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SELECTION THEOREMS FOR G_δ-VALUED MULTIFUNCTIONS

BY

S. M. SRIVASTAVA

ABSTRACT. In this paper we establish under suitable conditions the existence of measurable selectors for G_δ-valued multifunctions. In particular we prove that a measurable partition of a Polish space into G_δ sets admits a Borel selector.

1. Introduction. In recent years a large number of articles have been devoted to proving the existence of measurable selectors for multifunctions taking closed values in a Polish space. An extensive bibliography of such results is to be found in Wagner [7].

The present article initiates the study of the existence of measurable selectors for multifunctions taking G_δ values in a Polish space. Our main result asserts the existence of measurable selectors for such multifunctions under fairly mild restrictions. We then deduce from our main result that a measurable partition of a Polish space into G_δ sets admits a Borel selector. This answers a question raised by Kallman and Mauldin [3].

The paper is organized as follows. §2 introduces the basic definitions and notation. In §3 we record some preliminary results which are used in the proofs of the selection theorems. The main result is established in §4. §5 deals with the existence of selectors for partitions of Polish spaces.

2. Definitions and notation. The set of positive integers will be denoted by N. We use P to denote the set of all finite sequences of positive integers, including the empty sequence e. For each $k \geqslant 0$, we denote by P_k the set of elements of P of length k. If $p, q \in P$, pq will denote the catenation of the two finite sequences p and q. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N, Σ becomes a homeomorph of the space of irrationals.

Let $(X, \mathcal{Q})$, $(Y, \mathfrak{B})$ be measurable spaces. We denote by $\mathcal{Q} \otimes \mathfrak{B}$ the product of the σ-fields $\mathcal{Q}$ and $\mathfrak{B}$. Let f be a function on X to Y. The class $\{f^{-1}(B) \colon B \in \mathfrak{B}\}$ will be denoted by $f^{-1}(\mathfrak{B})$. If $f^{-1}(\mathfrak{B}) = \mathcal{Q}$ and f is onto Y, we note that $f(A) \in \mathfrak{B}$ for each $A \in \mathcal{Q}$. This fact will be used without

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explicit mention in the sequel. We say that the σ-field $\mathcal{Q}$ is *countably generated* if there exist subsets A_n, $n \geq 1$, of X such that $\mathcal{Q}$ is generated by $\{A_n: n \geq 1\}$. If A_n, $n \geq 1$, is a sequence of subsets of X, then by the *characteristic function* of the sequence $\{A_n\}$ is meant the function f: $X \to [0, 1]$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{A_n}(x),$$

where I_{A_n} denotes the indicator of the set A_n. If Z is a metric space, we denote by $\mathcal{B}_Z$ the Borel σ-field of Z. Here is an important fact about countably generated σ-fields. Let $\mathcal{Q}$ be a σ-field on X generated by the sequence $\{A_n\}$ and let f be the characteristic function of the sequence $\{A_n\}$. If $Z = f(X)$, then $\mathcal{Q} = f^{-1}(\mathcal{B}_Z)$. A measurable space $(X, \mathcal{Q})$ is said to be *standard Borel* if there is a function g on X onto a Borel subset Z of $[0, 1]$ such that $\mathcal{Q} = g^{-1}(\mathcal{B}_Z)$. If $E \subseteq X \times Y$ and $x \in X$, the set $\{y \in Y: (x, y) \in E\}$ is denoted by E^x. We use π_X to denote the projection of $X \times Y$ to X. A subset G of E is said to *uniformize* E if $\pi_X(G) = \pi_X(E)$ and G^x is at most a singleton for every x in X. If X, Y are metric spaces, then a function f: $X \to Y$ is said to be of *class* 1 if $f^{-1}(V)$ is an F_σ in X for each open set V in Y.

A *multifunction* F: $T \to X$ is a function whose domain is T and whose values are nonempty subsets of X. A multifunction whose values are singleton sets can therefore be identified with an ordinary function. Let F: $T \to X$ be a multifunction. If $E \subseteq X$, we denote the set $\{t \in T: F(t) \cap E \neq \varnothing\}$ by $F^{-1}(E)$. If $\mathcal{Q}$ is a σ-field on T and X is a metric space, we say that F is *$\mathcal{Q}$-measurable* if $F^{-1}(V) \in \mathcal{Q}$ for each open set V in X. In particular, a function f: $T \to X$ is *$\mathcal{Q}$-measurable* if $f^{-1}(V) \in \mathcal{Q}$ for each open set V in X. A function f: $T \to X$ is said to be a *selector* for a multifunction F: $T \to X$ if $f(t) \in F(t)$ for each $t \in T$. For a multifunction F: $T \to X$, we denote the set $\{(t, x) \in T \times X: x \in F(t)\}$ by $\mathrm{Gr}(F)$; we call it the *graph* of F.

Let X be a metric space. By a *partition* $\mathbf{Q}$ of X is meant a family of nonempty, disjoint subsets of X whose union is X. If $\mathbf{Q}$ is a partition of X, $R(\mathbf{Q})$ will denote the equivalence relation on X which induces $\mathbf{Q}$, that is,

$$R(\mathbf{Q}) = \cup\{E \times E: E \in \mathbf{Q}\}.$$

If $A \subseteq X$, then the set

$$\cup\{E \in \mathbf{Q}: E \cap A \neq \varnothing\}$$

is denoted by $A^{*\mathbf{Q}}$ or simply by A^*, if there is no ambiguity. A set $A \subseteq X$ is said to be $\mathbf{Q}$-*invariant* if $A = A^*$. We denote by $\mathcal{Q}(\mathbf{Q})$ the σ-field of $\mathbf{Q}$-invariant Borel subsets of X. We say that $\mathcal{Q}(\mathbf{Q})$ is the *σ-field induced by the partition* $\mathbf{Q}$. Say that the partition $\mathbf{Q}$ of X is *measurable* if $V^* \in \mathcal{Q}(\mathbf{Q})$ for every open set V in X. If $\mathcal{Q}$ is an atomic σ-field on X and $\mathbf{Q}$ is the set of atoms of $\mathcal{Q}$, we say that $\mathbf{Q}$ is *the partition of X induced by* $\mathcal{Q}$. If $\mathbf{Q}$ is a partition of X whose elements are Borel subsets of X, then $\mathbf{Q}$ is just the partition of X induced by $\mathcal{Q}(\mathbf{Q})$. A subset S of X is said to be a *selector* for a partition $\mathbf{Q}$ of X if S meets each element of $\mathbf{Q}$ in exactly one point.

The rest of our terminology is from [4].

3. Auxiliary results. In this section we set down some results which will be found useful in the sequel. We begin with an observation of Kallman and Mauldin [3] about sets which are simultaneously F_σ and G_δ in a Polish space. The proof is included here for completeness.

LEMMA 3.1. *Let X be a Polish space and let V_n, $n \geq 1$, be an open base for X. If E is simultaneously an F_σ and a G_δ subset of X, then $E \cap V_n = \bar{E} \cap V_n \neq \varnothing$ for some $n \geq 1$.*

PROOF. Express $E = \bigcup_{m \geq 1} F_m$, where the F_m, $m \geq 1$, are closed in X. Since E is a dense G_δ in $\bar{E}$, the Baire category theorem implies that E is nonmeager relative to $\bar{E}$. It follows that there is an F_m with nonempty interior relative to $\bar{E}$. Consequently, there is a V_n such that $\varnothing \neq \bar{E} \cap V_n \subseteq F_m \subseteq E$, from which the desired conclusion follows.

Next we turn to the study of properties of σ-fields induced by partitions.

LEMMA 3.2. *Let $\mathcal{Q}$ be a countably generated sub-σ-field of a Polish space X. Let $\mathbf{Q}$ be the partition of X induced by $\mathcal{Q}$. Then $R(\mathbf{Q}) \in \mathcal{Q} \otimes \mathcal{Q}$ and consequently, $R(\mathbf{Q})$ is a Borel subset of $X \times X$.*

PROOF. Let A_n, $n \geq 1$, generate the σ-field $\mathcal{Q}$. Denote by f the characteristic function of the sequence $\{A_n\}$. Then it is easy to see that

$$R(\mathbf{Q}) = \{(x, y) \in X \times X : f(x) = f(y)\},$$

so that $R(\mathbf{Q}) \in \mathcal{Q} \otimes \mathcal{Q}$. The second assertion is now obvious.

The next is a result of Blackwell [1, Theorem 3]. A proof is outlined for simplicity.

LEMMA 3.3. *Let $\mathcal{Q}$ be a countably generated sub-σ-field of the Borel σ-field of a Polish space X. Let $\mathbf{Q}$ be the partition of X induced by $\mathcal{Q}$. Then $\mathcal{Q} = \mathcal{Q}(\mathbf{Q})$.*

PROOF. Fix a sequence of sets in X which generates the σ-field $\mathcal{Q}$ and let f: $X \to [0, 1]$ be the characteristic function of the sequence. Let $A \in \mathcal{Q}(\mathbf{Q})$. Then $f(A)$ and $f(X - A)$ are two disjoint analytic sets in $[0, 1]$. Let B be a Borel set in $[0, 1]$ such that $f(A) \subseteq B$ and $B \cap f(X - A) = \varnothing$ [4, p. 485]. Then $A = f^{-1}(B)$ and so $A \in \mathcal{Q}$. Thus, $\mathcal{Q}(\mathbf{Q}) \subseteq \mathcal{Q}$. The inclusion $\mathcal{Q} \subseteq \mathcal{Q}(\mathbf{Q})$ is obvious.

LEMMA 3.4. *Suppose that $\mathbf{Q}$ is a measurable partition of a Polish space X such that each element of $\mathbf{Q}$ is a G_δ in X. Then $\mathcal{Q}(\mathbf{Q})$ is countably generated.*

PROOF. Let V_n, $n \geqslant 1$, be an open base for X. We shall show that if E_1, E_2 are distinct elements of $\mathbf{Q}$, then there is a V_i such that $E_1 \subseteq V_i^* \subseteq (X - E_2)$ or $E_2 \subseteq V_i^* \subseteq (X - E_1)$. Two cases arise. First suppose that $E_2 - \overline{E}_1 \neq \varnothing$. So if $x \in E_2 - \overline{E}_1$, we can find V_i such that $x \in V_i$ and $V_i \cap E_1 = \varnothing$. It follows that $E_2 \subseteq V_i^* \subseteq (X - E_1)$. Next assume that $E_2 \subseteq \overline{E}_1$. By the first principle of separation for G_δ sets [4, p. 350], there is a set B, which is simultaneously an F_σ and a G_δ subset of X, such that $E_1 \subseteq B$ and $E_2 \cap B = \varnothing$. We can assume $B \subseteq \overline{E}_1$ by replacing B by $B \cap \overline{E}_1$ if necessary. By Lemma 3.1, we can find V_i such that $B \cap V_i = \overline{B} \cap V_i \neq \varnothing$. Since $\overline{B} = \overline{E}_1$, it follows that $\overline{E}_1 \cap V_i \neq \varnothing$ and so $E_1 \cap V_i \neq \varnothing$, as V_i is open. Also $E_2 \subseteq \overline{E}_1 = \overline{B}$, so $E_2 \cap V_i \subseteq (\overline{B} - B) \cap V_i = \varnothing$. It now follows that $E_1 \subseteq V_i^* \subseteq (X - E_2)$.

To complete the proof of the lemma, let $\mathcal{Q}$ be the σ-field on X generated by the sets V_n^*, $n \geqslant 1$. Since $\mathbf{Q}$ is a measurable partition, it follows that $\mathcal{Q}$ is a countably generated sub-σ-field of the Borel σ-field of X. Moreover, the assertion made at the beginning of the previous paragraph implies that $\mathbf{Q}$ is just the partition of X induced by $\mathcal{Q}$. So by Lemma 3.3, $\mathcal{Q} = \mathcal{Q}(\mathbf{Q})$ and hence $\mathcal{Q}(\mathbf{Q})$ is countably generated.

The next result is an invariant version of Novikov's first multiple separation principle for analytic sets. The invariant version of the first separation principle for a pair of disjoint analytic sets can be found, for instance, in Burgess and Miller [2].

LEMMA 3.5. *Let $\mathbf{Q}$ be a partition of a Polish space X such that $R(\mathbf{Q})$ is an analytic subset of $X \times X$. If the A_n, $n \geqslant 1$, are $\mathbf{Q}$-invariant analytic subsets of X such that $\bigcap_{n \geqslant 1} A_n = \varnothing$, then there exist $\mathbf{Q}$-invariant Borel subsets B_n, $n \geqslant 1$, of X such that $A_n \subseteq B_n$ for each $n \geqslant 1$ and $\bigcap_{n \geqslant 1} B_n = \varnothing$.*

PROOF. By Novikov's first multiple separation principle for analytic sets ([4, p. 510], [6]), there exist Borel sets C_n in X such that $A_n \subseteq C_n$ and $\bigcap_{n \geqslant 1} C_n = \varnothing$. Since $R(\mathbf{Q})$ is analytic, it follows that $(X - C_n)^*$ is analytic. Now A_n and $(X - C_n)^*$ are $\mathbf{Q}$-invariant analytic sets such that $A_n \cap (X - C_n)^* = \varnothing$. So we can now invoke the invariant version of the first separation principle for a pair of disjoint invariant analytic sets to get a $\mathbf{Q}$-invariant Borel set B_n in X such that $A_n \subseteq B_n$ and $(X - C_n)^* \cap B_n = \varnothing$. But then $B_n \subseteq X - (X - C_n)^* \subseteq C_n$. Since $\bigcap_{n \geqslant 1} C_n = \varnothing$, it now follows that $\bigcap_{n \geqslant 1} B_n = \varnothing$.

LEMMA 3.6. *Let $\mathbf{Q}$ be a partition of a Polish space X such that $R(\mathbf{Q})$ is analytic in $X \times X$. If the Z_n, $n \geqslant 1$, are $\mathbf{Q}$-invariant coanalytic subsets of X such that $\bigcup_{n \geqslant 1} Z_n$ is Borel in X, then there exist $\mathbf{Q}$-invariant Borel subsets D_n, $n \geqslant 1$, of X such that $D_n \subseteq Z_n$ for each $n \geqslant 1$, $D_n \cap D_m = \varnothing$ for $n \neq m$ and $\bigcup_{n \geqslant 1} D_n = \bigcup_{n \geqslant 1} Z_n$.*

PROOF. Set $E = \bigcup_{n \geq 1} Z_n$ and $A_n = E - Z_n$, $n \geq 1$. Then the sets A_n are **Q**-invariant analytic sets such that $\bigcap_{n \geq 1} A_n = \varnothing$. So by Lemma 3.5, there exist **Q**-invariant Borel sets B_n in X such that $A_n \subseteq B_n$ and $\bigcap_{n \geq 1} B_n = \varnothing$. Define $D_1 = E \cap (X - B_1)$ and

$$D_n = E \cap (X - B_n) \cap \bigcap_{i<n} B_i$$

for $n \geq 2$. It is easy to verify that the sets D_n have the desired properties.

LEMMA 3.7. *Let $(T, \mathcal{Q})$ and $(X, \mathcal{B})$ be measurable spaces. Let $\mathcal{Q}$ be atomic and $G \in \mathcal{Q} \otimes \mathcal{B}$. Then $G^t = G^{t'}$ holds for t and t' belonging to the same atom of $\mathcal{Q}$.*

PROOF. Let t and t' belong to the same atom of $\mathcal{Q}$ and $x \in X$. The set $G_x = \{t \in T\colon (t, x) \in G\} \in \mathcal{Q}$. Therefore, $t \in G_x$ if and only if $t' \in G_x$. That is, $x \in G^t$ if and only if $x \in G^{t'}$. As $x \in X$ was arbitrary, it follows that $G^t = G^{t'}$.

The next result, which will play a key role in the proof of our main result, seems to be of independent interest. It is closely related to and will be deduced from the following result of Saint-Raymond [**6**]: *Let X, Y be compact metric spaces and let E, F be disjoint analytic subsets of $X \times Y$ such that E^x is σ-compact for each $x \in X$. Then there exist Borel sets B_n, $n \geq 1$, in $X \times Y$ such that B_n^x is compact for each $x \in X$ and, for $n \geq 1$,*

$$E \subseteq \bigcup_{n \geq 1} B_n \quad \textit{and} \quad F \cap \left(\bigcup_{n \geq 1} B_n \right) = \varnothing.$$

LEMMA 3.8. *Let T, X be Polish spaces and let $\mathcal{Q}$ be a countably generated sub-σ-field of the Borel σ-field of T. Suppose $G \in \mathcal{Q} \otimes \mathcal{B}_X$ and G^t is a G_δ in X for each $t \in T$. Then there exist sets $G_n \in \mathcal{Q} \otimes \mathcal{B}_X$ such that G_n^t is open in X for each $t \in T$ and $n \geq 1$ and $G = \bigcap_{n \geq 1} G_n$.*

PROOF. Let Y be a metric compactification of X. By a well-known result of Alexandrov and Hausdorff, X is a G_δ in Y. Next fix a sequence of sets in T which generates the σ-field $\mathcal{Q}$ and denote by f the characteristic function of the sequence. Define $g\colon T \times X \to [0, 1] \times Y$ as follows: $g(t, x) = (f(t), x)$. Let $Z = f(T)$, so that Z is analytic. Observe further that $g^{-1}(\mathcal{B}_Z \otimes \mathcal{B}_X) = \mathcal{Q} \otimes \mathcal{B}_X$.

Put $H = g(G)$. Since $G \in \mathcal{Q} \otimes \mathcal{B}_X$, it follows that $H \in \mathcal{B}_Z \otimes \mathcal{B}_X$. As X is a G_δ in Y, we have $\mathcal{B}_Z \otimes \mathcal{B}_X \subseteq \mathcal{B}_Z \otimes \mathcal{B}_Y$. Hence $H \in \mathcal{B}_Z \otimes \mathcal{B}_Y$. Set $M = (Z \times Y) - H$, so $M \in \mathcal{B}_Z \otimes \mathcal{B}_Y$. Since H and M are relatively Borel subsets of the analytic set $Z \times Y$, it follows that H and M are disjoint analytic subsets of $[0, 1] \times Y$. Again, since X is a G_δ in Y, by Lemma 3.7, it follows that H^z is a G_δ in Y for each $z \in [0, 1]$. So M^z is σ-compact in Y for each $z \in [0, 1]$. We now invoke the result of Saint-Raymond quoted above to get Borel subsets B_n, $n \geq 1$, of $[0, 1] \times Y$ such that B_n^z is compact for each

$z \in [0, 1]$ and $n \geqslant 1$, $M \subseteq \bigcup_{n \geqslant 1} B_n$ and $H \cap (\bigcup_{n \geqslant 1} B_n) = \varnothing$.

To complete the proof, let $H_n = (Z \times X) - B_n$, $n \geqslant 1$. Then each $H_n \in \mathfrak{B}_Z \otimes \mathfrak{B}_X$ and H_n^z is open in X for each $z \in Z$.

Furthermore, $H \subseteq ([0, 1] \times Y) - \bigcup_{n \geqslant 1} B_n$ and since $H \subseteq Z \times X$, we have

$$H \subseteq (Z \times X) - \bigcup_{n \geqslant 1} B_n = \bigcap_{n \geqslant 1} H_n.$$

To go the other way, observe that

$$H = (Z \times Y) - M \supseteq (Z \times Y) - \bigcup_{n \geqslant 1} B_n$$

$$\supseteq (Z \times X) - \bigcup_{n \geqslant 1} B_n = \bigcap_{n \geqslant 1} H_n.$$

Thus we have proved that $H = \bigcap_{n \geqslant 1} H_n$. Finally, let $G_n = g^{-1}(H_n)$, $n \geqslant 1$. It is now easy to see that the sets G_n have the desired properties.

4. Selectors for multifunctions. Our main result can now be stated as follows.

THEOREM 4.1. *Let T, X be Polish spaces and let $\mathcal{Q}$ be a countably generated sub-σ-field of the Borel σ-field of T. Suppose that F: $T \to X$ is a multifunction such that F is $\mathcal{Q}$-measurable,* $\mathrm{Gr}(F) \in \mathcal{Q} \otimes \mathfrak{B}_X$ *and $F(t)$ is a G_δ in X for each $t \in T$. Then there is a $\mathcal{Q}$-measurable selector f for F.*

PROOF. The idea of the proof is very simple. Given a nonempty G_δ, there is an effective procedure for selecting a point from it (see, for example [**4**, p. 418]). What we have to do is to apply this procedure to each $F(t)$ uniformly.

Let, then, d be a metric on X such that d-diameter $(X) < 1$. Fix a system $\{V(p), p \in P\}$ of nonempty open subsets of X such that

(a) $V(e) = X$,

(b) d-diameter $(V(p)) < 1/2^k, p \in P_k, k \geqslant 0$,

(c) $\{V(pm): m \geqslant 1\}$ is an open base for $V(p), p \in P$, and

(d) $\overline{V(pm)} \subseteq V(p), p \in P, m \geqslant 1$.

Let $G = \mathrm{Gr}(F)$. By Lemma 3.8, there exist sets $G_n \in \mathcal{Q} \otimes \mathfrak{B}_X$, $n \geqslant 1$, such that G_n^t is open in X for each $t \in T$ and $n \geqslant 1$ and $G = \bigcap_{n \geqslant 1} G_n$. Put $G_0 = T \times X$. Let $\mathbf{Q}$ be the partition of T induced by $\mathcal{Q}$. We shall now prove that there is a system $\{B(p), p \in P\}$ of subsets of T satisfying the following conditions:

(i) $B(p) = \bigcup_{m \geqslant 1} B(pm), p \in P$, and $B(e) = T$,

(ii) $p, q \in P_k$ and $p \neq q \to B(p) \cap B(q) = \varnothing$,

(iii) $p \in P_k$ and $t \in B(p) \to G^t \cap V(p) \neq \varnothing$ and $\overline{V(p)} \subseteq G_k^t$, and

(iv) $B(p) \in \mathcal{Q}(\mathbf{Q}), p \in P$.

To see that such a system can be defined, we proceed inductively. First define $B(e) = T$. Next suppose that the $B(p)$, $p \in P_i$, $i \leqslant k$, have been defined in such a way that the above conditions are satisfied. We shall now define the sets $B(q)$, $q \in P_{k+1}$. Fix $p \in P_k$ and set

$$Z_m = \left\{t \in B(p): G^t \cap V(pm) \neq \varnothing \text{ and } \overline{V(pm)} \subseteq G^t_{k+1}\right\},$$

$m \geqslant 1$. It is easy to see that

$$Z_m = B(p) \cap F^{-1}(V(pm)) \cap \left(T - \pi_T\left(\left(T \times \overline{V(pm)}\right) - G_{k+1}\right)\right).$$

It follows that the sets Z_m are coanalytic subsets of T. Moreover, since $B(p) \in \mathcal{Q}(\mathbf{Q})$, $F^{-1}(V(pm)) \in \mathcal{Q}$ and $G_{k+1} \in \mathcal{Q} \otimes \mathcal{B}_X$, by Lemma 3.7, the sets Z_m are $\mathbf{Q}$-invariant. Next we check that $\bigcup_{m\geqslant 1} Z_m = B(p)$. To see this, let $t \in B(p)$, so that $G^t \cap V(p) \neq \varnothing$. Choose $x \in G^t \cap V(p) \subseteq G^t_{k+1} \cap V(p)$. Since G^t_{k+1} is open in X, we can find m such that $x \in V(pm) \subseteq \overline{V(pm)} \subseteq G^t_{k+1} \cap V(p)$. It now follows that $t \in Z_m$, which proves the inclusion $B(p) \subseteq \bigcup_{m\geqslant 1} Z_m$. The reverse inclusion is obvious. By Lemma 3.2, $R(\mathbf{Q})$ is a Borel, and therefore analytic, subset of $T \times T$. So Lemma 3.6 can be applied to the sets Z_m. We will then get sets $D_m \in \mathcal{Q}(\mathbf{Q})$, $m \geqslant 1$, such that $D_m \subseteq Z_m$, $D_n \cap D_m = \varnothing$ for $n \neq m$ and $\bigcup_{m\geqslant 1} D_m = B(p)$. Define $B(pm) = D_m$, $m \geqslant 1$. Since $p \in P_k$ was fixed but arbitrary, this completes the definition of the sets $B(q)$, $q \in P_{k+1}$. It is now an easy matter to verify that these sets satisfy the required conditions.

For the final step in the proof, define

$$C_k = \bigcup \left\{B(p) \times \overline{V(p)} : P \in P_k\right\}, \qquad k \geqslant 0,$$

and

$$C = \bigcap_{k\geqslant 0} C_k.$$

Using conditions (a)–(d) and (i)–(iv), one checks that each $C_k \in \mathcal{Q}(\mathbf{Q}) \otimes \mathcal{B}_X$ and so $C \in \mathcal{Q}(\mathbf{Q}) \otimes \mathcal{B}_X$, that C^t contains exactly one point for each $t \in T$, and that $C \subseteq G$. The set C defines uniquely a function f on T to X whose graph is C. Since C is a Borel subset of $T \times X$, f is Borel measurable [**4**, p. 489]. It follows from this, the fact that $C \in \mathcal{Q}(\mathbf{Q}) \otimes \mathcal{B}_X$ and from Lemmas 3.3 and 3.7 that f is $\mathcal{Q}(\mathbf{Q})$-measurable. So, by Lemma 3.3, f is $\mathcal{Q}$-measurable. As $C \subseteq G$, f is a selector for F. This completes the proof.

Next we relax somewhat the requirement in Theorem 4.1 that T be a Polish space.

THEOREM 4.2. *Let T be an analytic set and let $\mathcal{Q}$ be a countably generated sub-σ-field of the Borel σ-field of T. Let X be a Polish space. Suppose F: $T \to X$ is a multifunction such that F is $\mathcal{Q}$-measurable,* $\mathrm{Gr}(F) \in \mathcal{Q} \otimes \mathcal{B}_X$ *and $F(t)$ is a G_δ in X for each $t \in T$. Then there is an $\mathcal{Q}$-measurable selector f for F.*

PROOF. Let h be a continuous function on Σ onto T. Set $\mathcal{C}' = h^{-1}(\mathcal{C})$, so $\mathcal{C}'$ is a countably generated sub-σ-field of the Borel σ-field of Σ. Define a multifunction $F'\colon \Sigma \to X$ by $F'(\sigma) = F(h(\sigma))$. It is easy to check that F' is $\mathcal{C}'$-measurable, $\mathrm{Gr}(F') \in \mathcal{C}' \otimes \mathcal{B}_X$ and $F'(\sigma)$ is a G_δ in X for each $\sigma \in \Sigma$. So by Theorem 4.1, there exists an $\mathcal{C}'$-measurable selector g for F'. Finally, define a function f on T to X by the formula $f(h(\sigma)) = g(\sigma)$, $\sigma \in \Sigma$. As g is $\mathcal{C}'$-measurable, g is constant on atoms of $\mathcal{C}'$. Further, for each $t \in T$, $h^{-1}(t)$ is a subset of an atom of $\mathcal{C}'$. For if A is the atom of $\mathcal{C}$ containing t, then $h^{-1}(A)$ must be an atom of $\mathcal{C}'$. Therefore, f is well defined. Plainly f is a selector for F. That f is $\mathcal{C}$-measurable follows now from the fact that g is $\mathcal{C}'$-measurable and $\mathcal{C}' = h^{-1}(\mathcal{C})$.

REMARK. In Theorem 4.2 the condition that $\mathcal{C}$ be countably generated can be dropped. Indeed, let $\mathcal{C}$ be any sub-σ-field of the Borel σ-field of T and let F be as in the statement of Theorem 4.2. Since $\mathrm{Gr}(F) \in \mathcal{C} \otimes \mathcal{B}_X$, there exist rectangles $A_i \times B_i \in \mathcal{C} \otimes \mathcal{B}_X$, $i \geqslant 1$, such that $\mathrm{Gr}(F)$ is already in the σ-field generated by $A_i \times B_i$, $i \geqslant 1$. Next, if V_n, $n \geqslant 1$, is an open base for X, set $C_i = F^{-1}(V_i)$, $i \geqslant 1$. Now let $\mathcal{C}'$ be the σ-field on T generated by the sets A_i, C_i, $i \geqslant 1$. Then, as is easy to verify, $\mathcal{C}'$ is a countably generated sub-σ-field of $\mathcal{C}$, F is $\mathcal{C}'$-measurable and $\mathrm{Gr}(F) \in \mathcal{C}' \otimes \mathcal{B}_X$. By Theorem 4.2, there is a $\mathcal{C}'$-measurable selector f for F. Clearly f is $\mathcal{C}$-measurable.

An important consequence of Theorem 4.2 is the following result on the uniformization of Borel sets.

COROLLARY 4.3. *Let T, X be Polish spaces. Suppose B is a Borel subset of $T \times X$ such that B^t is a G_δ in X for each $t \in T$ and $\pi_T(B \cap (T \times V))$ is relatively Borel in $\pi_T(B)$ for each open set V in X. Then B can be uniformized by a Borel subset of $T \times X$.*

PROOF. An application of Theorem 4.2 yields a Borel measurable function f on $\pi_T(B)$ to X such that C is a uniformization of B, where C is the graph of f. By a result of Kuratowski [4, p. 434], there is a Borel measurable function g on T to X which extends f. Let D be the graph of g. Then D is a Borel subset of $T \times X$ and $C = D \cap B$, which proves that C is a Borel subset of $T \times X$. This completes the proof.

It should be noted that the above result on the uniformization of Borel sets cannot be improved upon. Indeed, if B is a G_δ in the plane whose projection to the first coordinate is not Borel, then B cannot be uniformized by a Borel subset of the plane. This shows that the condition requiring $\pi_T(B \cap (T \times V))$ to be relatively Borel in $\pi_T(B)$ cannot be dropped from Corollary 4.3. On the other hand, Kallman and Mauldin [3] have shown that there is an F_σ subset B of $\Sigma \times \Sigma$ such that the projection to the first coordinate of $B \cap (\Sigma \times V)$ is Borel in Σ for each open set V in Σ, but which does not admit a Borel uniformization. This shows that the condition requiring B^t to be a G_δ for each $t \in T$ (actually, for a cocountable set of t's) cannot be dropped from Corollary 4.3.

We conclude this section by showing that the condition in Theorem 4.1 that $\mathrm{Gr}(F)$ belong to $\mathcal{Q} \otimes \mathcal{B}_X$ cannot be dropped. Indeed, let $T = X = [0, 1]$ and $\mathcal{Q} = \mathcal{B}_T$. Let Φ be the set of Borel measurable functions of T to X and let φ map T onto Φ. Define a multifunction F: $T \to X$ by : $F(t) = [0, 1] - \{\varphi(t)(t)\}$. Then $F(t)$ is open in X for each $t \in T$. Clearly F is $\mathcal{Q}$-measurable as $F(t)$ is dense in X for each $t \in T$. Now suppose f is a $\mathcal{Q}$-measurable selector for F. Then $f \in \Phi$, so $f = \varphi(t_0)$ for some $t_0 \in T$. It now follows that $\varphi(t_0)(t_0) \neq \varphi(t_0)(t_0)$, which is a contradiction. So F does not admit a $\mathcal{Q}$-measurable selector.

5. Selectors for partitions. We now turn our attention to the existence of measurable selectors for measurable partitions of a Polish space.

THEOREM 5.1. *Let* $\mathbf{Q}$ *be a measurable partition of a Polish space* X *such that each element of* $\mathbf{Q}$ *is a* G_δ *in* X. *Then there is a Borel selector for* $\mathbf{Q}$.

PROOF. Define a multifunction F: $X \to X$ as follows: $F(x) =$ the element of Q containing x. Then $F(x)$ is a G_δ in X for each $x \in X$. Since $\mathbf{Q}$ is a measurable partition, it follows that F is $\mathcal{Q}(\mathbf{Q})$-measurable.

By Lemma 3.4, $\mathcal{Q}(\mathbf{Q})$ is countably generated. Also the partition of X induced by $\mathcal{Q}(\mathbf{Q})$ is just $\mathbf{Q}$. So by Lemma 3.2, $R(\mathbf{Q}) \in \mathcal{Q}(\mathbf{Q}) \otimes \mathcal{Q}(\mathbf{Q})$. But $R(\mathbf{Q}) = \mathrm{Gr}(F)$, so $\mathrm{Gr}(F) \in \mathcal{Q}(\mathbf{Q}) \otimes \mathcal{B}_X$. It is therefore seen that Theorem 4.1 can be applied to the multifunction F. This will yield an $\mathcal{Q}(\mathbf{Q})$-measurable selector f for F. Now let $S = \{x \in X: f(x) = x\}$. Then S is a Borel selector for $\mathbf{Q}$.

COROLLARY 5.2. *Let* T *be a Borel subset of a Polish space* X. *Suppose* $\mathbf{Q}$ *is a measurable partition of* T *such that each element of* $\mathbf{Q}$ *is a* G_δ *in* X. *Then there is a Borel selector for* $\mathbf{Q}$.

PROOF. Define a partition $\mathbf{Q}'$ of X as follows: $\mathbf{Q}' = \mathbf{Q} \cup \{\{x\}: x \in X - T\}$. Then, as is easy to check, $\mathbf{Q}'$ is a measurable partition of X. Also each element of $\mathbf{Q}'$ is a G_δ in X. So by Theorem 5.1, there is a Borel selector S for $\mathbf{Q}'$. Plainly $S \cap T$ is a Borel selector for $\mathbf{Q}$.

Kallman and Mauldin [3] have proved the special case of Corollary 5.2 where each element of the partition $\mathbf{Q}$ is simultaneously a F_σ and G_δ subset of X.

COROLLARY 5.3. *Let T be a Borel subset of a Polish space X. Suppose $\mathbf{Q}$ is a measurable partition of T such that each element of $\mathbf{Q}$ is a G_δ in X. Then $(T, \mathcal{Q}(\mathbf{Q}))$ is a standard Borel space.*

PROOF. If $\mathbf{Q}'$ is as in the proof of Corollary 5.2, then $\mathcal{Q}(\mathbf{Q}) = \{E \cap T: E \in \mathcal{Q}(\mathbf{Q}')\}$. By Lemma 3.4, $\mathcal{Q}(\mathbf{Q}')$ is countably generated and hence so is $\mathcal{Q}(\mathbf{Q})$. Now let f be the characteristic function (on T) of a sequence of subsets of T which generates $\mathcal{Q}(\mathbf{Q})$. By Corollary 5.2, there is a Borel selector S for $\mathbf{Q}$. So f restricted to S is one-to-one and $f(S) = f(T)$. Hence, by a known result [**4**, p. 489], $f(S)$ is a Borel subset of [0, 1]. So $f(T)$ is Borel in [0, 1], which completes the proof.

COROLLARY 5.4. *Let X, Y be Polish spaces and let f be a function of class 1 on X into Y. If $f(V)$ is relatively Borel in $f(X)$ for each open set V in X, then $f(X)$ is a Borel subset of Y. Moreover, there is a Borel measurable function g on $f(X)$ into X such that $f(g(y)) = y$ for every y in $f(X)$.*

PROOF. Let $\mathbf{Q} = \{f^{-1}(y): y \in f(X)\}$. Then $\mathbf{Q}$ is a measurable partition of X and each element of $\mathbf{Q}$ is a G_δ in X. So by Theorem 5.1, there is a Borel selector S for $\mathbf{Q}$. Now f restricted to S is one-to-one and $f(S) = f(X)$. It therefore follows from [**4**, p. 489] that $f(X)$ is a Borel subset of Y. Now define g on $f(X)$ into X by $g(y) =$ the unique element of $S \cap f^{-1}(y)$. To see that g is Borel measurable, we observe that for a Borel set E in X, $g^{-1}(E) = f(E \cap S)$, which by [**4**, p. 489] is a Borel set in Y. This completes the proof.

It has been observed by Kallman and Mauldin [**3**] that if $\mathbf{Q}$ is a measurable partition of a Polish space X into F_σ sets, then $\mathbf{Q}$ need not admit a Borel selector. Indeed, they have examples to show that any of the following may then occur:

(i) $\mathcal{Q}(\mathbf{Q})$ need not be countably generated; (ii) even if $\mathcal{Q}(\mathbf{Q})$ is countably generated, $(X, \mathcal{Q}(\mathbf{Q}))$ need not be standard Borel; and (iii) even if $(X, \mathcal{Q}(\mathbf{Q}))$ is standard Borel, $\mathbf{Q}$ need not admit a Borel selector. Our Theorem 5.1 is therefore seen to be the best possible result for measurable partitions of a Polish space into Borel sets.

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Statistics-Mathematics Division, Indian Statistical Institute, Calcutta, India