LOCAL H-MAPS OF CLASSIFYING SPACES

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ABSTRACT. Let BU denote the localization at an odd prime \( p \) of the classifying space for stable complex bundles, and let \( f: BU \rightarrow BU \) be an \( H \)-map with fiber \( F \). In this paper the Hopf algebra \( H^*(F, \mathbb{Z}/p) \) is computed for any such \( f \). For certain \( H \)-maps \( f \) of geometric interest the \( p \)-local cohomology of \( F \) is given by means of the Bockstein spectral sequence. A direct description of \( H^*(F, \mathbb{Z}_p) \) is also given for an important special case. Applications to the classifying spaces of surgery will appear later.

0. Introduction. This paper presents a computation of the cohomology of the fibers of maps \( f: BU \rightarrow BU \), where \( BU \) denotes the localization at a fixed odd prime \( p \) of the classifying space for stable complex bundles, which preserve the \( H \)-multiplication induced by Whitney sum of bundles. Using the \( p \)-local \( H \) space equivalence \( BU \simeq BO \times \Omega^2 BO \) a computation of the cohomology of the fibers of \( H \)-maps \( BO \rightarrow BO \) is also obtained. In a subsequent paper these results will be applied to a detailed study of certain classifying spaces of surgery which arise in this fashion.

The viewpoint of this paper is motivated by a result of Adams that an \( H \) map \( f: BU \rightarrow BU \) is determined up to homotopy by the induced homomorphism \( f_*: \pi_*(BU) \rightarrow \pi_*(BU) \). If we write \( \mathbb{Z}(p) \) for the integers localized at \( p \) and define the characteristic sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( f \) by the condition that \( f_* \) is multiplication by \( \lambda_j \in \mathbb{Z}(p) \) on \( \pi_j(BU) = \mathbb{Z}(p) \), then in principle the cohomology of the fiber can be completely described in terms of \( \lambda \). We thus begin by studying the characteristic sequence, using as our main tool a natural decomposition of Hopf algebras \( H^*(BU, \mathbb{Z}(p)) \cong \bigotimes_{n \text{ prime to } p} A_n^* \) in which \( A_n^* \) is a polynomial Hopf algebra on generators \( a^*_{n,k} \) of degree \( 2np^k \), \( k = 0, 1, \ldots \).

THEOREM A. If \( f: BU \rightarrow BU \) is an \( H \)-map with characteristic sequence \( \lambda \) and \( n_1 \equiv n_2 \mod(p - 1) \), then \( p^{j+1} \) divides \( \lambda_{n_1p^j} - \lambda_{n_2p^j} \).

Define the surplus sequence \( s(f) \) of \( f \) by \( s(f)_m = \nu(\lambda_m) - \nu(m) \) where \( \nu(x) \) is the exponent of the highest power of \( p \) dividing \( x \). An important step in the
proof of Theorem A is the fact that if \( n \) is prime to \( p \) and \( j \) is minimal such that \( s(f)_{np^j} < 0 \), then \( s(f)_{np^{j+1}} = -k \) for all \( k > 0 \). Thus for any \( n \) prime to \( p \) we may define \( \delta_n = j \) if \( s(f)_{np^j} = 0 \) and \( \delta_n = \infty \) if no such \( j \) exists. For any integer \( \delta > 0 \) let \( \xi^\delta \) denote the Frobenius map \( x \to x^{p^\delta} \), and let \( \xi^\infty \) be the augmentation.

**Theorem B.** Suppose \( f: BU \to BU \) is an \( H \)-map with fiber \( F \) and index \( \delta_n \) defined as above. There is an isomorphism of Hopf algebras

\[
H^*(F, \mathbb{Z}/p) \cong \bigotimes_{n \text{ prime to } p} E \left\{ \alpha^*_a \mid 0 < j < \delta_n \right\} \otimes \left( A^*_a / \xi^j A^*_a \otimes \mathbb{Z}/p \right)
\]

in which \( \alpha: H^q(BU, \mathbb{Z}/p) \to H^{q-1}(F, \mathbb{Z}/p) \) is the cohomology suspension. Moreover, the map induced on cohomology by \( F \to BU \) is given by the natural projections \( A^*_a \otimes \mathbb{Z}/p \to A^*_a / \xi^j A^*_a \otimes \mathbb{Z}/p \).

Perhaps the most transparent way to display the \( p \)-local cohomology is by means of the Bockstein spectral sequence. This is the graded \( E_1 \) spectral sequence associated to the exact couple given by the cohomology sequence of \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0 \). Of particular interest for geometric reasons is the Bockstein spectral sequence of the fiber of a "two-step function" \( f \). This is an \( H \)-map such that \( s(f)_n = s(f)_{pn} = \cdots = s(f)_{pn^{s_n}} = s_n > 0 \) for each \( n \) prime to \( p \) such that \( \delta_n > 0 \).

**Theorem C.** If \( f: BU \to BU \) is a two-step function with fiber \( F \), then the \( E_r \) term of the Bockstein spectral sequence of \( F \) is given by

\[
\bigotimes_{n \text{ prime to } p} E \left\{ \alpha^*_a, r - s_n + j \mid 0 < j < \delta_n \right\} \otimes \left( A^*_a \left( r - s_n \right) / \xi^j A^*_a \left( r - s_n \right) \otimes \mathbb{Z}/p \right)
\]

where \( r - s_n = \max\{0, r - s_n\} \) and

\[
A^*_a(k) = \mathbb{Z}(p)[a^*_{a_k}, a^*_{a_{k+1}}, \ldots] \quad \text{for any } k \geq 0.
\]

In an application of this work to smoothing theory it will be necessary to have an explicit description of the \( \mathbb{Z}(p) \) cohomology of the fiber of an \( H \)-map \( f: BU \to BU \) whose index \( \delta_n \) never exceeds 1. Let \( S \) denote the set of all \( n \) prime to \( p \) such that \( \delta_n = 1 \), and write \( T \) for the set of all sequences \( \alpha = (\alpha_1, \alpha_2, \ldots) \) of nonnegative integers each of whose nonzero entries is \( < p - 1 \) and lies in one of the subsequences \( \alpha_n, \alpha_{np}, \ldots \) for some \( n \in S \). For each \( \alpha \in T \) write \( l(\alpha) \) for the number of such subsequences containing a nonzero entry, and let \( \text{ind}(\alpha) = \min_{n \in S} p^{\alpha(i)+i} \) where \( \alpha_{np} \) is the first nonzero entry of \( \alpha_n, \alpha_{np}, \ldots \) (if none exists, \( i = \infty \)). Let \( w(\alpha) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots \).
Theorem D. If \( f : BU \to BU \) is an \( H \)-map with fiber \( F \) such that \( \delta_n < 1 \) for each \( n \) prime to \( p \), then for each \( m > 0 \)

\[
H^m(F, \mathbb{Z}_p) = \bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{\infty} \bigoplus_{\alpha \in T} (\mathbb{Z}/\text{ind}(\alpha))^{\binom{j-1}{i-1}}
\]

where the right-hand term denotes the direct sum of \( \binom{j-1}{i-1} \) copies of \( \mathbb{Z}/\text{ind}(\alpha) \).

The paper is arranged as follows. In §1 we discuss the arithmetic of characteristic sequences and the basic congruence of Theorem A. This depends on the decomposition \( H^*(BU, \mathbb{Z}_p) = \otimes \text{n prime to } p A_n^* \) which is established in §2. §3 is devoted to a detailed study of the homomorphism \( f^* : H^*(BU, \mathbb{Z}_p) \to H^*(BU, \mathbb{Z}_p) \) induced by an \( H \)-map \( f \). This permits us to calculate torsion products for various \( p \) local rings \( R \) which, by some remarkable collapse theorems for the Eilenberg-Moore spectral sequence, is equivalent to computing the \( R \) cohomology of the fiber of \( f \). This is done for \( R = \mathbb{Z}/p \) in §4 to obtain Theorem B and for \( R = \mathbb{Z}_p \) in §5 to obtain Theorem C. In §6 we carry out the calculation Theorem D.

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1. The characteristic sequence. All spaces in this paper are assumed to be localized at a fixed odd prime \( p \) and thus have \( p \)-local homotopy groups and reduced homology groups ([25], [26]). Write \( \mu \) for the localization at \( p \) of the \( \pi^* \)-multiplication (or an appropriate iterate) induced by Whitney sum of bundles; \( \pi \) corresponds to loop multiplication under the Bott equivalence \( BU \simeq \Omega^\infty BU \). One immediate benefit of localizing is a geometric splitting of \( BU \).

1.1 Theorem (Adams-Peterson). There exist equivalences of infinite loop spaces

\[
BU \to W \times \Omega^2W \times \cdots \times \Omega^{2p-4}W \quad \text{and} \quad BO \to W \times \Omega^4W \times \cdots \times \Omega^{2p-6}W
\]

where \( \pi_{2k(p-1)}(W) = \mathbb{Z}_p, k = 1, 2, \ldots, \) and \( \pi_k(W) = 0 \) otherwise.

\( W \) may be defined as the bottom space of a spectrum associated to a bordism theory with singularities [19]. The \( H \)-space decomposition can be found in [19] or [1], and the infinite loop equivalence then follows from [2]. Notice in particular that \( BU \simeq BO \times \Omega^2BO \), so any \( H \)-map \( f : BO \to BO \) may be regarded as a factor of the \( H \)-map \( f \times 1 : BU \to BU \) with the same fiber. Similarly, 1.1 and Bott periodicity yield an equivalence \( BO \simeq BSp \).

For any map \( f : BU \to BU \) we define a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) in \( \mathbb{Z}_p \) by the condition that \( f_\lambda \) equals multiplication by \( \lambda_j \) on \( \pi_j(BU) = \mathbb{Z}_p \). If \( f \) is an
H-map (that is, if \( f_\mu = \mu(f \times f) \)) we refer to \( \lambda \) as the characteristic sequence of \( f \) because of the following.

1.2 Lemma [15, p. 100]. If two H-maps \( f, g : BU \to BU \) have the same characteristic sequence they are homotopic.

Using 1.1 we may define infinite loop maps \( \Omega^{2i}W \to BU \) and \( BU \to \Omega^{2j}W \) such that the composite \( \Omega^{2i}W \to BU \to \Omega^{2j}W \) is the identity or constant depending on whether or not \( i = j \). Using these the following decomposition of H-maps follows directly from 1.2.

1.3 Corollary. For any H-map \( f : BU \to BU \) let \( \tilde{f}_{2k} \) denote the composite \( \Omega^{2k}W \to BU \to \tilde{f} : BU \to \Omega^{2k}W \). Then the diagram below homotopy commutes.

\[
\begin{array}{ccc}
BU & \xrightarrow{f} & BU \\
\approx & \downarrow & \approx \\
p-2 \prod_{j=0}^{p-2} \Omega^{2j}W & \xrightarrow{\tilde{f}_0 \times \tilde{f}_2 \times \cdots \times \tilde{f}_{2p-4}} & p-2 \prod_{j=0}^{p-2} \Omega^{2j}W
\end{array}
\]

1.4 Operations on characteristic sequences. Given H-maps \( f, g : BU \to BU \) with characteristic sequences \( \lambda \) and \( \eta \) we may form new H-maps \( f + g = \mu \circ (f \times g) \circ \Delta \) and \( f \circ g \) whose characteristic sequences are the pointwise sum \( \lambda + \eta \) and product \( \lambda \cdot \eta \), respectively. The loop space inverse \( -1 \) has characteristic sequence \((-1, -1, \ldots)\), so that \(-f = (-1) \circ f \) has sequence \(-\lambda\). Adding copies of 1 or \((-1)\) we obtain maps with characteristic sequences of any constant integer value \( n \). These are homotopy invertible for \( n \) prime to \( p \), and thus for any \( \rho \in \mathbb{Z}(\rho) \) there is an H-map with constant sequence \( \rho \).

The splitting 1.3 induces a shearing operation. Let \( S \) be some subset of \( \{0, 2, \ldots, 2p - 4\} \) and let

\[
\tilde{f}_S : \prod_{j=0}^{p-2} \Omega^{2j}W \to \prod_{j=0}^{p-2} \Omega^{2j}W
\]

be the product whose \( 2j \)th component is \( \tilde{f}_{2j} \) if \( 2j \in S \) and 1 otherwise. By 1.3, \( \tilde{f}_S \) induces a unique H-map \( f_S : BU \to BU \) with characteristic sequence \( \lambda(S) \) defined by \( \lambda(S)_n = \lambda_n \) or 1 depending on whether or not \( 2n + 2j \equiv 0 \) mod \( 2(p - 1) \) for some \( 2j \in S \). Setting \( S = \{0, 4, \ldots, 2p - 6\} \) we may thus identify an H-map \( f : BO \to BO \) with \( f_S : BU \to BU \) with characteristic sequence \( (1, \lambda_2, 1, \lambda_4, \ldots) \) and fiber(\( f \)) = fiber(\( f_S \)). We can also mix characteristic sequences. If \( S_1 \) and \( S_2 \) partition \( \{0, 2, \ldots, 2p - 4\} \), for example, then \( f_{S_1} \circ g_{S_2} = g_{S_2} \circ f_{S_1} \) has characteristic sequence \( \lambda(S_1) \cdot \eta(S_2) \), a mixture of \( \lambda \) and \( \eta \). Finally, note that by Bott periodicity we have a loop map \( \Omega^{2k}f : BU \to BU \) whose characteristic sequence \( (\lambda_{k+1}, \lambda_{k+2}, \ldots) \) is a left shift of \( \lambda \).
1.5 Definition. For any $\beta \in \mathbb{Z}(p)$ let $v(\beta)$ denote the exponent of $p$ in a prime power decomposition of the numerator of $\beta$ (let $v(0) = \infty$). We say that $\beta_1 \equiv \beta_2 \mod p^k$ if $v(\beta_1 - \beta_2) \geq k$. Define the surplus sequence $s(f)$ of an $H$-map $f : BU \to BU$ with characteristic sequence $\lambda$ by $s(f)_n = v(\lambda_n) - v(n)$. A positive surplus sequence is necessary and sufficient for the triviality of $f^* : H^*(BU, \mathbb{Z}/p) \to H^*(BU, \mathbb{Z}/p)$ (3.7), and in general surplus measures the torsion in the cohomology of the fiber (compare §5). In §3 we will also establish the following.

1.6 Lemma. For any $n$ prime to $p$, if $j$ is minimal such that $s(f)_np^j < 0$, then $s(f)_{np^j+k} = -k$.

1.7 Theorem. Let $f : BU \to BU$ be an $H$-map with characteristic sequence $\lambda$. Then $\lambda_{np^k} \equiv \lambda_{np^j} \mod p^k+1$ whenever $m \equiv n \mod (p-1)$.

Proof. If we replace $f$ by $g = f - \lambda_{np^k} \cdot 1$ as in 1.4, the resulting characteristic sequence has 0 as its $np^k$th term. In particular, $s(f)_{np^k} > 0$ and thus $s(f)_{np^j} > 0$ for $j < k$ by 1.6. But this means that $p^{j+1}$ divides $\lambda_{np^j} - \lambda_{np^k}$ for $j < k$. Using these congruences it thus suffices to check that $\lambda_{np^j} \equiv \lambda_{(n+p-1)p^j} \mod p^{j+1}$ for any $n, j$. But if $k = p^j(n-1)$, it follows by 1.4 that $\Omega^{2j}f$ has a characteristic sequence with $p^j$th term $\lambda_{np^j}$ and $p^j+1$st term $\lambda_{(n+p-1)p^j}$. The desired congruence now follows from 1.6 as above. □

1.8 Remark. Suppose $f : BU \to BU$ is any map (not necessarily an $H$-map) with characteristic sequence $\lambda$. Then $\Omega^{2j}f : BU \to BU$ is an $H$-map with characteristic sequence $(\lambda_{j+1}, \lambda_{j+2}, \ldots)$. Applying 1.7 for various values of $j$ it follows easily that $\lambda_{np^j} \equiv \lambda_{np^k} \mod p^{k+1}$ if $m \equiv n \mod (p-1)$ and $m \neq 1 \neq n$. I conjecture that the latter restriction on $m$ and $n$ is not necessary. But in any case, by the shearing construction of 1.4 we may identify any map $f : BO \to BO$ with a map $f_2 : BU \to BU$ with characteristic sequence $(1, \lambda_2, 1, \lambda_4, \ldots)$ so that 1.7 holds for $\lambda_2, \lambda_4, \ldots$. These congruences may be viewed as a generalized Kummer congruence; in a subsequent paper we study a cannibalistic class $\rho : BO \to BO \otimes$ for which 1.7 reduces to the classical congruences between Bernoulli numbers.

2. Homology of local classifying spaces. Let $R$ be a commutative principal ideal domain which is $p$-local. This means that $R \simeq R \otimes \mathbb{Z}(p)$ as a group or, equivalently, that multiplication by $m$ is bijective for $m$ prime to $p$. We write $c_n \in H^{2n}(BU, R)$ for the class associated by localization to the $n$th Chern class, and note that cup product and the coproduct induced by the Whitney sum map $\mu$ give $H^*(BU, R)$ the following familiar Hopf algebra structure ([13], [16]).
2.1 THEOREM. \( H^*(BU, R) \) is a polynomial Hopf algebra \( R[c_1, c_2, \ldots] \) with coproduct \( \mu^* \), \( c_n = \Sigma c_i \otimes c_{n-i} \). If \( d_n \in H_{2n}(BU, R) \) is dual (in the basis of monomials) to \( c_n \), then the correspondence \( c_n \rightarrow d_n \) defines an isomorphism of Hopf algebras \( H^*(BU, R) \rightarrow H^*(BU, R) \).

For any \( n \)-tuple \( a = (a_1, \ldots, a_n) \) of weight \( w(a) = a_1 + 2a_2 + \cdots + na_n \) write \( c^a \) for the cup product \( c_1^{a_1} \cdots c_n^{a_n} \in H^{2w(a)}(BU, R) \) with dual class \( d^a \in H_{2w(a)}(BU, R) \), and let \( d^a = d_1^{a_1} \cdots d_n^{a_n} \) be the Pontrjagin product induced by \( \mu_a \). Going full circle, write \( c_a \) for the dual of \( d^a \) in the monomial basis in \( H_*(BU, R) \) (the Chern class \( c_a \) is in fact dual to \( d^a \)). By 2.1 it follows that the primitives in \( H^*(BU, R) \) and \( H_*(BU, R) \) are generated as \( R \) modules by \( \{c_1, c_2, \ldots\} \) and \( \{d_1, d_2, \ldots\} \), respectively (where \( e_n = (0, \ldots, 0, 1) \) is the \( n \)th unit vector), and \( c_a \rightarrow d_a \) under the isomorphism of 1.2. By [5] we may choose generators \( \omega_n \in \pi_{2n}(BU) = Z(p) \) which are carried by the Hurewicz map to \( (m-1)!d_n \). We can also describe the primitives directly. For any \( a \) let \( |a| = a_1 + a_2 + \cdots + a_n \) and

\[
\{a\} = (a_1 + \cdots + a_n)!/a_1! \cdots a_n!.
\]

2.2 THEOREM. \( d_a = \Sigma \omega_n \cdot (-1)^{|n|} n(a)d^a/|a| \).

An identical formula holds for \( c_a \). If we identify \( c_n \) with the \( n \)th elementary symmetric function as in [4], then 2.2 is a consequence of Waring's formula for the Newton polynomials [13].

Given indeterminates \( t_0, t_1, \ldots \) define the \( k \)th Witt polynomial \( T_k \) by \( T_k(t) = t_0^k + pt_1^{k-1} + \cdots + p^k t_k \). Let \( t^p = (t_0^p, t_1^p, \ldots) \).

2.3 LEMMA. Let \( P_0, P_1, \ldots \) be polynomials in \( t_0, t_1, \ldots \) with coefficients in \( R \) (respectively, \( Z \)) such that \( P_k(t) - P_{k-1}(t^p) \) vanishes mod \( p^k \). Then the equations \( P_k(t) = T_k(\varphi_0(t), \varphi_1(t), \ldots, \varphi_k(t)) \) inductively define polynomials \( \varphi_0, \varphi_1, \ldots \) with coefficients in \( R \) (respectively, \( Z \)).

PROOF. Suppose the polynomials \( P_i \) are integral, the proof for \( R \) being identical. Since \( \varphi_0 = P_0 \), we assume inductively that \( \varphi_0, \ldots, \varphi_{k-1} \) are integral for some \( k > 0 \). Since \( T_k(t) = T_{k-1}(t^p) + p^k t_k \), we must verify that \( P_k(t) - T_{k-1}(\varphi_0(t), \ldots, \varphi_k(t)) \) vanishes mod \( p^k \).

Evidently the integral polynomials \( (\varphi_j(t))^p \) and \( \varphi_j(t^p) \) are congruent mod \( p \). Applying the binomial theorem inductively, it follows that \( (\varphi_j(t))^p \varphi_{j+1} \) and \( \varphi_j(t^p) \varphi_j \) are congruent mod \( p^{j+1} \). Consequently,

\[
T_{k-1}(\varphi_0(t), \ldots, \varphi_{k-1}(t)) \equiv T_{k-1}(\varphi_0(t^p), \ldots, \varphi_{k-1}(t^p)) \mod p^k.
\]

But \( P_{k-1}(t) = T_{k-1}(\varphi_0(t), \ldots, \varphi_{k-1}(t)) \) by definition, and so \( P_{k-1}(t^p) = T_{k-1}(\varphi_0(t^p), \ldots, \varphi_{k-1}(t^p)) \). The lemma now follows from the congruence \( P_k(t) \equiv P_{k-1}(t^p) \mod p^k \). \( \square \)
2.4 Corollary. For each \( n \) prime to \( p \) and \( k > 0 \) there exists \( a_{n,k} \in H_{2np^k}(BU, R) \) defined inductively by \( d_{np^k} = T_k(a_{n,0}, \ldots, a_{n,k}) \).

Proof. Let \( S_n \) denote the polynomial of 2.2. By 2.3 we must show that

\[
S_{np^k}(d_1, \ldots, d_{np^k}) - S_{np^{k-1}}(d_1, \ldots, d_{np^{k-1}})
\]

vanishes mod \( p^k \). For any \( \alpha \) of weight \( w(\alpha) = np^k \) with some entry prime to \( p \), the coefficient of \( d^\alpha \) in (*) is \( np^k(|\alpha|/|\alpha|) \). We must check that \( \nu(|\alpha|/|\alpha|) > \nu(|\beta|) \) where \( \nu(\alpha) \) is the exponent of \( p \) in a prime power decomposition. But \( \{\alpha\} = \{\alpha_1, \alpha_2 + \cdots + \alpha_k\} : \{\alpha_2, \ldots, \alpha_k\} \), so this follows from the familiar \( \nu(\{\beta\}) > \nu(|\beta|) \) where \( \beta = (\beta_1, \beta_2) \) with \( \beta_1 \) prime to \( p \).

In fact, the inequality \( \nu(\{p^m\beta\}) > \nu(|\beta|) \) holds for any \( \beta = (\beta_1, \ldots, \beta_s) \). As before, it suffices to show this for \( s = 2 \) with \( \beta_1 \) prime to \( p \). Let \( \nu(i) \) denote the product of all natural numbers \( < i \) which are prime to \( p \). By some simple bookkeeping we obtain \( \{p\beta\}/\{\beta\} = \pi(p\beta_1 + p\beta_2)/\pi(p\beta_1)\pi(p\beta_2) \) and hence

\[
\prod_{i=1}^m \pi(p\beta_1)\pi(p\beta_2) = \{\beta\} \prod_{i=1}^m \pi(p\beta_1 + p\beta_2).
\]

But \( \nu(\{\beta\}) > \nu(|\beta|) \) from above, while \( \nu(\{p^m\beta\}) = \nu(\{|\beta|\}) \) since \( \nu \) takes values prime to \( p \).

Returning to the polynomial (*), if \( \alpha = p\beta \) then by 2.2 the coefficient of \( d^\alpha \) is

\[
\pm \left( \frac{np^k}{|p\beta|} \frac{np^k - 1}{|\beta|} \right) \frac{np^k - 1}{|\beta|} (\{p\beta\} - \{\beta\})
\]

But since \( \{p\beta\}/\{\beta\} = \pi(|p\beta|)/\pi(p\beta_1) \cdots \pi(p\beta_{np^k}) \) it follows that

\[
\pi(p\beta_1) \cdots \pi(p\beta_{np^k})(\{p\beta\} - \{\beta\}) = \{\beta\}(\pi(p\beta_1) - \pi(p\beta_1) \cdots \pi(p\beta_{np^k})).
\]

If \( q = \min(\nu(\beta_1), \ldots, \nu(\beta_{np^k})) \), then \( \nu(\{\beta\}) > \nu(|\beta|) - q \) by the paragraph above. By Wilson's Theorem (see e.g. [20]), \( \nu(|p\beta|) = (-1)^{|p\beta|/p^k} \mod p^{q+1} \) and \( \nu(p\beta_i) = (-1)^{|p\beta_i|/p^k} \mod p^{q+1} \). Thus \( \nu(p\beta_1 - \pi(p\beta_1) \cdots \pi(p\beta_{np^k}) \equiv 0 \mod p^{q+1} \) so that \( p^{\nu(|\beta|) + 1} \) divides \( \{p\beta\} - \{\beta\} \), and the coefficients of \( d^\alpha \) in (*) vanishes mod \( p^k \). □

Using 2.2 it follows that \( a_{n,k} = nd_{np^k} + \text{decomposables} \) and thus \( \{a_{n,k}|n \) is prime to \( p, k > 0\} \) is a polynomial basis of \( H_*(BU, R) \). For each \( n \) prime to \( p \) let \( A_n \) denote the subalgebra of \( H_*(BU, R) \) generated by \( a_{n,0}, a_{n,1}, \ldots \). Since \( A_n \) is pure in \( H_*(BU, Z(p)) \) (if \( mx \in A_n \) for some integer \( m \) then \( x \in A_n \)) and \( d_{np^k} \) is primitive, it follows from the defining relation for \( a_{n,k} \) that \( A_n \) is a Hopf subalgebra for \( R = Z(p) \) and hence for any \( p \)-local \( R \) by universal coefficients. This proves the first part of the following result which was first established by Husemoller using different methods [11].
2.5 Theorem. There are isomorphisms of Hopf algebras

\[ H_\ast(BU, R) \cong \bigotimes_{n \text{ prime to } p} A_n \text{ and } \]

\[ H_\ast(\Omega^{2k}W, R) \cong \bigotimes_{n \text{ prime to } p} A_n. \]

When \( R = \mathbb{Z}(p) \) this decomposition is best possible: given Hopf subalgebras \( B, C \subseteq H_\ast(BU, \mathbb{Z}(p)) \) such that \( H_\ast(BU, \mathbb{Z}(p)) = B \otimes C \), then for any \( n \) prime to \( p \) either \( A_n \subseteq B \) or \( A_n \subseteq C \).

Proof. We prove the last statement first. Since \( B \) and \( C \) are pure, it follows from [17, 6.16] that any primitive \( d_{n,0} \) is in \( B \) or in \( C \). Suppose inductively that \( d_{n,0}, d_{n,1}, \ldots, d_{n,j-1} \in B \). By 2.4, \( x = d_{n,0}^p - T_j((a_{n,0})^p, \ldots, (a_{n,j-1})^p) \) is \( p \)-divisible and thus so is its image \([x]\) in \( H_\ast(BU, \mathbb{Z}(p))//B = C \). But if \( d_{n,0} \in C \), \([x]\) = \([d]\) which is not \( p \)-divisible by 2.2. Thus \( d_{n,0} \) and hence \( a_{n,j} \) and \( A_n \) lie in \( B \).

Because the Hurewicz map sends some generator \( \omega_m \in \pi_{2m}(BU) = \mathbb{Z}(p) \) to \((m-1)!d_{n,0}\), it follows from 1.1 that \( H_\ast(\Omega^{2k}W, \mathbb{Z}(p)) \) is the smallest pure Hopf subalgebra of \( H_\ast(BU, \mathbb{Z}(p)) \) containing the primitives \( d_{n,0} \) where \( n + k \equiv 0 \mod(p - 1) \). Then from the argument above we have

\[ H_\ast(\Omega^{2k}W, R) = \bigotimes_{n \text{ prime to } p} A_n \]

for \( R = \mathbb{Z}(p) \) and hence for any \( p \)-local \( R \) by universal coefficients. □

By the same arguments we have a decomposition in cohomology.

2.6 Theorem. For \( n \) prime to \( p \) and \( j > 0 \) define \( a_{n,j}^* \in H^{2np}(BU, R) \) inductively by \( a_{n,j}^* = T_j(a_{n,0}^*, \ldots, a_{n,j}^*) \). Let \( A_n^* \) denote the polynomial Hopf subalgebra \( R[a_{n,0}^*, a_{n,1}^*, \ldots] \). Then there exist isomorphisms of Hopf algebras

\[ H^\ast(BU, R) \cong \bigotimes_{n \text{ prime to } p} A_n^* \text{ and } \]

\[ H^\ast(\Omega^{2k}W, R) \cong \bigotimes_{n \text{ prime to } p} A_n^*. \]

The homology and cohomology decompositions of \( BU \) (and of \( \Omega^{2k}W \)) are dual. If we give \( H_\ast(BU, R) \) the basis of monomials in \( \{a_{n,j}\} \), then \( c_{n,k}/n \) is dual to \( a_{n,k}^* \). The subalgebras \( A_n \) and \( A_n^* \) are in some sense the minimal bipolynomial Hopf algebras [21].

3. The homology of \( H \)-maps. We can now describe the induced homomorphism of an \( H \)-map \( f : BU \rightarrow BU \). Since the Hurewicz map induces an isomorphism between \( \pi_\ast(BU) \otimes \mathbb{Q} \) and the primitives of \( H_\ast(BU, \mathbb{Q}) \) ([5],[17]), the characteristic sequence \( \lambda \) of \( f \) may be defined by \( f_\ast(d_{n,0}) = \lambda_m d_{m,0} \).
where $d_{an}$ is as in 2.2. Thus for each $n$ prime to $p$ and $k > 0$ we may apply $f_*$ to the defining relation 2.4 to obtain

$$\lambda_{np^*} T_k(a_{n,0}, \ldots, a_{n,k}) = T_k(f_*(a_{n,0}), \ldots, f_*(a_{n,k})). \quad (3.1)$$

We illustrate the use of this equation in the ring $H_*(BU, \mathbb{Z}(p))$ where the $a_{n,k}$ are not $p$-divisible. Suppose $v(\lambda_{np^*}) = j > 0$. Then $(f_*(a_{n,0}))^{p^j}$ is divisible by $p$ and hence by $p^j$. If $j > 1$ divides both sides of 3.1 by $p$, conclude that $(f_*(a_{n,1}))^{p^j - 1}$ is divisible by $p^{j - 1}$, and continue inductively. It follows that $f_*(a_{n,0}), \ldots, f_*(a_{n,\min(j - 1, k)})$ all vanish mod $p$ and, if $(j - 1) < k$, that $f_*(a_{n,j}) \equiv 0$.

Restating this in terms of surplus (1.5), we have that $s(f)_{np^*} > 0$ iff $f_*(a_{n,0}), \ldots, f_*(a_{n,k})$ all vanish mod $p$. Moreover, if $s(f)_{np^*} < 0$ and $j = k + s(f)_{np^*}$, then $f_*(a_{n,0}), \ldots, f_*(a_{n,j - 1})$ all vanish mod $p$ while $f_*(a_{n,j})$ does not. In particular, if $f_*(a_{n,k})$ is the first nonvanishing (mod $p$) term of $f_*(a_{n,0}), f_*(a_{n,1}), \ldots$, then we must have $s(f)_{np^*} = 0$, $s(f)_{np^{*+1}} = -1, \ldots$. This proves Assertion 1.6.

3.2 Definition. For any $H$-map $f: BU \to BU$ and $n$ prime to $p$ define the $n$th index of $f$, denoted $\delta_n(f) = \delta_n$, by $\delta_n = j$ if $s(f)_{np^*} = 0$ and $\delta_n = \infty$ if no such $j$ exists.

By 1.7 the index $\delta_n$ is constant as $n$ varies within a residue class mod $(p - 1)$. The discussion above may be summarized by saying that $f_*(a_{n,i})$ vanishes mod $p$ for $j < \delta_n$ while $f_*(a_{n,i}) \equiv 0 \mod p$.

3.3 Theorem. For each $n$ prime to $p$ and $k < \delta_n$ we have

$$f_*(a_{n,k}) = \left(\lambda_{np^*} / p^k\right) T_k(a_{n,0}, \ldots, a_{n,k}) + px$$

for some $x$ in the ideal in $A_n$ generated by all $f_*(a_{n,i}), i < k$. If $f$ satisfies the growth condition $s(f)_{np^*} < ps(f)_{np^{*-1}} - 1$ for $i < k$, then $v(px) > v(\lambda_{np^*} / p^k) = s(f)_{np^*}$ and thus $v(f_*(a_{n,k})) = s(f)_{np^*}$.

For the final statement we assume that the ground ring $R$ is $\mathbb{Z}(p)$. For $x \in H_*(BU, R)$ the symbol $v(x)$ then denotes the maximal $j$ such that $x = p^j y$ for some $y \in H_*(BU, R)$. The growth condition is not satisfied in general (examples can be easily constructed using 1.4) but is satisfied by a number of maps $f$ of geometric interest. The last statement often permits an explicit description of the torsion in the $\mathbb{Z}(p)$ cohomology of the fiber of $f$.

Proof. If we solve the defining equation 3.1 for $f_*(a_{n,k})$ we obtain

$$\left(\lambda_{np^*} / p^k\right) T_k(a_{n,0}, \ldots, a_{n,k}) - \left(1 / p^k\right) T_{k-1}(f_*(a_{n,0}))^p, \ldots, (f_*(a_{n,k-1}))^p).$$

But $(f_*(a_{n,i}))^{p^{k-i}} / p^{k-i} = ((f_*(a_{n,i}))^{p^{k-i-1}} / p^{k-i}) f_*(a_{n,i})$ when $i < k$ where the first factor is evidently $p$-divisible since $f_*(a_{n,i})$ is. Thus the second polynomial above has the desired form $px$. 

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To prove the second assertion we may suppose inductively that \( \nu(f_\bullet(a_{n,i})) = s(f)\nu_p \) for \( i < k \) and thus
\[
\nu\left( \left( f_\bullet(a_{n,i}) \right)^{p^{k-i}} / p^{k-i} \right) = p^{k-i}s(f)\nu_p - (k - i).
\]
But the growth condition implies that \( ps(f)\nu_p^{k-1} - 1 < p^2s(f)\nu_p^{k-2} - 2 < \ldots \)
and hence
\[
\nu(px) = \nu\left( \left( f_\bullet(a_{n,k-1}) \right)^p / p \right) = ps(f)\nu_p^{k-1} - 1 > s(f)\nu_p^k. \quad \square
\]
Evaluating \( f_\bullet(a_{n,k}) \) for \( k > \delta_n \) is more difficult since \( \lambda_{np} \) and \( f_\bullet(a_{n,i}) \) have insufficient \( \nu_p \)-divisibility to carry out the manipulations above. A less direct approach seems more fruitful. Fix some \( n \) prime to \( p \) with finite \( \delta_n \) and for each \( k > \delta_n \) let \( u_k = \lambda_{np}/p^{\delta_n} \), a unit in \( \mathbb{Z}/p^k \). By 1.7 it follows that \( (1/u_k) = (1/u_{k-1}) \mod p^{\delta_n-\delta_k} \). Then using 2.3 we may inductively define polynomials \( \rho_{k-\delta_k}(t) = \rho_{k-\delta_k}(t_0, \ldots, t_{k-\delta_k}) \) and \( \xi_{k-\delta_k}(t) = \xi_{k-\delta_k}(t_1, \ldots, t_k) \) for \( k > \delta_n \) as follows.
\[
(1/u_k)T_{k-\delta_k}(t) = T_{k-\delta_k}(\rho_0(t), \ldots, \rho_{k-\delta_k}(t)),
\]
\[
- p^{k-\delta_k+1}T_{k-\delta_k-1}(t_0, t_1, \ldots, t_{k-\delta_k+1}) = T_{k-\delta_k}(\xi_0(t), \ldots, \xi_{k-\delta_k}(t)). \quad (3.4)
\]
It follows immediately that \( \xi_{k-\delta_k} \) is always \( p \)-divisible and that \( \rho_{k-\delta_k}(t) = (1/u_k)t_k + \text{decomposables} \). Hence the following result includes a partial description of \( f_\bullet(a_{n,k}) \).

3.5 Theorem. If \( n \) is prime to \( p \) and \( k > \delta_n \) then
\[
a_{n,k}^{\nu_p} = \rho_{k-\delta_k}(f_\bullet(a_{n,k})), \ldots, f_\bullet(a_{n,k})) + \xi_{k-\delta_k}(a_{n,1}, \ldots, a_{n,k}) + px
\]
for some \( x \) in the ideal in \( A_n \) generated by all \( f_\bullet(a_{n,i}) \), \( i < k \).

Proof. For the proof we abbreviate \( a_{n,i} = a_i \), \( \xi_{i-\delta_i}(a_1, \ldots, a_i) = \xi_{i-\delta_i} \), and \( \rho_{i-\delta_i}(f_\bullet(a_{n,i})), \ldots, f_\bullet(a_i)) = \rho_{i-\delta_i} \). For \( k = \delta_n \) the result is established in 3.3, so suppose inductively that 3.5 holds for all \( i \) such that \( \delta_n < i < k \). Then using the relations \( a_{n,k}^{\nu_p} = \rho_{k-\delta_k} + \xi_{k-\delta_k} + px \), the \( p \)-divisibility of \( \xi_{k-\delta_k} \), and the multinomial theorem it follows (compare the proof of 2.4) that
\[
T_{k-\delta_k-1}(a_0^{p^{k-1}}, \ldots, a_{k-\delta_k}^{p^{k-1}}) = T_{k-\delta_k-1}(\rho_0, \ldots, \rho_{k-\delta_k-1})
\]
\[
+ T_{k-\delta_k-1}(\xi_0, \ldots, \xi_{k-\delta_k-1}) + p^{k-\delta_k+1}\bar{x} \quad (*)
\]
for some \( \bar{x} \) in the ideal generated by \( f_\bullet(a_0), \ldots, f_\bullet(a_{k-1}) \). If we multiply the defining equations 3.1 by the unit \( p^{\delta_k}/\lambda_k = 1/u_k \) and substitute the identities (*), the left-hand side of the resulting equation is
\[
p^{\delta_k}T_{k-\delta_k-1}(\rho^p) + p^{\delta_k}T_{k-\delta_k-1}(\xi^p) + p^{k+1}\bar{x}
\]
\[
+ p^kq_{k-\delta_k}^{\nu_p} + p^{k+1}T_{k-\delta_k-1}(a_{k-\delta_k+1}, \ldots, a_k).
\]
Substituting the defining relation 3.4 for $\xi_{k-\delta}$ and simplifying via the identity $T_{k-\delta}(\xi) = T_{k-\delta-1}(\xi^p) + p^{k-\delta} \xi_{k-\delta}$ this becomes

$$p^{k+1}T_{k-\delta-1}(\rho^p) + p^{k+1}x + p^k \xi_{k-\delta} - p^{k-\delta} \xi_{k-\delta}.$$  (***)

Using the arguments of 3.3 it follows that the right-hand side of 3.1 (multiplied by $p^{\delta}/\lambda_{mp^k} = 1/u_k$ as before) may be written as

$$p^{k+1}x + (p^{\delta}/u_k)T_{k-\delta}(f_*(a_{\delta}), \ldots, f_*(a_k))$$

for some $x$ in the ideal generated by $f_*(a_0), \ldots, f_*(a_k)$. Then substituting the defining equation 3.4 for $\rho_{k-\delta}$ and the identity $T_{k-\delta}(\rho) = T_{k-\delta-1}(\rho^p) + p^{k-\delta} \rho_{k-\delta}$ this becomes

$$p^{k+1}x + p^\delta T_{k-\delta-1}(\rho^p) + p^k \rho_{k-\delta}.$$  (***)

The desired result follows by equating (***) and (***)

The above work made no use of the properties of the $H$-map $f$ beyond the relation 3.1. Since the cohomology homomorphism $f^*: H^*(BU, R) \to H^*(BU, R)$ sends each primitive $c_e$ to some multiple of itself and since $\langle c_e, de \rangle = \pm n$, it follows that $f^*(c_e) = \lambda_n c_e$ and 3.3 and 3.5 hold for cohomology.

3.6 Theorem. For each $n$ prime to $p$ we have

$$f^*(a_{n,k}) = (\lambda_{mp^k}/p^k)T_k(a_{n,0}, \ldots, a_{n,k}) + px^*_k \quad \text{for } k < \delta_n,$$

\[ (a_{n,k-\delta})^{p^k} = \rho_{k-\delta}(f^*(a_{n,\delta}), \ldots, f^*(a_{n,k})) + \xi_{k-\delta} (a_{n,1}, \ldots, a_{n,k}) + px^*_k \]

if $k > \delta_n$

where $x^*_k$ lies in the ideal in $A^*_n$ generated by all $f^*(a_{n,i})$, $i < k$, and $\xi_{k-\delta}$, $\rho_{k-\delta}$ are as in 3.4. If $f$ satisfies the additional growth condition $s(f)_{mp^k} < ps(f)_{mp^{k-1}} - 1$ for $i < \delta_n$, then $\nu(f^*(a_{n,k})) = s(f)_{mp^k}$ for $k < \delta_n$.

For the last statement we again assume that the ground ring is $\mathbb{Z}(p)$. The above result (and the homology analogue) simplifies considerably if we reduce mod $p$.

3.7 Corollary. The Hopf algebra homomorphism $f^*: H^*(BU, \mathbb{Z}/p) \to H^*(BU, \mathbb{Z}/p)$ induced by an $H$-map $f$ satisfies

$$f^*(a_{n,k}^*) = \begin{cases} 0 & \text{if } k < \delta_n, \\ u_k (a_{n,k-\delta})^{p^k} + x_k & \text{if } k > \delta_n, \end{cases}$$

where $x_k$ lies in the ideal in $A^*_n$ generated by all $(a_{n,j}^*)^{p^k}$, $j < k - \delta_n - 1$, and $u_k \in \mathbb{Z}(p)$ is a unit.
4. The mod $p$ cohomology of the fiber. For any commutative Noetherian
ring $R$ and for any simple fibration $F \to E \to B$ with $E$ and $B$ connected and
of finite type over $R$ there is an Eilenberg-Moore spectral sequence $\{E_r\}$
converging from $E_2 = \text{Tor}_{H^*(B, R)}(R, H^*(E, R))$ to $E_\infty = E_0 H^*(F, R)$, the
bigraded module associated to some filtration of $H^*(F, R)$ ([9], [10]). This
spectral sequence is natural for morphisms of fibrations, and the filtration on
$H^*(F, R)$ behaves well with respect to products. Applying the diagonal map
we obtain a spectral sequence of bigraded rings. If $E \to B$ is an $H$-map there
is an induced multiplication on $F$ and, when $R$ is a field, we have a spectral
sequence of bigraded Hopf algebras.

Any map $f: BU \to BU$ may be regarded as a simple fibration [24, §2.8] with
$H^*(BU, R)$ of finite type over $R$ if $R$ is $p$-local. For such fibrations we may
apply the beautiful Eilenberg-Moore spectral sequence collapse theorems of
May ([14], [10]), Munkholm [18], and others ([3], [12], [22], [27]). The following
special case of May's result [10, Theorem B] will suffice. We use the
notation $\bigoplus M$ to denote the graded module associated to a bigraded module
$M$.

4.1 Theorem. If $f: BU \to BU$ is a fibration with fiber $F$, then there is a
natural isomorphism of bigraded algebras

$$E_0 H^*(F, R) \cong \text{Tor}_{H^*(BU, R)}(R, H^*(BU, R))$$

for some natural, nonpositive, decreasing filtration of $H^*(F, R)$. If $f$ is an
$H$-map and $R$ is a field then this is an isomorphism of bigraded Hopf algebras.
In either case, the map in filtration zero is induced by the canonical map

$$H^*(BU, R) \otimes_{H^*(BU, R)} R \to H^*(F, R)$$

induced by inclusion of the fiber. Finally, there is no additive extension problem
and we obtain an isomorphism of graded $R$ modules

$$H^*(F, R) \cong \bigoplus \text{Tor}_{H^*(BU, R)}(R, H^*(BU, R)).$$

4.2. To determine the torsion products above we first compute
$\text{Tor}_{A_n^*}(R, A_n^*)$ for each $n$ prime to $p$ by means of the classical Koszul resolution. Let $\sigma: H^1(BU, R) \to H^1(U, R)$ denote the cohomology suspension; this is the composite

$$H^1(BU, R) \xrightarrow{\sigma} H^1(PBU, U; R) \xrightarrow{\delta^{-1}} H^{i-1}(U, R)$$

where $U \to PBU \to BU$ is the path fibration. Then $\sigma(A_n^*) \subseteq H^*(U, R)$ is an
exterior Hopf subalgebra $\bigl\{a_{n,j}^*\bigr| j > 0\}$ on primitive generators $a_{n,j}^* \in
H^{2np^j-1}(U, R)$. The module $\sigma(A_n^*) \otimes_R A_n^*$ with differential generated by
$\sigma a_{n,j}^* \to a_{n,j}^*$ is an $A_n^*$-free resolution of $R$ (compare [3, §2]). Moreover, the
Hopf algebra structures on $A_n^*$ and $\sigma(A_n^*)$ induce one on the resolutions.
Applying $\bigotimes RA_n^*$ and checking the definition of product and coproduct for
Tor, it follows that \( \bigoplus \text{Tor}_{\mathbb{A}^*}(R, A_n^*) = H(\sigma A_n^* \otimes A_n^*, d) \) as a graded ring (and as a Hopf algebra if \( f \) is an \( H \)-map and \( R \) is a field) with differentials given by \( a_n^* \to f^*(a_n^*) \). When \( R \) is a field the external product on Tor is also an isomorphism \([18]\) so we obtain a natural isomorphism
\[
\bigoplus \text{Tor}_{H^*(BU, R)}(R, H^*(BU, R)) \cong \bigotimes_{n \text{ prime to } p} \text{Tor}_{\mathbb{A}^*}(R, A_n^*).
\]

4.3 Theorem. Let \( f: BU \to BU \) be an \( H \)-map, \( R = \mathbb{Z}/p \), and regard \( A_n^* \) as a module over itself via \( f^* \). Then
\[
\bigoplus \text{Tor}_{\mathbb{A}^*}(\mathbb{Z}/p, A_n^*) \cong E \{ a_n^* | 0 < j < \delta_n \} \otimes_{\mathbb{Z}/p} (A_n^* / \xi^k A_n^*)
\]
as Hopf algebras where \( \xi: x \to x^p \) is the Frobenius map.

Proof. By [23, 1.5] there is an isomorphism of Hopf algebras
\[
\bigoplus \text{Tor}_{\mathbb{A}^*}(\mathbb{Z}/p, A_n^*) \cong \text{Tor}_{\text{subker}}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes (A_n^* / f^* A_n^*)
\]
where subker \( f^* \) is the unique Hopf subalgebra of \( A_n^* \) generating \( \ker f^*: A_n^* \to A_n^* \). But by 3.7, subker \( f^* = \mathbb{Z}/p[a_n^*0, \ldots, a_n^*\delta_n-1] \) and \( f^* A_n^* = \xi^k A_n^* \), and the result follows by inspection of the Koszul resolution.

Thus by the Kunneth theorem there is an isomorphism of Hopf algebras
\[
\bigoplus E_0 H^*(F, \mathbb{Z}/p) \cong \bigotimes_{n \text{ prime to } p} E \{ a_n^* | 0 < j < \delta_n \} \otimes (A_n^* / \xi^k A_n^*).
\]
Comparing this with the Eilenberg-Moore spectral sequence for the universal principal fibration (see e.g. [22]) and applying some standard Hopf algebra arguments, we may replace the term on the left with \( H^*(F, \mathbb{Z}/p) \). This proves the following.

4.4 Theorem. Let \( f: BU \to BU \) be an \( H \)-map with fiber \( F \). Then \( H^*(F, \mathbb{Z}/p) \)
\[
= \bigotimes_{n \text{ prime to } p} E \{ a_n^* | 0 < j < \delta_n \} \otimes (A_n^* / \xi^k A_n^*)
\]
as Hopf algebras. The induced homomorphism of the inclusion \( i: F \to B \) is given by the natural projections \( A_n^* \to A_n^* / \xi^k A_n^* \), while \( a_n^* \) pulls back via the canonical map \( U \to F \) to the cohomology suspension of \( a_n^* \).

By 1.3 the fiber \( F \) of an \( H \)-map \( f: BU \to BU \) decomposes as an \( H \) space into a product \( F(0) \times F(2) \times \cdots \times F(2p - 4) \) where (in the notation of 1.4) \( F(2k) \) may be defined as the fiber of \( f_{2k}: BU \to BU \) or of \( f_{2k}: \Omega^{2k} W \to \Omega^{2k} W \). Applying 4.4 to \( f_{2k} \) we obtain the following.

4.5 Corollary. If \( F = F(0) \times F(2) \times \cdots \times F(2p - 4) \) is the natural \( H \) space decomposition of \( F \), then
\[
H^*(F(2k), \mathbb{Z}/p) = \bigotimes_{n + k \equiv 0 \mod(p - 1)} E \{ a_n^* | 0 < j < \delta_n \} \otimes (A_n^* / \xi^k A_n^*).
\]
5. The Bockstein spectral sequence of the fiber. For any chain complex $(C, \partial)$ of abelian groups the long exact sequence in homology associated to the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$ may be regarded as an exact couple whose underlying graded spectral sequence is the Bockstein spectral sequence. The $E_1$ term is $E_1(C) = H_*(C, \mathbb{Z}/p)$ and the differential for $E_r$ is the $r$th order Bockstein $\beta_r$, which may be described as follows. Given $\{c\} \in H_*(C, \mathbb{Z}/p)$ where $c \in C$ is a chain satisfying $\partial(c) = p^r c'$, define $\beta_r(c) = \{c'\} \in H_*(C, \mathbb{Z}/p)$. This spectral sequence is discussed in considerable detail in [6] and [7]. We recall some useful facts.

5.1. If $x$ generates a direct summand $\mathbb{Z}/p^r$ in $H_*(C)$, then $j_\partial(x) \neq 0$ in $E_r(C)$ where $j_\partial = H_*(C) \to E_r(C)$ is the map induced by $j : H_*(C) \to H_*(C, \mathbb{Z}/p)$.

5.2. Kernel$(j_\partial) = pH_*(C) + T_{r-1}$ where $T_{r-1}$ is the subgroup of $H(C)$ annihilated by $p^{r-1}$.

5.3. $E_\infty(C) = (H_*(C)/\text{Torsion}) \otimes \mathbb{Z}/p$.

5.4. Given complexes $C$ and $D$, the canonical map $H_*(C, \mathbb{Z}/p) \otimes H_*(D, \mathbb{Z}/p) \to H_*(C \otimes D, \mathbb{Z}/p)$ induces an isomorphism of chain complexes $E_r(C) \otimes E_r(D) \to E_r(C \otimes D)$ for all $r > 1$.

Now suppose $f : BU \to BU$ is an $H$-map with fiber $F$. Using 4.1 and 4.2 with $R = \mathbb{Z}(p)$ it follows that $H^*(F, \mathbb{Z}(p))$ is the homology of the chain complex $\otimes_{n \text{ prime to } p} \sigma(A_n^*) \otimes A_n^*$ with degree one differential defined by $\partial(a_n^*(c)) = f^*(a_n^*(c))$.

by 5.4, to describe the Bockstein spectral sequence of the space $F$ it will suffice to describe that of the component complexes $\sigma(A_n^*) \otimes A_n^*$. Of particular interest for geometric applications is a two-step complex of surplus $s$. This is a chain complex $\sigma(A_n^*) \otimes A_n^*$ in which the $H$-map $f$ inducing the differential as in 4.2 satisfies $s(f)_{n+k} = s > 0$ for all $k < \delta_n$.

5.5 Theorem. Let $\sigma(A_n^*) \otimes A_n^*$ be a two-step complex of surplus $s$. For $r < s$ the Bockstein spectral sequence is given by

$$E_r \cong E \{a_n^* \otimes \cdots \otimes a_n^* \otimes A_n^* / / \xi^s A_n^* \otimes \mathbb{Z}/p.$$

If $r > s$ we have

$$E_r \cong E \{a_n^* \otimes \cdots \otimes a_n^* \otimes A_n^*(r-s) / / \xi^s A_n^*(r-s) \otimes \mathbb{Z}/p$$

where $A_n^*(k) = Z(\mathbb{Z}(p)[a_n^* k, \ a_n^* k+1, \cdots]$. 

Proof. For convenience we may first suppose that the characteristic subsequence $\lambda, \lambda^p, \ldots$ used to define the differential on $\sigma(A_n^*) \otimes A_n^*$ is given by $p^s, p^{s+1}, \ldots, p^{s+\delta_n}, p^\delta, p^\delta, \ldots$. By 3.6 we may choose elements...
\[
c_k = a_{a,k}^* - \sum_{i<k} \xi(a_{i,n}^*) \otimes x_i \text{ for suitable } x_i \in A_{i,n}^* \text{ such that } \partial(c_k) = p^n T_k(a_{n,k}^*, \ldots, a_{n,i}^*) \text{ if } k < \delta_n \text{ and } \partial(c_k) = (a_{n,k-\delta_n}^*)^{p^n} - \xi_{k-\delta_n}(a_{n,k}^*, \ldots, a_{n,k}^*) \text{ for } k > \delta_n.\]

Then setting \(b_0 = c_0, b_k = c_k - b_0(a_{n,0}^*) p^{k-1} \cdot \ldots \cdot \xi_{k-\delta_n}(a_{n,k}^*, \ldots, a_{n,0}^*) \text{ for } k > \delta_n,\) it follows that we may rewrite \(\sigma(A_n^*) \otimes A_*^*\) as a complex \(C(0) = E(b_0, b_1, \ldots) \otimes A_*^*\) with differential \(\partial_{0,t}\) defined by \(\partial_{0,t}(b_k) = p^{k+t} a_{n,k}^* \text{ for } k < \delta_n \text{ and } \partial_{0,t}(b_k) = (a_{n,k-\delta_n}^*)^{p^n} - \xi_{k-\delta_n}(a_{n,k}^*, \ldots, a_{n,0}^*) \text{ if } k > \delta_n.\)

For each \(t > 0\) let \(C(k)\) denote the free \(Z_p\) module \(E\{b_k, b_{k+1}, \ldots\} \otimes A_*^*(k)\) where \(b_j\) is, as above, an exterior generator of degree \(np^j - 1.\) On \(C(0)\) we have already defined differentials \(\partial_{0,t}\) for each \(s > 0.\) We may similarly define \(\partial_{t,s}\) on \(C(k)\) by

\[
\partial_{t,s}(b_j) = p^{j-k} a_{s,j}^* \text{ if } k < j < k + \delta_n, \quad \partial_{t,s}(b_j) = (a_{n,j-\delta_n}^*)^{p^n} - \xi_{j-\delta_n}(a_{n,j+1}^*, \ldots, a_{n,j}^*) \text{ if } j > k + \delta_n.
\]

**Assertion 1.** \(E_r(C(0), p^i \partial_{0,t-r}) \cong E_r(C(0), p^{i-1} \partial_{0,t-r+1})\) for all \(i = 1, \ldots, s - 1\) and \(r > i.\)

**Assertion 2.** \(E_r(C(k), p^k \partial_{k,i}) \cong E_r(C(k + 1), p^k \partial_{k+1,i})\) if \(r > k + 1.\)

Assuming these for the moment the proof proceeds as follows. Clearly \(E_t(C(k), p^t \partial_{k,s}) = C(k)\) with \(i\)th Bockstein the mod \(p\) reduction of \(\partial_{k,s}.\) Thus

\[
E_{i+1}(C(k), p^i \partial_{k,s}) = E\{b_k, \ldots, b_{k+\delta_n-1}\} \otimes A_*^*(k)/\xi^k A_*^*(k) \otimes Z/p
\]

by the argument of 4.3. From the definition of the elements \(b_j\) this is isomorphic to \(E(\sigma a_{n,k}^*, \ldots, \sigma a_{n,k+\delta_n-1}^*) \otimes A_*^*(k)/\xi^k A_*^*(k) \otimes Z/p.\) Thus if \(r < s\) it follows by Assertion 1 that

\[
E_r(C(0), \partial_{0,s}) \cong E_r(C(0), p^{r-1} \partial_{0,s-r+1}) \cong E\{\sigma a_{n,0}^*, \ldots, \sigma a_{n,\delta_n-1}^*\} \otimes A_*^*/\xi^k A_*^* \otimes Z/p.
\]

When \(r > s\) it follows by Assertions 1 and 2 that

\[
E_r(C(0), \partial_{0,s}) \cong E_r(C(0), p^r \partial_{0,1}) \cong E_{r-s+1}(C(0), \partial_{0,1}) \cong E_{r-s+1}(C(r - s), p^{r-s} \partial_{r-s,1}) \cong E\{\sigma a_{n,-r}, \ldots, \sigma a_{n,-r+\delta_n-1}^*\} \otimes A_*^*(r - s)/\xi^k A_*^*(r - s) \otimes Z/p.
\]

To prove Assertion 1 it suffices to check that \(E_t(C(0), p \partial_{0,t-i}) \cong E_t(C(0), \partial_{0,t-i+1})\) for \(r > 1\) and \(i = 1, \ldots, s - 1.\) Define a chain map \(\varphi: (C(0), \partial_{t-i+1}) \to (C(0), p \partial_{0,t-i})\) by setting \(\varphi = 1\) on \(A_*^*, \varphi(b_i) = b_i\) if \(i < \delta_n,\) and \(\varphi(b_i) = p b_i\) if \(i > \delta_n.\) Then \(\varphi\) induces an isomorphism on \(E_2.\) For as noted
above,
\[ E_2(C(0), p \partial_{\sigma_{n-1}}) = E_1(C(0), \partial_{\sigma_{n-1}}) \]
\[ = E \{ b_0, \ldots, b_{\delta_n-1} \} \otimes A_n^* / \xi \delta A_n^* \otimes \mathbb{Z}/p. \]

But by the same argument we have \( E_1(C(0), \partial_{\sigma_{n-1}+1}) = E \{ b_0, \ldots, b_{\delta_n-1} \} \otimes A_n^* / \xi \delta A_n^* \otimes \mathbb{Z}/p \) with the entire homology group lying in the kernel of the first Bockstein.

The second assertion is proved similarly. We will define a chain map \( \varphi: (C(k+1), p \partial_{k+1,1}) \to (C(k), \partial_{k,1}) \) which induces an isomorphism on \( E_2 \). Let \( \varphi|_{A(k+1)} \) be the canonical inclusion \( A(k+1) \subseteq A(k) \), define \( \varphi(b) = b_j \) for \( k+1 < j < k + \delta_n \), and
\[ \varphi(b_{k+\delta}) = pb_{k+\delta} - b_k \otimes (a_{n,k}^*)^{p^\delta_n - 1} - \cdots - b_{k+\delta-1} \otimes (a_{n,k+\delta-1}^*)^{p^\delta_n - 1}. \]

Next abbreviate \( \xi(a_{n,k+1}^*, \ldots, a_{n,k+\delta}) = \xi \) and \( \xi_{j-1}(a_{n,k+2}^*, \ldots, a_{n,k+\delta}) = \hat{\xi}_{j-1} \), and note that the defining relation 3.4 yields
\[ pT_{k-\delta_n-1} \xi_0, \ldots, \hat{\xi}_{k-\delta_n-1} = T_k \cdot (\xi_0, \ldots, \xi_{k-\delta_n}). \]

Then using the \( p \)-divisibility of \( \xi_0 \) and the binomial theorem (compare the proof of 2.4) it follows inductively that \( \hat{\xi}_j = \xi + \eta \xi_0 \) for some \( \eta \in A_n^*(k) \). We complete the definition of \( \varphi \) by setting \( \varphi(b_{k+\delta} + x) = pb_{k+\delta} + \eta \xi_0 \) for all \( j > 0 \) where \( x = pb_{k+\delta} - b_k(a_{n,k}^*)^{p^\delta_n - 1} \) satisfies \( \partial_{k,1}(x) = p \xi_0 \).

To compute \( \varphi \) on \( E_2 \) recall first that
\[ E_2(C(k+1), p \partial_{k+1,1}) \]
\[ = E \{ b_{k+1}, \ldots, b_{k+\delta_n} \} \otimes A_n^*(k+1) / \xi \delta A_n^*(k+1) \otimes \mathbb{Z}/p \]
and
\[ E_1(C(k), \partial_{k,1}) = E \{ b_k, \ldots, b_{k+\delta_n} \} \otimes A_n^*(k) / \xi \delta A_n^*(k) \otimes \mathbb{Z}/p \]
as before. The first Bockstein on the latter ring is defined by the conditions
\[ \beta_1(b_k) = a_k \text{ and } \beta_1(b_{k+1}) = \cdots = \beta_1(b_{k+\delta_n-1}) = \beta_1(A_n^*(k) / \xi \delta A_n^*(k)) = 0. \]

It follows that Ker \( \beta_1 \) is the subring generated by \( b_{k+1}, \ldots, b_{k+\delta_n-1}, b_k \otimes (a_{n,k}^*)^{p^\delta_n - 1}, \) and \( A_n^*(k) \), while \( \text{Im}(\beta_1) = a_{n,k}^* \cdot (\text{Ker} \beta_1) \). By a change of basis we may thus write \( \text{Ker} \beta_1 = E \{ b_{k+1}', \ldots, b_{k+\delta_n}' \} \otimes A_n^*(k) / \xi \delta A_n^*(k) \otimes \mathbb{Z}/p \) where \( b_j' = b_j \) if \( j < k + \delta_n \) and \( b_{k+\delta_n}' = pb_{k+\delta_n} - b_k \otimes (a_{n,k}^*)^{p^\delta_n - 1} \). Thus \( E_2(C(k), \partial_{k,1}) = E \{ b_{k+1}', \ldots, b_{k+\delta_n}' \} \otimes (A_n^*(k+1) / \xi \delta A_n^*(k+1)) \) with \( \varphi \) inducing an isomorphism on \( E_2 \). □

The Bockstein spectral sequence for the complex \( C = \sigma(A_n^*) \otimes A_n^* \) with arbitrary surplus \( s_0, s_{np}, \ldots, s_{np^{s-1}} \) is, unfortunately, still a mess. If \( s = \min \{ s_0, \ldots, s_{np^{s-1}} \} \) and \( C_1 = \sigma(A_n^*) \otimes A_n^* \) is the complex of constant surplus \( s \) obtained by dividing the characteristic sequence for \( C \) by appropriate
powers of $p$, then applying 5.1, 5.2, and 5.3 to the computation 5.5 yields reasonably explicit information about the torsion group $H(C^1)$. In particular cases one can then profitably compare $H(C)$ and $H(C^1)$ by studying, via the mapping cone construction, the homology of the chain map $\varphi: C \to C^1$ defined by

$$
\varphi(aa_n^{*}) = p^s w^{-1} aa_n^{*} \quad \text{for} \ j < \delta_n,
$$

$$
\varphi(aa_n^{*}) = aa_n^{*} \quad \text{for} \ j \geq \delta_n,
$$

and $\varphi|_{A^*_n} = 1$. Such considerations are omitted here in the hope the general case will soon be brought under control.

6. An important special case. The $\mathbb{Z}/(p)$ cohomology of the fiber of an $H$-map $f: BU \to BU$ with index never exceeding 1 admits a direct description. Such an explicit computation, though less transparent than the Bockstein spectral sequence computation of the previous section, will be necessary in a subsequent study of the classifying space of smoothing theory.

We first suppose that $f: BU \to BU$ is an $H$-map with characteristic sequence $\lambda$ such that, for a fixed $n$ prime to $p$, $\delta_n(f) = 1$ and $\lambda_n = p^s w$ for some unit $w \in \mathbb{Z}/(p)$. Regarding $A^*_n$ as a module over itself via $f^*$ as usual, we describe $\bigoplus \text{Tor}_{A_n^*}(\mathbb{Z}/(p), A^*_n)$ as both a $\mathbb{Z}/(p)$ module and an algebra.

6.1 Theorem. There is an isomorphism of graded algebras

$$
\bigoplus \text{Tor}_{A_n^*}(\mathbb{Z}/(p), A^*_n) \cong A_n^*/(f^* A_n^*) = A_n^*/I
$$

where $I$ is the $A^*_n$ ideal generated by

$$
\lambda_n a_n^{*0} (a_n^{*1})^p - pw_1 a_n^{*1}, (a_n^{*2})^p - pw_2 a_n^{*2}, \ldots
$$

for certain units $w_1, w_2, \ldots \in \mathbb{Z}/(p)$. The submodule of elements of degree $2r$ is isomorphic to $\bigoplus \alpha \mathbb{Z}/\text{ind}(\alpha)$ where $\alpha = (\alpha_1, \alpha_2, \ldots)$ ranges over all sequences of nonnegative integers of weight $r$ each of whose nonzero entries is $< (p - 1)$ and lies in the subsequence $\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots$, and where $\text{ind}(\alpha) = p^m + j$ if $\alpha_{n+p}$ is the first nonzero entry of $\alpha$.

Proof. By the definition of torsion product (compare e.g. [3]) we have that $\text{Tor}_{A_n^*}(\mathbb{Z}/(p), A^*_n) = \mathbb{Z}/(p) \otimes_{f^*} A^*_n = A_n^*/(f^* A_n^*)$ where the quotient module $A_n^*/(f^* A_n^*)$ is torsion because $f^*$ is a rational isomorphism. But since $(A_n^*/(f^* A_n^*)) \otimes \mathbb{Z}/p \cong A_n^* \otimes \mathbb{Z}/p \otimes (f^* A_n^*) \otimes \mathbb{Z}/p$ is a vector space with one generator in each dimension $nj$ by 3.7, it follows by 4.4 and inspection that

$$
\text{Tor}_{A_n^*}(\mathbb{Z}/p, A^*_n \otimes \mathbb{Z}/p) \cong E \{ \sigma a_n^{*0} \} \otimes A^*_n/(\xi A^*_n \otimes \mathbb{Z}/p)
$$

$$
\cong (A_n^*/(f^* A_n^*) \otimes \mathbb{Z}/p) \oplus \text{Tor}_{1}(\mathbb{Z}/(p), (f^* A_n^*) \otimes \mathbb{Z}/p).
$$

But by universal coefficients $\text{Tor}_{A_n^*}(\mathbb{Z}/p, A^*_n \otimes \mathbb{Z}/p)$ is isomorphic to
(\text{Tor}_{A^*_n}(\mathbb{Z}_p), A^*_n) \otimes \mathbb{Z}/p) \oplus \text{Tor}_1(\text{Tor}_{A^*_n}(\mathbb{Z}_p), A^*_n), \mathbb{Z}/p). \text{ Thus } \text{Tor}_{A^*_n}(\mathbb{Z}_p), A^*_n) = 0 \text{ if } s \neq 0 \text{ and hence } \\

\text{Tor}_{A^*_n}(\mathbb{Z}_p), A^*_n) = \bigoplus \text{Tor}_{A^*_n}(\mathbb{Z}_p), A^*_n) = A^*_n/p^* A^*_n.

For each } j > 0 \text{ let } I_j \text{ denote the } A^*_n \text{ ideal generated by } f^*(a^{*}_{n_0}), \ldots, f^*(a^{*}_{n_j}) \text{ and assume inductively that } I_{j-1} \text{ is generated by } \lambda_n a^{*}_{n_0}, (a^{*}_{n_0})^p - w_1 a^{*}_{n_1}, \ldots, (a^{*}_{n_j-2})^p - w_{j-1} p a^{*}_{n_{j-1}} \text{ for some units } w_1, \ldots, w_{j-1} \in \mathbb{Z}_p. \text{ It follows from 3.4 and 3.5 that } f^*(a^{*}_{n_j}) = v_j((a^{*}_{n_j-1})^p - \xi_{j-1}(a^{*}_{n_1}, \ldots, a^{*}_{n_j})) + x \text{ where } x \in I_{j-1} \text{ and } v_j = p/\lambda_n p^i, \text{ a unit in } \mathbb{Z}_p. \text{ By definition } \xi_{j-1} \text{ is a } p\text{-divisible polynomial given by } \xi_{j-1}(a^{*}_{n_1}, \ldots, a^{*}_{n_j}) = - p a^{*}_{n_j} - p^{1-j}T_{j-2}(\xi^p_0, \ldots, \xi^p_{j-2}). \text{ If } j = 1 \text{ the last term on the right vanishes, and if } j = 2 \text{ this becomes } \xi_1(a^{*}_{n_1}, a^{*}_{n_2}) = - p a^{*}_{n_2} + p^{p-1}(a^{*}_{n_1})^p. \text{ In general, since } \xi_{j-1} \text{ is homogeneous of degree } np^i \text{ and since the degrees of } a^{*}_{n_2}, \ldots, a^{*}_{n_j} \text{ are multiples of } np^2, \text{ we may regard } \xi_{j-1} \text{ as a polynomial in } (a^{*}_{n_1}), a^{*}_{n_2}, \ldots, a^{*}_{n_j}. \text{ Thus } \xi_{j-1} \equiv - p a^{*}_{n_j} + p \xi((a^{*}_{n_j}), \ldots, a^{*}_{n_{j-1}}) \mod I_{j-1} \text{ for some polynomial } \xi. \text{ As before, regard } \xi \text{ as a polynomial in } (a^{*}_{n_2}), a^{*}_{n_3}, \ldots, a^{*}_{n_j} \text{ and continue inductively. Thus we may write } \\

\xi_{j-1}(a^{*}_{n_1}, \ldots, a^{*}_{n_j}) \equiv - p a^{*}_{n_j} + p \eta((a^{*}_{n_j-1})^p) \mod I_{j-1}

\text{for some } \eta \in \mathbb{Z}_p. \text{ It follows that } I_j \text{ is generated over } A^*_n \text{ by } I_{j-1} \text{ and } (a^{*}_{n_j-1})^p - w_k p a^{*}_{n_j} \text{ where } w_k = 1/(\eta p - 1) \text{ and the first part of 6.1 is established.}

It follows from the work above that any monomial in } A^*_n \text{ is congruent modulo } I = I_1 \cup I_2 \cup \ldots \text{ to some multiple of a monomial in "reduced form"}

\left( (a^{*}_{n_1})^{k_0} \cdots (a^{*}_{n_j})^{k_j} \right) = (a^{*}_{n_0})^{k_0} \cdots (a^{*}_{n_j})^{k_j} \text{ where } k_i < p \text{ for all } i.

Moreover this monomial is unique since distinct reduced monomials have different degrees. We must show that } (a^{*}_{n_1})^{k_0} \cdots (a^{*}_{n_j})^{k_j} \text{ has order precisely } p^{m+i} \text{ if } k_i \neq 0. \text{ From the congruences } \\
p^i(a^{*}_{n_1})^{0, \ldots, 0, k_i, \ldots, k_j} \equiv (p^i - 1/w_j)(a^{*}_{n_1})^{0, \ldots, 0, p^k-1, \ldots, k_j} \equiv \ldots \equiv (1/w_1 \cdots w_j)(a^{*}_{n_1})^{p^i, p^i-1, \ldots, p^i-1, k_i, \ldots, k_j}

\text{mod } I \text{ it will suffice to show that any monomial } (a^{*}_{n_1})^{k_0} \cdots (a^{*}_{n_j})^{k_j} \text{ with } 0 < k_0 < p \text{ and } k_i < p \text{ for } i > 0 \text{ has order } p^m \text{ in } A^*_n/I. \text{ Thus suppose } \\

\eta(a^{*}_{n_1})^{k_0} \cdots (a^{*}_{n_j})^{k_j} = \lambda_n a^{*}_{n_0} x_0 + \sum_{i=0}^j ((a^{*}_{n_i})^p - w_i p a^{*}_{n_i})x_i \tag{*}

\text{for some } x_0, \ldots, x_j \in A^*_n \text{ and } \eta \in \mathbb{Z}_p, \text{ and consider the evaluation map } E: A^*_n \to \mathbb{Z}_p, \text{ sending } \\
a^{*}_{n,k} \to (p^{p^k-p^{k-1}-\cdots-p-1}/(w_1^{p^{k-1}} w_2^{p^{k-2}} \cdots w_{k-1} w_k).
This map is defined so that \( E((a_{n,k})^p - w_{k,p}a_{n,k}^*) = 0 \) for all \( k > 0 \). This implies that among all monomials \( (a_n^*)^{(l_0, \ldots, l_j)} \) of a given degree the one whose \( E \) image has the lowest \( p \) divisibility is the monomial in reduced form. Applying \( E \) to (*) above we obtain \( \eta E((a_n^*)^{(k_0, \ldots, k_j)}) = \lambda_j E(a_{n,0}x_0) \). Thus if \( k_0 < p, \eta \) must be divisible by at least \( p^m \). If \( k_0 = p \), then both \( (a_n^*)^{(k_0, \ldots, k_j)} \) and \( a_{n,0}x_0 \) have a factor \( (a_n^*)^p \), so again \( p^m \) divides \( \eta \). □

Using universal coefficients we can now combine the calculation above for various values of \( n \) to prove Theorem D of the introduction.

**Proof of Theorem D.** For each finite subset \( S' \subseteq S \) let \( C_{S'} \) denote the complex \( \bigotimes_{n \in S'} A_n^* \otimes A_n^* \) with differential defined via \( f^* \) as in 4.2, and let \( T_{S'} \subseteq T \) consist of those sequences \( \alpha \) whose nonzero entries lie in some subsequence \( \alpha_n, \alpha_{n+1}, \ldots \) for \( n \in S' \). We will show by induction on the size of \( S' \) that

\[
H_m(C_{S'}) \cong \bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-1} \bigoplus_{\alpha \in T_{S'}} \langle \mathbb{Z}/\text{ind } \alpha \rangle^{(i)}.
\]

(6.2)

This proves 6.2 since, for each \( m > 0 \), \( H^m(F, \mathbb{Z}/p) = H_m(C_{S'}) \) for some finite \( S' \subseteq S \).

If \( S' \) consists of a single element, then \( l(\alpha) = 1 \) for any \( \alpha \in T_{S'} \) and the formula 6.2 reduces to that of 6.1. Thus suppose \( S' \subseteq S \) is some finite set for which (6.2) holds and choose \( n \in S \setminus S' \). By the Künneth formula

\[
H_m(C_{S' \cup \{n\}}) = H_m(C_{S'}) \otimes H_m(C_{\{n\}})
\]

\[
\cong \bigoplus_{m' + m'' = m} H_m(C_{S'}) \otimes H_m(C_{\{n\}}) \quad \text{(a)}
\]

\[
\oplus_{m' + m'' = m + 1 \atop m' \neq 0} H_m(C_{S'}) \otimes H_{m'}(C_{\{n\}}) \quad \text{(b)}
\]

This uses the fact that the reduced groups \( \tilde{H}_*(C_{S'}) \) and \( \tilde{H}_*(C_{\{n\}}) \) are torsion. But for any \( \alpha \in T_{S' \cup \{n\}} \) there are isomorphisms

\[
\mathbb{Z}/\text{ind } \alpha \cong \mathbb{Z}/\text{min}(\text{ind } \alpha', \text{ind } \alpha'')
\]

\[
\cong \mathbb{Z}/\text{ind } \alpha' \oplus \mathbb{Z}/\text{ind } \alpha''
\]

where \( \alpha = \alpha' + \alpha'' \) for some unique \( \alpha' \in T_S, \alpha'' \in T_{\{n\}} \).

Then applying the inductive assumption, the subgroup of (a) above corresponding to \( m' = 0 \) or \( m'' = 0 \) is given by
When \( \alpha'' \neq 0 \) we have \( l(\alpha') = l(\alpha) - 1 \) so that the remaining contribution from (a) is

\[
\bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-1} \bigoplus_{\alpha \in \mathcal{T}_{S \cup \{n\}}, \alpha'' = 0 \text{ or } \alpha'' = 0} (\mathbb{Z}/\text{ind } \alpha)^{\ell - 1}. \tag{a_1}
\]

Similarly, applying the inductive assumption to (b) yields

\[
\bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-2} \bigoplus_{\alpha \in \mathcal{T}_{S \cup \{n\}}, \alpha'' \neq 0 \neq \alpha''} (\mathbb{Z}/\text{ind } \alpha)^{\ell - 2}. \tag{a_2}
\]

Then adding (a_1), (a_2), and (b_1) and substituting the identity

\[
\binom{j-2}{i} + \binom{j-2}{i-1} = \binom{j-1}{i}
\]

we obtain the desired formula 6.2 for the set \( S' \cup \{n\} \). \( \square \)

We conclude with a remark about nontorsion elements. Most of the results in this paper involve an \( H \)-map \( f: BU \to BU \) whose characteristic sequence has only nonzero entries. Equivalently, the reduced \( \mathbb{Z}_p \) cohomology of the fiber of \( f \) is torsion. If \( \lambda \) has trivial entries, however, there is a reasonably simple algorithm for computing the rank of the cohomology. For any set \( S \) of positive integers and for any positive integers \( k \) and \( n \), denote by \( p_S(k, n) \) the number of distinct ways of writing \( n \) as a sum of integers from the set \( S \) (an element may be used more than once) in which exactly \( k \) different elements from \( S \) are used. For example, if \( S \) is the set of all positive integers then \( p_S(1, n) + p_S(2, n) + \ldots \) is the number of partitions of \( n \). Let \( p_S(k, n) = 0 \) if \( n \) is not an integer.

6.4 Theorem. Let \( F: BU \to BU \) be an \( H \)-map with characteristic sequence \( \lambda \) and fiber \( F \), and write \( S \) for the set of all indices \( n \) such that \( \lambda_n = 0 \). Then for every \( m > 0 \) the \( \mathbb{Z}_p \) rank of \( H^m(F, \mathbb{Z}_p) \) is given by

\[
\sum_{i,j \geq 0} \binom{i+j}{i} p_S(i+j, (i+m)/2).
\]

We omit the proof which is a straightforward counting argument involving the rational Eilenberg-Moore spectral sequence of \( f \).
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