DIXMIER'S REPRESENTATION THEOREM OF CENTRAL DOUBLE CENTRALIZERS ON BANACH ALGEBRAS

BY

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Abstract. The present paper is devoted to a representation theorem of central double centralizers on a complex Banach algebra with a bounded approximate identity. In particular, our result implies the representation theorem of the ideal center of an arbitrary C*-algebra established by J. Dixmier.

1. Introduction. R. C. Busby [2] has noted that every central double centralizer on any C*-algebra can be represented as a bounded continuous complex-valued function on its structure space, which is equivalent to Dixmier's representation theorem [6, Theorem 5]. Let A be a complex Banach algebra with a bounded approximate identity and Prim A the structure space of A. A central double centralizer T on A may be identified with a bounded linear operator T on A such that (Tx)y = x(Ty) for all x, y ∈ A. In this paper, we show that, if the ideal center of A has a Hausdorff structure space, every central double centralizer T on A can be represented as a bounded continuous complex-valued function ϕ_T on Prim A such that Tx + P = ϕ_T(P)(x + P) for all x ∈ A and P ∈ Prim A. Here x + P for P ∈ Prim A denotes the canonical image of x in A/P. In particular, if A is a C*-algebra then our representation theorem implies the Busby's or, equivalently, Dixmier's. In this way we get a proof of Dixmier's theorem, quite different from that given in [6]. This was inspired by Davenport's representation theorem of multiplier algebras on Banach algebras with bounded approximate identity [5, Theorem 2.8]. We also obtain a similar representation theorem of central double centralizers on a quasi-central Banach algebra with a completely regular center.

2. Davenport's representation theorem of Z(M(A)). In this paper, a complex Banach algebra with a bounded approximate identity will be denoted by A and the central double centralizer-algebra on A will be denoted by Z(M(A)), that is the center of the double centralizer-algebra M(A) on A. Let {e_a} be the approximate identity on A and A# the set of all elements f in the dual space A* of A such that \[ \lim_a \| f \cdot e_a - f \| = 0, \] where f \cdot a(x) = f(ax) for

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229
each \( a, x \in A \) and \( f \in A^* \). The set \( A^\# \) is a closed subspace of \( A^* \) and \( A^\# = \{f \cdot a : f \in A^*, a \in A\} \) (cf. [3], [5]). Then the dual space \((A^\#)^* \) of \( A^\# \) becomes a Banach algebra under the restriction to \( A^\# \) of the Arens product on the second dual space \( A^{**} \) of \( A \). In fact, the restriction to \( A^\# \) of the Arens product on \( A^{**} \) can be described as follows:

\[
[ G, f ](a) = G(f \cdot a), \quad F \cdot G(f) = F[ G, f ]
\]

for each \( a \in A, f \in A^\# \) and \( F, G \in (A^\#)^* \). Let \( \pi \) be the canonical embedding of \( A \) into \( A^{**} \) and \( i \) the inclusion map of \( A^\# \) into \( A^* \). Put \( \tau = i^* \pi \), where \( i^* \) denotes the dual map induced by \( i \). Then the map \( \tau \) is a norm reducing isomorphism of \( A \) into \( (A^\#)^* \) [5, Lemma 2.5]. Furthermore, we can easily verify that \( \|x\| \leq M\|\tau(x)\| \) for each \( x \in A \), where \( M \) is the bound on \( \{e_a\} \). Then \( \tau(A) \) is uniformly closed in \( (A^\#)^* \). We now define \( D(A) \) to be the set

\[
D(A) = \{ F \in (A^\#)^*: F \cdot \tau(A) + \tau(A) \cdot F \subset \tau(A) \}.
\]

Then \( D(A) \) is a Banach subalgebra of \( (A^\#)^* \) and \( \tau(A) \) is a closed two-sided ideal of \( D(A) \).

**Definition 2.1.** Let \( Z(D(A)) \) be the center of \( D(A) \). The algebra \( Z(D(A)) \) is said to be the ideal center of \( A \).

By Lemma 2.7 in [5], the algebra \( (A^\#)^* \) has an identity \( J \). If \( A \) is a \( C^* \)-algebra, then \( A^\# = A^* \) and so \( (A^\#)^* = A^{**} \) (cf. [5, Proof of Corollary 2.10]). Then \( Z(D(A)) \) becomes the ideal center of \( A \) in the sense of Dixmier [6].

J. Davenport has proved that there exists a continuous, algebraic isomorphism \( \mu \) of \( Z(M(A)) \) into \( (A^\#)^* \) such that \( \tau(Tx) = (\mu T) \cdot \tau x = \tau x \cdot (\mu T) \) for all \( x \in A \) and \( T \in Z(M(A)) \) [5, Theorem 2.8]. In fact, the map \( \mu \) is given by

\[
\mu T = \text{weak}^*\text{-lim}_a \tau(T e_a)
\]

for each \( T \in Z(M(A)) \). We then, from this Davenport result, obtain the following

**Lemma 2.2.** Let \( A \) be a Banach algebra with a bounded approximate identity \( \{e_a\} \) and \( \mu \) the isomorphism of \( Z(M(A)) \) into \( (A^\#)^* \) given by Davenport. Then \( \mu(Z(M(A))) = Z(D(A)) \).

**Proof.** Let \( T \in Z(M(A)) \). Since \( \tau(A) \cdot (\mu T) = (\mu T) \cdot \tau(A) = \tau(T(A)) \subset \tau(A) \), we have \( \mu T \in D(A) \). It is well known that \( \pi(A) \) is weak*-dense in \( A^{**} \). Moreover, \( i^* \) is a weak*-continuous map of \( A^{**} \) onto \( (A^\#)^* \). These facts imply that \( \tau(A) \) is weak*-dense in \( (A^\#)^* \). Then for each \( F \in (A^\#)^* \), there exists a net \( \{x_\lambda\} \) in \( A \) such that \( F = \text{weak}^*\text{-lim}_\lambda \tau(x_\lambda) \). Let \( T^* \) be the dual map induced by \( T \). Note that \( T^*(f \cdot a) = f \cdot (Ta) \) for each \( a \in A \) and \( f \in A^* \), so
that $T^*(A^*) \subset A^*$. Put $T^* = T^* | A^*$. Then we have
\[
\lim_{\lambda} \tau(Tx_\lambda)(f) = \lim_{\lambda} \tau(x_\lambda)(T^* f) = F(T^* f)
\]
\[
= \lim_{\alpha} F(T^* (f \cdot e_\alpha)) = \lim_{\alpha} \tau(Te_\alpha)[F, f]
\]
\[
= (\mu T)[F, f] = (\mu T) \cdot F(f)
\]
for all $f \in A^*$. On the other hand, we have
\[
\lim_{\lambda} \tau(Tx_\lambda)(f) = \lim_{\lambda} \tau(x_\lambda) \cdot (\mu T)(f)
\]
\[
= \lim_{\lambda} \tau(x_\lambda)[\mu T, f] = F(\mu T, f) = F(\mu T)(f)
\]
for all $f \in A^*$. We thus obtain $(\mu T) \cdot F = F \cdot (\mu T)$, so that $\mu T \in Z(D(A))$.

Conversely, let $G \in Z(D(A))$. Put $\hat{G}(x) = \tau^{-1}(G \cdot \tau x)$ for each $x \in A$. Then $\hat{G}$ is a bounded linear operator on $A$. Furthermore, $\hat{G} \in Z(M(A))$. In fact, for each $x, y \in A$, $\tau((\hat{G}x)y) = \tau(\tau^{-1}(G \cdot \tau x) y) = (G \cdot \tau x) \cdot \tau y = \tau x \cdot (G \cdot \tau y)$, so that $(\hat{G}x)y = x \tau^{-1}(G \cdot \tau y) = x((\hat{G}y))$. We also have
\[
\mu \hat{G}(f) = \lim_{\alpha} \tau(G e_\alpha)(f) = \lim_{\alpha} G \cdot \tau e_\alpha(f)
\]
\[
= \lim_{\alpha} \tau e_\alpha \cdot G(f) = \lim_{\alpha} [G, f](e_\alpha)
\]
\[
= \lim_{\alpha} G(f \cdot e_\alpha) = G(f)
\]
for all $f \in A^*$. We thus get $\mu \hat{G} = G$ and our result is proved.

Furthermore, the following stronger result can be proved, and it is similar to one established by K. Saito [7], in which he has given a characterization of double centralizer-algebras on Banach algebras under some conditions. This was pointed out by the referee.

**Proposition 2.3.** Let $A$ be a Banach algebra with a bounded approximate identity $\{e_\alpha; \alpha \in \Lambda\}$. Then there exists a continuous, algebraic isomorphism $\nu$ of the double centralizer-algebra $M(A)$ on $A$ onto $D(A)$ which is an extension of $\mu$ to $M(A)$.

**Proof.** Let $(T, S) \in M(A)$. Since $\{\tau(Te_\alpha); \alpha \in \Lambda\}$ is bounded, it has a weak*-convergent subnet $\{T(e_\alpha')\}; \alpha' \in \Lambda' \}$ in $(A^*)^\ast$. We now define the map $\nu$ from $M(A)$ to $(A^*)^\ast$ by
\[
\nu(T, S) = \lim_{\alpha} \tau(Te_\alpha)
\]
for each $(T, S) \in M(A)$. Then $\nu$ is well defined. In fact, let $\{\tau(Te_\alpha); \alpha' \in \Lambda' \}$ converge to $F$ and $\{\tau(Te_\alpha); \alpha'' \in \Lambda'' \}$ converge to $G$, each in the weak*-topology. Then for each $x \in A$ and $f \in A^*$, we have
\[
F \cdot \tau x(f) = \lim_{\alpha'} \tau(T(e_\alpha' x))(f) = \lim_{\alpha'} f(T(e_\alpha' x))
\]
\[
= \lim_{\alpha'} \tau(T(e_\alpha x))(f) = f(Tx) = \tau(Tx)(f).
\]
Similarly, \( G \cdot \tau x(f) = \tau(Tx)(f) \) for each \( x \in A \) and \( f \in A^{\#} \). Hence \((F - G) \cdot \tau x = 0\) for each \( x \in A \). It follows by [5, 2.6.3] that \( F = G \). Clearly, \( \nu \) is linear. Also, since \( \nu(T, S) \cdot \tau x = \tau(Tx) \) for each \( x \in A \) as was seen in the above argument, we have \( \nu(T, S) \cdot \tau(A) \subset \tau(A) \). Furthermore, we have

\[
\tau x \cdot \nu(T, S)(f) = \lim_{a} \tau x \cdot (Te_a)(f) \quad \text{(from [5, 2.6.2])}
\]

\[
= \lim_{a} \tau((Sx)e_a)(f) = \lim_{a} f((Sx)e_a)
\]

\[
= f(Sx) = \tau(Sx)(f)
\]

for each \( x \in A \) and \( f \in A^{\#} \). Then \( \tau x \cdot \nu(T, S) = \tau(Sx) \) for each \( x \in A \), so that \( \tau(A) \cdot \nu(T, S) \subset \tau(A) \). In other words, \( \nu(M(A)) \subset D(A) \). Now let \( (T, S), (T', S') \in M(A) \). Set \( F = \nu(T, S), \ F' = \nu(T', S') \) and \( F'' = \nu(TT', S'S) \). Then \( F'' \cdot \tau x = \tau(TT'x) \) for each \( x \in A \). On the other hand, for each \( x \in A, F \cdot F' \cdot \tau x = F \cdot \tau(T'x) = \tau(TT'x) \), and hence \((F'' - F \cdot F') \cdot \tau x = 0\). It follows by [5, 2.6.3] that \( F'' = F \cdot F' \). In other words, \( \nu((T, S)(T', S')) = \nu(T, S) \cdot \nu(T', S') \). If \( F = F' \), then \( \nu(T - T')x = (F - F') \cdot \tau x = 0 \) for each \( x \in A \), and so \( T = T' \) (and hence \( S = S' \)) since \( \tau \) is one-to-one. Thus \( \nu \) is an algebraic isomorphism of \( M(A) \) into \( D(A) \). To show that \( \nu \) is onto, let \( F \in D(A) \) and set

\[
L_F(x) = \tau^{-1}(F \cdot \tau x), \quad R_F(x) = \tau^{-1}(\tau x \cdot F)
\]

for each \( x \in A \). Then \( (L_F, R_F) \) is an element of \( M(A) \). We further have

\[
(\nu(L_F, R_F) - F) \cdot \tau x = \tau(L_Fx) - F \cdot \tau x = 0
\]

for each \( x \in A \). It follows that \( \nu(L_F, R_F) = F \). Thus \( \nu \) is onto. Let \( (T, S) \in M(A) \). Since \( \nu(T, S) = \text{weak}^* \)-lim _\( a \) \( \tau(Te_a) \) and \( ||\tau(Te_a)|| < ||Te_a|| < M(||(T, S)|| \), it follows that \( ||\nu(T, S)|| < M(||(T, S)|| \). Finally we can easily see that the restriction to \( Z(M(A)) \) of \( \nu \) is equal to \( \mu \) from the definition of \( \nu \).

REMARK 2.4. Let \( A \) be as in Proposition 2.3 and let \( (T, S) \) be any element of \( M(A) \). Then \( \nu(T, S) \cdot \tau x = \tau(Tx) \) and \( \tau x \cdot \nu(T, S) = \tau(Sx) \) as was seen in the proof of Proposition 2.3. Therefore \( ||(T, S)|| < M||\nu(T, S)|| \) since \( \tau \) is norm reducing and \( ||x|| \leq M||\tau x|| \) for all \( x \in A \). If \( M = 1 \), then \( \nu \) is isometric (cf. [5, Corollary 2.9]).

3. Main theorems. If \( B \) is any algebra, then Prim \( B \) will always denote the structure space of \( B \), that is the set of all primitive ideals in \( B \), with hull-kernel topology. Let \( P \in \text{Prim} A \). Then by Theorem 2.6.6 in [8], there exists a unique element \( P' \) in Prim \( D(A) \) such that \( P' \cap \tau(A) = \tau(P) \). If \( T \in Z(M(A)) \), then \( \mu T \in Z(D(A)) \) from Lemma 2.2, so that \( \mu T + P' \) belongs to the center of \( D(A)/P' \). Notice that \( D(A)/P' \) is a primitive Banach algebra and so its center reduces to the complex field. Therefore there exists a unique complex number \( \Phi_{\tau}(P) \) such that \( \mu T + P' = \Phi_{\tau}(P)(J + P') \).
Moreover,
\[ |\Phi_T(P)| < \|\Phi_T(P)(J + P')\| = \|\mu T + P'\| \leq \|\mu T\| \leq \|\mu\| \|T\|. \]

We thus obtain a bounded complex-valued function \(\Phi_T\) on Prim \(A\) for each \(T\) in \(Z(M(A))\). Let \(R(A)\) be the radical of \(A\), that is the intersection of all primitive ideals in \(A\), and \(ZM_R(A)\) the set of all \(T \in Z(M(A))\) such that \(T(A) \subset R(A)\). Then \(ZM_R(A)\) is a closed two-sided ideal of \(Z(M(A))\). Let \(C^b(\text{Prim } A)\) be the algebra of all bounded continuous complex-valued functions on \(\text{Prim } A\).

The following result can be seen in the proof of Theorem 2.7.5 in [8].

**Lemma 3.1.** Let \(A\) be a Banach algebra and \(Z(A)\) its center. Then for each \(P \in \text{Prim } A\) with \(Z(A) \subset P\), \(P \cap Z(A) \in \text{Prim } Z(A)\). If \(A\) has an identity, then \(P \rightarrow P \cap Z(A)\) is a continuous map of \(\text{Prim } A\) into \(\text{Prim } Z(A)\).

We are now in a position to state and prove our main theorems.

**Theorem 3.2.** Let \(A\) be a complex Banach algebra with a bounded approximate identity. If the ideal center of \(A\) has a Hausdorff structure space, then the map \(T \rightarrow \Phi_T\) is a continuous homomorphism of \(Z(M(A))\) into \(C^b(\text{Prim } A)\) such that \(Tx + P = \Phi_T(P)(x + P)\) for all \(x \in A\) and \(P \in \text{Prim } A\), the kernel of the homomorphism being equal to \(ZM_R(A)\).

**Proof.** By the construction of \(\Phi_T\), the map \(T \rightarrow \Phi_T\) is a continuous homomorphism of \(Z(M(A))\) into the algebra of all bounded complex-valued functions on \(\text{Prim } A\). We first show that \(\Phi_T \in C^b(\text{Prim } A)\) for each \(T \in Z(M(A))\). By Theorem 2.6.6 in [8], the map \(P \rightarrow P'\) is a homeomorphism of \(\text{Prim } A\) into \(\text{Prim } D(A)\). Moreover, \(M \rightarrow M \cap Z(D(A))\) is a continuous map of \(\text{Prim } D(A)\) into \(\text{Prim } Z(D(A))\) by Lemma 3.1. Since \(\text{Prim } Z(D(A))\) is Hausdorff, \(Z(D(A))\) is completely regular (cf. [8, Definition 2.7.1]). It follows by Theorem 3.7.1 in [8] that the map \(Q \rightarrow \chi_Q\) is a homeomorphism of \(\text{Prim } Z(D(A))\) onto \(\text{Hom } Z(D(A))\). Here \(\chi_Q\) denotes the nonzero homomorphism of \(Z(D(A))\) onto the complex field induced by \(Q \in \text{Prim } Z(D(A))\) and \(\text{Hom } Z(D(A))\) denotes the carrier space of \(Z(D(A))\) with \(Z(D(A))\)-topology. We thus observe the map \(P \rightarrow \chi_{P \cap Z(D(A))}(z)\) is continuous on \(\text{Prim } A\) for each \(z \in Z(D(A))\). Let \(T \in Z(M(A))\) and \(P \in \text{Prim } A\). Since \(\mu T + P' = \Phi_T(P)(J + P')\), we have \(\mu T - \Phi_T(P)J \in P' \cap Z(D(A))\). It follows that \(\Phi_T(P) = \chi_{P \cap Z(D(A))}(\mu T)\). Then \(\Phi_T \in C^b(\text{Prim } A)\) by the above argument.

We next show that if \(T \in Z(M(A))\), then \(Tx + P = \Phi_T(P)(x + P)\) for all \(x \in A\) and \(P \in \text{Prim } A\). Let \(T \in Z(M(A))\), \(x \in A\) and \(P \in \text{Prim } A\). Then we have
\[ \tau(Tx) + P' = (\mu T) \cdot \tau x + P' = (\mu T + P')(\tau x + P') \]

\[ = \Phi_T(P)(J + P')(\tau x + P') = \Phi_T(P)(\tau x + P'). \]

It follows that \( \tau(Tx - \Phi_T(P)x) \in P' \cap \tau(A) = \tau(P). \) Since \( \tau \) is injective, we obtain that \( Tx + P = \Phi_T(P)(x + P). \)

Finally we can easily see that the kernel of the map \( T \to \Phi_T \) is equal to \( ZM_R(A) \) from the equation \( Tx + P = \Phi_T(P)(x + P), \ x \in A. \)

**Remark 3.3.** Let \( A \) be as in Theorem 3.2. If the approximate identity of \( A \) is uniformly bounded by one, then the map \( \mu \) is isometric [5, Corollary 2.9] and so \( T \to \Phi_T \) is a norm reducing homomorphism of \( Z(M(A)) \) into \( C^b(\text{Prim } A) \). If \( A \) is semisimple, then the map \( T \to \Phi_T \) is injective. Furthermore, if \( A \) is arbitrary \( C^* \)-algebra, then the ideal center \( Z(D(A)) \) of \( A \) has necessarily a Hausdorff structure space because \( Z(D(A)) \) is also a commutative \( C^* \)-algebra.

The following result is equivalent to the Dixmier’s representation theorem [6, Theorem 5].

**Corollary 3.4.** If \( A \) is an arbitrary \( C^* \)-algebra, then the map \( T \to \Phi_T \) is an isometric \(*\)-isomorphism of \( Z(M(A)) \) onto \( C^b(\text{Prim } A) \).

**Proof.** Note that the map \( T \to \Phi_T \) is isometric from Remark 3.3. The map is also surjective because the Dauns and Hofmann theorem [4] has showed that every function in \( C^b(\text{Prim } A) \) can be realized uniquely in this way.

The following definition can be seen in the Archbold’s paper [1], in case that \( A \) is a \( C^* \)-algebra.

**Definition 3.5.** Let \( A \) be a Banach algebra and \( Z(A) \) its center. Then \( A \) is said to be quasi-central if hull \( Z(A) = \emptyset. \) Here hull \( Z(A) \) denotes the set of all primitive ideals \( P \) in \( A \) such that \( Z(A) \subset P. \)

**Theorem 3.6.** Let \( A \) be a quasi-central Banach algebra with a bounded approximate identity and \( Z(A) \) its center. If \( Z(A) \) is completely regular, then the map \( T \to \Phi_T \) is a continuous homomorphism of \( Z(M(A)) \) into \( C^b(\text{Prim } A) \).

**Proof.** As in the proof of Theorem 3.2, we only show that \( \Phi_T \in C^b(\text{Prim } A) \) for each \( T \in Z(M(A)). \) To see this, we first show that each \( P \in \text{Prim } A \) and \( T \in Z(M(A)) \) satisfy the following properties:

\[ \chi_{P \cap Z(D(A))} \tau|Z(A) = \chi_{P \cap Z(A)}, \]

\[ \Phi_T(P) = \chi_{P \cap Z(D(A))}(\mu T), \]

\[ \Phi_T(P)\chi_{P \cap Z(A)} = \chi_{P \cap Z(A)} T|Z(A). \]

In fact, let \( z \in Z(A). \) Since hull \( Z(A) = \emptyset, \) it follows that there exists \( z_0 \in Z(A) \) with \( z_0 \notin P, \) and hence \( \chi_{P \cap Z(A)}(z_0) \neq 0. \) Note that \( \tau(Z(A)) \subset Z(D(A)), \) so that \( \tau(\chi_{P \cap Z(D(A))}(Tz))z_0 - zz_0 \in P' \cap \tau(Z(A)). \) Since \( P' \cap \tau(Z(A)) = \tau(P \cap Z(A)), \) we have \( \chi_{P \cap Z(D(A))}(Tz)z_0 - zz_0 \in P \cap Z(A) \) and
so \( \chi_{P \cap Z(D(A))}(\tau z) = \chi_{P \cap Z(A)}(z) \). Hence (1) holds. Notice that \( \mu T - \Phi_T(P) J \in P' \cap Z(D(A)) \). Then \( \chi_{P' \cap Z(D(A))}(\mu T) = \Phi_T(P) \chi_{P' \cap Z(D(A))}(J) = \Phi_T(P) \) and hence (2) holds. Note that \( T(Z(A)) \subset Z(A) \). Then, by (1) and (2), we have

\[
\chi_{P \cap Z(A)}(\tau z) = \chi_{P' \cap Z(D(A))}(\tau(Tz)) = \chi_{P' \cap Z(D(A))}(\mu T) \chi_{P \cap Z(A)}(z) = \Phi_T(P) \chi_{P \cap Z(A)}(z).
\]

Thus (3) has been shown.

By Theorem 2.7.5 in [8], \( P \to P \cap Z(A) \) is a continuous map of \( \text{Prim } A \) into \( \text{Prim } Z(A) \). Moreover, \( I \to \chi_I \) is a homeomorphism of \( \text{Prim } Z(A) \) onto \( \text{Hom } Z(A) \) by Theorem 3.7.1 in [8]. Therefore, by (3), \( P \to \Phi_T(P) \chi_{P \cap Z(A)}(z) \) is a continuous complex-valued function on \( \text{Prim } A \) for each \( z \in Z(A) \). Suppose that \( \Phi_T \) is discontinuous at some point \( P_0 \) in \( \text{Prim } A \). Then there exists a positive number \( \varepsilon_0 \) and a net \( \{ P_\lambda : \lambda \in \Lambda \} \) in \( \text{Prim } A \) which converges to \( P_0 \) such that

\[
|\Phi_T(P_\lambda) - \Phi_T(P_0)| > \varepsilon_0 \tag{4}
\]

for all \( \lambda \in \Lambda \). Since \( A \) is quasi-central, we can choose an element \( z_0 \) in \( Z(A) \) such that \( \chi_{P_0 \cap Z(A)}(z_0) = 1 \). By the above argument, the complex-valued function \( P \to \chi_{P \cap Z(A)}(z_0) \) and \( P \to \Phi_T(P) \chi_{P \cap Z(A)}(z_0) \) on \( \text{Prim } A \) are continuous. Then for any \( \varepsilon > 0 \), there exists a neighbourhood \( U_{P_0} \) of \( P_0 \) such that

\[
|\chi_{P \cap Z(A)}(z_0) - 1| < \varepsilon \tag{5}
\]

and

\[
|\Phi_T(P) \chi_{P \cap Z(A)}(z_0) - \Phi_T(P_0)| < \varepsilon \tag{6}
\]

for all \( P \in U_{P_0} \). In particular, choose \( \varepsilon \) such that

\[
\varepsilon = \min\left\{ 1/2, (4 + 4|\Phi_T(P_0)|)^{-1}\varepsilon_0 \right\}.
\]

Furthermore, choose \( \lambda_0 \in \Lambda \) such that \( P_{\lambda_0} \in U_{P_0} \) and set \( \delta = \chi_{Q_0}(z_0) \), where \( Q_0 = P_{\lambda_0} \cap Z(A) \). Note that \( |\delta| > 1/2 \) by (5). We then have

\[
|\Phi_T(P_{\lambda_0}) - \Phi_T(P_0)| < |\delta|^{-1}(|\Phi_T(P_{\lambda_0}) \delta - \Phi_T(P_0)| + |\Phi_T(P_0) - \Phi_T(P_0) \delta|) < |\delta|^{-1}(\varepsilon + |\Phi_T(P_0)|\varepsilon) \quad \text{(from (5) and (6))}
\]

\[
< \varepsilon_0/2.
\]

This contradicts (4). We thus obtain that \( \Phi_T \) is continuous on \( \text{Prim } A \) and the proof is complete.

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