DIXMIER'S REPRESENTATION THEOREM OF CENTRAL DOUBLE CENTRALIZERS ON BANACH ALGEBRAS

BY

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ABSTRACT. The present paper is devoted to a representation theorem of central double centralizers on a complex Banach algebra with a bounded approximate identity. In particular, our result implies the representation theorem of the ideal center of an arbitrary C*-algebra established by J. Dixmier.

1. Introduction. R. C. Busby [2] has noted that every central double centralizer on any C*-algebra can be represented as a bounded continuous complex-valued function on its structure space, which is equivalent to Dixmier's representation theorem [6, Theorem 5]. Let A be a complex Banach algebra with a bounded approximate identity and Prim A the structure space of A. A central double centralizer T on A may be identified with a bounded linear operator T on A such that (Tx)y = x(Ty) for all x, y ∈ A. In this paper, we show that, if the ideal center of A has a Hausdorff structure space, every central double centralizer T on A can be represented as a bounded continuous complex-valued function ΦT on Prim A such that Tx + P = ΦT(P)(x + P) for all x ∈ A and P ∈ Prim A. Here x + P for P ∈ Prim A denotes the canonical image of x in A/P. In particular, if A is a C*-algebra then our representation theorem implies the Busby's or, equivalently, Dixmier's. In this way we get a proof of Dixmier's theorem, quite different from that given in [6]. This was inspired by Davenport's representation theorem of multiplier algebras on Banach algebras with bounded approximate identity [5, Theorem 2.8]. We also obtain a similar representation theorem of central double centralizers on a quasi-central Banach algebra with a completely regular center.

2. Davenport's representation theorem of Z(M(A)). In this paper, a complex Banach algebra with a bounded approximate identity will be denoted by A and the central double centralizer-algebra on A will be denoted by Z(M(A)), that is the center of the double centralizer-algebra M(A) on A. Let \{e_α\} be the approximate identity on A and A# the set of all elements f in the dual space A* of A such that \lim_α\|f·e_α - f\| = 0, where f·a(x) = f(ax) for...
each $a, x \in A$ and $f \in A^*$. The set $A^\#$ is a closed subspace of $A^*$ and $A^\# = \{ f \cdot a : f \in A^*, a \in A \}$ (cf. [3], [5]). Then the dual space $(A^\#)^*$ of $A^\#$ becomes a Banach algebra under the restriction to $A^\#$ of the Arens product on the second dual space $A^{**}$ of $A$. In fact, the restriction to $A^\#$ of the Arens product on $A^{**}$ can be described as follows:

$$[G, f](a) = G(f \cdot a), \quad F \cdot G(f) = F[G, f]$$

for each $a \in A, f \in A^\#$ and $F, G \in (A^\#)^*$. Let $\pi$ be the canonical embedding of $A$ into $A^{**}$ and $i$ the inclusion map of $A^\#$ into $A^*$. Put $\tau = i^* \pi$, where $i^*$ denotes the dual map induced by $i$. Then the map $\tau$ is a norm reducing isomorphism of $A$ into $(A^\#)^*$ [5, Lemma 2.5]. Furthermore, we can easily verify that $\|x\| \leq M\|\tau(x)\|$ for each $x \in A$, where $M$ is the bound on $\{e_a\}$. Then $\tau(A)$ is uniformly closed in $(A^\#)^*$. We now define $D(A)$ to be the set

$$D(A) = \{ F \in (A^\#)^*: F \cdot \tau(A) + \tau(A) \cdot F \subset \tau(A) \}.$$  

Then $D(A)$ is a Banach subalgebra of $(A^\#)^*$ and $\tau(A)$ is a closed two-sided ideal of $D(A)$.

**Definition 2.1.** Let $Z(D(A))$ be the center of $D(A)$. The algebra $Z(D(A))$ is said to be the ideal center of $A$.

By Lemma 2.7 in [5], the algebra $(A^\#)^*$ has an identity $J$. If $A$ is a $C^*$-algebra, then $A^* = A^\#$ and so $(A^\#)^* = A^{**}$ (cf. [5, Proof of Corollary 2.10]). Then $Z(D(A))$ becomes the ideal center of $A$ in the sense of Dixmier [6].

J. Davenport has proved that there exists a continuous, algebraic isomorphism $\mu$ of $Z(M(A))$ into $(A^\#)^*$ such that $\tau(Tx) = (\mu T) \cdot \tau x = \tau x \cdot (\mu T)$ for all $x \in A$ and $T \in Z(M(A))$ [5, Theorem 2.8]. In fact, the map $\mu$ is given by

$$\mu T = \lim_{a} \tau(Te_a)$$

for each $T \in Z(M(A))$. We then, from this Davenport result, obtain the following

**Lemma 2.2.** Let $A$ be a Banach algebra with a bounded approximate identity $\{e_a\}$ and $\mu$ the isomorphism of $Z(M(A))$ into $(A^\#)^*$ given by Davenport. Then $\mu(Z(A^\#)) = Z(D(A))$.

**Proof.** Let $T \in Z(M(A))$. Since $\tau(A) \cdot (\mu T) = (\mu T) \cdot \tau(A) = \tau(T(A)) \subset \tau(A)$, we have $\mu T \in D(A)$. It is well known that $\pi(A)$ is weak*-dense in $A^{**}$. Moreover, $i^*$ is a weak*-continuous map of $A^{**}$ onto $(A^\#)^*$. These facts imply that $\tau(A)$ is weak*-dense in $(A^\#)^*$. Then for each $F \in (A^\#)^*$, there exists a net $\{x_\lambda\}$ in $A$ such that $F = \lim_\lambda \tau(x_\lambda)$. Let $T^*$ be the dual map induced by $T$. Note that $T^*(f \cdot a) = f \cdot (Ta)$ for each $a \in A$ and $f \in A^*$, so
that \( T^*(A^*) \subset A^* \). Put \( T^* = T^*|A^* \). Then we have

\[
\lim_{\lambda} \tau(Tx_{\lambda})(f) = \lim_{\lambda} \tau(x_{\lambda})(T^*f) = F(T^*f)
\]

\[
= \lim_{\lambda} F(T^*(f \cdot e_\alpha)) = \lim_{\lambda} \tau(\tau_\alpha(f)) \tau_\alpha(f)
\]

\[
= (\mu T)[F, f] = (\mu T) \cdot F(f)
\]

for all \( f \in A^* \). On the other hand, we have

\[
\lim_{\lambda} \tau(Tx_{\lambda})(f) = \lim_{\lambda} \tau(x_{\lambda}) \cdot (\mu T)(f)
\]

\[
= \lim_{\lambda} \tau(x_{\lambda})[\mu T, f] = F[\mu T, f] = F(\mu T)(f)
\]

for all \( f \in A^* \). We thus obtain \((\mu T) \cdot F = F \cdot (\mu T)\), so that \( \mu T \in Z(D(A)) \).

Conversely, let \( G \in Z(Z(A)) \). Put \( \hat{G}(x) = \tau^{-1}(G \cdot \tau x) \) for each \( x \in A \). Then \( \hat{G} \) is a bounded linear operator on \( A \). Furthermore, \( \hat{G} \in Z(M(A)) \).

Proposition 2.3. Let \( A \) be a Banach algebra with a bounded approximate identity \( \{e_\alpha: \alpha \in \Lambda\} \). Then there exists a continuous, algebraic isomorphism \( \nu \) of the double centralizer-algebra \( M(A) \) on \( A \) onto \( D(A) \) which is an extension of \( \mu \) to \( M(A) \).

Proof. Let \((T, S) \in M(A)\). Since \( \{\tau(\tau_{\alpha_\alpha}): \alpha \in \Lambda\} \) is bounded, it has a weak*-convergent subnet \( \{\tau(\tau_{\alpha_\alpha}): \alpha' \in \Lambda'\} \) in \( (A^*)^* \). We now define the map \( \nu \) from \( M(A) \) to \( (A^*)^* \) by

\[
\nu(T, S) = \lim_{\alpha} \tau(\tau_{\alpha_\alpha}) = \lim_{\alpha} G \tau_{\alpha_\alpha} = G(f)
\]

for all \( f \in A^* \). We thus get \( \nu \hat{G} = G \) and our result is proved.

Furthermore, the following stronger result can be proved, and it is similar to one established by K. Saito [7], in which he has given a characterization of double centralizer-algebras on Banach algebras under some conditions. This was pointed out by the referee.
Similarly, $G \cdot \tau x(f) = \tau(Tx)(f)$ for each $x \in A$ and $f \in A^*$. Hence $(F - G) \cdot \tau x = 0$ for each $x \in A$. It follows by [5, 2.6.3] that $F = G$. Clearly, $\nu$ is linear. Also, since $\nu(T, S) \cdot \tau x = \tau(Tx)$ for each $x \in A$ as was seen in the above argument, we have $\nu(T, S) \cdot \tau(A) \subset \tau(A)$. Furthermore, we have
\[
\tau x \cdot \nu(T, S)(f) = \lim_{a} \tau x \cdot (Te_a)(f) \quad \text{(from [5, 2.6.2])}
\]
\[
= \lim_{a} \tau((Sx)e_a)(f) = \lim_{a} f((Sx)e_a)
\]
\[
= f(Sx) = \tau(Sx)(f)
\]
for each $x \in A$ and $f \in A^*$. Then $\tau x \cdot \nu(T, S) = \tau(Sx)$ for each $x \in A$, so that $\tau(A) \cdot \nu(T, S) \subset \tau(A)$. In other words, $\nu(M(A)) \subset D(A)$. Now let $(T, S), (T', S') \in M(A)$. Set $F = \nu(T, S), F' = \nu(T', S')$ and $F'' = \nu(T'T', S'S)$. Then $F'' \cdot \tau x = \tau(T'T'x)$ for each $x \in A$. On the other hand, for each $x \in A, F \cdot F' \cdot \tau x = F \cdot \tau(T'x) = \tau(T'T'x)$, and hence $(F'' - F \cdot F') \cdot \tau x = 0$. It follows by [5, 2.6.3] that $F'' = F \cdot F'$. In other words, $\nu((T, S)(T', S')) = \nu(T, S) \cdot \nu(T', S')$.

If $F = F'$, then $\tau((T - T')x) = (F - F') \cdot \tau x = 0$ for each $x \in A$, and so $T = T'$ (and hence $S = S'$) since $\tau$ is one-to-one. Thus $\nu$ is an algebraic isomorphism of $M(A)$ into $D(A)$. To show that $\nu$ is onto, let $F \in D(A)$ and set
\[
L_F(x) = \tau^{-1}(F \cdot \tau x), \quad R_F(x) = \tau^{-1}(\tau x \cdot F)
\]
for each $x \in A$. Then $(L_F, R_F)$ is an element of $M(A)$. We further have
\[
(\nu(L_F, R_F) - F) \cdot \tau x = \tau(L_Fx) - F \cdot \tau x = 0
\]
for each $x \in A$. It follows that $\nu(L_F, R_F) = F$. Thus $\nu$ is onto. Let $(T, S) \in M(A)$. Since $\nu(T, S) = \text{weak}^*\text{-lim}_{a} \tau(Te_a)$ and $||\tau(Te_a)|| < ||Te_a|| < M||T, S||$, it follows that $||\nu(T, S)|| < M||T, S||$.

Finally we can easily see that the restriction to $Z(M(A))$ of $\nu$ is equal to $\mu$ from the definition of $\nu$.

**Remark 2.4.** Let $A$ be as in Proposition 2.3 and let $(T, S)$ be any element of $M(A)$. Then $\nu(T, S) \cdot \tau x = \tau(Tx)$ and $\tau x \cdot \nu(T, S) = \tau(Sx)$ as was seen in the proof of Proposition 2.3. Therefore $||(T, S)|| < M||\nu(T, S)||$ since $\tau$ is norm reducing and $||x|| < M||\tau x||$ for all $x \in A$. If $M = 1$, then $\nu$ is isometric (cf. [5, Corollary 2.9]).

3. **Main theorems.** If $B$ is any algebra, then Prim $B$ will always denote the structure space of $B$, that is the set of all primitive ideals in $B$, with hull-kernel topology. Let $P \in \text{Prim } A$. Then by Theorem 2.6.6 in [8], there exists a unique element $P'$ in Prim $D(A)$ such that $P' \cap \tau(A) = \tau(P)$. If $T \in Z(D(A))$, then $\mu T \in Z(D(A))$ from Lemma 2.2, so that $\mu T + P'$ belongs to the center of $D(A)/P'$. Notice that $D(A)/P'$ is a primitive Banach algebra and so its center reduces to the complex field. Therefore there exists a unique complex number $\Phi_T(P)$ such that $\mu T + P' = \Phi_T(P)(J + P')$. 

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Moreover,
\[
|\Phi_T(P)| < \|\Phi_T(P)(J + P')\| = \|\mu T + P'\|
\leq \|\mu T\| < \|\mu\| \|T\|.
\]

We thus obtain a bounded complex-valued function \(\Phi_T\) on \(\text{Prim} A\) for each \(T\) in \(Z(M(A))\). Let \(R(A)\) be the radical of \(A\), that is the intersection of all primitive ideals in \(A\), and \(ZM_R(A)\) the set of all \(T \in Z(M(A))\) such that \(T(A) \subseteq R(A)\). Then \(ZM_R(A)\) is a closed two-sided ideal of \(Z(M(A))\). Let \(C^b(\text{Prim} A)\) be the algebra of all bounded continuous complex-valued functions on \(\text{Prim} A\).

The following result can be seen in the proof of Theorem 2.7.5 in [8].

**Lemma 3.1.** Let \(A\) be a Banach algebra and \(Z(A)\) its center. Then for each \(P \in \text{Prim} A\) with \(Z(A) \not\subseteq P\), \(P \cap Z(A) \in \text{Prim} Z(A)\). If \(A\) has an identity, then \(P \rightarrow P \cap Z(A)\) is a continuous map of \(\text{Prim} A\) into \(\text{Prim} Z(A)\).

We are now in a position to state and prove our main theorems.

**Theorem 3.2.** Let \(A\) be a complex Banach algebra with a bounded approximate identity. If the ideal center of \(A\) has a Hausdorff structure space, then the map \(T \rightarrow \Phi_T\) is a continuous homomorphism of \(Z(M(A))\) into \(C^b(\text{Prim} A)\) such that \(Tx + P = \Phi_T(P)(x + P)\) for all \(x \in A\) and \(P \in \text{Prim} A\), the kernel of the homomorphism being equal to \(ZM_R(A)\).

**Proof.** By the construction of \(\Phi_T\), the map \(T \rightarrow \Phi_T\) is a continuous homomorphism of \(Z(M(A))\) into the algebra of all bounded complex-valued functions on \(\text{Prim} A\). We first show that \(\Phi_T \in C^b(\text{Prim} A)\) for each \(T \in Z(M(A))\). By Theorem 2.6.6 in [8], the map \(P \rightarrow P'\) is a homeomorphism of \(\text{Prim} A\) into \(\text{Prim} D(A)\). Moreover, \(M \rightarrow M \cap Z(D(A))\) is a continuous map of \(\text{Prim} D(A)\) into \(\text{Prim} Z(D(A))\) by Lemma 3.1. Since \(\text{Prim} Z(D(A))\) is Hausdorff, \(Z(D(A))\) is completely regular (cf. [8, Definition 2.7.1]). It follows by Theorem 3.7.1 in [8] that the map \(Q \rightarrow \chi_Q\) is a homeomorphism of \(\text{Prim} Z(D(A))\) onto \(\text{Hom} Z(D(A))\). Here \(\chi_Q\) denotes the nonzero homomorphism of \(Z(D(A))\) onto the complex field induced by \(Q \in \text{Prim} Z(D(A))\) and \(\text{Hom} Z(D(A))\) denotes the carrier space of \(Z(D(A))\) with \(Z(D(A))\)-topology. We thus observe the map \(P \rightarrow \chi_{P \cap Z(D(A))}(z)\) is continuous on \(\text{Prim} A\) for each \(z \in Z(D(A))\). Let \(T \in Z(M(A))\) and \(P \in \text{Prim} A\). Since \(\mu T + P' = \Phi_T(P)(J + P')\), we have \(\mu T - \Phi_T(P)J \in P' \cap Z(D(A))\). It follows that \(\Phi_T(P) = \chi_{P \cap Z(D(A))}(\mu T)\). Then \(\Phi_T \in C^b(\text{Prim} A)\) by the above argument.

We next show that if \(T \in Z(M(A))\), then \(Tx + P = \Phi_T(P)(x + P)\) for all \(x \in A\) and \(P \in \text{Prim} A\). Let \(T \in Z(M(A))\), \(x \in A\) and \(P \in \text{Prim} A\). Then we have
\[ \tau(Tx) + P' = (\mu T) \cdot \tau x + P' = (\mu T + P')(\tau x + P') = \Phi_T(P)(\tau x + P'). \]

It follows that \( \tau(Tx - \Phi_T(P)x) \in P' \cap \tau(A) = \tau(P) \). Since \( \tau \) is injective, we obtain that \( Tx + P = \Phi_T(P)(x + P) \).

Finally we can easily see that the kernel of the map \( T \rightarrow \Phi_T \) is equal to \( ZM_{\mu}(A) \) from the equation \( Tx + P = \Phi_T(P)(x + P), x \in A \).

**Remark 3.3.** Let \( A \) be as in Theorem 3.2. If the approximate identity of \( A \) is uniformly bounded by one, then the map \( \mu \) is isometric [5, Corollary 2.9] and so \( T \rightarrow \Phi_T \) is a norm reducing homomorphism of \( Z(M(A)) \) into \( C^b(\text{Prim } A) \). If \( A \) is semisimple, then the map \( T \rightarrow \Phi_T \) is injective. Furthermore, if \( A \) is arbitrary \( C^* \)-algebra, then the ideal center \( Z(D(A)) \) of \( A \) has necessarily a Hausdorff structure space because \( Z(D(A)) \) is also a commutative \( C^* \)-algebra.

The following result is equivalent to the Dixmier's representation theorem [6, Theorem 5].

**Corollary 3.4.** If \( A \) is an arbitrary \( C^* \)-algebra, then the map \( T \rightarrow \Phi_T \) is an isometric *-isomorphism of \( Z(M(A)) \) onto \( C^b(\text{Prim } A) \).

**Proof.** Note that the map \( T \rightarrow \Phi_T \) is isometric from Remark 3.3. The map is also surjective because the Dauns and Hofmann theorem [4] has showed that every function in \( C^b(\text{Prim } A) \) can be realized uniquely in this way.

The following definition can be seen in the Archbold's paper [1], in case that \( A \) is a \( C^* \)-algebra.

**Definition 3.5.** Let \( A \) be a Banach algebra and \( Z(A) \) its center. Then \( A \) is said to be quasi-central if \( \text{hull } Z(A) = \emptyset \). Here \( \text{hull } Z(A) \) denotes the set of all primitive ideals \( P \) in \( A \) such that \( Z(A) \subset P \).

**Theorem 3.6.** Let \( A \) be a quasi-central Banach algebra with a bounded approximate identity and \( Z(A) \) its center. If \( Z(A) \) is completely regular, then the map \( T \rightarrow \Phi_T \) is a continuous homomorphism of \( Z(M(A)) \) into \( C^b(\text{Prim } A) \).

**Proof.** As in the proof of Theorem 3.2, we only show that \( \Phi_T \in C^b(\text{Prim } A) \) for each \( T \in Z(M(A)) \). To see this, we first show that each \( P \in \text{Prim } A \) and \( T \in Z(M(A)) \) satisfy the following properties:

1. \[ \chi_{P \cap Z(A)} T|Z(A) = \chi_{P \cap Z(A)}, \]
2. \[ \Phi_T(P) = \chi_{P \cap Z(D(A))}(\mu T), \]
3. \[ \Phi_T(P) \chi_{P \cap Z(A)} = \chi_{P \cap Z(A)} T|Z(A). \]

In fact, let \( z \in Z(A) \). Since \( \text{hull } Z(A) = \emptyset \), it follows that there exists \( z_0 \in Z(A) \) with \( z_0 \notin P \), and hence \( \chi_{P \cap Z(A)}(z_0) \neq 0 \). Note that \( \tau(Z(A)) \subset Z(D(A)) \), so that \( \tau(\chi_{P \cap Z(D(A))}(\tau z)z_0 - zz_0) \in P' \cap \tau(Z(A)) \). Since \( P' \cap \tau(Z(A)) = \tau(P \cap Z(A)) \), we have \( \chi_{P \cap Z(D(A))}(\tau z)z_0 - zz_0 \in P \cap Z(A) \) and
so \( \chi_{P' \cap Z(D(A))}(\tau z) = \chi_{P' \cap Z(A)}(z) \). Hence (1) holds. Notice that \( \mu T - \Phi_T(P)J \in P' \cap Z(D(A)) \). Then \( \chi_{P' \cap Z(D(A))}(\mu T) = \Phi_T(P)\chi_{P' \cap Z(D(A))}(J) = \Phi_T(P) \) and hence (2) holds. Note that \( T(Z(A)) \subset Z(A) \). Then, by (1) and (2), we have

\[
\chi_{P \cap Z(A)}(Tz) = \chi_{P' \cap Z(D(A))}(\tau(Tz)) \\
= \chi_{P' \cap Z(D(A))}(\mu T)\chi_{P' \cap Z(D(A))}(Tz) \\
= \Phi_T(P)\chi_{P \cap Z(A)}(z).
\]

Thus (3) has been shown.

By Theorem 2.7.5 in [8], \( P \to P \cap Z(A) \) is a continuous map of \( \text{Prim } A \) into \( \text{Prim } Z(A) \). Moreover, \( I \to \chi_I \) is a homeomorphism of \( \text{Prim } Z(A) \) onto \( \text{Hom } Z(A) \) by Theorem 3.7.1 in [8]. Therefore, by (3), \( P \to \Phi_T(P)\chi_{P \cap Z(A)}(z) \) is a continuous complex-valued function on \( \text{Prim } A \) for each \( z \in Z(A) \). Suppose that \( \Phi_T \) is discontinuous at some point \( P_0 \) in \( \text{Prim } A \). Then there exists a positive number \( \epsilon_0 \) and a net \( \{ P_\lambda : \lambda \in \Lambda \} \) in \( \text{Prim } A \) which converges to \( P_0 \) such that

\[
|\Phi_T(P_\lambda) - \Phi_T(P_0)| \geq \epsilon_0 \quad (4)
\]

for all \( \lambda \in \Lambda \). Since \( A \) is quasi-central, we can choose an element \( z_0 \) in \( Z(A) \) such that \( \chi_{P_0 \cap Z(A)}(z_0) = 1 \). By the above argument, the complex-valued function \( P \to \chi_{P \cap Z(A)}(z_0) \) and \( P \to \Phi_T(P)\chi_{P \cap Z(A)}(z_0) \) on \( \text{Prim } A \) are continuous. Then for any \( \epsilon > 0 \), there exists a neighbourhood \( U_{P_0} \) of \( P_0 \) such that

\[
|\chi_{P \cap Z(A)}(z_0) - 1| < \epsilon \quad (5)
\]

and

\[
|\Phi_T(P)\chi_{P \cap Z(A)}(z_0) - \Phi_T(P_0)| < \epsilon \quad (6)
\]

for all \( P \in U_{P_0} \). In particular, choose \( \epsilon \) such that

\[
\epsilon = \min\{1/2, (4 + 4|\Phi_T(P_0)|)^{-1}\epsilon_0\}.
\]

Furthermore, choose \( \lambda_0 \in \Lambda \) such that \( P_{\lambda_0} \in U_{P_0} \) and set \( \delta = \chi_{Q_0}(z_0) \), where \( Q_0 = P_{\lambda_0} \cap Z(A) \). Note that \( |\delta| > 1/2 \) by (5). We then have

\[
|\Phi_T(P_{\lambda_0}) - \Phi_T(P_0)| < |\delta|^{-1}(|\Phi_T(P_{\lambda_0})\delta - \Phi_T(P_0)\delta| + |\Phi_T(P_0) - \Phi_T(P_0)\delta|) \\
< |\delta|^{-1}(\epsilon + |\Phi_T(P_0)\epsilon|) \quad (\text{from (5) and (6)}) \\
< \epsilon_0/2.
\]

This contradicts (4). We thus obtain that \( \Phi_T \) is continuous on \( \text{Prim } A \) and the proof is complete.

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