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THE ATOMIC DECOMPOSITION FOR PARABOLIC H^p SPACES

BY

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ABSTRACT. The theorem of A. P. Calderón giving the atomic decomposition for certain parabolic H^p spaces is extended to all such spaces. The proof given also applies to Hardy spaces defined on the Heisenberg group.

Introduction. The purpose of this note is to extend the atomic decomposition to two examples of Hardy spaces. In the first case we will obtain the atomic decomposition for the parabolic H^p spaces of A. P. Calderón and A. Torchinsky [2], [3] extending a result of A. P. Calderón [1]. The second example we will consider is a classical Hardy space of holomorphic functions defined on a domain in complex n-space whose boundary may be identified with the Heisenberg group $\mathbf{H}^n$. The methods we will use in both cases are very similar. This is because both $\mathbf{R}^n$ with a parabolic dilation structure and $\mathbf{H}^n$ with its natural dilation structure are examples of spaces of homogeneous type.

Earlier examples of atomic decomposition theorems may be found in [4], [11], [1], and [6]. A good exposition of the general theory of atomic Hardy spaces is [5] to which the reader may refer for many applications.

1. Parabolic H^p spaces. We begin with a brief review of the basic material in [2] and [3], to which the reader should refer for further details.

Let $A_t = t^P$ $(0 < t < \infty)$ be a group of linear transformations on $\mathbf{R}^n$ with infinitesimal generator P satisfying $(Px, x) \geqslant (x, x)$ where $(\ ,\ )$ is the usual inner product in $\mathbf{R}^n$. For each $x \in \mathbf{R}^n$ let $\rho(x)$ denote the unique t such that $|A_t^{-1}x| = 1$ where, as usual, $|x| = (x, x)^{1/2}$. The function $\rho\colon \mathbf{R}^n \to \mathbf{R}$ is a norm which satisfies $\rho(A_t x) = t\rho(x)$ $(t > 0)$. Let $d(x, y) = \rho(x - y)$ denote the metric associated with ρ and, for $r > 0$, put $B_r(x) = \{y\colon d(x, y) < r\}$. A change of variables shows that the measure of $B_r(x)$ is $|B_r(x)| = \omega_n \det A_t = \omega_n t^\gamma$ where $\gamma = \operatorname{tr} P$ and ω_n is the volume of the unit ball in $\mathbf{R}^n$. Thus we see that $\mathbf{R}^n$ endowed with the metric d and Lebesgue measure is a space of homogeneous type (see [5]).

Let $\mathcal{S}$ denote Schwartz space. If $\varphi \in \mathcal{S}$, $t > 0$, define $\varphi_t(x) = t^{-\gamma}\varphi(A_t^{-1}x)$.

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If $\int \varphi \neq 0$ and if f is a distribution define

$$F(x, t) = (f * \varphi_t)(x), \qquad x \in \mathbf{R}^n, \quad t > 0,$$

and a maximal function

$$M_\alpha f(x) = \sup_{\rho(y) < \alpha t} |F(x + y, t)|, \qquad \alpha > 0.$$

We say $f \in H^p$ $(0 < p < \infty)$ if $M_\alpha f \in L^p$. Also, $\|f\|_{H^p} = \|M_\alpha f\|_{L^p}$. For any other choice of φ and α we obtain the same space H^p and an equivalent norm $\|\cdot\|_{H^p}$.

Let

$$\mathcal{Q}_N = \left\{ \varphi \in \mathcal{S}: \sup_{|J|, |K| < N} |x^J D^K \varphi(x)| < 1 \right\}$$

and

$$f^*(x) = \sup_{\varphi \in \mathcal{Q}_N} |M_\alpha f(x)|$$

where J, K denote multi-indices and $|J|$, $|K|$ their orders. The proof of Theorem 4.6 in [2] shows that if we choose N sufficiently large (depending only on P and p), then $f \in H^p$ if and only if $f^* \in L^p$, and, moreover, $\|f^*\|_{L^p}$ defines a norm on H^p equivalent to $\|\cdot\|_{H^p}$.

In this context an atom is defined as follows: A *p-atom* $(0 < p < 1)$ is a function a which is supported on a ball $B_r(x_0)$ and which satisfies

(i) $|a(x)| < |B_r(x_0)|^{-1/p}$;

(ii) $\int_{\mathbf{R}^n} x^J a(x)\, dx = 0$, $|J| < [\gamma(1/p - 1)]$.

We are now ready to state our theorem.

THEOREM 1. *Let $f \in H^p$ $(0 < p < 1)$. Then there exist a sequence a_i of p-atoms and a sequence $\lambda_i > 0$ such that*

$$\sum_{i=1}^{\infty} \lambda_i^p < B\|f\|_{H^p}^p \tag{1.1}$$

and

$$f = \sum_{i=1}^{\infty} \lambda_i a_i. \tag{1.2}$$

Conversely, if $f = \sum_{i=1}^{\infty} \lambda_i a_i$ where each a_i is a p-atom and $\{\lambda_i\} \in l^p$, then $f \in H^p$ and

$$A\|f\|_{H^p}^p < \sum_{i=1}^{\infty} |\lambda_i|^p. \tag{1.3}$$

The constants A and B depend only on the choice of norm for H^p.

A. P. Calderón [1] has obtained this theorem in the case where P is diagonalizable over the complex numbers.

PROOF. The converse is quite easy. One need only show that for each

p-atom a, $\|M_1 a\|_{L^p} \leq C$ for some choice of $\varphi \in \mathcal{S}$. See, for example, [11].

We begin the proof of the hard direction by noticing that $L^1 \cap H^p$ is dense in H^p. Thus, a limiting argument (see [11]) allows us to assume $f \in L^1 \cap H^p$. Then $f^* \in L^p$. Put $\Omega_k = \{x\colon f^*(x) > 2^k\}$, $k = 0, \pm 1, \pm 2, \ldots$. Using Lemma 1.6 of [2] we obtain for each k a sequence of balls $B_i^k = B_{r_i^k}(x_i^k)$ which satisfy, for each k,

$$\Omega_k = \bigcup_{i=1}^{\infty} B_i^k, \tag{1.4}$$

$$\begin{gathered}\text{there are constants } \alpha > \beta > 1 \text{ such that}\\ B_{\beta r_i^k}(x_i^k) \subset \Omega_k \text{ and } B_{\alpha r_i^k}(x_i^k) \cap \Omega^c \neq \varnothing,\end{gathered} \tag{1.5}$$

$$\begin{gathered}\text{there is a constant } c < 1 \text{ such that}\\ \text{the balls } B_{c r_i^k}(x_i^k) \text{ are disjoint,}\end{gathered} \tag{1.6}$$

$$\begin{gathered}\text{there is a constant } M \text{ such that no point of } \mathbf{R}^n\\ \text{lies in more than } M \text{ of the balls } B_{\beta r_i^k}(x_i^k).\end{gathered} \tag{1.7}$$

Here the constants α, β, c, and M depend only on P.

Let Ψ be a C^∞ function, $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ on $B_1(0)$ and $\Psi \equiv 0$ off $B_\beta(0)$. Let $\psi_i^k(x) = \Psi(A_{r_i^k}^{-1}(x_i^k - x))$, and let

$$\varphi_i^k(x) = \psi_i^k(x) \Big/ \sum_{j=1}^{\infty} \psi_j^k(x).$$

We note the following properties of φ_i^k:

$$\varphi_i^k \text{ is } C^\infty, \qquad 0 \leq \varphi_i^k \leq 1, \tag{1.8}$$

$$\varphi_i^k \text{ is supported on } B_{\beta r_i^k}(x_i^k), \tag{1.9}$$

and

$$\sum_{i=1}^{\infty} \varphi_i^k = \chi_{\Omega_k}. \tag{1.10}$$

Denote by V_i^k the Hilbert space of polynomials of degree $\leq [\gamma(1/p - 1)]$ with the norm

$$\|P\|_{\varphi_i^k}^2 = \left(\int \varphi_i^k\right)^{-1} \int_{R^n} |P(x)|^2 \varphi_i^k(x)\,dx.$$

For each i and k let P_i^k be the projection of f into V_i^k; i.e.,

$$\int \left[A_{r_i^k}(x - x_i^k)\right]^J P_i^k \varphi_i^k\,dx = \int \left[A_{r_i^k}(x - x_i^k)\right]^J f \varphi_i^k\,dx$$

($|J| \leq [\gamma(1/p - 1)]$). Also let P_{ij}^{k+1} be the projection of $(f - P_j^{k+1})\varphi_i^k$ into V_j^{k+1}. Notice that $\sum_{i=1}^{\infty} P_{ij}^{k+1}\varphi_j^{k+1} = 0$.

We will now show that $|P_i^k \varphi_i^k| \leq C 2^k$. Fix i and k. Replacing x by $(A_{r_i^k}x) + x_i^k$ allows us to assume φ_i^k is supported on $B_\beta(0)$. Let $\pi_0, \ldots, \pi_L$ be an orthonormal basis for V_i^k. An elementary argument shows that the coefficients of the π_j are all bounded by a constant depending only on P. It follows that

$$C_N\left(\int \varphi_i^k\right)^{-1} \pi_j(x)\varphi_i^k(x) = \Phi_j(y_i^k - x)$$

where $y_i^k \in B_{\alpha r_i^k}(x_i^k) \cap \Omega^c$ and $\Phi_j \in \mathcal{Q}_N$. Thus

$$\left|\left(\int \varphi_i^k\right)^{-1} \int_{R^n} f\pi_j\varphi_i^k\, dx\right| < C_N f^*(y_i^k) < C2^k.$$

Because

$$P_i^k = \sum_{j=0}^{L}\left(\left(\int \varphi_i^k\right)^{-1} \int f\pi_j\varphi_i^k\right)\pi_j$$

we see $|P_i^k\varphi_i^k| < C2^k$ as required. In the same way we may show $|P_{ij}^{k+1}\varphi_j^{k+1}| < C2^{k+1}$.

For $k = 0, \pm 1, \ldots$ we write

$$\begin{aligned} f &= \left(f\chi_{\Omega_k^c} + \sum_{i=1}^{\infty} P_i^k\varphi_i^k\right) + \sum_{i=1}^{\infty}(f - P_i^k)\varphi_i^k \\ &= g_k + \sum_{i=1}^{\infty}(f - P_i^k)\varphi_i^k. \end{aligned} \tag{1.11}$$

(This decomposition for $f \in H^p(\mathbf{R}^n)$ may be found in [6].) Because $g_k \to 0$ as $k \to -\infty$ and $g_k \to f$ a.e. as $k \to +\infty$ we see

$$f = \sum_{k=-\infty}^{\infty}(g_{k+1} - g_k) \quad \text{a.e.} \tag{1.12}$$

Now, by (1.11),

$$\begin{aligned} g_{k+1} - g_k &= \sum_{i=1}^{\infty}(f - P_i^k)\varphi_i^k - \sum_{j=1}^{\infty}(f - P_j^{k+1})\varphi_j^{k+1} \\ &= \sum_{i=1}^{\infty}\left[(f - P_i^k)\varphi_i^k - \sum_{j=1}^{\infty}(f - P_j^{k+1})\varphi_j^{k+1}\varphi_i^k\right] \\ &= \sum_{i=1}^{\infty}\left\{(f - P_i^k)\varphi_i^k - \sum_{j=1}^{\infty}\left[(f - P_j^{k+1})\varphi_i^k - P_{ij}^{k+1}\right]\varphi_j^{k+1}\right\} \\ &= \sum_{i=1}^{\infty}\beta_i^k. \end{aligned} \tag{1.13}$$

Notice that $|\beta_i^k| < C2^{k+1}$, β_i^k is supported on $B_{Cr_i^k}(x_i^k)$, and $\int x^J\beta_i^k = 0$ whenever $|J| < [\gamma(1/p - 1)]$. Thus we may write

$$\beta_i^k = C2^k|B_i^k|^{1/p}a_i^k = \lambda_i^k a_i^k \tag{1.14}$$

where a_i^k is a p-atom. Combining (1.14), (1.13), and (1.12), we obtain (1.2).

Finally,

$$\sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} (\lambda_i^k)^p = C \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} (2^k)^p |B_i^k| = C \sum_{k=-\infty}^{\infty} 2^{k-1}(2^k)^{p-1}|\Omega_k|$$
$$\leqslant C\int_0^{\infty} \lambda^{p-1}|\{f^*(x) > \lambda\}|\, d\lambda \leqslant C\|f^*\|_{L^p}^p$$
$$\leqslant C\|f\|_{H^p}^p$$

which gives (1.1).

The inequality (1.1) shows that the sum in (1.2) converges to f in H^p. That it also does so in the sense of distributions follows from the discussion below.

Thus Theorem 1 is proved.

REMARK 1. Notice that the proof shows that the number of moments which we require to vanish in our definition of p-atom may be increased or decreased. The number $[\gamma(1/p - 1)]$ is the minimum number needed to make (1.3) true. But, in some cases, we may not need to require even this many vanishing moments to obtain (1.3). For example, if P is a diagonal matrix with eigenvalues $k_1 \geqslant \cdots \geqslant k_n \geqslant$ le1then we need only require that $\int x^J a(x) = 0$ if $J = (j_1, \ldots, j_n)$ where $k_1 j_1 + \cdots + k_n j_n \leqslant [\gamma(1/p - 1)]$.

We now introduce some Lipschitz spaces. Let $\beta > 0$ and $K = (k_1, \ldots, k_n) \in \mathbf{R}^n$ satisfy $k_j \geqslant 1$ $(j = 1, \ldots, n)$. Let $\mathcal{G}_K$ denote the collection of all linear combinations of monomials x^J where $(J, K) \leqslant \beta$ and define $\mathcal{L}(\beta, K)$ to be those $\varphi \in L^1_{\text{loc}}$ for which

$$\|\varphi\|_{\mathcal{L}(\beta, K)} = \sup_{(x,t)\in\mathbf{R}^{n+1}_+} \left[t^{-\beta} \inf_{P\in\mathcal{G}_K} \frac{1}{|B_t(x)|} \int_{B_t(x)} |\varphi(y) - P(y)|\, dy \right] < \infty.$$

Notice that if $P \in \mathcal{G}_K$, then $\|P\|_{\mathcal{L}(\beta, K)} = 0$. Thus the elements of $\mathcal{L}(\beta, K)$ are the equivalence classes obtained by identifying two elements if they differ by a polynomial in $\mathcal{G}_K$. If $\beta = 0$, $\mathcal{L}(\beta, K) = \text{BMO}$. In any case, if $\varphi \in \mathcal{L}(\beta, K)$, then φ grows no faster at ∞ than a polynomial of sufficiently large degree. If $K_0 = (1, \ldots, 1)$ we denote $\mathcal{L}(\beta, K_0)$ by $\mathcal{L}_\beta$.

The following corollary follows easily from Theorem 1.

COROLLARY 1. *Let* $0 < p \leqslant 1$. *The dual space of* H^p *is* $\mathcal{L}_\beta$ *where* $\beta = \gamma(1/p - 1)$. *That is, if L is a continuous linear functional on* H^p, *then there exists a unique* $\varphi \in \mathcal{L}_\beta$ *such that* $\|\varphi\|_{\mathcal{L}_\beta} \leqslant A_p\|L\|$ *and*

$$L\varphi = \int f\varphi, \qquad f \in \mathfrak{D}_p, \tag{1.15}$$

where $\mathfrak{D}_p$ *is the set of all finite linear combinations of p-atoms. Conversely, let* $\varphi \in \mathcal{L}_\beta$. *Then, if* $f \in \mathfrak{D}_p$,

$$\left|\int f\varphi\right| \leqslant B_p\|f\|_{H^p}\|\varphi\|_{\mathcal{L}_\beta}.$$

Thus $Lf = \int f\varphi$ *extends to a continuous linear functional on* H^p.

For $p = 1$, this is a result of Calderón and Torchinsky [3] who also obtained a somewhat different characterization of $(H^p)^*$ $(p < 1)$.

In case P is diagonalizable over the complex numbers we may use Remark 1 to obtain a better characterization of $(H^p)^*$ $(p < 1)$. Calderón [1] has shown that in this case P may be replaced by a symmetric matrix whose eigenvalues are the real parts of the eigenvalues of P without changing the corresponding H^p. For simplicity we will assume P is diagonal and has eigenvalues $k_1 \geqslant \cdots \geqslant k_n \geqslant 1$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy $0 < \alpha_1 \leqslant \cdots \leqslant \alpha_n$. Let $J = (j_1, \ldots, j_n)$ be a multi-index. We say J is *maximally admissible* for α if

$$\gamma_\alpha(J) = 1 - \sum_{i=1}^{n} \frac{j_i}{\alpha_i} > 0,$$

but if $J' \neq J$ satisfies $j_i' \geqslant j_i$ $(i = 1, \ldots, n)$ then $\gamma_\alpha(J') < 0$. We say $\varphi \in L(\alpha_1, \ldots, \alpha_n, \text{loc})$ if whenever J is maximally admissible for α, then $D^J\varphi$ exists, is continuous, and

$$\sup_{\substack{x \in \mathbf{R}^n \\ i = 1, \ldots, n}} \left\| D^J\varphi(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) \right\|_{\Lambda_{\alpha_i \gamma_\alpha(J)}(\mathbf{R})} < \infty$$

where the Lipschitz spaces Λ_α are defined in [13]. As with $\mathcal{L}(\beta, K)$, the elements of $L(\alpha_1, \ldots, \alpha_n, \text{loc})$ are equivalence classes.

Now using Remark 1 we obtain the result that if $K = (k_1, \ldots, k_n)$, then the dual space of H^p is $\mathcal{L}(\beta, K)$ where $\beta = \gamma(1/p - 1)$. By adapting arguments of Krantz [9] to our case one may show that if $\varphi \in \mathcal{L}(\beta, K)$, then φ may be redefined on a set of measure zero so that $\varphi \in L(\alpha_1, \ldots, \alpha_n, \text{loc})$ where $\alpha_j = \beta/k_j$ $(j = 1, \ldots, n)$. That $L(\alpha_1, \ldots, \alpha_n, \text{loc}) \subset \mathcal{L}(\beta, K)$ is easy. These arguments also give the equivalence of norms. We thus have the following result.

THEOREM 2. *Let P be diagonalizable over the complex numbers. Let $k_1 \geqslant \cdots \geqslant k_n \geqslant 1$ be the real parts of the eigenvalues of P. Then there is a change of coordinates ρ in $\mathbf{R}^n$ such that the dual space of H^p $(p < 1)$ may be identified with* $\{\varphi: \varphi \circ \rho^{-1} \in L(\alpha_1, \ldots, \alpha_n, \text{loc})\}$ *where* $\alpha_j = (\gamma/k_j)(1/p - 1)$. *The pairing is given by* (1.15).

Details of the proof are left to the interested reader.

2. The Heisenberg group. Let

$$U_n = \{z = (z_1, z') \in \mathbf{C}^n: \operatorname{Im} z_1 - |z'|^2 > 0\}.$$

U_n is equivalent to the unit ball in $\mathbf{C}^n$ via a linear fractional transformation. $\mathbf{R} \times \mathbf{C}^{n-1}$ acts on U^n in the following manner: If $(\xi, \zeta) \in \mathbf{R} \times \mathbf{C}^{n-1}$, then

$$(\xi, \zeta) \cdot (z_1, z') = \left(z_1 + \xi + 2i \sum_{j=2}^{n} \bar{\zeta}_j z_j + i|\zeta|^2, z' + \zeta \right). \tag{2.1}$$

This action turns $\mathbf{R} \times \mathbf{C}^{n-1}$ into a group with the group law

$$(\xi, \zeta) \cdot (\xi', \zeta') = (\xi + \xi' - 2 \operatorname{Im} \zeta' \cdot \bar{\zeta}, \zeta + \zeta') \tag{2.2}$$

where $z \cdot \bar{w} = \sum_{i=1}^{n-1} z_i \bar{w}_i$. This group is the Heisenberg group $\mathbf{H}^n$. Notice that $\mathbf{H}^n$ acts simply transitively on ∂U_n so that ∂U_n may be identified with $\mathbf{H}^n$ by $g \Leftrightarrow g \cdot 0$. If $g = (\xi, \zeta) \in \mathbf{H}^n$, define the dilation group $A_t g = (t^2\xi, t\zeta)$ ($t > 0$). Then $(A_t g) \cdot 0 \in \partial U_n$ if $g \in \mathbf{H}^n$, $t > 0$. For $g \in \mathbf{H}^n$ define $\rho(g)$ to be the unique t such that $|(\xi/t^2, \zeta/t)| = 1$ as in §1. ρ is the homogeneous norm on $\mathbf{H}^n$. This norm, along with Haar measure (which is Lebesgue measure on $\mathbf{R} \times \mathbf{C}^{n-1}$) makes $\mathbf{H}^n$ into a space of homogeneous type. More details of the above may be found in **[10]**.

The Hardy spaces in U_n are defined as follows. If $z = (z_1, z') \in U_n$, let $h(z) = \operatorname{Im} z_1 - |z'|^2$. Notice that if $g \in \mathbf{H}^n$ and $z \in U^n$, then $h(g \cdot z) = h(z)$. If $t > 0$, the level set $\{z: h(z) = t > 0\}$ may be identified with $\{g \cdot (te): g \in \mathbf{H}^n\}$ where $e = (i, 0, \ldots, 0) \in U^n$. We define $\mathcal{H}^p(U_n)$ $(0 < p < \infty)$ to be those holomorphic functions F on U_n for which

$$\|F\|_{\mathcal{H}^p} = \sup_{t>0} \left[\int_{\mathbf{H}^n} |F(g \cdot (te))|^p \, dg \right]^{1/p} < \infty. \tag{2.3}$$

We remark that these spaces are not equivalent to the spaces $\mathcal{H}^p(B_n)$ on the unit ball.

If $F \in \mathcal{H}^p$, then $F(g) = \lim_{t \to 0} F(g \cdot (te))$ exists a.e. on $\mathbf{H}^n$ and is a subspace of $L^p(\mathbf{H}^n)$. We identify $\mathcal{H}^p$ with this subspace. $\mathcal{H}^2$ is a Hilbert space, and we denote by $P: L^2(\mathbf{H}^n) \to \mathcal{H}^2$ the orthogonal projection.

Atoms on $\mathbf{H}^n$ are defined much as in the case of parabolic H^p spaces only our definition must reflect the group structure of $\mathbf{H}^n$. If $0 < p \leq 1$, a *p-atom* for $\mathbf{H}^n$ is a function a on $\mathbf{H}^n$ which is supported on a ball $B_r(g) = \{h: \rho(h^{-1}g) < r\}$ and which satisfies

(i) $|a(h)| \leq |B_r(g)|^{-1/p}$, $h \in \mathbf{H}^n$,

(ii) $\int_{\mathbf{H}^n} (h^{-1}g)^{(j, J_1, J_2)} a(h) \, dh = 0$ for all (j, J_1, J_2) satisfying $2j + |J_1| + |J_2| \leq [2n(1/p - 1)]$ where if $g = (\xi, \zeta) \in \mathbf{H}^n$, $g^{(j, J_1, J_2)} = \xi^j \zeta^{J_1} \bar{\zeta}^{J_2}$.

If $A = Pa$ where a is a p-atom then A is called a *holomorphic p-atom*. It is not difficult to show (using the results of **[11]**) that $A \in \mathcal{H}^p$ and $\|A\|_{\mathcal{H}^p} \leq C_{p,n}$.

The maximal function associated with $\mathcal{H}^p$ is defined by

$$MF(g) = \sup_{\rho(h^{-1}g)^2 < t} |F(h \cdot (te))|$$

whenever F is a function on U_n. Koranyi **[10]** has shown that if $F \in \mathcal{H}^p$, then

$$\|MF\|_{L^p} \leq B_{p,n} \|F\|_{\mathcal{H}^p} \qquad (0 < p < \infty). \tag{2.4}$$

We are now ready to state the main theorem of this section.

THEOREM 3. *Let $0 < p \leq 1$. If $F \in \mathcal{H}^p$, then there exist a sequence A_i of holomorphic p-atoms and a sequence $\lambda_i > 0$ such that*

$$\sum_{i=1}^{\infty} \lambda_i^p \leq B_{p,n} \|F\|_{\mathcal{H}^p}^p \tag{2.5}$$

and

$$\left\| F - \sum_{i=1}^{n} \lambda_i A_i \right\|_{\mathcal{H}^p} \to 0 \quad as\ n \to \infty. \tag{2.6}$$

Conversely, if $F = \sum_{i=1}^{\infty} \lambda_i A_i$ *where each* A_i *is a holomorphic p-atom and* $\{\lambda_i\} \in l^p$, *then* $F \in \mathcal{H}^p$ *and*

$$C_{p,n} \|F\|_{\mathcal{H}^p}^p < \sum_{i=1}^{\infty} |\lambda_i|^p. \tag{2.7}$$

COROLLARY 2. *Let F be holomorphic on* U_n. *Then* $F \in \mathcal{H}^p$ $(0 < p < \infty)$ *if and only if* $MF \in L^p(\mathbf{H}^n)$. *Moreover,*

$$A_{n,p} \|MF\|_{L^p} < \|F\|_{\mathcal{H}^p} < B_{n,p} \|MF\|_{L^p}. \tag{2.8}$$

The case $p > 1$ of Corollary 2 is well known (see Koranyi [**10**]).

PROOF OF THEOREM 3. Note that the proof of Theorem 1 depends in no essential way on the group structure of $\mathbf{R}^n$. We thus may obtain a decomposition of F into atoms by the same methods. The proof that $MF \in L^p$ implies the fact that a suitable "grand maximal function" $F^* \in L^p$ may be found in Geller [**8**]. The proof is completed by the same methods used in [**7**].

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