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ON THE GLOBAL ASYMPTOTIC BEHAVIOR OF BROWNIAN LOCAL TIME ON THE CIRCLE

BY

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ABSTRACT. The asymptotic behavior of the local time of Brownian motion on the circle is investigated. For fixed time point t this is a (random) continuous function on S^1. It is shown that after appropriate norming the distribution of this random element in $C(S^1)$ converges weakly as $t \to \infty$. The limit is identified as $2(B(x) - \int B(y)\,dy)$ where B is the Brownian bridge. The result is applied to obtain the asymptotic distribution of a Cramer-von Mises type statistic for the global deviation of the local time from the constant t on S^1.

1. Introduction. Let $(\Omega, \mathfrak{F}, \xi_t, P^x)$ be a continuous Brownian motion on $\mathbf{R}$; that is for $t \in [0, \infty)$, $\xi_t\colon \Omega \to \mathbf{R}$ is $\mathfrak{F}$-measurable, for each $\omega \in \Omega$, $t \mapsto \xi_t(\omega)$ is continuous and ξ_t under the law P^x is Brownian motion starting in $x \in \mathbf{R}$. We assume that $\mathfrak{F}$ contains all sets of P^x-measure 0 for all x. We assume that there are transition operators on Ω, that are mappings $\theta_s\colon \Omega \to \Omega$ such that $\xi_t(\theta_s\omega) = \xi_{t+s}(\omega)$ for all $t,s \in [0, \infty)$.

Let $\mathfrak{F}_t$ be the σ-algebra generated by the ξ_s, $s \leqslant t$, and the common P^x-null sets. If the set where some event is not true is of P^x measure 0 for all $x \in \mathbf{R}$ then we say that this event holds almost sure.

Let S^1 be a circle with circumference 1. We shall identify S^1 with the half-open interval $[0, 1) \subset \mathbf{R}$. Addition on S^1 is defined in the usual way. If $x,y \in S^1$ then $|x - y| = \min((x - y) \bmod 1, (y - x) \bmod 1)$. Brownian motion ζ_t on S^1 can be defined through $\zeta_t = \xi_t \bmod 1 = \xi_t - [\xi_t]$ where $[x] = \max\{k \in \mathbf{Z}\colon k \leqslant x\}$. For $x \in [0, 1)$, P^x governs ξ_t starting at x, so it governs ζ_t too.

Any 'starting' probability measure μ on $\mathbf{R}$ gives in the usual way a law P^μ for ξ_t formally by $P^\mu = \int P^x\mu(dx)$. If μ is concentrated on $[0, 1)$ P^μ is a law for ζ_t.

For Brownian motion ξ_t there exists continuous local time l which is a mapping $(0, \infty) \times \mathbf{R} \times \Omega \to [0, \infty)$ with

(1.1) For each $(t, x) \in (0, \infty) \times \mathbf{R}$, $l(t, x)$ is $\mathfrak{F}_t$ measurable.

(1.2) Almost surely l is joint continuous in (t, x).

(1.3) Almost surely for $a < b$ and arbitrary t

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$$\int_a^b l(t, x)\, dx = \int_0^t 1_{[a,b]}(\xi_s)\, ds.$$

Brownian local time starts growing after the first hitting of x that is if $x \in \mathbf{R}$ and $T_x = \inf\{s: \xi_s = x\}$ then almost surely $l(u, x, \omega) = 0$ for $u \leqslant T_x(\omega)$. For each $x \in \mathbf{R}$ Brownian local time at x is a perfect additive functional that is a.s. for each x, s, t, $l(t + s, x, \omega) = l(t, x, \omega) + l(s, x, \theta_t(\omega))$ (see Blumenthal and Getoor [3, p. 225]); especially if T is a stopping time, one has $l(T(\omega) + s, x, \omega) = l(T(\omega), x, \omega) + l(s, x, \theta_{T(\omega)}(\omega))$ and so for each y the P^y law of $l(T + s, x) - l(T)$ conditioned on $\mathcal{F}_T$ is the P^{ξ_T} law of $l(s, x)$. So one may say that the local time starts afresh at stopping times.

Local time for Brownian motion on the circle exists too, which follows from general theorems about existence of local times [3, p. 216] but we shall construct it directly from l. Let $T_n = \inf\{t: |\xi_t| = n\}$ and $A_n(t) = \{\omega: |\xi_0| < n,\ T_n > t\}$. $\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} A_n(k)$ has P^x measure 1 for each x. For $x \in [0, 1)$, $k \in \mathbf{N}$ and $\omega \in A_n(k)$ one has

$$\sum_{j=-\infty}^{\infty} l(s, x + j, \omega) = \sum_{j=-n}^{n-1} l(s, x + j, \omega) \quad \text{for } s \leqslant k,$$

so for $\omega \in \Lambda$, $\sum_{j=-\infty}^{\infty} l(s, x + j, \omega)$ is a finite sum. We introduce local time for ζ_t by

$$\lambda(s, x, \omega) = \sum_{k=-\infty}^{\infty} l(s, x + k, \omega) \tag{1.4}$$

which is then a.s. well defined and fulfills statements analogue to (1.1)–(1.3). For each fixed s, $X_s(x) = \sqrt{s}\,(\lambda(s, x)/s - 1)$ is a random element in $(C(S^1), \|\ \|)$ where $C(S^1)$ is the space of continuous functions on S^1 and $\|\ \|$ the sup-norm. The main result of this paper is the following

THEOREM 1. *For each starting measure μ on S^1 and for $s \to \infty$, X_s converge weakly to the continuous centered Gaussian process on S^1 with covariance*

$$\Gamma(x, y) = 4(\min(x, y) - xy) - 2(x(1 - x) + y(1 - y)) + \tfrac{1}{3}.$$

REMARKS. (1) It is easy to see that $\Gamma(x, x + y) = \Gamma(0, y)$ and further $\Gamma(0, 1) = \Gamma(0, 0)$. So $\Gamma(x, y)$ is actually continuous on S^1. Let $Z(x)$ be the process corresponding to this kernel. It is then easy to see that $Z(x)$ may be represented as $2(B(x) - \int_0^1 B(x)\, dx)$ where $(B(x))_{x \in [0,1)}$ is the Brownian bridge.

(2) A central limit theorem for the local time at a single point and for a large class of 1-dimensional ergodic diffusions has been obtained by Tanaka [11]. Asymptotic behavior of the global local time seems not to have been considered in the literature as yet.

As an application one obtains the following corollary which has also been proved by Baxter and Brosamler [1], [4] using different methods.

COROLLARY 1. *If ν is a signed measure with finite total variation on S^1 then for an arbitrary starting distribution (for ζ_t) $(\int_0^1 \lambda(t, x)\nu(dx) - t\lambda(S^1))/\sqrt{t}$ is asymptotically normally distributed with mean* 0 *and variance* $\int_0^1 \int_0^1 \Gamma(x, y)\nu(dx)\nu(dy)$.

Baxter and Brosamler obtained this theorem for a quite large class of Markov-processes on compact spaces.

The main advantage of Theorem 1 however is that it enables one to consider nonlinear functionals on the local time. An application of this type is the following corollary which gives the asymptotic distribution of a measure of discrepancy between the local time and the density of the ergodic distribution on S^1 (which is of course Lebesgue measure).

COROLLARY 2. *For each starting distribution on S^1, $t\int_{S^1}(\lambda(t, x)/t - 1)^2\, dx$ converges in distribution to a probability on $\mathbf{R}^+$ with characteristic function*

$$\chi(t) = \prod_{j=1}^{\infty}\left(1 - \frac{2it}{\pi^2 j^2}\right)^{-1}.$$

PROOF. We consider the compact integral operator Γ on $L_2(S^1)$ where $(\Gamma f)(x) = \int \Gamma(x, y) f(y)\, dy$. The nonzero eigenvalues of this operator are $1/\pi^2 j^2$, $j \in \mathbf{N}$, each with multiplicity 2 and normalized eigenfunctions $\sqrt{2}\, \sin(2\pi jx)$, $\sqrt{2}\, \cos(2\pi jx)$. In fact an elementary calculation shows that these functions are eigenfunctions to $1/\pi^2 j^2$ and as they form a complete system in $\overline{\mathrm{Rg}\,(\Gamma)} = \{f \in L_2(S^1)\colon \int f(x)\, dx = 0\}$ there are no others.

It follows that the Karhunen-Loéve expansion for Z is

$$Z(x) = \sum_{j=1}^{\infty} \frac{1}{\pi j}\left[\sqrt{2}\, N_j \sin(2\pi jx) + \sqrt{2}\, N_j' \cos(2\pi jx)\right]$$

where N_j, N_j' are independent standard Gauss variables. So $\int_0^1 Z(x)^2\, dx = \sum_{j=1}^{\infty} (N_j^2 + N_j'^2)/\pi^2 j^2$ and from this expression the corollary follows.

Let $\mathfrak{M}(S^1)$ be the space of signed measures on S^1, $\|\ \|$ the total variation norm on $\mathfrak{M}(S^1)$. The following map is continuous $(C(S^1), \|\ \|) \xrightarrow{\Phi} (\mathfrak{M}(S^1), \|\ \|)$ where $\Phi(f)(A) = \int_A f(x)\, dx$. If $\mu_t(dx, \omega) = \Phi(\lambda(t, \cdot, \omega))$ then μ_t is simply the measure of sojourn times $\mu_t(A) = \int_0^1 1_A(\zeta_s)\, ds$. If ρ is Lebesgue measure on S^1 one obtains from Theorem 1

COROLLARY 3. *For $t \to \infty$, $\sqrt{t}\,(\mu_t/t - \rho)$ converges weakly in $(\mathfrak{M}(S^1), \|\ \|)$ to the $\mathfrak{M}(S^1)$-valued random variable $\omega \to \Phi(Z(\cdot, \omega))$.*

One should compare this with results obtained by Donsker and Varadhan in [5]. They evaluated for a large class of Markov processes on a compact set

K the leading term in the asymptotic expansion of $P(\mu_t/t \in A)$ for special sets $A \subset \mathcal{P}(K)$ (the space of probability measures on K) with $\rho \notin A$ (ρ in the general case being the ergodic distribution). In the special case of Brownian motion on S^1 their result stands in the same relation to the above corollary as stand classical large deviation results to the classical central limit theorem

2. Applications of the Tanaka formula. The basic idea of the Tanaka formula is to represent Brownian local time $l(t, x)$ by $\int_0^t \delta_x(\xi_s)\, ds$ where δ_x is the Dirac impulse function at x. δ_x is not a function in the usual sense, but is formally the second derivative of $f(y) = x \vee y$. Applying the Ito formula in a formal way one obtains

$$(\xi_t \vee x) - (\xi_0 \vee x) = \int_0^t 1_{[x,\infty)}(\xi_s)\, d\xi_s + \frac{1}{2}\int_0^t \delta_x(\xi_s)\, ds.$$

This suggests to declare $\xi_t \vee x - \xi_0 \vee x - \int_0^t 1_{[x,\infty)}(\xi_s)\, d\xi_s$ as half the local time at x. McKean proved in [8] that this process has a version which fulfills (1.1)–(1.3), so this is indeed justified. (McKean's local time is half ours.) Thus

$$\frac{1}{2}\, l(t, x) = (\xi_t \vee x) - (\xi_0 \vee x) - \int_0^t 1_{[x,\infty)}(\xi_s)\, d\xi_s. \tag{2.1}$$

Let T be a Brownian stopping time with

$$\int_0^\infty P^y(T > s, \xi_s \geqslant x)\, ds < \infty \quad \text{for all } y \in \mathbf{R}. \tag{2.2}$$

For each $t > 0$,

$$\int_0^{t \wedge T} 1_{[x,\infty)}(\xi_s)\, d\xi_s = \int_0^\infty 1_{[0,T\wedge t]}(s) 1_{[x,\infty)}(\xi_s)\, d\xi_s$$

(see e.g. [6, p. 72]). If (2.2) is fulfilled then, letting $t \to \infty$, the right side of the above equality goes to $\int_0^\infty 1_{[0,T]}(s)1_{[x,\infty)}(\xi_s)\, d\xi_s$ in probability. So one obtains for a stopping time T with (2.2):

$$\frac{1}{2}\, l(T, x) = (\xi_T \vee x) - (\xi_0 \vee x) - \int_0^\infty 1_{[0,T]}(s) 1_{[x,\infty)}(\xi_s)\, d\xi_s. \tag{2.3}$$

As an application one obtains Lemma 1 below where the following two well-known relations are used: Let f, g be two $\mathcal{F}_t$ adapted processes with $\int_0^\infty E^y(f(t)^2)dt < \infty$ and the same for g. Then

$$E^y \int_0^\infty f(t)\, d\xi_t = 0, \tag{2.4}$$

$$E^y \int_0^\infty f(t)\, d\xi_t \int_0^\infty g(t)\, d\xi_t = \int_0^\infty E^y(f(t)g(t))\, dt. \tag{2.5}$$

LEMMA 1. *For* $\varepsilon > 0$ *let* $T^\varepsilon = \inf\{t\colon |\xi_t| = \varepsilon\}$ *then*

$$E^0(l(T^\varepsilon, 0)) = \varepsilon, \tag{2.6}$$

$$\operatorname{var}^0(l(T^\varepsilon, 0)) = \varepsilon^2. \tag{2.7}$$

PROOF.

$$\int_0^\infty P^0(T \geqslant s, \xi_s \geqslant x)\, ds \leqslant \int_0^\infty P^0(T \geqslant s)\, ds = E(T) < \infty.$$

So (2.3)–(2.5) are applicable. From (2.3) and (2.4) one gets $E^0(l(T, 0)) = 2E(\xi_T \vee 0) = \varepsilon$. So (2.6) is proved.

$$\frac{1}{4} E^0\big(l(T, 0)^2\big) = E^0\big((\xi_T \vee 0)^2\big) - 2E^0\left((\xi_T \vee 0)\int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right)$$
$$+ E^0\left(\left(\int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right)^2\right).$$

$$E^0\left((\xi_T \vee 0)\int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right) = \frac{1}{2} E^0\left(|\xi_T|\int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right)$$
$$+ \frac{1}{2} E^0\left(\int_0^T d\xi_s \int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right).$$

The first summand is zero and the second equals

$$\frac{1}{2}\int_0^\infty E^0\big(1_{[0,T]}(s) 1_{[0,\infty)]}(\xi_s)\big)\, ds = \frac{1}{2} E^0\left(\int_0^T 1_{[0,\infty)}(\xi_s)\, d\xi_s\right)^2.$$

So $\frac{1}{4}E^0(l(T, 0)^2) = E^0((\xi_T \vee 0)^2) = \frac{1}{2}\varepsilon^2$. Thus (2.7) follows.

We are going to investigate $\lambda(t, x) = \sum_{k=-\infty}^{\infty} l(t, x + k)$ on $\Lambda = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty A_n(k)$ ($A_n(k)$ was $\{\omega\colon |\xi_0| < n, T^n > k\}$), $x \in [0, 1)$. Let $x, y \in S^1$ with $|x - y| = \delta < \frac{1}{2}$. There are two cases. Either $x = y + \delta \bmod 1$ or $y = x + \delta \bmod 1$. We assume the second to be true which is no restriction to the ensuing considerations. Set on $\mathbf{R}^1$: $y_0 = y$, $x_0 = y - \delta$ (it may be that $x_0 = x - 1$) and $y_k = y_0 + k$, $x_k = x_0 + k$. Then $\lambda(t, x) - \lambda(t, y) = \sum_{k=-\infty}^\infty l(t, x_k) - l(t, y_k)$ which on $A_n(k)$ is $\sum_{j=-n}^n l(t, x_j) - l(t, y_j)$ for $t \leqslant k$.

Let $\tilde{F}_n(z) = \sum_{j=-n}^n (z \vee x_j) - (z \vee y_j)$. From the Tanaka formula, one obtains on $A_n(k)$ and for $t \leqslant k$:

$$\lambda(t, x) - \lambda(t, y) = \tilde{F}_n(\xi_t) - \tilde{F}_n(\xi_0) - \int_0^t \tilde{F}_n'(\xi_s)\, d\xi_s.$$

Now $\delta\xi_t - \delta\xi_0 = \int_0^t \delta\, d\xi_s$, so if $F_n(z) = \tilde{F}_n(z) - \delta(z - x_0) + (n + 1)\delta$ one has on $A_n(k)$ and for $t \leqslant k$:

$$\lambda(t, x) - \lambda(t, y) = F_n(\xi_t) - F_n(\xi_0) - \int_0^t F_n'(\xi_s)\, d\xi_s.$$

It follows from an elementary calculation that for $m > n$, F_m restricted to $[-n, n]$ is F_n. So if $F\colon \mathbf{R} \to \mathbf{R}$ is defined by $F|_{[-n,n]} = F_n$ one has

$$\lambda(t, x) - \lambda(t, y) = F(\xi_t) - F(\xi_0) - \int_0^t F'(\xi_s)\, d\xi_s \tag{2.8}$$

on $A_n(k)$ and $t \leqslant k$ for each n, so on Λ and for all t. One easily obtains that

$$\sup_{u \in \mathbf{R}} |F(u)| \leqslant \delta \quad \text{and} \tag{2.9}$$

$$F'(u) = \begin{cases} 1 - \delta & \text{for } x_k \leqslant u \leqslant y_k, \\ -\delta & \text{else.} \end{cases} \tag{2.10}$$

Lemma 2. *For each starting distribution μ on S^1 which has bounded density with respect to Lebesgue measure, there is a $c > 0$ such that for each $t \geqslant 1$, $0 < \delta < \frac{1}{2}$ and $x,y \in S^1$ with $|x - y| = \delta$ one has*

$$E^\mu \left(\int_0^t F'(\xi_n)\, d\xi_n \right)^4 \leqslant ct^2\delta^2 \tag{2.11}$$

(*where F is constructed from x, y as above*).

Proof. For the sake of notational convenience we assume that $0 \leqslant x < y = x + \delta < 1$. We use Problem 4 on p. 40 of [**8**]. Thus

$$\begin{aligned} E^\mu \left(\int_0^t F'(\xi_u)\, d\xi_u \right)^4 &\leqslant 36 E^\mu \left(\int_0^t (F'(\xi_u))^2\, du \right)^2 \\ &\leqslant 72 E^\mu \left(\int_0^t \hat{g}(\xi_u)\, du \right)^2 + c_1 t^2 \delta^2 \end{aligned} \tag{2.12}$$

where

$$\hat{g}(u) = \begin{cases} 1 & \text{for } x + k \leqslant u \leqslant y + k,\ k \in \mathbf{Z}, \\ 0 & \text{else.} \end{cases}$$

If $g\colon S^1 \to \mathbf{R}$ is $\hat{g}$ restricted to $[0, 1)$ one obtains

$$\begin{aligned} E^\mu \left(\int_0^t \hat{g}(\xi_u)\, du \right)^2 &= E^\mu \left(\int_0^t g(\zeta_u)\, du \right)^2 \\ &= \int_0^t \int_0^t E^\mu (g(\zeta_u) g(\zeta_v))\, du\, dv \\ &= 2 \iint_{u \leqslant v} E^\mu (g(\zeta_u) g(\zeta_v))\, du\, dv \\ &= 2 \int_0^t \int_0^{t-u} E^\mu (g(\zeta_u) g(\zeta_{u+s}))\, ds\, du \\ &= 2 \int_0^t \int_0^{t-u} \left(\int_{S^1} \int_{S^1} \int_{S^1} p_u(y, z) p_s(z, v) g(v) g(z)\, \mu(dy)\, dz\, dv \right) ds\, du \end{aligned}$$

where $p_t(x, y)$ is the transition density for Brownian motion on S^1. By the assumption on μ there is a c_2 with $\int_{S^1} p_u(y, z)\,\mu(dy) \leqslant c_2$ for all $z \in S^1$ and $u > 0$.

One thus obtains

$$\begin{aligned} E^{\mu}\left(\int_0^t g(\zeta_u)\,du\right)^2 & \\ &\leqslant 2c_2\int_0^t \int_{S^1}\int_{S^1} g(v)g(z)\int_0^{t-u}(p_s(z, v) - 1)\,ds\,dz\,dv\,du \\ &\quad + 2c_2\int_0^t\int_0^{t-u}\int_{S^1}\int_{S^1} g(v)g(z)\,dv\,dz\,du\,ds. \end{aligned} \tag{2.13}$$

The second summand in (2.13) is simply $c_2 t^2\delta^2$. From Theorem (2.15) of **[1]** there is an $\alpha > 0$ such that for $s \geqslant 1$ and $z,v \in S^1$, $|p_s(z, v) - 1| \leqslant e^{-\alpha s}$. Using this and the fact that for $s \to 0$, $p_s(z, v)$ behaves as $(1/\sqrt{2\pi s}\,)\exp(-|z - v|^2/2s)$ one obtains that $\int_0^w |p_s(z, v) - 1|\,ds$ is bounded uniformly in $w \in [0, \infty)$, $z,v \in S^1$. Combining this with (2.13) and (2.12) one obtains the statement of the lemma.

3. Asymptotic normality of the finite dimensional distributions. It is convenient to prove asymptotic normality and to evaluate the covariance in the special case where the points lie symmetrically on S^1. Together with tightness (which is proved in §4) this is clearly sufficient for Theorem 1.

Let $m \in \mathbf{N}$, $m \geqslant 2$ and set $x_0 = 0$, $x_1 = 1/m, \ldots, x_{m-1} = (m-1)/m$ on S^1. $T_0 = \inf\{t\colon \zeta_t = 0\}$ and inductively $T_{k+1} = \inf\{t\colon |\zeta_{T_k + t} - \zeta_{T_k}| = 1/m\}$. $\tau_k = \sum_{i=0}^k T_i$, $X_k = \zeta_{\tau_k}$, for $k \geqslant 0$, $L_k = \lambda(\tau_k, \zeta_{\tau_{k-1}}) - \lambda(\tau_{k-1}, \zeta_{\tau_{k-1}})$ for $k \geqslant 1$. For any starting distribution μ on S^1 the following statements are evident from the strong Markov property.

(3.1) $\{X_k\}_{k=0,1,\ldots}$ is a symmetric random walk on the cyclic group $\{k/m \bmod 1\} = G_m$ with

$$P\left(X_{k+1} = \frac{i}{m}\,\middle|\,X_k = \frac{j}{m}\right) = \begin{cases} 1/2 & i = j+1 \text{ or } i = j-1 \bmod m, \\ 0 & \text{else.} \end{cases}$$

X_0 is of course 0. The chain X_k has recurrence times with moments of any order. The ergodic distribution gives measure $1/m$ to any state.

(3.2) The sequence $\{X_k\}_{k=0,1,\ldots}$ and all pairs (T_i, L_i), $i = 1, 2, \ldots$ are independent and the pairs (T_i, L_i) are identically distributed. (Here it is essentially used that the distances between neighbouring points are all equal.)

As is well known $E(T_i) = 1/m^2$ and $E(T_i^k) = O(1/m^{2k})$ for $k \in \mathbf{N}$. From Lemma 1 $E(L_i) = 1/m$ and $E(L_i^2) = 2/m^2$.

$$\lambda(\tau_n, x_k) - \lambda(\tau_0, x_k) = \sum_{i=1}^n 1_{x_k}(X_{i-1})L_i. \tag{3.3}$$

We further define a sequence of $\mathbf{N} \cup \{0\}$ valued random variables $S_0 = 0$, and inductively $S_{k+1} = \inf\{j \geq S_k + 1, X_j = 0\}$.

(3.4) $\{S_i - S_{i-1}, X_{S_{i-1}}, X_{S_{i-1}+1}, \ldots, X_{S_i - 1}\}_{i=1,2,\ldots}$ is a sequence of independent $(\mathbf{N} \times (G_m)^{\mathbf{N}}$ valued) random variables.

Set $\sigma_i = \tau_{S_i}$ for $i \geq 0$ and $\sigma_{-1} = 0$. For $t \geq 0$ let $\rho_t = \inf\{k: \sigma_k > t\}$ and set $\sigma_t = \sigma_{\rho_t}$.

LEMMA 3. *For each starting measure μ on S^1 one has*

(a) *For $n \geq 1$ the $\sigma_n - \sigma_{n-1}$ are independent and identically distributed.*

(b) *For fixed k, $\lambda(\sigma_n, x_k) - \lambda(\sigma_{n-1}, x_k)$, $n \geq 1$, are independent and identically distributed.*

(c) *$E(\lambda(\sigma_n, x_k) - \lambda(\sigma_{n-1}, x_k))$ and $E(\sigma_n - \sigma_{n-1})$ are finite and both $1/m$ (for $n \geq 1$).*

(d) *For $n \geq 0$ the $\sigma_n - \sigma_{n-1}$ have finite fourth moments.*

(e) *For $n \geq 0$ the $\lambda(\sigma_n, x_k) - \lambda(\sigma_{n-1}, x_k)$ have finite second moments.*

(f) *$E(\sigma_t - t)$ is bounded uniformly in t.*

(g) *$E(\lambda(\sigma_t, x_k) - \lambda(t, x_k))$ is bounded uniformly in t.*

(h) *ρ_t/t converges in the first mean to m.*

PROOF. (a) and (b) follow from (3.2) and (3.4).

(c): $\lambda(\sigma_1, x_k) - \lambda(\sigma_0, x_k) = \sum_{j=1}^{S_1} 1_{x_k}(X_{j-1})L_j$. From (3.2) one obtains

$$E(\lambda(\sigma_1, x_k) - \lambda(\sigma_0, x_k)) = E(L_1)E\left(\sum_{j=1}^{S_1} 1_{x_k}(X_{j-1})\right).$$

From Corollary 6-21 of [7] one has $E(\sum_{j=1}^{S_1} 1_{x_k}(X_{j-1})) = 1$ for each k. $E(L_1) = 1/m$ from Lemma 1. On the other hand $\sigma_1 - \sigma_0 = \sum_{j=1}^{S_1} T_j$ and S_j and the T_j are independent, so $E(\sigma_1 - \sigma_0) = E(S_1)E(L_1) = m \cdot 1/m^2 = 1/m$.

(d): follows from the fact that S_1 and the T_j have moments of any order.

(e): Let first $n \geq 1$. It suffices to consider the case $n = 1$. $\lambda(\sigma_1, x_k) - \lambda(\sigma_0, x_k) \leq \sum_{j=1}^{S_1} L_j$. From Wald's inequality one has

$$E\left(\sum_{j=1}^{S_1}\left(L_j - \frac{1}{m}\right)\right)^2 = E(S_1)E\left(L_j - \frac{1}{m}\right)^2 < \infty.$$

$E(\lambda(\sigma_1, x_k) - \lambda(\sigma_0, x_k))^2 < \infty$ clearly follows. For $n = 0$, $\lambda(\sigma_0, x_k) - \lambda(\sigma_{-1}, x_k) = \lambda(\sigma_0, x_k)$. σ_0 is the first hitting of 0 for ζ_t which is the first hitting of $\{0, 1\}$ for ξ_t.

$E^{\mu}(\lambda(\sigma_0, x_k)^2) \leq E^{x_k}(\lambda(\sigma_0, x_k)^2) \leq E^{x_k}(l(R, x_k)^2)$ where R is the first hitting of $\{x_k + 1, x_k - 1\}$ for ξ_t and this last expression is $< \infty$ by Lemma 1.

(f): is proved similarly as (g) below.

(g): Let $\hat{\sigma}_t = \sigma_{\rho_t - 1}$.

$$\lambda(\sigma_t, x_k) - \lambda(t, x_k) \leqslant \lambda(\sigma_t, x_k) - \lambda(\hat{\sigma}_t, x_k). \tag{3.5}$$

Let $\varphi(t) = E(\lambda(\sigma_t, x_k) - \lambda(\hat{\sigma}_t, x_k))$.

$$\begin{aligned}\varphi(t) &= \int_0^\infty E(\lambda(\sigma_t, x_k) - \lambda(\hat{\sigma}_t, x_k)|\sigma_0 = s)P(\sigma_0 \in ds)\\ &= \int_0^t \varphi(t-s)P(\sigma_0 \in ds) + \int_t^\infty E(\lambda(s, x_k)|\sigma_0 = s)P(\sigma_0 \in ds).\end{aligned} \tag{3.6}$$

$$\begin{aligned}\int_t^\infty E(\lambda(s, x_k)|\sigma_0 = s)P(\sigma_0 \in ds) &= E(\lambda(\sigma_0, x_k); \sigma_0 \geqslant t)\\ &\leqslant \left(E(\lambda(\sigma_0, x_k))^2 P(\sigma_0 \geqslant t)\right)^{1/2} \leqslant c \cdot t^{-2}\end{aligned} \tag{3.7}$$

from (d) and (e) where $c > 0$ is some constant. Clearly $\varphi(t)$ is finite on finite intervals. We assume for the moment that $P(\sigma_0 = 0) < 1$ (which is equivalent with $\mu \neq \delta_0$). Then there is a $\Delta > 0$ with $P(\sigma_0 \leqslant \Delta) = 1 - \varepsilon < 1$. Let $s \leqslant \Delta$. From (3.6) and (3.7) one has

$$\varphi(t + \Delta) \leqslant (1 - \varepsilon) \sup_{t<u<t+\Delta} \varphi(u) + \sup_{0<u<t} \varphi(u) + ct^{-2}.$$

$$\varepsilon \sup_{t<u<t+\Delta} \varphi(u) \leqslant \sup_{0<u<t} \varphi(u) + ct^{-2}.$$

So it easily follows that $\varphi(t)$ is bounded on $[0, \infty)$. If $\sigma_0 = 0$ a.s. one can copy the argument with σ_1 instead of σ_0.

(h): $\sigma_t - \sigma_0 = \sum_{i=1}^{\rho_t}(\sigma_i - \sigma_{i-1})$. From Wald's identity $E(\rho_t)E(\sigma_1 - \sigma_0) = E(\sigma_t - \sigma_0)$, $E(\sigma_1 - \sigma_0) = 1/m$ and thus from (f) $E(\rho_t)/t \to m$ as $t \to \infty$.

$$\begin{aligned}&E\left(((\sigma_t - \sigma_0)/t - \rho_t E(\sigma_1 - \sigma_0)/t)^2\right)\\ &\qquad = \frac{1}{t^2} E\left(\sum_{i=1}^{\rho_t} ((\sigma_i - \sigma_{i-1}) - E(\sigma_i - \sigma_{i-1}))\right)\end{aligned}$$

and using again Wald's identity this is $(1/t^2)E(\rho_t)$ var $(\sigma_1 - \sigma_0)$ which tends to 0 as $t \to \infty$. So $\rho_t E(\sigma_i - \sigma_0)/t - (\sigma_t - \sigma_0)/t \to 0$ in L_2 and from (f) $\sigma_t/t \to 1$ and $\sigma_0/t \to 0$ in L_1. (h) is proved.

LEMMA 4. *Let t_n be a sequence of positive real numbers increasing to ∞. Then for any starting distribution μ on S^1 $\{(1/\sqrt{t_n})(\lambda(t_n, x_k) - t_n)\}_{0 \leqslant k \leqslant m-1}$ is asymptotically joint normally distributed.*

PROOF.

$$\begin{aligned}\lambda(t_n, x_k) - t_n &= \sum_{i=1}^{\zeta_{t_n}} (\lambda(\sigma_i, x_k) - \lambda(\sigma_{i-1}, x_k) - (\sigma_i - \sigma_{i-1}))\\ &\quad + (\sigma_{t_n} - t_n) - (\lambda(\sigma_{t_n}, x_k) - \lambda(t_n, x_k)) + \lambda(\sigma_0, x_k) - \sigma_0.\end{aligned}$$

From (d), (e), (f) and (g) of Lemma 3 it follows that $(1/\sqrt{t_n})(\lambda(t_n, x_k) - t_n)$

is asymptotically equivalent to

$$\frac{1}{\sqrt{t_n}} \sum_{i=1}^{\zeta_{t_n}} (\lambda(\sigma_i, x_k) - \lambda(\sigma_{i-1}, x_k) - (\sigma_i - \sigma_{i-1})).$$

Let $(f_0, \ldots, f_{m-1})$ be an arbitrary vector in $\mathbf{R}^m$. Then

$$\sum_{k=0}^{m-1} f_k (\lambda(\sigma_i, x_k) - \lambda(\sigma_{i-1}, x_k) - (\sigma_i - \sigma_{i-1}))$$

are independent random variables (by 3.4) with expectation 0 (by Lemma 3 (c)) and finite variance (by Lemma 3 (d) and (e)). So from Lemma 3 (h) and Rényi's random central limit theorem (**[10]**) it follows that

$$\frac{1}{\sqrt{t_n}} \sum_{k=0}^{m-1} f_k (\lambda(t_n, x_k) - t_n))$$

is asymptotically normally distributed. But $(f_0, \ldots, f_{m-1})$ was arbitrary. So the lemma follows.

REMARK. The above Lemma 4 clearly follows from Corollary 2 a proof of which has been sketched by Brosamler in **[4]** using a central limit theorem for mixing sequences, but details are given only for μ with bounded derivative in **[1]**. The renewal approach used here seems more elementary and applicable also for other recurrent one dimensional diffusions without compactness of the state space.

4. Termination of the proof of Theorem 1. Let $t_n \geq 0$, $t_n \uparrow \infty$. We want to show that for arbitrary starting distribution μ on S^1

$$X_{t_n}(x) = \frac{1}{\sqrt{t_n}} (\lambda(t_n, x) - t_n)$$

is a tight sequence of processes on S^1. We may assume that $t_n > 1$. Then

$$\lambda(t_n, x, \omega) = \lambda(1, x, \omega) + \lambda(t_n - 1, x, \theta_1(\omega)).$$

$$X_{t_n}(x, \omega) = ((t_n - 1)/t_n)^{1/2}\Big((\lambda(t_n - 1, x, \theta_1(\omega)) - (t_n - 1))/(t_n - 1)^{1/2} + \big(\lambda(1, x, \omega) - 1)/(t_n - 1)^{1/2}\big).$$

$\sup_x \lambda(1, x) < \infty$ a.s., so $\sup_x \lambda(1, x)/(t_n - 1)^{1/2} \to 0$ a.s. It therefore suffices to prove the tightness of $(\lambda(t_{n-1}, x, \theta_1(\omega)) - (t_n - 1))/(t_n - 1)^{1/2}$. For notational convenience we replace $t_n - 1$ by t_n. Distribution of $\lambda(t_n, x, \theta_1)$ under starting measure μ is the same as distribution of $\lambda(t_n, x)$ under starting measure $P^{\mu}(\zeta_1 \in dx)$ which has bounded density with respect to Lebesgue measure. In order to prove tightness of X_{t_n} it therefore suffices to consider the case where μ has bounded density.

It follows from Lemma 4 that $X_{t_n}(0)$ converges in distribution. In order to obtain tightness it therefore remains to show that for each $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\Big(\sup_{|x-y|<\delta} |X_{t_n}(x) - X_{t_n}(y)| > \varepsilon\Big) = 0.$$

By our modified Tanaka formular (2.8)

$$X_{t_n}(x) - X_{t_n}(y) = \Big(F(\xi_{t_n}) - F(\xi_0) - \int_0^{t_n} F'(\xi_s)\, d\xi_s\Big)\Big/ t_n^{1/2}.$$

F is bounded and it therefore remains to show that for $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\Big(\sup_{|x-y|<\delta} \Big|\int_0^{t_n} F'(\xi_s)\, d\xi_s\Big| \Big/ t_n^{1/2}\Big) = 0.$$

This now follows from Lemma 2 and standard techniques for weak convergence (see e.g. [2, Theorem 12.3]). We have therefore proved that for each sequence $t_n \uparrow \infty$ X_{t_n} is a tight sequence of continuous processes on S^1. From Lemma 4 it then follows that X_{t_n} converges weakly to a Gaussian measure on $C(S^1)$. It remains to identify this measure with the measure induced by $Z(x)$. To this end it is clearly sufficient to show that for $f\colon S^1 \to R$ bounded and measurable $\int_0^1 X_{t_n}(x) f(x)\, dx$ converges weakly to $\int_0^1 Z(x) f(x)\, dx$. But as was remarked above this has already been obtained by Baxter and Brosamler in [1]. We have proved that for each sequence $t_n \uparrow \infty$ X_{t_n} converges weakly to Z. So for $t \to \infty$ X_t converges weakly to Z.

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REFERENCES

1. J. R. Baxter and G. A. Brosamler, *Energy and the law of iterated logarithm*, Math. Scand. **38** (1976), 115–136.
2. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.
3. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Academic Press, New York, 1968.
4. G. A. Brosamler, *A probabilistic solution of the Neumann problem*, Math. Scand. **38** (1976), 137–147.
5. M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time*. I, Comm. Pure. Appl. Math. **28** (1975), 1–47.
6. A. Friedman, *Stochastic differential equations and applications*, Vol. I, Academic Press, New York, 1975.
7. J. G. Kemeney, J. L. Snell and A. W. Knapp, *Denumerable Markov chains*, Springer-Verlag, Berlin, 1976.
8. H. P. McKean, *Brownian local times*, Topics in Probability Theory, D. W. Stroock and S. R. S. Vardahan (editors), Courant Inst. Math. Sci., New York Univ., New York, 1973, pp. 59–92.
9. ______, *Stochastic integrals*, Academic Press, New York, 1969.

10. A. Rényi, *On the central limit theorem for the sum of a random number of independent random variables*, Acta Math. Sci. Hungar. **11** (1960), 97–102.

11. H. Tanaka, *Certain limit theorems concerning one dimensional diffusion processes*, Mem. Fac. Sci. Kyushu Univ. **12** (1958), 1–11.

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