}{t}\int_0^t \frac{|f(e^{ix})|}{\rho(x)}\,dx$$

$$\leq \frac{1}{t}\left(\int_0^t |f(e^{ix})|^2\,dx\right)^{1/2}\left(\int_0^t \frac{1}{\rho(x)^2}\,dx\right)^{1/2}.$$

Since $1/\rho(x) \leqslant x^p$, and since

$$\left(\int_0^t |f(e^{ix})|^2\,dx\right)^{1/2} \leqslant \sqrt{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(e^{ix})|^2\,dx\right)^{1/2} = \sqrt{2\pi}\,\|f\|_2,$$

we obtain

$$\begin{aligned}\frac{F(t)-F(0)}{t} &\leqslant \frac{1}{t}\sqrt{2\pi}\,\|f\|_2\left(\int_0^t x^{2p}\,dx\right)^{1/2}\\ &= \sqrt{2\pi}\,\|f\|_2 t^{(2p-1)/2}/\,(2p+1).\end{aligned}$$

It is easy to see that for $t < 0$ the same inequality holds if on the right-hand side t is replaced by $|t|$. The last term above has limit zero as $t \to 0$, since $p > \frac{1}{2}$. Thus $F'(0) = 0$, and the proof of the lemma is complete.

LEMMA 4. *Let γ be a proper closed subarc of the unit circle and let γ' be its complementary arc. There exists an outer $\mathcal{H}^\infty$ function ϕ such that*

(a) *ϕ has a continuous extension to γ', and*

(b) *if $f \in \mathcal{H}^2$ and $f \neq 0$, then ϕf cannot be continuously extended to any open subarc of γ.*

PROOF. Let $\{\theta_n\}$ be a sequence such that $\{e^{i\theta_n}: n = 1, 2, \ldots\}$ is dense in γ, and let $\rho_n(x) = \rho(x - \theta_n)$, where ρ is the function of the preceding lemma extended by periodicity. Let $\{a_n\}$ be a summable sequence of positive numbers and put $\sigma(x) = \sum_{n=1}^{\infty} a_n\rho_n(x)$. Since $\{a_n\}$ is summable and all the ρ_n have the same $L^1(0, 2\pi)$ norm, it follows from the monotone convergence theorem that the preceding series converges a.e., and σ is in $L^1(0, 2\pi)$. Thus $\log 1/\sigma$ is also in $L^1(0, 2\pi)$, and it is bounded above.

We claim that σ has a continuous derivative on any compact subset K of $(0, 2\pi)$ that is at a positive distance δ from $\{\theta_n: n = 1, 2, \ldots\}$. For it is easy to see that, on K, $|\rho_n'(x)| \leqslant |\rho'(\delta)|$. Consequently, summability of $\{a_n\}$ implies that $\sum_{n=1}^{\infty} a_n\rho_n'(x)$ converges uniformly and absolutely on K. Thus σ has a continuous derivative on K.

Since $\log 1/\sigma$ is integrable and bounded above, we can define an outer $\mathcal{H}^\infty$ function ϕ by

$$\phi(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} H_z(e^{i\theta})\log\frac{1}{\sigma(\theta)}\,d\theta\right],$$

and

$$|\phi(e^{i\theta})| = 1/\sigma(\theta) \qquad \text{a.e.}$$

Because of the differentiability of σ on the complement of $\{\theta_n: n = 1, 2, \ldots\}^-$, it follows that ϕ has a continuous extension to γ' (cf. [**8**, p. 79]).

It remains to show that if $f \in \mathcal{H}^2$ and ϕf has a continuous extension to an open subarc of γ, then $f = 0$. We will verify that, for each n,

$$\lim_{r \to 1-} \phi(re^{i\theta_n}) f(re^{i\theta_n}) = 0. \tag{$*$}$$

For this will imply that if ϕf has a continuous extension to an open subarc of γ, then it vanishes identically on that subarc. The F. and M. Riesz theorem then implies ϕf is identically zero, from which it follows that $f = 0$.

To verify ($*$) we define ϕ_n and f_n by

$$\phi_n(z) = \phi(ze^{i\theta_n})$$

and

$$f_n(z) = f(ze^{i\theta_n}),$$

so ($*$) becomes

$$\lim_{r \to 1-} \phi_n(r) f_n(r) = 0.$$

Since

$$|\phi(e^{i\theta})| = 1/\sigma(\theta) \leq 1/a_n \rho_n(\theta) \qquad \text{a.e.,}$$

we have

$$|\phi_n(e^{i\theta})| = |\phi(e^{i(\theta+\theta_n)})| \leq 1/a_n \rho_n(\theta + \theta_n) = 1/a_n \rho(\theta).$$

Thus the desired limit relation follows from an application of Lemma 3 to $a_n \phi_n$.

THEOREM 4. *The unilateral shift operator has an uncountable set of dense invariant operator ranges each pair of which intersect in* $\{0\}$.

PROOF. Let $a \in (0, 1)$ and define the operator D on $\mathcal{H}^2$ by

$$D \sum_0^\infty b_n z^n = \sum_0^\infty a^n b_n z^n.$$

Thus, with respect to the usual basis for $\mathcal{H}^2$, D is the diagonal operator determined by the sequence $\{a^n\}_{n=0}^\infty$. Clearly D is in the trace class, and a simple calculation shows that if S is the unilateral shift on $\mathcal{H}^2$, then $SD = (1/a) DS$. Further, the range of D contains all the functions z^n, is dense, and consists of functions which are analytic at least on the disc of radius $1/a$.

Choose a proper closed subarc γ of the unit circle and let ϕ be a corresponding function defined as in Lemma 4. Let T_ϕ be the analytic Toeplitz operator determined by ϕ, and let $A_\gamma = T_\phi D$. Then

$$SA_\gamma = ST_\phi D = T_\phi SD = (1/a) T_\phi DS = (1/a) A_\gamma S,$$

and consequently the range of A_γ is invariant under S. Since ϕ is outer, the range of T_ϕ is dense. Thus A_γ is the product of two operators each having dense range; hence the range of A_γ is also dense. From Lemma 4 and from the fact that functions in the range of D are analytic on the closed unit disc, it follows that all nonzero functions in the range of A_γ have continuous extensions to γ' and cannot be continuously extended to any open subarc of γ. Thus distinct arcs give rise to operators whose ranges are invariant under S and intersect only in the trivial subspace $\{0\}$. Clearly there are uncountably many distinct arcs γ, and thus the proof of Theorem 4 is complete.

The following appears as Proposition 4 of **[16]** and is a special case of the "Intertwining Lemma" of [3]. We include a proof for completeness.

LEMMA 5. *If $\|T\| < 1$ and T has a cyclic vector, then there exists an injective dense-range operator X such that $XS = TX$, where S is the unilateral shift.*

PROOF. Let f be a cyclic vector for T. Define the operator X: $\mathcal{H}^2 \to \mathcal{H}$ by

$$X\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} a_n T^n f.$$

Since

$$\left\|\sum a_n T^n f\right\| \leq \sum |a_n|\, \|T^n\|\, \|f\| \leq \left(\sum |a_n|^2\right)^{1/2}\left(\sum \|T\|^{2n}\right)^{1/2} \|f\|,$$

X is bounded with norm at most $(\Sigma\|T\|^{2n})^{1/2}\|f\|$. Then $XSz^n = Xz^{n+1} = T^{n+1}f = TXz^n$, so $XS = XT$. Obviously the range of X is dense, so we need only show that X is injective. If $\Sigma a_n T^n f = 0$ for $\{a_n\} \in l^2$ and $\{a_n\} \neq 0$, then $\psi(T) = 0$ for ψ the analytic function $\psi(z) = \Sigma_{n=0}^{\infty} a_n z^n$, because f is a cyclic vector (since $\|T\| < 1$, $\psi(T)$ is defined by the Riesz functional calculus). Now ψ has at most finitely many zeros in $\{z\colon |z| \leq \|T\|\}$; let p be the product of the corresponding linear factors and let $\phi = \psi/p$. Then $1/\phi$ is analytic in $\sigma(T)$, so $\phi(T)$ is invertible. Thus $\psi(T) = 0$ implies $p(T) = 0$. (We have proven the folk result that an "analytically zero operator" is algebraic.) But this contradicts the fact that T is cyclic.

THEOREM 5. *Every cyclic operator has an uncountable set of dense invariant operator ranges each pair of which intersect in* $\{0\}$.

PROOF. Let $\{A_\gamma\}$ be an uncountable collection of operators whose ranges are invariant under the unilateral shift S and are such that any pair intersects only in $\{0\}$; Theorem 4 gives the existence of such a collection. Any given operator T has the same invariant linear manifolds as its multiples, so we can assume $\|T\| < 1$. Let T be cyclic, and using Lemma 5, choose an injective operator X with dense range such that $XS = TX$. Then, for each γ, $T(XA_\gamma\mathcal{H}) = XSA_\gamma\mathcal{H} \subset XA_\gamma\mathcal{H}$. Thus the ranges of the operators $\{XA_\gamma\}$ satisfy the conclusion of the theorem.

Theorem 5 need not hold for noncyclic operators. For example, if F is a projection of rank 1 then every dense linear manifold invariant under F

contains the range of F. If we do not require density, however, the result still holds.

THEOREM 6. *Every operator has an uncountable collection of infinite-dimensional invariant operator ranges each pair of which intersects in* $\{0\}$.

PROOF. Let T be a given operator. If $\bigvee_{n=0}^{\infty}\{T^n f\}$ is finite-dimensional for all vectors f then T is locally algebraic, and Kaplansky's well-known theorem **[10]** implies T is algebraic. Then T has an infinite-dimensional eigenspace $\mathfrak{M}$; in this case any uncountable collection of operator ranges contained in $\mathfrak{M}$ which have trivial pairwise intersection will serve.

If $\bigvee_{n=0}^{\infty}\{T^n f\}$ is infinite-dimensional for some f, then Theorem 5 applied to the restriction of T to $\bigvee_{n=0}^{\infty}\{T^n f\}$ gives the result.

The following theorem shows that the lattice of invariant operator ranges for an operator is even richer than the above indicates.

THEOREM 7. *Let $\mathfrak{M}$ be any infinite-dimensional operator range invariant under T. Then there exist uncountably many invariant operator ranges for T, all included in $\mathfrak{M}$, each pair of which intersect in* $\{0\}$.

PROOF. We can assume, with no loss of generality, that $\overline{\mathfrak{M}} = \mathfrak{H}$ and that $\mathfrak{M} = K\mathfrak{H}$ with K injective. Thus $TK = KX$ for some operator X. Now X has uncountably many invariant operator ranges $\mathfrak{L}_\alpha$ as in Theorem 6; then each $K\mathfrak{L}_\alpha$ is an invariant operator range for T. Also, by the injectivity of K, we have $K\mathfrak{L}_\alpha \cap K\mathfrak{L}_\beta = \{0\}$ for $\alpha \neq \beta$.

4. Remarks and questions. (i) The above results establish the existence of a wealth of compact operator ranges invariant under a given operator A, and thus invariant under all polynomials in A. Of course, these ranges do not have to be invariant under the weakly (or even uniformly) closed algebra $\mathfrak{A}$ generated by A. The question arises: When does $\mathfrak{A}$ leave a nonzero compact operator range invariant? The answer should be interesting in view of the conjecture given in **[11]** which states that if $\mathfrak{A}$ leaves a compact operator range invariant, then it has a nontrivial invariant subspace.

(ii) Foiaş [7] calls an operator range a "strange" invariant operator range for the operator T if it is invariant under the commutant $\mathfrak{A}$ of T, but is not the range of any operator in $\mathfrak{A}$. He shows that the unilateral shift, its adjoint, and certain $C_0(1)$ operators have such operator ranges. At least in the case of the backward shift our structure result (Theorem 3) yields an easier proof of existence: Represent the backward shift T by an upper triangular matrix and let $\mathfrak{A}$ be the commutant of T. With the notation of Theorem 3 we have $\mathfrak{A} \subseteq \mathfrak{T}(P, \lambda) \subseteq \mathfrak{A}(P)$, where P is the diagonal compact operator with eigenvalues λ^n, and where λ is an arbitrary number in (0, 1). Thus $\mathfrak{A}$ leaves the range of P invariant. But the only compact operator commuting with $\mathfrak{A}$ is 0, and hence $P\mathcal{H}$ is not the range of a commuting operator.

There exist operators with no "strange" invariant operator ranges; all selfadjoint projections, for example, have this property. The following question seems to be unsettled: Exactly which operators have "strange" operator ranges?

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