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SMOOTH ORBIT EQUIVALENCE OF ERGODIC $\mathbf{R}^d$ ACTIONS, $d > 2$

BY

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ABSTRACT. We show here that any two free ergodic finite measure preserving actions of $\mathbf{R}^d$, $d > 2$, are orbit equivalent by a measure preserving map which on orbits is C^∞.

I. Preliminaries. What we will demonstrate here is that any two free ergodic finite measure preserving actions of $\mathbf{R}^d$, $d > 2$, are orbit equivalent by a nonsingular map, which when restricted to a single orbit, is C^∞ and measure and orientation preserving.

For $\mathbf{R}$ actions this kind of smooth orbit equivalence is identical to Kakutani equivalence of cross sections which, as is shown in [2], [4] and [5], has a much richer structure. For example, there are uncountable infinitely many equivalence classes.

For higher dimensional actions the structure is trivial, hence much more like the structure of orbit equivalence for finite measure preserving ergodic $\mathbf{Z}$ actions (1). This parallel extends even to the structure of the proof. The reason for this parallel, and the idea at the core of our argument, is that homeomorphisms of $\mathbf{R}^d$, $d > 2$, are much more diverse than $\mathbf{R}$. A homeomorphism of $\mathbf{R}$ must be monotone, but for $\mathbf{R}^d$, $d > 2$, small disjoint regions can be rigidly shifted at will by a homeomorphism much as a permutation acts on $\mathbf{Z}$.

There is hope that the rich structure of Kakutani equivalence for $\mathbf{R}$ can be lifted to $\mathbf{R}^d$ by putting further restrictions on the orbit maps allowed. We will discuss these ideas when they arise later.

Let $T = \{T_v\}_{v\in\mathbf{R}^d}$ and $T' = \{T'_v\}_{v\in\mathbf{R}^d}$ be two ergodic free measure preserving actions on the Lebesgue probability spaces $\{\Omega, \mathfrak{F}, \mu\}$, $\{\Omega', \mathfrak{F}', \mu'\}$ respectively. A "smooth orbit equivalence" of these two systems is a measurable a.e. invertible map ϕ; $\Omega \to \Omega'$ so that for every $\omega \in \Omega$, $v \in \mathbf{R}^d$, $\phi(T_v(\omega)) = T'_{f(\phi(\omega), v)}(\phi(\omega))$ where for a.e. $\omega' \in \Omega'$, $f(\omega', \cdot)$; $\mathbf{R}^d \circlearrowleft$ is C^∞ and orientation preserving.

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In this case it must be true that

$$f(\omega', v + v') = f(\omega', v) + f(T'_{f(\omega', v)}(\omega'), v'). \tag{1.1}$$

Any map f; $\Omega' \times \mathbf{R}^d \to \mathbf{R}^d$ satisfying (1.1) which for a.e. ω' has $f(\omega', \cdot)$ a C^∞ orientation preserving diffeomorphism of $\mathbf{R}^d$ to itself we will call a "smooth time change" of T'. In this case

$$\hat{T}_v(\omega) = T_{f(\omega, v)}(\omega)$$

is a well-defined action of $\mathbf{R}^d$, not necessarily preserving μ. We will write this action T_f. Thus two systems are smoothly orbit equivalent if one can be smoothly time changed to be isomorphic to the other.

For f a smooth time change, let $J(f)$; $\Omega' \to \{2 \times 2$ matrices over $\mathbf{R}\}$ be the Jacobean of $f(\omega, \cdot)$ evaluated at 0. Notice $f(\omega, \cdot)$ preserves orientation on orbits iff $\det(J(f)) > 0$. As T' is ergodic, this is either true a.e. or false a.e.

Now as

$$\det(J(f)) = \frac{d\mu'}{d\phi(\mu)},$$

it must be in $L^1(\mu')$, and ϕ is measure preserving iff $\det(J(f)) \equiv 1$.

For $\mathbf{R}$ actions, then, $J(f) \in L^1(\mu')$. It is not true, though, that for $\mathbf{R}^d$ actions, $d \geqslant 2$, that $J(f) \in L^1(\mu')$, even if $\det(J(f)) \equiv 1$. In fact the assumption that $J(f) \in L^1(\mu')$ should make this higher dimensional equivalence parallel the one dimensional theory. D. Nadler [3] has recently proven a first step in this direction by showing that if we add a slightly stronger condition that $J(f)$ be bounded, then there exist at least two nonequivalent $\mathbf{R}^2$ actions.

What we will prove here is the following.

PROPOSITION 1.1. *Any two free ergodic $\mathbf{R}^d$ actions, $d \geqslant 2$, which preserve a probability measure, are smoothly orbit equivalent by a measure preserving map.*

We break the proof of this into a number of steps. The first is to develop some preliminary geometry on $\mathbf{R}^d$ which will show why it differs from $\mathbf{R}$, and will provide the basic tools for our argument.

Our first lemma is virtually obvious. Figure 1 illustrates it, and we leave an explicit proof as an amusing exercise for the reader.

LEMMA 1.2. *Let $A > B > C > 0$ be positive real numbers. Let $\bar{A} = [0, A] \times [0, B]^{d-1} \subset \mathbf{R}^d$ and $\bar{B} = [(B - C)/2, C + (B - C)/2]^d \subset \bar{A}$, $d \geqslant 2$. For any $0 \leqslant t \leqslant A - (B - C)/2$, there exists a C^∞ measure preserving diffeomorphism f_t: $\mathbf{R}$ ↺ such that $f_t/\bar{A}^c$ is the identity map, and $f_t/\bar{B}$ is linear translation by $(t, 0 \ldots 0)$.* □

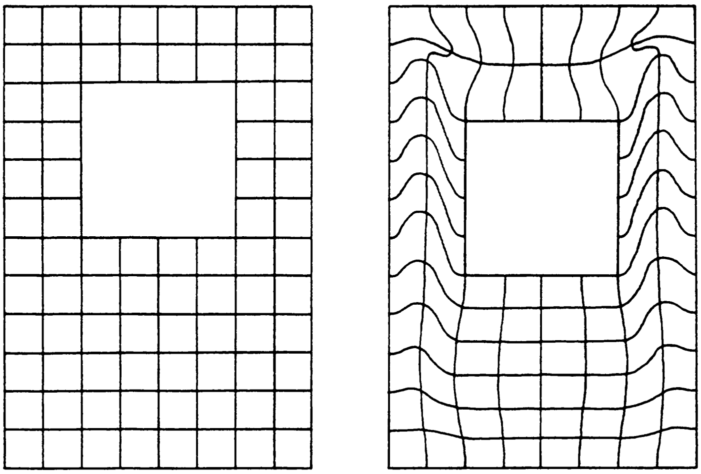

FIGURE 1

Such functions will allow us to modify orbits of one $\mathbf{R}^d$ action to look like those of another through an inductive procedure on large squares of orbit. Here is what the inductive step will look like. By an "A-block" we will mean some linear translate in $\mathbf{R}^d$ of $[0, A)^d$, and if $\bar{B}$ is an A-block, by B_ε, $\varepsilon > 0$, we will mean the $(A - 2\varepsilon)$-block of points of $\bar{B}$ at least ε away from $\bar{B}^c$. For those well acquainted with Dye's Theorem on orbit equivalence of finite measure preserving ergodic $\mathbf{Z}$ actions [1] our next lemma will make Proposition 1.1 almost obvious.

LEMMA 1.3. *Let $A > B > 0$ and $\varepsilon > 0$ be real numbers with $\varepsilon A/B > 3B$. Let $\bar{B}_1 \ldots \bar{B}_k \subset [0, A]^d = \bar{A}$ be disjoint B-blocks, as are $\bar{B}'_1 \ldots \bar{B}'_k \subset \bar{A}$. There is, then, a C^∞ measure preserving diffeomorphism f; $\mathbf{R}^d \circlearrowleft$ which on $\bar{A}^c$ is the identity, and on each $\bar{B}_{i,\varepsilon}$ is linear translation onto $\bar{B}'_{i,\varepsilon}$.*

PROOF. We will work this through for $d = 2$, the argument for larger d being a simple generalization.

We say $\bar{B}_{i,\varepsilon}$ is above $\bar{B}_{j,\varepsilon}$ if any point of $\bar{B}_i$ is directly above a point of $\bar{B}_j$. Partition the $\bar{B}_{i,\varepsilon}$'s into sets $L_1, L_2 \ldots L_s$ where L_1 are those $\bar{B}_{i,\varepsilon}$'s with no $\bar{B}_{j,\varepsilon}$ below it, and L_i are those not in $L_1 \ldots L_{i-1}$ and with only elements of $L_1 \ldots L_{i-1}$ below them.

Using applications of Lemma 1 move the $\bar{B}_{i,\varepsilon}$'s in L_1 to precisely $\varepsilon/2$ of the bottom of $\bar{A}$, then shift the leftmost to the left to $\varepsilon/2$ of the left edge of $\bar{A}$, and then one-by-one each successive block to exactly ε of the one to its left. Next shift the blocks in L_2 down to a line ε above the layer just built, and again shift left. Continue through all the L_i. This moves $\bar{B}_{l,\varepsilon} \ldots \bar{B}_{k,\varepsilon}$ by an m.p. C^∞ map f_1 to a lattice of blocks spaced uniformly ε apart. As $\varepsilon A/B > 3B$, above and to the right of these new block positions are empty strips at least $3B$ wide.

In the same manner move $\bar{B}'_{l,\varepsilon} \ldots \bar{B}'_{k,\varepsilon}$ by a map f_2 to a grid in the lower left-hand corner of $\bar{A}$. Let $\tilde{B}_{i,\varepsilon} = f_1(\bar{B}_{i,\varepsilon})$ and $\tilde{B}'_{i,\varepsilon} = f_2(\bar{B}'_{i,\varepsilon})$. Now the $\tilde{B}_{i,\varepsilon}$'s and the $\tilde{B}'_{i,\varepsilon}$'s may not fill out the same set of lattice positions in $\bar{A}$, so to simplify matters add further blocks $\tilde{B}_{k+1,\varepsilon} \ldots \tilde{B}_{l,\varepsilon}$ and $\tilde{B}'_{k+1,\varepsilon} \ldots \tilde{B}'_{l,\varepsilon}$ so that the arrays are square of the same size, and still have empty strips at least $2B$ wide at the top and right.

Our two lattices are now identical except for the order of the blocks. What we need, then, is to show that we can permute the positions of the $\tilde{B}_{i,\varepsilon}$'s to match the $\tilde{B}'_{i,\varepsilon}$'s. To do this it is sufficient to show that we can interchange any two blocks which are side by side, or one above the other, as such generate all permutations. We treat the side by side case, the other being symmetric to it.

Suppose $\tilde{B}_{i,\varepsilon}$ and $\tilde{B}_{j,\varepsilon}$ sit next to each other in layer p of the lattice. Move the first $(p-1)$ layers upward leaving a gap B wide between layers $(p-1)$ and p. Move $\tilde{B}_{i,\varepsilon}$ upward into the gap and then sideways until it lies directly over $\tilde{B}_{j,\varepsilon}$. Move $\tilde{B}_{j,\varepsilon}$ sideways into $\tilde{B}_{i,\varepsilon}$'s old position. Move $\tilde{B}_{i,\varepsilon}$ down into $\tilde{B}_{j,\varepsilon}$'s old position. Move the upper $(p-1)$ layers back into place. Thus we can permute the grid of $\tilde{B}_{i,\varepsilon}$'s to be arranged exactly like the $\tilde{B}'_{i,\varepsilon}$'s. Now shift them by f_2^{-1} to their proper places. □

Our proof of Proposition 1.1 will proceed from here as follows. We will construct an $\mathbf{R}^d$ action T analogous to the adding machine for $\mathbf{Z}$ actions. Next, for an arbitrary $\mathbf{R}^d$ action S we will construct a sequence of partitions $\{P'_i\}_{i=1}^\infty$ which generate under S, which on an S trajectory look like blocks nested inside blocks, the P'_i-blocks sitting inside P'_{i+1}-blocks as the $\bar{B}_{i,\varepsilon}$'s in Lemma 1.3 sat inside $\bar{A}$. Finally, using Lemma 1.3 inductively, we will simultaneously build a sequence of partitions $\{P_i\}_{i=1}^\infty$ and a smooth measure preserving time change f of T, modifying the $(T_f, \bigvee_{i=1}^\infty P_i)$-names to look exactly like $(S, \bigvee_{i=1}^\infty P_i)$-names, and so that $\{P_i\}_{i=1}^\infty$ generates under T_f (it is this last fact that uses the special form of T). This will complete our proof.

II. Construction of T. Set $\Omega = [0, 1)^{d+1}$ with Lebesgue measure. For $\omega \in \Omega$ write $\omega = (x, v)$, $v \in \mathbf{R}^d$. If (x, v) and $(x, v + v')$ are both in Ω, set $T_{v'}(x, v) = (x, v + v')$.

Now subdivide Ω into 2^d sets $A_1^l = [l/2^d, (l+1)/2^d) \times [0, 1)^d$. Stack these to form a $1/2^d \times 2 \times \ldots \times 2$ rectangle. Let g_1 be the map of $\Omega \leftrightarrow [0, 1/2^d) \times [0, 2)^d = \Omega_1$, linear on each A_1^l which accomplishes this stacking. If (x, v) and $(x, v + v')$ are both in Ω_1 set

$$T_{v'}(g^{-1}(x, v)) = g^{-1}(x, v + v').$$

This extends the definition of T_v to more of Ω. Figure 2 illustrates the stacking for $d = 2$.

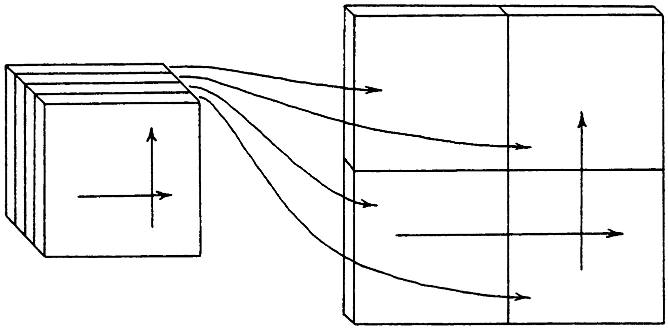

FIGURE 2

Inductively suppose $g_k;\ \Omega \leftrightarrow \Omega_k = [0, 1/2^{dk}) \times [0, 2^k)^d$ has been defined, linear on each $A_k^l = [l/2^{dk}, (l+1)/2^{dk}) \times [0, 1)^d$, and if (x, v) and $(x, v + v')$ are both in Ω_k,

$$T_{v'}\left(g_k^{-1}(x, v)\right) = g_k^{-1}(x, v + v').$$

Now cut Ω_k into 2^d sets of the form

$$\bar{A}_{k+1}^l = \left[\frac{l}{2^{d(k+1)}}, \frac{l+1}{2^{d(k+1)}}\right) \times \left[0, 2^k\right)^d, \quad l = 0 \ldots 2^d - 1.$$

Now the sets

$$A_{k+1}^l = \left[\frac{l}{2^{d(k+1)}}, \frac{l+1}{2^{d(k+1)}}\right) \times \left[0, 1\right)^d, \qquad l = 0 \ldots 2^{d(k+1)} - 1,$$

are precisely sets of the form $A_k^{l'} \cap g_k^{-1}(\bar{A}_{k+1}^{l''})$.

Let $\bar{g}_{k+1}$ be a map $\Omega_k \leftrightarrow \Omega_{k+1}$, linear on each $\bar{A}_{k+1}^l$, stacking them to fill Ω_{k+1}. Let $g_{k+1} = \bar{g}_{k+1} \circ g_k$. For any (x, v) and $(x, v + v') \in \Omega_{k+1}$, set

$$T_{v'}\left(g_{k+1}^{-1}(x, v)\right) = g_{k+1}^{-1}(x, v + v'),$$

further extending the definition of T_v. Continue inductively.

For any $M > 0$, consider the set S^M of $(x, v) \in \Omega$ for which $T_{v'}$ is undefined, some v' with $|v'| < M$. Now $S^M = \bigcap_{k=1}^{\infty} S_k^M$, where

$$S_k^M = \left\{(x, v) | d\left(g_{k+1}(x, v), \partial(\Omega_k)\right) < M\right\}.$$

Now $\mu(S_k^M) < M2d/2^k \underset{k}{\rightarrow} 0$. Thus $\mu(S^M) = 0$ and $T = \{T_v\}_{v \in \mathbf{R}^d}$ defined a.e. on Ω.

What is useful for us in this action is that the sets $g_k^{-1}([0, 1/2^{kd}) \times \{0\}^d)$ form bases for Rochlin towers of size $[0, 2^k)^d$, and also generate under T.

III. Building the nice partitions of an $\mathbf{R}^d$ action. Let $S = \{S_v\}_{v \in \mathbf{R}^d}$ be a free ergodic measure preserving action of $\mathbf{R}^d$ on $\{\Omega', \mathcal{F}', \mu'\}$. We will now build in Ω' a collection of partitions $\{P_i'\}_{i=1}^{\infty}$ where each $P_i' = \{P_{i,j}'\}_{j=0}^{l(i)}$ is a finite partition and

$$\mathcal{F}' = \bigvee_{i=1}^{\infty} \bigvee_{v \in \mathbf{R}^d}^{\infty} S_v(P_i').$$

The partitions are "nice" in that the P_i, S-name of a.e. $\omega' \in \Omega'$ will be made of blocks, all of the same size, each in $P_{i,1}' \ldots P_{i,l(i)}'$, surrounded by $P_{i,0}'$. We will call these blocks "labeling blocks." All the labeling blocks of P_i' will lie inside labeling blocks for P_{i+1}', and furthermore the subscript "label" on a P_{i+1}'-labeling block will determine all the positions and labels of the P_i'-labeling blocks in it. The partition names will have a bit more structure which we will indicate when we introduce it.

First set $\varepsilon_i = 2^{-(i+2)}$, and pick a refining sequence of partitions $\tilde{P}_1 \subset \tilde{P}_2 \subset \ldots \subset \tilde{P}_i \ldots, \bigvee_{i=1}^{\infty} \tilde{P}_i = \mathcal{F}'$. Let F_1' be the base of a Rochlin tower of size $B_1 = [0, 1)^d$, with error set of size ε_1. For any point ω in the tower, i.e. $S_{-v_1(\omega)}(\omega) \in F_1'$ some unique $v_1(\omega) \in [0, 1)^d$, let $B_1(\omega) = \bigvee_{v \in B_i} S_{v - v_1(\omega)}(\omega)$, the trajectory of ω in the 1-cube of the tower. Now let $\tilde{P}_1'$ be such that $|\tilde{P}_1, \tilde{P}_1'| < \varepsilon_1$ and

$$\{Q_i\}_{i=1}^{l(1)} = \bigvee_{v \in B_i} S_{-v}(\tilde{P}_i')/F_1'$$

is a finite partition. Let $P_1^1 = \{P_{1,i}^1\}_{i=1}^{l(1)}$ be a partition with $\omega \in P_{1,i}^1$ iff $S_{-v_1(\omega)}(\omega) \in Q_i$, $v(\omega) \in B_{1,\varepsilon_1}$, otherwise $\omega \in P_{1,0}^1$. Figure 3 illustrates this partition.

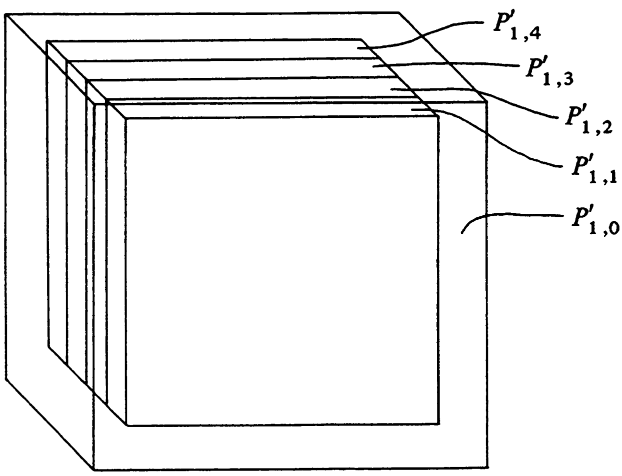

FIGURE 3

As we continue, this partition will be modified successively to $P_1^2, P_1^3, \ldots,$ converging to P_1'.

Now let $a_1 = \inf \mu(P_{1,i}^1)$. Pick $n(2)$ so large that for all but $\varepsilon_2/4$ of the $\omega \in \Omega'$, the density in

$$\bigcup_{v \in B_2} S_v(\omega), \quad B_2 = \left[0, 2^{n(2)}\right)^d,$$

of sets in P_1^1 is within $\varepsilon_2 a_1/4d$ of their measure. Furthermore we want $2^{-n(2)} < \varepsilon_2 a_1/10d$. Using this, let F_2' be the base of a Rochlin tower of size B_2, with error set of size ε_2, where for every $\omega \in F_2'$, the density of any set in P_1^1 in $\bigcup_{v \in B_2} S_v(\omega)$ is within $\varepsilon_2 a_1/2d$ of its measure.

For any ω in this tower, where $S_{-v_2(\omega)}(\omega) \in F_2'$, some unique $v_2(\omega) \in B_2$, let

$$B_2(\omega) = \bigcup_{v \in B_2} S_{v - v_2(\omega)}(\omega),$$

the trajectory through the tower containing ω.

Now modify P_1^1 as follows. Any ω for which $B_1(\omega)$ does not lie more than $\varepsilon_2 a_1/2$ from the boundary of $B_2(\omega)$, relabel to lie in $P_{1,0}^1$, i.e., erase the labeling cubes for P_1^1 too near the boundary. Call the new partition $\hat{P}_1^1$. Next, for every $\omega \in F_2'$, let $\bar{v}_1(\omega), \bar{v}_2(\omega) \ldots \bar{v}_{s(\omega)}(\omega)$ be those points of B_2 for which $S_{\bar{v}_i(\omega)}(\omega) \in F_1'$, and let $p_i(\omega)$ be the $\hat{P}_1^1$ label on the cube over ω. Let $R_2 = \{R_{2,i}\}_{i=1}$ partition F_2' so that two points ω_1, ω_2 are in the same set iff $|\bar{v}_i(\omega_1) - \bar{v}_i(\omega_2)| < \varepsilon_2 a_1/4d$ and $p_i(\omega_1) = p_i(\omega_2)$, $i = 1 \ldots s(\omega_1) = s(\omega_2)$. Let $\omega_i \in R_{2,i}$ be one point from each set in R_2. Further modify $\hat{P}_1^1$ to P_1^2 by setting, for any ω in the tower with $S_{\bar{v}_2(\omega)}(\omega) \in R_{2,i}$,

$$P_1^2(\omega) = \hat{P}_1^1\big(S_{v_2(\omega)}(\omega_i)\big),$$

i.e., shift the $\hat{P}_1^1$ names a little to be constant over each $R_{2,i}$. It follows that $|P_1^1, P_1^2| < 2\varepsilon_1$.

Let $\tilde{P}_2'$ be such that $|\tilde{P}_2, \tilde{P}_2'| < \varepsilon_2$ and

$$\bar{Q}_2 = \bigvee_{v \in B_2} S_{-v}(\tilde{P}_2')/F_2'$$

is finite. Let $Q_1 = \bar{Q}_2 \vee R_2$, $Q_2 = \{Q_{2,i}\}_{i=1}^{l(2)}$. Build P_2^2 by putting $\omega \in P_{2,i}^2$ if $S_{-v_2(\omega)}(\omega) \in Q_{2,i}$, and $v_2(\omega) \in B_{2,(\varepsilon_2 a_1/8)}$, otherwise $\omega \in P_{2,0}^2$. This builds the labeling blocks for P_2^2 with

(a) every labeling block for P_1^2 is at least $\varepsilon_2 a_1/8$ inside a P_2^2-labeling block,

(b) the label on a P_2^2-labeling block determines all the positions and labels of the P_1^2-labeling blocks inside it, and

(c) inside any P_2^2-labeling block the density of occurrence of the various labels for P_1^2-labeling blocks is within $2\varepsilon_1$ of its measure.

Continuing inductively, suppose we have constructed towers of sizes $B_k = [0, 2^{n(k)})^d$, $k = 1 \ldots \kappa$, with bases $F_1' \ldots F_\kappa'$, and partitions $P_1^\kappa \ldots P_\kappa^\kappa$, where

$P_j^\kappa = \{P_{j,i}^\kappa\}_{i=1}^{l(j)}$ where each $P_{j,k}^\kappa$, $i \neq 0$, intersects a trajectory in $(2^{n(k)} - \varepsilon_k a_{k-1}/8)$-blocks,

$$a_{k-1} = \inf_i \mu\left(P_{k-1,i}^{k-1}\right),$$

with

(a) every labeling block of P_j^κ is at least $\varepsilon_{j+1}a_j/8$ inside a P_{j+1}^κ-labeling block,

(b) the label on a P_{j+1}^κ-labeling block determines all the positions and labels of the P_j^κ-labeling blocks inside it and

(c) inside any P_{j+1}^κ-labeling block the density of occurrence of the various labels for P_j^κ-labeling blocks is within $\sum_{i=j}^{\kappa} 2\varepsilon_i$ of its measure.

We now set $a_\kappa = \inf_i \mu(P_{\kappa,i}^\kappa)$, choose $n(\kappa + 1)$ so large that for all but $(\varepsilon_\kappa + 1)/4$ of the $\omega \in \Omega'$, the density in $\bigcup_{v \in B_{\kappa+1}} S_v(\omega)$, $B_{\kappa+1} = [0, 2^{n(\kappa+1)})^d$, of sets in P_κ^κ is within $\varepsilon_{\kappa+1}a_\kappa/4d$ of their measure. Furthermore, make sure

$$2^{n(\kappa+1)} > 3 \cdot 2^{2n(\kappa)}(8/\varepsilon_2 a_1).$$

Build a Rochlin tower of size $B_{\kappa+1}$, so that for all $\omega \in F'_{\kappa+1}$, the densities of P_κ^κ-labeling blocks is within $\varepsilon_{\kappa+1}a_\kappa/2d$ of their measure.

As for $\kappa = 2$, modify $P_1^\kappa \dots P_\kappa^\kappa$ to $P_1^{\kappa+1} \dots P_\kappa^{\kappa+1}$ with $|P_i^\kappa, P_i^{\kappa+1}| < 2\varepsilon_\kappa$, and build $P_{\kappa+1}^{\kappa+1}$, using $P_{\kappa+1}$, satisfying (a), (b) and (c) for $\kappa = \kappa + 1$.

Now as $|P_i^\kappa, P_i^{\kappa+1}| < 2\varepsilon_\kappa$, $\lim_{\kappa\to\infty} P_i^\kappa = P_i'$ exists, and furthermore as

$$\tilde{P}_i \overset{\varepsilon_i}{\subset} \bigvee_{v \in \mathbf{R}^d} S_v(P_i^i), \qquad \mathfrak{F}' = \bigvee_{k=1}^{\infty} \bigvee_{v \in \mathbf{R}^d} S_v(P_k').$$

Finally, (a), (b) and (c) extend to the limit partitions so that

(A) every labeling block of P_j' is at least $\varepsilon_{j+1}a_j/8$ inside a P_{j+1}'-labeling block,

(B) the label on a P_{j+1}'-labeling block determines all the positions and labels of the P_j'-labeling blocks inside it and

(C) inside a P_{j+1}'-labeling block, the density of occurrence of the various labels of P_j'-labeling blocks is within $\sum_{i=j}^{\infty} 2\varepsilon_i = 4\varepsilon_j$ of its measure.

Figure 4 illustrates how the P_j' partition a segment of an orbit for $d = 2$. Notice that since the P_j'-name across a P_{j+1}'-labeling block is the P_j^j-name of some point ω across the midsection of the $(j + 1)$st tower, any two P_j'-labeling blocks are separated by a gap at least $\varepsilon_2 a_1/4$ wide, and as

$$\left(\frac{\varepsilon_2 a_1}{8}\right)\frac{2^{n(j)}}{2^{n(j-1)}} > 3 \cdot 2^{n(j)-1},$$

Lemma 1.3 applies to P_j'-labeling blocks sitting inside a P_{j+1}'-labeling block. We will now prove *Proposition* 1.1 by using Lemma 1.3 to label and smoothly modify the blocks in the orbits of T to look like the labeling blocks in an orbit of S.

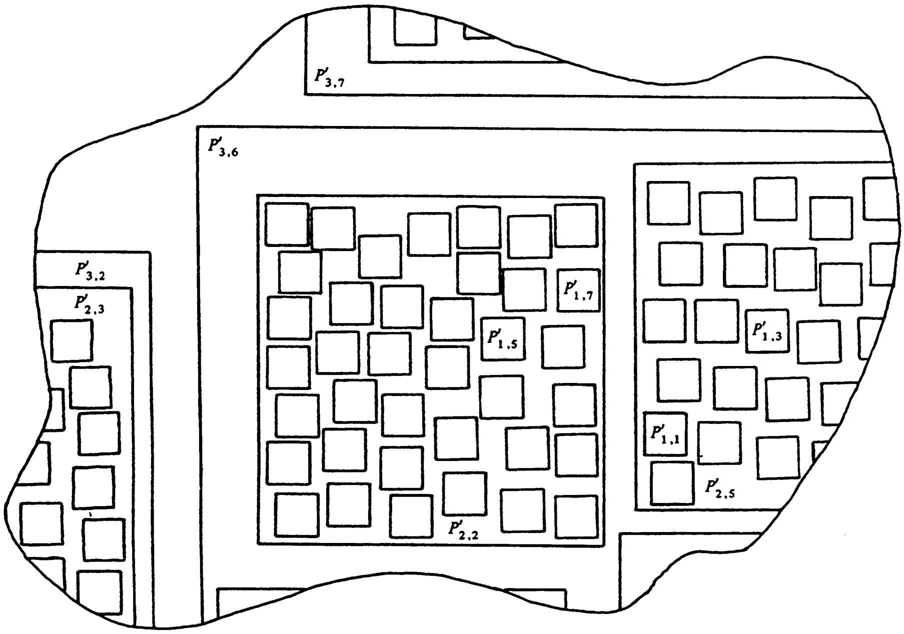

FIGURE 4

IV. Proof of Proposition 1.1. We construct inductively partitions P_i and a smooth time change f so that $(T_f, \bigvee_{i=1}^{\infty} P_i)$ is the same process as $(S, \bigvee_{i=1}^{\infty} P'_i)$, and

$$\mathfrak{F} = \bigvee_{i=1}^{\infty} \bigvee_{v \in \mathbf{R}^d} T_{f(\cdot, v)}(P_i).$$

We first construct a partition $\overline{P}_1^1$, as an initial approximation to P_1. In $\Omega = [0, 1) \times [0, 1)^d$, write, for $i \neq 0$,

$$\overline{P}_{1,i}^1 = \left[\frac{k_i}{2^{n(2)}}, \frac{k_{i+1}}{2^{n(2)}} \right) \times \left[\varepsilon_i, 1 - \varepsilon_i \right)^d, \quad 0 = k_1 < \cdots < k_{I(1)+1} \leqslant 1,$$

with $|\mu(\overline{P}_{1,i}^1) - \mu(P'_{1,i})| < \varepsilon_1 a_1$. We have chosen $n(2)$ large enough to do this. The rest of Ω is in $\overline{P}_{1,0}^1$. As we will see later, it is very important that each $\overline{P}_{1,i}^1$, $i \neq 0$, intersects the first coordinate in an interval.

Now $\overline{P}_1^1$ on Ω looks almost exactly like the first Rochlin tower for S. Look at $g_{n(2)}(\overline{P}_1^1)$ as a partition of $\Omega_{n(2)}$. Every block of trajectory $x \times [0, 2^{n(2)})^d$ in $\Omega_{n(2)}$ has the same $g_{n(2)}(\overline{P}_1^1)$-name, and hence in this name the density of any particular labeling block $\overline{P}_{1,i}^1$ is within $\varepsilon_1 a_1$ of $\mu'(P'_i)$.

In each labeling block for P'_2, the density of labeling blocks from P'_1 is within $4\varepsilon_1$ of its measure, by (C). Furthermore by (B), the P'_2 label on the block determines the positions and labels of the P'_1-blocks in it.

Partition $\Omega_{n(2)}$ into sets

$$\bar{P}^2_{2,i} = \left[\frac{k_i}{2^{n(3)}}, \frac{k_{i+1}}{2^{n(3)}}\right) \times \left[\frac{\varepsilon_2 a_1}{8}, 2^{n(2)} - \frac{\varepsilon_2 a_1}{8}\right),$$

with $|\mu(\bar{P}^2_{2,i}) - \mu(P_{2,i})| < \varepsilon_2 a_2$. We chose $n(3)$ large enough to do this. Fix a set $A_i = [k_i/2^{n(3)}, k_{i+1}/2^{n(3)})$, and let $B_1 \ldots B_{2^{n(3)}}$ be the $g_{n(2)}(\bar{P}^1_1)$-labeling blocks in $x \times [0, 2^{n(\,)})^d$, $x \in A_i$.

By relabeling or deleting a subset of these blocks of density at most $5\varepsilon_1$, modify $\bar{P}^1_1/(A_i \times [0, 2^{n(2)})^d)$ to $\bar{P}^2_1/(A_i \times [0, 2^{n(2)})^d)$ so that the number of $\bar{P}^2_1$-labeling blocks B_j labeled $\bar{P}^2_{1,k}$ is the same as the number of blocks labeled $P'_{1,k}$ in a $P'_{2,i}$-labeling block. Using Lemma 1.3, let $\tilde{f}_i$; $[0, 2^{n(2)}) \circlearrowleft$ be a C^∞ measure preserving homeomorphism, which on $([\varepsilon_2 a_1/8, 2^{n(2)} - \varepsilon_2 a_1/8)^d)^C$ is the identity, and on $\bar{P}^2_1$-labeling blocks is a linear translation to new positions so that $\tilde{f}_{1,i}(g_{n(2)}(\bar{P}^2_1))$ partitions $\Omega_{n(2)}$ exactly as P'_1 partitions a $P'_{2,i}$-labeling block. Let $f_1(x, v) = (x, \tilde{f}_i(v))$ if $x \in A_i$.

Continue inductively, at stage κ, on $g_{n(\kappa)}(\Omega)$, first construct a first approximation $\bar{P}^\kappa_\omega$ of P'_κ, then modify $\bar{P}^{\kappa-1}_1 \ldots \bar{P}^{\kappa-1}_{\kappa-1}$ to $\bar{P}^\kappa_1 \ldots \bar{P}^{\kappa-1}_{\kappa-1}$, changing by at most $5\varepsilon_{\kappa-1}$, so that the density in a $\bar{P}^\kappa_{\kappa,i}$-labeling block of $\bar{P}^\kappa_{\kappa-1}$-labeling blocks is precisely that of $P'_{\kappa-1}$-labeling blocks in a $P'_{\kappa,i}$-labeling block, modifying the $\bar{P}^{\kappa-1}_1 \ldots \bar{P}^{\kappa-1}_{\kappa-2}$-blocks to preserve the coding from the $\bar{P}^{\kappa-1}_{\kappa-1}$ label on a block to the positions and labels of the $\bar{P}^{\kappa-1}_1 \ldots \bar{P}^{\kappa-1}_{\kappa-2}$-blocks in it.

Next use Lemma 1.3 to construct a function f_κ; $\Omega_{n(\kappa)}$, a C^∞ measure preserving homeomorphism, which outside $g_{n(\kappa)}(\bar{P}^\kappa_\kappa)$-labeling blocks is the identity, is linear translation on $g_{n(\kappa)}(\bar{P}^\kappa_{\kappa-1})$-labeling blocks, and so that $f_\kappa(g_{n(\kappa)}(\bar{P}^\kappa_{\kappa-1}))$ partitions a $P^\kappa_{\kappa,i}$-labeling block exactly as $P'_{\kappa-1}$ partitions a $P'_{\kappa,i}$-labeling block.

Now set $P_i = \lim_{\kappa\to\infty} \bar{P}^\kappa_i$. This is well defined as $|\bar{P}^\kappa_i, P^{\kappa+1}_i| < 3\varepsilon_\kappa$. Also define $\bar{f}_k$; $\Omega \circlearrowleft$ by

$$\bar{f}_k = \prod_{i=1}^{k} g^{-1}_{n(i)} f_i g_{n(i)}.$$

What $\bar{f}_k$ does is to map each P_k-labeling block to itself, but inside a $P_{k,i}$-labeling block it modifies the $\bigvee_{j=1}^{k-1} P_j$-name to look exactly like the $\bigvee_{j=1}^{k-1} P'_j$-name across a $P'_{k,i}$-labeling block. Also notice $\bar{f}_k \circ \bar{f}^{-1}_{k'}$, for $k' < k$, maps a $P_{k'}$-labeling block by a linear translation within its T orbit.

We can now write down the smooth time change from T to S. Let $\bar{\Omega} = \{(x, v)|$ for any $v' \in \mathbf{R}^d$, for some k, both $\bar{f}_k(x, v)$ and $T_{v'}(\bar{f}_k(x, v))$ belong to the same P_k-labeling block in the orbit of $(x, v)\}$. It is easy to check that $\bar{\Omega}$ is a T invariant set of full measure. For any $v' \in \mathbf{R}^d$, $(x, v) \in \bar{\Omega}$, set

$$T_{f((x,v),v')}(x, v) = \bar{f}^{-1}_k\big(T_{v'}\big(\bar{f}_k(x, v)\big)\big),$$

for k such that $\bar{f}_k(x, v)$ and $T_{v'}(\bar{f}_k(x, v))$ are both in the same P_k-labeling block.

Just to check the obvious, if $k' < k$ and $\bar{f}_{k'}(x, v)$ and $T_{v'}(\bar{f}_k(x, v))$ are both in the same $P_{k'}$-labeling block, then they are also in the same P_k-labeling block and

$$T_{v'}\left(\bar{f}_k\bar{f}_{k'}^{-1}\left(\bar{f}_k(x, v)\right)\right) = \bar{f}_k\bar{f}_{k'}^{-1}\left(T_{v'}\left(\bar{f}_k(x, v)\right)\right)$$

as $f_k f_{k'}^{-1}$ is linear translation on $P_{k'}$-labeling blocks, so

$$f_k^{-1}T_{v'}f_k(x, v) = f_{k'}^{-1}T_{v'}f_{k'}(x, v)$$

and f is well defined.

It is obvious from its form that f is a smooth measure and orientation preserving time change and that

$$\left(T_f, \bigvee_{i=1}^{\infty} P_i\right) \quad \text{and} \quad \left(S, \bigvee_{i=1}^{\infty} P_i'\right)$$

are identical processes.

We need now only that

$$\bigvee_{i=1}^{\infty} \bigvee_{v\in\mathbf{R}^d} T_{f(\cdot,v)}(P_i) = \mathfrak{F}.$$

What we will check is that the T_f, $\bigvee_{i=1}^{\infty}P_i$-name of a.e. point (x, v) determines (x, v) uniquely. Let A_l be the set (x, v) in a P_l^l-labeling block whose label is never modified. Now $\mu(A_l^c) \leqslant 10\varepsilon_l$, and

$$g_{n(l)}((x, v)) \in \left[\frac{k_j}{2^{n(l)}}, \frac{k_{j+1}}{2^{n(l)}}\right) \times \left[\frac{\varepsilon_l a_{l-1}}{8}, 2^{n(l)} - \frac{\varepsilon_l a_{l-1}}{8}\right).$$

The label $P_{l,j}^l$ on (x, v) now tells us the function $f_l(x, v)$, as it depends only on the P_l label of (x, v). Thus, from the T_f, $\bigvee_{i=1}^{l}P_l$-name of such an (x, v) we know its T, $\bigvee_{i=1}^{l}P_l$-name across the P_l-labeling block it lies in.

Let $A = \overline{\lim}\, A_l$, a set of full measure. Let $(x, v) \in A$, and thus for l large enough $(x, v) \in A_l$, and as we saw above, from the T_f, $\bigvee_{i=1}^{l}P_l$-name of (x, v) we know its T, $\bigvee_{i=1}^{l}$ P-name across the P_l-labeling block containing it, for all l large enough, hence the entire T_1 $\bigvee_{i=1}^{l}P_l$-name of (x, v). Thus

$$\bigvee_{i=1}^{\infty} \bigvee_{v\in\mathbf{R}^d} (T_v(P_i)) = \bigvee_{i=1}^{\infty} \bigvee_{v\in\mathbf{R}^d} (T_{f(\cdot,v)}(P_i)),$$

and we only need check $\bigvee_{i=1}^{\infty}P_i$ generates for T.

But now for $(x, v) \in A$, for arbitrarily large l, the T, P_l-name of (x, v) gives us a j and a vector v' with

$$g_l(x, v) \in \left[\frac{k_i}{2^{n(l)}}, \frac{k_{i+1}}{2^{n(l)}}\right) \times v',$$

hence

$$(x, v) \in \left[\frac{k}{2^{n(l)}}, \frac{k + (\bar{k}_{i+1} - \bar{k}_i)}{2^{n(l)}} \right) \times v''$$

some k and v''. But now $(\bar{k}_{i+1} - \bar{k}_i)/2^{n(l)}$ is bounded above by the maximum size of any set in P_l, which goes to zero in l. Hence $\bigvee_{i=1}^{\infty} \bigvee_{v \in \mathbf{R}^d} T_v(P_i)$ separates points, and hence is $\mathfrak{F}$. This completes Proposition 1.1.

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