

ISOMETRIES OF L_p -SPACES ASSOCIATED WITH
 SEMIFINITE VON NEUMANN ALGEBRAS

BY

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ABSTRACT. The paper determines the structure of (classes of) linear isometries between L_p -spaces associated with semifinite normal faithful traces on von Neumann algebras, generalizing works of M. Broise and B. Russo. Also established are some auxiliary results on L_p norm inequalities which are of interest by themselves.

1. Introduction. Let ϕ (resp., ψ) be a semifinite normal faithful trace on a von Neumann algebra \mathfrak{A} (resp., \mathfrak{B}), $L_p(\phi)$ (resp., $L_p(\psi)$) be the L_p -space associated with ϕ (resp., ψ) (cf. [1], [2], [3] or [4]), and Θ a linear isometry from $L_p(\phi)$ onto $L_p(\psi)$. In [5], Russo proved that if $\phi = \psi$ is finite on $\mathfrak{A} = \mathfrak{B}$ and if $p = 1$, then for each Θ there are a Jordan $*$ -automorphism Φ of \mathfrak{A} , a positive selfadjoint $Z \in L_1(\phi)$ affiliated with the center of \mathfrak{A} , and a unitary $W \in \mathfrak{A}$ such that for any $X \in \mathfrak{A}$ we have $\Theta(X) = \Phi(X)ZW$ (strong product, cf. [2]). When $\mathfrak{A} = \mathfrak{B}$ is abelian and $\phi = \psi$, similar results hold for arbitrary $p \neq 2$, $p \in (1, \infty)$, and can be deduced from [7]. In [6] Broise showed that if $p = 2$ and if $\Theta[L_2^+(\phi)] = L_2^+(\psi)$ (equivalently $\Theta[L_2^+(\phi)] \subset L_2^+(\psi)$, where L_2^+ denotes the positive part of L_2), then there are uniquely a Jordan $*$ -isomorphism Φ of \mathfrak{A} onto \mathfrak{B} , and a positive selfadjoint operator $Z \in L_2(\psi)$ affiliated with the center of \mathfrak{B} such that

$$\Theta(X) = \Phi(X)Z \quad \text{for every } X \in \mathfrak{A} \cap L_2(\phi).$$

It is the purpose of this paper to establish similar results for a rather general class of $(\mathfrak{A}, \mathfrak{B}, \phi, \psi, p, \Theta)$. More precisely, we shall prove in §3 the following two theorems.

THEOREM. *Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$, $p \in [1, \infty)$. Then there exist uniquely a Jordan $*$ -isomorphism Φ of \mathfrak{A} onto \mathfrak{B} , and a positive selfadjoint operator $Z \in L_p(\psi)$ affiliated with the center of \mathfrak{B} such that*

$$\Theta(X) = \Phi(X)Z, \quad X \in \mathfrak{A} \cap L_p(\phi).$$

THEOREM. *Suppose $\mathfrak{A}, \mathfrak{B}$ are finite (ϕ, ψ may be semifinite) and*

$$\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi), \quad p \in [1, \infty) \setminus \{2\}.$$

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Then there are a Jordan $*$ -isomorphism Φ of \mathfrak{A} onto \mathfrak{B} , a positive selfadjoint operator $Z \in L_p(\psi)$ affiliated with the center of \mathfrak{B} , and a unitary $W \in \mathfrak{B}$ such that

$$\Theta(X) = \Phi(X)ZW, \quad X \in \mathfrak{A} \cap L_p(\phi).$$

Some other results in [5] and [6] are also generalized in §3 to the present context.

In order to obtain the above theorems, we shall prove in §2 some results on L_p norm inequalities which are of interest by themselves [cf. Theorems 1 and 2 of §2 below].

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2. L_p norm inequalities. In this section we shall establish some auxiliary results needed in §3, of which some special cases are known (cf. [7] and [12]).

THEOREM 1. Let $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$.

(i) Let $p > 1$. Then $\phi(S^p + T^p) \leq \phi[(S + T)^p]$, and equality holds iff $ST = 0$.

(ii) Let $p > 1$. Then $\phi[(S + T)^p] \leq 2^{p-1}\phi(S^p + T^p)$, and equality holds iff $S = T$.

(iii) Let $p \in (0, 1)$. Then $\phi(S^p + T^p) \geq \phi[(S + T)^p]$. If $ST = 0$, then we have equality. On the other hand, if we have equality, if ϕ is finite, and if $S + T$ either has a bounded inverse or is of finite rank, then $ST = 0$.

(iv) Let $p \in (0, 1)$. Then $\phi[(S + T)^p] \geq 2^{p-1}\phi(S^p + T^p)$, and equality holds iff $S = T$.

REMARKS. (1) The inequalities obtained here for $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$ can be easily extended to $S, T \in L_p^+(\phi)$ by using results in [2]. As to when equality holds [for $S, T \in L_p^+(\phi)$], the conditions are easily seen to be sufficient by [4]. Since we have not decided whether the conditions are also necessary and shall not use them in §3, we omit the details of such a generalization.

(2) It is plausible that for $p \in (0, 1)$ and $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$, $\phi(S^p + T^p) = \phi[(S + T)^p]$ implies $ST = 0$. As evident from the proof of (ii) below, this assertion will follow from its special case where $S + T$ is injective.

We shall need the following lemmas in proving the above theorem. ($\mathbb{C}, \mathbb{R}, \mathbb{N}$ denote, respectively, the set of all complex, real, strictly positive integral numbers.)

LEMMA 1. Let the map $t \in I \mapsto S_t \in \mathfrak{A}^+ \cap L_p(\phi)$ be differentiable (with respect to the norm on \mathfrak{A}) on the real open interval I . Suppose for some $a \in I$, $S'_a \equiv dS_t/dt|_{t=a} \in \mathfrak{A} \cap L_p(\phi)$.

(i) For any $p \in [1, \infty)$ we have

$$\frac{d}{dt} \phi(S_t^p) \Big|_{t=a} = \phi[pS_a^{p-1} S'_a]. \quad (1)$$

(ii) Suppose $p \in (0, 1)$. If ϕ is finite and if S_a is invertible in \mathfrak{A} , then (1) holds.

PROOF. Ad (i). Fix $p \in [1, \infty)$ and fix a compact subinterval I_1 of I with $a \in I_1$. Choose a branch of the complex function z^p , and a sector

$$\Omega \equiv \{re^{i\theta}: r \in [0, \infty), \theta \in (-\theta_1, \theta_1)\} \quad (\theta_1 > 0)$$

of the right half-plane of \mathbf{C} so that the branch is a single-valued holomorphic function on the sector. For $\varepsilon > 0$ let Γ_ε be the positively oriented contour consisting of the segments

$$\{re^{-i\theta_2}: r \in [\varepsilon, b)\}, \quad \{re^{i\theta_2}: r \in [\varepsilon, b)\}$$

and the circular arc $\{be^{i\theta}: \theta \in [-\theta_2, \theta_2]\}$, where $\theta_2 \equiv \theta_1/2$, and $b > \text{Max}\{\|S_t\|: t \in I_1\}$. Let Γ be similarly defined with ε replaced by zero. Since (cf. [8, p. 272]) for any $\lambda \in \Gamma \setminus \{0\}$,

$$\|\lambda^p(\lambda - S_t)^{-1}\| < |\lambda|^p |\text{Im } \lambda|^{-1} = |\lambda|^{p-1} \sin^{-1} \theta_2$$

($\text{Im } \lambda \equiv$ imaginary part of λ), we see that $\int_{\Gamma_\varepsilon} \lambda^p(\lambda - S_t)^{-1} d\lambda$ converges in norm as $\varepsilon \rightarrow 0$; we shall denote the limit by $\int_{\Gamma} \lambda^p(\lambda - S_t)^{-1} d\lambda$. It is then easy to see that, for every $t \in I_1$,

$$S_t^p = \frac{1}{2\Pi i} \int_{\Gamma} \lambda^p(\lambda - S_t)^{-1} d\lambda.$$

Now the above construction works for the function z^{p-1} as well, provided that the sector Ω and Γ have been chosen carefully. Thus we also have

$$S_t^{p-1} = \frac{1}{2\Pi i} \int_{\Gamma} \lambda^{p-1}(\lambda - S_t)^{-1} d\lambda.$$

Similarly we see that

$$(2\Pi i)^{-1} \int_{\Gamma_\varepsilon} \lambda^p(\lambda - S_a)^{-1} S'_a(\lambda - S_a)^{-1} d\lambda$$

converges in norm as $\varepsilon \rightarrow 0$; we denote the limit by D . Now we claim that the map $t \rightarrow S_t^p$ is differentiable (with respect to the norm) and

$$\frac{d}{dt} S_t^p \Big|_{t=a} = D. \quad (2)$$

In fact we have

$$\|(t-a)^{-1}(S_t^p - S_a^p) - D\| < l_1(t) + l_2(t)$$

where

$$l_1(t) \equiv \lim_{\varepsilon \rightarrow 0} (2\Pi i)^{-1} \left\| \int_{\Gamma_\varepsilon} \lambda^p (\lambda - S_a)^{-1} (D_t - S'_a) (\lambda - S_a)^{-1} d\lambda \right\|,$$

$$l_2(t) \equiv \lim_{\varepsilon \rightarrow 0} (2\Pi i)^{-1} \left\| \int_{\Gamma_\varepsilon} \lambda^p [(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}] D_t (\lambda - S_a)^{-1} d\lambda \right\|$$

with $D_t \equiv (t - a)^{-1}(S_t - S_a)$. It is easy to see that $\lim_{t \rightarrow a} l_1(t) = 0$. For $l_2(t)$, choose a constant k and a compact subinterval I_2 of I with $a \in I_2$ and

$$\sup\{\|D_t\|; t \in I_2, t \neq a\} < k.$$

For $\varepsilon_1 > 0$ and $t \in I_2 \setminus \{a\}$ we have

$$\begin{aligned} & \left\| \int_{\Gamma_{\varepsilon_1}} \lambda^p [(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}] D_t (\lambda - S_a)^{-1} d\lambda \right\| \\ &= \left\| \int_{\Gamma_{\varepsilon_1}} \lambda^p (\lambda - S_t)^{-1} (S_t - S_a) (\lambda - S_a)^{-1} D_t (\lambda - S_a)^{-1} d\lambda \right\| \\ &< m_{\varepsilon_1} k \|S_t - S_a\| \end{aligned}$$

where m_{ε_1} is a constant which depends on ε_1 only. On the other hand, for $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} & \left\| \int_{\Gamma_\varepsilon \setminus \Gamma_{\varepsilon_1}} \lambda^p [(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}] D_t (\lambda - S_a)^{-1} d\lambda \right\| \\ &< 4k \int_\varepsilon^{\varepsilon_1} r^{p-2} [\sin \theta_2]^{-2} dr < 4k\varepsilon_1^{p-1} [(p-1)\sin^2 \theta_2]^{-1}. \end{aligned}$$

From these estimates it is clear that $\lim_{t \rightarrow a} l_2(t) = 0$. Thus our claim (2) is established.

By the spectral theorem and the semifiniteness of ϕ , there exists an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of projections in \mathfrak{A} with $\phi(E_n) < \infty$, $E_n S_a = S_a E_n$, and $\lim_{n \rightarrow \infty} E_n = I$. But we have in the sense of norm convergence:

$$\left(\frac{d}{dt} S_t^p \Big|_{t=a} \right) E_n = (2\Pi i)^{-1} \int_\Gamma \lambda^p (\lambda - S_a)^{-1} S'_a (\lambda - S_a)^{-1} E_n d\lambda.$$

Hence

$$\phi \left[\left(\frac{d}{dt} S_t^p \Big|_{t=a} \right) E_n \right] = (2\Pi i)^{-1} \int_\Gamma \lambda^p \phi [(\lambda - S_a)^{-2} S'_a E_n] d\lambda.$$

Now, since

$$\frac{d}{d\lambda} [\lambda^p (\lambda - S_t)^{-1}] = p\lambda^{p-1} (\lambda - S_t)^{-1} - \lambda^p (\lambda - S_t)^{-2}, \quad \lambda \in \Gamma_\varepsilon,$$

we have

$$\begin{aligned} pS_i^{p-1} &= (2\Pi i)^{-1} \int_{\Gamma} p\lambda^{p-1}(\lambda - S_i)^{-1} d\lambda \\ &= (2\Pi i)^{-1} \int_{\Gamma} \lambda^p(\lambda - S_i)^{-2} d\lambda, \end{aligned}$$

and

$$\phi(pS_a^{p-1}S'_a E_n) = (2\Pi i)^{-1} \int_{\Gamma} \lambda^p \phi[(\lambda - S_a)^{-2} S'_a E_n] d\lambda,$$

i.e.,

$$\phi\left[\left(\frac{d}{dt} S_i^p \Big|_{t=a}\right) E_n\right] = \phi(pS_a^{p-1}S'_a E_n).$$

Letting $n \rightarrow \infty$ we obtain by (2)

$$\frac{d}{dt} \phi(S_i^p) \Big|_{t=a} = \phi\left(\frac{d}{dt} S_i^p \Big|_{t=a}\right) = \phi(pS_a^{p-1}S'_a).$$

Ad (ii). For t sufficiently close to a , S_t is invertible in \mathfrak{A} so that the contour Γ can be replaced by one lying in a domain of analyticity for a branch of the complex function z^p , and containing the spectrum of such S_t in its interior. Thus our assertion follows from a well-known result [9, p. 108].

LEMMA 2. (i) Let $p \in (0, \infty)$ and let $A_1, A_2 \in \mathfrak{A}^+ \cap L_p(\phi)$ with $A_1 \succ A_2$. Then $\phi(A_1^p) \succ \phi(A_2^p)$.

(ii) Let $p \in (1, \infty)$, $n \in \mathbb{N}$ and let $A, B \in \mathfrak{A}^+ \cap L_{np}(\phi)$. Denote $C_p \equiv B^{1/2}(A + B)^{p-1}B^{1/2}$. Then $\phi(C_p^n) \succ \phi(B^{np})$.

(iii) Suppose ϕ is finite and let $A, B \in \mathfrak{A}^+$ be such that $A + B$ is invertible in \mathfrak{A} . Then, for any $p \in (0, 1)$,

$$\phi[(A + B)^{p-1}B] \prec \phi(B^p).$$

PROOF. Ad (i). We prove first that the assertion is valid for $p = 2^n$, $n \in \mathbb{N}$. Indeed this can be shown by induction on n with the following observations:

$$A_1^{2^n} = (A_1^{1/2}A_1A_1^{1/2})^{2^{n-1}}, \quad \phi[(A_1^{1/2}A_2A_1^{1/2})^{2^n}] = \phi[(A_2^{1/2}A_1A_2^{1/2})^{2^n}].$$

Now suppose $p \in (0, \infty)$ and choose $n \in \mathbb{N}$ so large that $p_1 \equiv p2^{-n} < 1$. Then

$$\phi(A_1^p) = \phi[(A_1^{p_1})^{2^n}] \succ \phi[(A_2^{p_1})^{2^n}] = \phi(A_2^p).$$

Ad (ii). We proceed by induction. If $p \in (1, 2]$, then $C_p \succ B^p$; so by (i) we have $\phi(C_p^n) \succ \phi(B^{np})$ for all $n \in \mathbb{N}$ and for all $A, B \in \mathfrak{A}^+ \cap L_{np}(\phi)$. Now suppose that for some integral $m > 2$, (ii) is valid for all $p \in (1, m]$ and all $n \in \mathbb{N}$. Let $l = m + \alpha$ with $\alpha \in (0, 1]$, $n \in \mathbb{N}$ and let $A, B \in \mathfrak{A}^+ \cap L_{nl}(\phi)$. Then $C_l \equiv B^{1/2}(A + B)^{l-1}B^{1/2} \succ D^2$ where $D \equiv B^{1/2}(A + B)^{l/2-1}B^{1/2}$.

Thus by (i), $\phi(C_l^n) > \phi(D^{2n})$. But $l/2 \in (1, m]$; by the induction hypothesis $\phi(D^{2n}) > \phi(B^{nl})$. Hence $\phi(C_l^n) > \phi(B^{nl})$ and (ii) is established.

Ad (iii). For a strictly positive number ε , let $B_\varepsilon \equiv \varepsilon + B$. Then

$$(A + B_\varepsilon)^{p-1} < B_\varepsilon^{p-1}, \quad B_\varepsilon^{1/2}(A + B_\varepsilon)^{p-1}B_\varepsilon^{1/2} < B_\varepsilon^p.$$

Letting $\varepsilon \rightarrow 0$ we obtain $B^{1/2}(A + B)^{p-1}B^{1/2} < B^p$; so

$$\phi[(A + B)^{p-1}B] < \phi(B^p).$$

PROOF OF THEOREM 1. Ad (i). Define the function g on \mathbf{R}^+ by

$$g(t) \equiv \phi[(S + tT)^p] - \phi(S^p + t^pT^p), \quad t \in \mathbf{R}^+.$$

Then $g(0) = 0$, and by the preceding lemmas (also [1, Corollary 3 of Theorem 6]),

$$g'(t) = \phi[pT^{1/2}(S + tT)^{p-1}T^{1/2}] - \phi(pt^{p-1}T^p) > 0, \quad t \in \mathbf{R}^+.$$

Thus $g(1) > g(0)$, i.e., $\phi[(S + T)^p] > \phi(S^p + T^p)$. We proceed to demonstrate the assertion on the equality. If $ST = 0$, then obviously $TS = ST = 0$. Appealing to well-known commutative results we have $\phi[(S + T)^p] = \phi(S^p + T^p)$.

Suppose now $\phi[(S + T)^p] = \phi(S^p + T^p)$. To show $ST = 0$ we let

$$K \equiv i[T(S + T)^{p-1} - (S + T)^{p-1}T]$$

(which is a selfadjoint element of \mathfrak{A}), and define the function h on \mathbf{R} by:

$$h(t) \equiv \phi[(S + e^{-itK}Te^{itK})^p], \quad t \in \mathbf{R}.$$

Then the result we just obtained implies that h attains its absolute minimum at $t = 0$. By Lemma 1, we have

$$\begin{aligned} 0 &= ip^{-1}h'(0) = \phi[(S + T)^{p-1}(KT - TK)] \\ &= \phi\{[T(S + T)^{p-1} - (S + T)^{p-1}T]K\}. \end{aligned}$$

Hence $T(S + T)^{p-1} = (S + T)^{p-1}T$. By the spectral theorem, $T(S + T) = (S + T)T$, i.e., $ST = TS$. But for any two nonnegative real numbers s, t : $(s + t)^p = s^p + t^p$ only if $st = 0$. As we have $\phi[(S + T)^p] = \phi(S^p + T^p)$ where S and T commute, we conclude that $ST = 0$.

Ad (ii). By Minkowski's inequality,

$$\phi[(S + T)^p] \leq [\phi(S^p)^{1/p} + \phi(T^p)^{1/p}]^p.$$

But as $p > 1$, we have, for any $s, t \in \mathbf{R}^+$, $(s + t)^p < 2^{p-1}(s^p + t^p)$. Hence

$$\phi[(S + T)^p] < 2^{p-1}\phi(S^p + T^p).$$

If $S = T$ then obviously $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$. Suppose that $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$ for some $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$. In order to show that $S = T$, we define a function f on \mathbf{R} by

$$f(t) \equiv \phi[(S + e^{-itH}Te^{itH})^p], \quad t \in \mathbf{R},$$

where H is a selfadjoint element of \mathfrak{A} . By the result we just obtained, f attains its absolute maximum at $t = 0$. Therefore

$$0 = ip^{-1}f'(0) = \phi[(S + T)^{p-1}(HT - TH)].$$

As before, we deduce that $ST = TS$. Since we have

$$\phi\{[2^{-1}(S + T)]^p\} = 2^{-1}[\phi(S^p) + \phi(T^p)],$$

we conclude that $S = T$.

Ad (iii). Suppose at first that $S + T > \varepsilon$ (a positive number) and that ϕ is finite. Define a function g_1 on \mathbf{R}^+ by

$$g_1(t) \equiv \phi[(S + tT)^p] - \phi(S^p + t^pT^p), \quad t \in \mathbf{R}^+.$$

Then $g_1(0) = 0$, and by the preceding lemmas, we have $g_1'(t) < 0$ for all $t \in (0, 1)$. Hence $g_1(1) < g_1(0)$, i.e.,

$$\phi[(S + T)^p] < \phi(S^p) + \phi(T^p).$$

The extra condition " $S + T > \varepsilon$ and ϕ is finite" can now be removed as follows. For given $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$, let $(E_n)_{n \in \mathbf{N}}$ be an increasing sequence of projections in \mathfrak{A} such that each E_n commutes with $S + T$, $(S + T)E_n > n^{-1}E_n$, and $(S + T)(I - E_n) < n^{-1}(I - E_n)$. As $E_n < n^p(S + T)^p$, we have $\phi(E_n) < \infty$. Then the result of the preceding paragraph implies that

$$\phi\{[(S + T)E_n]^p\} < \phi[(E_nSE_n)^p + (E_nTE_n)^p],$$

i.e.,

$$\|(S + T)^{1/2}E_n\|_{2p}^{2p} < \|S^{1/2}E_n\|_{2p}^{2p} + \|T^{1/2}E_n\|_{2p}^{2p}.$$

Taking adjoints we get

$$\|E_n(S + T)^{1/2}\|_{2p}^{2p} < \|E_nS^{1/2}\|_{2p}^{2p} + \|E_nT^{1/2}\|_{2p}^{2p},$$

i.e.,

$$\phi\{[(S + T)^{1/2}E_n(S + T)^{1/2}]^p\} < \phi[(S^{1/2}E_nS^{1/2})^p] + \phi[(T^{1/2}E_nT^{1/2})^p].$$

Since

$$(S^{1/2}E_nS^{1/2})^p < S^p, \quad (T^{1/2}E_nT^{1/2})^p < T^p,$$

since E_n commutes with $S + T$, and since $(S + T)(I - \lim_{n \rightarrow \infty} E_n) = 0$, we have

$$\begin{aligned} \phi[(S + T)^p] &= \lim_{n \rightarrow \infty} \phi\{[(S + T)^{1/2}E_n(S + T)^{1/2}]^p\} \\ &< \phi(S^p) + \phi(T^p). \end{aligned}$$

We proceed to prove the assertion on the equality. If $ST = 0$, then obviously $\phi[(S + T)^p] = \phi(S^p + T^p)$. Suppose now that $\phi(S^p + T^p) = \phi[(S + T)^p]$

and that ϕ is finite. Assume at first that $S + T$ has a bounded inverse. Then by arguments similar to those used in proving (i), we have $ST = 0$. Assume now that $S + T$ is of finite rank. Let E_1 be the support of $S + T$, $E_2 \equiv I - E_1$, $S_1 \equiv E_1SE_1$ and $T_1 \equiv E_1TE_1$. Then $E_2SE_2 = E_2TE_2 = 0$, $S^{1/2}E_2 = T^{1/2}E_2 = 0$, $S_1 = (S^{1/2}E_1)^*(S^{1/2}E_1) = S$ and $T_1 = T$. Thus

$$\phi[(S_1 + T_1)^p] = \phi(S_1^p + T_1^p).$$

As $S_1 + T_1 = (S + T)E_1$ is injective and of finite rank, $S_1 + T_1$ has a bounded inverse. Thus $ST = S_1T_1 = 0$.

Ad (iv). As $p \in (0, 1)$, the map $A \mapsto A^p$ on \mathfrak{A}^+ is concave and we have

$$S^p + T^p < 2^{1-p}(S + T)^p, \quad \phi[(S + T)^p] > 2^{p-1}\phi(S^p + T^p).$$

If $S = T$ then we obviously have equality. On the other hand, if we have equality $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$, then $(S + T)^p = 2^{p-1}(S^p + T^p)$. So $S + T = 2^{1-q}(S^p + T^p)^q$ where $q \equiv p^{-1}$. Let $S_1 \equiv S^p$, $T_1 \equiv T^p$. Then

$$\begin{aligned} \phi(S + T) &= 2^{1-q}\phi[(S_1 + T_1)^q] \\ &< 2^{1-q}2^{q-1}\phi(S_1^q + T_1^q) = \phi(S + T) \end{aligned}$$

by (ii) as $q > 1$. Hence

$$\phi[(S_1 + T_1)^q] = 2^{q-1}\phi(S_1^q + T_1^q).$$

By (ii) again, $S_1 = T_1$; so $S = T$. This completes the proof of Theorem 1.

THEOREM 2. Let $A, B \in L_p(\phi) \cap \mathfrak{A}$ ($\|\cdot\|_p$ the L_p norm).

(i) If $p \in [2, \infty)$, then

$$\|A + B\|_p^p + \|A - B\|_p^p > 2(\|A\|_p^p + \|B\|_p^p).$$

(ii) If $p \in [1, 2]$, then

$$\|A + B\|_p^p + \|A - B\|_p^p < 2(\|A\|_p^p + \|B\|_p^p).$$

(iii) Suppose $p \in [1, \infty) \setminus \{2\}$. Then

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p)$$

iff $AB^* = B^*A = 0$.

REMARK. Statement (i) is Lemma 3.5 of [1] in slightly different form. Statement (ii) can be obtained via Lemma 3.4 of [1] as Dixmier did for Lemma 3.5 of [1]. In fact they hold for any $A, B \in L_p(\phi)$. It suffices, therefore, to give:

PROOF OF THEOREM 2(iii). Suppose that $A, B \in \mathfrak{A} \cap L_p(\phi)$ satisfy

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p).$$

Let $q \equiv p/2$, $C \equiv A^*A + B^*B$, and $D \equiv A^*B + B^*A$. Then we have

$$\phi[(C + D)^q + (C - D)^q] = 2\phi[(A^*A)^q + (B^*B)^q].$$

By Theorem 1 we have for $q > 1$,

$$\phi[(C + D)^q + (C - D)^q] \geq 2\phi(C^q) \geq 2\phi[(A^*A)^q + (B^*B)^q],$$

and for $q \in (0, 1)$,

$$\phi[(C + D)^q + (C - D)^q] < 2\phi(C^q) < 2\phi[(A^*A)^q + (B^*B)^q].$$

Thus we have $\phi[(C + D)^q + (C - D)^q] = 2\phi(C^q)$ where $q \in (0, \infty) \setminus \{1\}$. By Theorem 1 (ii), (iv), $0 = D \equiv A^*B + B^*A$, i.e., $A^*B = -B^*A$. Since the equality

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p)$$

remains unchanged if (A, B) is replaced by (B^*, A^*) , we have $BA^* = -AB^*$ also. Then we have

$$A^*AB^*B = -A^*BA^*B = A^*BB^*A = B^*BA^*A,$$

i.e., A^*A commutes with B^*B . Since

$$\phi[(A^*A + B^*B)^q] = \phi(C^q) = \phi[(A^*A)^q + (B^*B)^q]$$

and since, for nonnegative real numbers s, t , $(s + t)^q = s^q + t^q$ only if $st = 0$, we conclude that $A^*AB^*B = 0$. Hence $(BA^*A)^*(BA^*A) = 0$, $BA^* = 0$ on the support of A^* and $AB^* = 0$. Replacing (A, B) by (B^*, A^*) we have $B^*A = 0$ also

Suppose $A, B \in \mathfrak{A} \cap L_p(\phi)$ satisfy $AB^* = B^*A = 0$. Then A^*A commutes with B^*B , $(A^*A)(B^*B) = 0$ and $AB^* + B^*A = 0$. Hence

$$\begin{aligned} 2\phi[(A^*A)^q + (B^*B)^q] &= 2\phi[(A^*A + B^*B)^q] \\ &= \phi\{[(A + B)^*(A + B)]^q + [(A - B)^*(A - B)]^q\}, \end{aligned}$$

i.e.,

$$2(\|A\|_p^p + \|B\|_p^p) = \|A + B\|_p^p + \|A - B\|_p^p.$$

This completes the proof.

3. L_p isometries. In this section we shall prove our main theorems introduced in §1, and also generalizations of some results of [5] and [6], by modifying the arguments of these papers. For clarity, however, we shall present the auxiliary results in full, though at places we shall refer the reader to [6] for more details.

Throughout this section, as before, we let ϕ [resp., ψ] be a semifinite normal faithful trace on \mathfrak{A} [resp., \mathfrak{B}], $L_p(\phi)$ [resp., $L_p(\psi)$] be the associated L_p space, and Θ a linear isometry from $L_p(\phi)$ into $L_p(\psi)$, $p \in [1, \infty) \setminus \{2\}$. (As in §1, the sum and product of two measurable unbounded operators will be in the strong sense; cf. [2].)

LEMMA 3. (i) Let $(A_\lambda)_{\lambda \in \Lambda}$ be a net in \mathfrak{A} , which converges to $A \in \mathfrak{A}$ ultraweakly. Then for any $X \in L_p(\phi)$, $(A_\lambda X)_{\lambda \in \Lambda}$ [resp., $(XA_\lambda)_{\lambda \in \Lambda}$] converges to AX (resp., XA) with respect to the $\sigma(L_p(\phi), L_q(\phi))$ topology on $L_p(\phi)$, where $q^{-1} + p^{-1} = 1$.

(ii) Let $(A_\lambda)_{\lambda \in \Lambda}$ be a net in $\mathfrak{A}^+ \cap L_p(\phi)$ bounded above by $A \in \mathfrak{A}^+ \cap L_p(\phi)$. Suppose $(A_\lambda)_{\lambda \in \Lambda}$ converges to A weakly and suppose $\phi(S) < \infty$ where S denotes the support of A . Then for any $B \in \mathfrak{B}$, $(\Theta(A_\lambda)B)_{\lambda \in \Lambda}$ [resp., $(B\Theta(A_\lambda))_{\lambda \in \Lambda}$] converges to $\Theta(A)B$ [resp., $B\Theta(A)$] with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology on $L_p(\psi)$.

PROOF. Ad (i). Let $X' \in L_q(\phi)$. Then $XX' \in L_1(\phi)$ and $\phi(A_\lambda XX')$ converges to $\phi(AXX')$ by [9, p. 82]. Hence $(A_\lambda X)_{\lambda \in \Lambda}$ converges to AX with respect to the $\sigma(L_p(\phi), L_q(\phi))$ topology on $L_p(\phi)$. Similarly by [4, Lemma 3.1] we prove that (XA_λ) converges to XA with respect to the $\sigma(L_p(\phi), L_q(\phi))$ topology on $L_p(\phi)$.

Ad (ii). Let $Y' \in L_q(\psi)$. Then, since the linear functional $\psi[\Theta(\cdot)BY']$ on $L_p(\phi)$ is continuous, there exists $X' \in L_q(\phi)$ such that $\psi[\Theta(Z)BY'] = \phi(ZX')$ for any $Z \in L_p(\phi)$. Since $AS = A$, we have $A_\lambda S = A_\lambda$. As $SX' \in L_1(\phi)$, $\phi(A_\lambda SX')$ converges to $\phi(ASX') = \phi(AX')$, i.e., $\psi[\Theta(A_\lambda)BY']$ converges to $\psi[\Theta(A)BY']$. Hence $\Theta(A_\lambda)B \rightarrow \Theta(A)B$ with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology on $L_p(\psi)$. Similarly $B\Theta(A_\lambda) \rightarrow B\Theta(A)$ with respect to the same topology on $L_p(\psi)$.

LEMMA 4. Let $A, B \in \mathfrak{A} \cap L_p(\phi)$. Suppose $\Theta(A), \Theta(B) \in \mathfrak{B}$. Then $AB^* = B^*A = 0$ iff $\Theta(A)\Theta(B)^* = \Theta(B)^*\Theta(A) = 0$.

PROOF. By Theorem 2(iii) of §2.

COROLLARY 1. Let P, P_1 be projections in $\mathfrak{A} \cap L_1(\phi)$ with $\Theta(P), \Theta(P_1) \in \mathfrak{B}$.

(i) Suppose $P_1P = PP_1$. Then $|\Theta(P_1)|$ commutes with $|\Theta(P)|$ [$|\Theta(P)|$ denotes the absolute value of $\Theta(P)$]. Consequently, if $\Theta(P), \Theta(P_1) \geq 0$, then

$$\Theta(P_1)\Theta(P) = \Theta(P)\Theta(P_1).$$

(ii) Suppose $PP_1P = P_1$. Then $|\Theta(P_1)| \leq |\Theta(P)|$ and $|\Theta(P_1)| = Q|\Theta(P_1)|Q$ where Q denotes the support of $|\Theta(P)|$.

PROOF (cf. [6, p. 95]). Ad (i). Let $Q_1 \equiv P_1P$, $Q_2 \equiv P - Q_1$, $Q_3 \equiv P_1 - Q_1$. Then Q_1, Q_2, Q_3 are mutually orthogonal projections with $Q_1 + Q_2 = P$ and $Q_1 + Q_3 = P_1$. A direct computation using Lemma 4 shows that

$$|\Theta(P)|^2|\Theta(P_1)|^2 = [\Theta(Q_1)^*\Theta(Q_1)]^2 = |\Theta(P_1)|^2|\Theta(P)|^2.$$

Ad (ii). Let $Q_0 \equiv P - P_1$. Then by Lemma 4 we have $|\Theta(P)|^2 = |\Theta(P_1)|^2 + |\Theta(Q_0)|^2$. The desired assertion then follows.

COROLLARY 2. Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+$. Let $M_{r,\phi}$ denote the two-sided ideal generated in \mathfrak{A} by projections in $L_1(\phi)$. Let $X, P \in M_{r,\phi}$ with P a projection.

- (i) Suppose $XP = PX$. Then $\Theta(X)\Theta(P) = \Theta(P)\Theta(X)$.
 (ii) Suppose $PXP = X$. Then $\Theta(X) = S\Theta(X)S$, where S denotes the support of $\Theta(P)$.

PROOF (cf. [6, p. 96]). Without loss of generality we can assume X to be positive. By making use of the spectral decomposition of X , Corollary 1 and Lemma 3 above, and the fact that the support of X belongs to $M_{r,\phi}$ and hence to $L_1(\phi)$ (cf. [10]), we establish (i). Statement (ii) is proved similarly.

LEMMA 5. Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+$. Then there exists a Jordan homomorphism Φ' of $\overline{M_{r,\phi}^u}$ (the closure of $M_{r,\phi}$ in \mathfrak{A} with respect to the uniform topology) into \mathfrak{B} such that:

- (i) $\Theta(X) = \Theta(P)\Phi'(X) = \Phi'(X)\Theta(P)$ for each $X \in M_{r,\phi}$ and projection $P \in M_{r,\phi}$ with $X = PXP$;
 (ii) $\Phi'(P) = S_{\Theta(P)}$ [the support of $\Theta(P)$] for each projection $P \in M_{r,\phi}$.

PROOF. This generalizes Lemma 3 of [6]. The proof given there can be carried through with some modification. For completeness and clarity, however, we shall sketch the proof.

For X, P as given and for an arbitrary, strictly positive real number $\epsilon > 0$, let $\Phi'_\epsilon(X, P) \equiv [\epsilon + \Theta(P)]^{-1}S\Theta(X)$, where S denotes the support of $\Theta(P)$. Then one can show that $0 < \Phi'_\epsilon(X, P) < \|X\|$ in case $X > 0$, that in general $\Phi''(X, P) \equiv (\text{strong}) \lim_{\epsilon \rightarrow 0} \Phi'_\epsilon(X, P)$ exists, and that the map $\Phi''(\cdot, P)$ from $P\mathfrak{A}P$ into \mathfrak{B} is linear, bounded and positive. Moreover we have $\Phi''(X, P_1) = \Phi''(X, P)$ for any projection $P_1 \in \mathfrak{A}$ satisfying $S_X < P_1 < P$ ($S_X \equiv$ the support of X). Define, for $X \in M_{r,\phi}$, $\Phi''(X) \equiv \Phi''(X, S_X)$. One can show that the map Φ'' from $M_{r,\phi}$ into \mathfrak{B} is linear, positive and bounded. Also $\Phi''(P) = S$ by considering $f_\epsilon[\Theta(P)]$ where $f_\epsilon(t) \equiv t(\epsilon + t)^{-1}$. Define Φ' to be the unique continuous extension of Φ'' to $\overline{M_{r,\phi}^u}$. Φ' can be shown to be a Jordan homomorphism by means of the fact $\Phi'(Q_1)\Phi'(Q_2) = 0$ for any two mutually orthogonal projections $Q_1, Q_2 \in M_{r,\phi}$. (i) is proved by observing that, for positive X ,

$$\begin{aligned} \Theta(P)\Phi''(X) &= \lim_{\epsilon \rightarrow 0} [\Theta(P)\Phi'_\epsilon(X, P)] \\ &= \lim_{\epsilon \rightarrow 0} [\Phi''(P, P)\Theta(X)] = S\Theta(X) = \Theta(X), \end{aligned}$$

where $\lim_{\epsilon \rightarrow 0}$ is taken in $L_p(\psi)$ with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology ($q^{-1} + p^{-1} = 1$).

LEMMA 6. Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$. Then:

(i) for any projection $Q \in \mathfrak{B}$, there exists a family of mutually orthogonal projections $(P_\lambda)_{\lambda \in \Lambda}$ in $M_{r,\phi}$ such that

$$Q = \sum_{\lambda \in \Lambda} \Phi'(P_\lambda) \quad \text{and} \quad \Phi'(P_\lambda) \in M_{r,\psi};$$

(ii) the weak closure of $\Phi'(M_{r,\phi})$ is \mathfrak{B} .

PROOF. As (ii) follows readily from (i), and as each projection in \mathfrak{B} is the direct sum of a family of mutually orthogonal projections in $M_{r,\psi}$, it suffices to prove (i) for a nonzero projection $Q \in M_{r,\psi}$. Then $\Theta^{-1}(Q) \in \mathfrak{A}^+ \cap L_p(\phi)$. By the spectral theorem there exist a positive number λ and a nonzero projection $P \in \mathfrak{A}$ such that $P \leq \lambda \Theta^{-1}(Q)$. Hence $P \in M_{r,\phi}$ and $\Theta(P) \leq \lambda Q$. Since $Q \in M_{r,\psi}$ we see that $\Theta(P)$ and its support $S_{\Theta(P)} = \Phi'(P)$ (by Lemma 5) belong to $M_{r,\psi}$; also $\Phi'(P) \leq Q$.

Now let $(P_n)_{n \in \mathbb{N}}$ be a maximal family of mutually orthogonal nonzero projections in $M_{r,\phi}$ such that $[\Phi'(P_n)]_{n \in \mathbb{N}}$ is a family of mutually orthogonal subprojections of Q with each $\Phi'(P_n) \in M_{r,\psi}$. The family is countable because ψ is faithful and $Q \in M_{r,\psi}$. Then $\sum_{n \in \mathbb{N}} \Phi'(P_n) = Q$. For if otherwise, $\bar{Q} \equiv Q - \sum_{n \in \mathbb{N}} \Phi'(P_n) \neq 0$. By the preceding paragraph, there is a nonzero projection $\bar{P} \in M_{r,\phi}$ such that $\Phi'(\bar{P}) \leq \bar{Q}$ and $\Phi'(\bar{P}) \in M_{r,\psi}$. Now as $S_{\Theta(\bar{P})} S_{\Theta(P_n)} = 0$, $\Theta(\bar{P})\Theta(P_n) = 0$ and $\bar{P}P_n = 0$. This contradicts the maximality of $(P_n)_{n \in \mathbb{N}}$; so $Q = \sum_{n \in \mathbb{N}} \Phi'(P_n)$.

LEMMA 7. Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$.

(i) There exist two projections Q_1, Q_2 in the center of \mathfrak{B} such that $Q_1 + Q_2 = I$ and such that the map $X \mapsto \Phi'(X)Q_1$ [resp., $X \mapsto \Phi'(X)Q_2$] of $\bar{M}_{r,\phi}^u$ into \mathfrak{B} is a homomorphism [resp., an antihomomorphism].

(ii) For every pair of projections Q_1, Q_2 satisfying (i), for every $X \in L_p(\phi)$ and for every $A \in \bar{M}_{r,\phi}^u$, we have

$$Q_1\Phi'(A)\Theta(X) = Q_1\Theta(A X) \quad [\text{resp.}, Q_2\Phi'(A)\Theta(X) = Q_2\Theta(XA)]$$

and

$$Q_1\Theta(X)\Phi'(A) = Q_1\Theta(XA) \quad [\text{resp.}, Q_2\Theta(X)\Phi'(A) = Q_2\Theta(A X)].$$

(iii) For each pair of projections Q_1, Q_2 satisfying (i), there exists a pair of projections P_1, P_2 in the center of \mathfrak{A} such that $P_1 + P_2 = I$ and such that, for every $X \in L_p(\phi)$, we have

$$Q_1\Theta(X) = \Theta(P_1 X) \quad [\text{resp.}, Q_2\Theta(X) = \Theta(P_2 X)].$$

(iv) The map Φ' from $\bar{M}_{r,\phi}^u$ into \mathfrak{B} is ultraweakly continuous and extends uniquely to a Jordan isomorphism Φ of \mathfrak{A} onto \mathfrak{B} .

PROOF. This generalizes Lemma 5 of [6], and statements (i) and (ii) are proved as in [6] with obvious modification (e.g., for (ii), apply Proposition 5 of [1]).

Ad (iii). By Lemma 6, we may let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of mutually orthogonal projections in $M_{r,\phi}$ such that $Q_1 = \sum_{\lambda \in \Lambda} \Phi'(P_\lambda)$. Put $P_1 \equiv \sum_{\lambda \in \Lambda} P_\lambda$. It is easy (cf. [6, p. 102]) to prove that for every $\lambda \in \Lambda$, $\Phi'(P_\lambda)\Theta(X) = \Theta(P_\lambda X)$, $X \in L_p(\phi)$. By Lemma 3, it follows that $Q_1\Theta(X) = \Theta(P_1 X)$. Similarly we have $\Theta(X)Q_1 = \Theta(XP_1)$. As Q_1 is in the center of \mathfrak{B} , we have $\Theta(P_1 Y) = \Theta(YP_1)$ for any $Y \in \mathfrak{A} \cap L_p(\phi)$. Thus P_1 belongs to the center of \mathfrak{A} . Similarly we prove that if $(P_\gamma)_{\gamma \in \Gamma}$ is a family of mutually orthogonal projections in $M_{r,\phi}$ such that $Q_2 = \sum_{\gamma \in \Gamma} \Phi'(P_\gamma)$, then $Q_2\Theta(X) = \Theta(XP_2)$, where $P_2 \equiv \sum_{\gamma \in \Gamma} P_\gamma$, and P_2 belongs to the center of \mathfrak{A} . Now

$$\Theta[X(P_1 + P_2)] = Q_1\Theta(X) + Q_2\Theta(X) = \Theta(X)$$

for any $X \in L_p(\phi)$, we conclude that $P_1 + P_2 = I$.

Ad (iv). In order to show that Φ' is ultraweakly continuous, it suffices (by Theorem 6 of [4]) to show that for every $Z \in L_1^+(\psi)$ the linear functional $A \mapsto \psi[\Phi'(A)Z]$ is ultraweakly continuous on $\overline{M}_{r,\phi}^\mu$. Since Θ is onto $L_p(\psi)$, it suffices, in turn, to show that for every $X \in L_p(\phi)$ and for every $W \in L_q(\psi)$ ($q^{-1} + p^{-1} = 1$), the linear functional $A \mapsto \psi[\Phi'(A)\Theta(X)W]$ is ultraweakly continuous on $\overline{M}_{r,\phi}^\mu$. By (ii) and (iii) above, it is easy to see that $\Phi'(A)\Theta(X) = \Theta(P_1 A X + P_2 X A)$. Hence

$$\psi[\Phi'(A)\Theta(X)W] = \psi[\Theta(P_1 A X + P_2 X A)W].$$

Since the linear functional $\psi[\Theta(\cdot)W]$ is continuous on $L_p(\phi)$, there exists $V \in L_q(\phi)$ such that $\psi[\Theta(Y)W] = \phi(YV)$ for every $Y \in L_p(\phi)$. Hence we have

$$\begin{aligned} \psi[\Phi'(A)\Theta(X)W] &= \phi[(P_1 A X + P_2 X A)V] \\ &= \phi(AXVP_1) + \phi(AVP_2 X). \end{aligned}$$

Since $AXVP_1$ and $AVP_2 X \in L_1(\phi)$, the maps $A \mapsto \phi(AXVP_1)$ and $A \mapsto \phi(AVP_2 X)$ are ultraweakly continuous. Thus Φ' is ultraweakly continuous on $\overline{M}_{r,\phi}^\mu$.

The rest of the proof is standard (cf. [6, p. 103]).

THEOREM 3. *Suppose $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$, $p \in [1, \infty)$. Then there exist uniquely a Jordan $*$ -isomorphism Φ of \mathfrak{A} onto \mathfrak{B} , and a positive selfadjoint operator $Z \in L_p(\psi)$ affiliated with the center of \mathfrak{B} such that*

$$\Theta(X) = \Phi(X)Z, \quad X \in \mathfrak{A} \cap L_p(\phi).$$

Suppose, in addition, that \mathfrak{A} or \mathfrak{B} is a factor. Then Φ is an isomorphism or an anti-isomorphism of \mathfrak{A} onto \mathfrak{B} , and Z is proportional to the identity of \mathfrak{B} ; so there exists uniquely a strictly positive number λ such that $\Theta(X) = \lambda\Phi(X)$ for every $X \in \mathfrak{A} \cap L_p(\phi)$.

PROOF. This “generalizes” Theorem 1 of [6]. The proof there carries through, with some modification, in the present context. For completeness and clarity, however, we shall sketch the proof.

Let $(P_\lambda)_{\lambda \in \Lambda}$ be a family of mutually orthogonal projections in $M_{r,\phi}$ such that $I = \sum_{\lambda \in \Lambda} P_\lambda$. Let S_λ be the support of $\Theta(P_\lambda)$. Let Z be the closure of the “direct sum” Z' of $\Theta(P_\lambda)|_{S_\lambda(\kappa)}$ (κ denotes the Hilbert space on which \mathfrak{B} acts). Then it follows from Lemmas 3 and 7 that

$$\Theta(X) = \sum_{\lambda \in \Lambda} \Theta(XP_\lambda) = \sum_{\lambda \in \Lambda} \Phi(X)\Theta(P_\lambda),$$

where $\sum_{\lambda \in \Lambda}$ is taken in $L_p(\psi)$ with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology ($q^{-1} + p^{-1} = 1$). Since

$$\Phi(X)\Theta(P_{\lambda_0}) = \left[\sum_{\lambda \in \Lambda} \Phi(X)\Theta(P_\lambda) \right] S_{\lambda_0} = \Theta(X)S_{\lambda_0},$$

it follows that $\Phi(X)Z' \subset \Theta(X)$ and, taking adjoints, it can be deduced that $\Theta(X) = Z\Phi(X) = \Phi(X)Z$. As the composite $\Phi(X) \circ Z$ is contained in the composite $Z \circ \Phi(X)$, Z is seen to be affiliated with the center of \mathfrak{B} . Φ is unique because $\Phi(P)$ has to be the support of $\Theta(P)$ for each projection $P \in M_{r,\phi}$. To see the uniqueness of Z , let Z_1 be another operator having the same properties as Z . If $Z_1 - Z \neq 0$, then by the spectral theorem and Lemma 6, there would be a projection $P \in M_{r,\phi}$ such that $(Z_1 - Z)\Phi(P) \neq 0$. Since $Z_1\Phi(P) = \Theta(P) = Z\Phi(P)$, we have $Z_1 = Z$. The rest of the theorem follows from well-known results on Jordan isomorphism (cf. [11]). (In the present context, Lemma 4 with $p = 2$ is Lemma 2 of [6], so the above outline works for $p = 2$ too.)

COROLLARY 1. *Let \mathfrak{A} be a finite factor with the canonical trace ϕ , and let Θ be a linear isometry from $L_p(\phi)$ ($1 \leq p < \infty$) onto itself with $\Theta(\mathfrak{A}^+) = \mathfrak{A}^+$. Then the restriction of Θ to \mathfrak{A} is either an automorphism or an anti-automorphism of \mathfrak{A} .*

COROLLARY 2. *Let \mathfrak{A} be a type I factor with the canonical trace ϕ , and let Θ be a linear isometry from $L_p(\phi)$ ($1 \leq p < \infty$) onto itself with $\Theta[L_p^+(\phi)] = L_p^+(\phi)$. Then there exists an automorphism or an anti-automorphism Φ of \mathfrak{A} such that $\Theta(X) = \Phi(X)$ for every $X \in L_p(\phi)$.*

PROOF. These corollaries follow readily from Theorem 3.

COROLLARY 3 (cf. [5, Theorem 2]). *Let Γ be a linear map from \mathfrak{A} onto \mathfrak{B} . Suppose for some $p \in [1, \infty)$, $\Gamma[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$ and suppose that $\|\Gamma(A)\|_p = \|A\|_p$ for every $A \in \mathfrak{A} \cap L_p(\phi)$. Then the following statements are equivalent:*

- (i) Γ is a Jordan *-isomorphism of \mathfrak{A} onto \mathfrak{B} ;
- (ii) $\Gamma(\mathfrak{A}^+) = \mathfrak{B}^+$ and $\Gamma(I) = I$.

PROOF. Γ can be extended to a linear isometry from $L_p(\phi)$ onto $L_p(\psi)$. Thus (ii) \Rightarrow (i) by Theorem 3 (recall that Φ is ultraweakly continuous). The implication (i) \Rightarrow (ii) is well known [11].

LEMMA 8. *Suppose there is a family $(P_\lambda)_{\lambda \in \Lambda}$ of mutually orthogonal projections in \mathfrak{A} such that $\sum_{\lambda \in \Lambda} P_\lambda = I$ and $\phi(P_\lambda) < \infty$, $\Theta(P_\lambda) \in \mathfrak{B}$ for each $\lambda \in \Lambda$. Then there exists a partial isometry $V \in \mathfrak{B}$ such that for any projection $P \in \mathfrak{A}$ with $\Theta(P) \in \mathfrak{B}$ and $P \leq P_{\lambda_0}$ (for some $\lambda_0 \in \Lambda$), we have $\Theta(P)V > 0$.*

PROOF. Denote $\Theta(P_\lambda) \equiv U_\lambda$, let $U_\lambda = W_\lambda|U_\lambda|$ be its polar decomposition, and let F_λ [resp., E_λ] be the initial [resp., final] projection of the partial isometry W_λ . By Lemma 4 above, $U_\lambda U_{\lambda'}^* = U_{\lambda'}^* U_\lambda = 0$ ($\lambda \neq \lambda'$); so $U_\lambda, U_{\lambda'}$ (resp., $U_\lambda^*, U_{\lambda'}^*$) have mutually orthogonal supports. Thus $E_\lambda E_{\lambda'} = 0$, $F_\lambda F_{\lambda'} = 0$. Let $V \equiv \sum_{\lambda \in \Lambda} W_\lambda^*$. Clearly $V \in \mathfrak{B}$ is a partial isometry. Let $\lambda_0 \in \Lambda$. Then $\Theta(P_{\lambda_0})V = U_{\lambda_0}V = |U_{\lambda_0}^*| > 0$ as $U_{\lambda_0}(1 - F_{\lambda_0}) = 0$. Let P be a projection in \mathfrak{A} majorised by P_{λ_0} , and let $Q \equiv P_{\lambda_0} - P$. Then $\Theta(P) + \Theta(Q) = U_{\lambda_0}$ and $\Theta(Q)^*\Theta(P) = 0$. Thus $[\Theta(Q)V]^*$ and $[\Theta(P)V]^*$ have mutually orthogonal supports. Since $|U_{\lambda_0}^*| = [\Theta(P)V]^* + [\Theta(Q)V]^*$ is nonnegative, we conclude that $\Theta(P)V > 0$.

PROPOSITION 1. *Suppose \mathfrak{A} is finite (ϕ may be semifinite) and $\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$. Then there is a partial isometry $V \in \mathfrak{B}$ such that $\Gamma(\cdot) \equiv \Theta(\cdot)V$ is a linear isometry from $L_p(\phi)$ into $L_p(\psi)$ with $\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+ \cap L_p(\psi)$, and $VV^* = I$.*

PROOF. By Proposition 10 of [9, §6, p. 100], there exists a family $(P_\lambda)_{\lambda \in \Lambda}$ of central projections in \mathfrak{A} satisfying the conditions of Lemma 8. Let V be constructed as in Lemma 8.

For $X \in M_{r,\phi}$ and $\lambda \in \Lambda$, let $X_\lambda \equiv P_\lambda X P_\lambda$, and let F_λ be the initial projection of W_λ introduced in the proof of Lemma 8. Then $X = (\text{weak}) \sum_{\lambda \in \Lambda} X_\lambda$ and $VV^*\Theta(X_\lambda)^* = VF_\lambda^*\Theta(X)^*$ (by Corollary 2 of Lemma 4) = $\Theta(X_\lambda)^*$. By Lemma 3, $VV^*\Theta(X)^* = \Theta(X)^*$. Since

$$\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$$

and since $M_{r,\phi}$ is dense in $L_p(\phi)$, we conclude that $VV^* = I$. As

$$\|\Theta(X)V\|_p = \|VV^*\Theta(X)^*\|_p = \|\Theta(X)^*\|_p = \|X\|_p,$$

$\Gamma(\cdot) \equiv \Theta(\cdot)V$ is a linear isometry from $L_p(\phi)$ into $L_p(\psi)$. By Lemma 8, it is clear that $\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+ \cap L_p(\psi)$.

The following theorem "generalizes" Theorem 1 of [5] with a completely different proof.

THEOREM 4. *Suppose \mathfrak{A} , \mathfrak{B} are finite (ϕ , ψ may be semifinite) and $\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$, $p \in [1, \infty) \setminus \{2\}$. Then there are a Jordan $*$ -isomorphism Φ of \mathfrak{A} onto \mathfrak{B} , a positive selfadjoint operator $Z \in L_p(\psi)$ affiliated with the center of \mathfrak{B} , and a unitary $W \in \mathfrak{B}$ such that $\Theta(X) = \Phi(X)ZW$, $X \in \mathfrak{A} \cap L_p(\phi)$.*

PROOF. Since \mathfrak{B} is finite, the partial isometry V in the preceding proposition is unitary (cf. [9, p. 217]) and $\Gamma(\cdot) \equiv \Theta(\cdot)V$ is onto $L_p(\psi)$ with

$$\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi).$$

A direct application of Theorem 3 then completes the proof.

REFERENCES

1. J. Dixmier, *Formes linéaires sur un anneau d'opérateurs*, Bull. Soc. Math. France **81** (1953), 9–39.
2. I. E. Segal, *A non-commutative extension of abstract integrations*, Ann. of Math. (2) **57** (1953), 401–457.
3. R. Kunze, *L_p Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. **89** (1958), 519–540.
4. T. Ogasawara and K. Yoshinaga, *A non-commutative theory of integration for operators*, J. Sci. Hiroshima Univ. Ser. A **18** (1955), 311–347.
5. B. Russo, *Isometries of L^p -spaces associated with finite von Neumann algebras*, Bull. Amer. Math. Soc. **74** (1968), 228–232.
6. M. M. Broise, *Sur les isomorphismes de certaines algèbres de von Neumann*, Ann. Sci. École Norm. Sup. **83** (1966), 91–111.
7. J. Lamperti, *On the isometries of certain function spaces*, Pacific J. Math. **8** (1958), 459–466.
8. T. Kato, *Perturbation theory for linear operators*, 2nd ed., Springer-Verlag, New York, 1976.
9. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
10. _____, *Application dans les anneaux d'opérateurs*, Compositio Math. **10** (1952), 1–55.
11. R. V. Kadison, *Isometries of operator algebras*, Ann. of Math. (2) **54** (1951), 325–338.
12. C. A. McCarthy, τ_p , Israel J. Math. **5** (1967), 249–271.

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