THE DIRICHLET NORM AND THE NORM OF SZEGÖ TYPE

BY

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Abstract. Let S be a smoothly bounded region in the complex plane. Let \( g(z, t) \) denote the Green's function of S with pole at \( t \). We show that

\[
\int_S |f(z)|^2 \, dz \, dt < \frac{1}{2} \int_{\partial S} |f(z)|^2 \left( \frac{\partial g(z, t)}{\partial n_z} \right)^{-1} |dz|
\]

holds for any analytic function \( f(z) \) on \( S \cup \partial S \). This curious inequality is obtained as a special case of a much more general result.

1. Introduction and preliminary facts. Let \( S \) denote an arbitrary compact bordered Riemann surface with boundary contours \( \{C_r\}_{r=1}^{2g+m-1} \) and of genus \( g \). Let \( \{C_r\}_{r=1}^{2g+m-1} \) be a canonical homology basis for \( S \). Let \( W(z, t) \) denote a meromorphic function whose real part is the Green's function \( g(z, t) \) with pole at \( t \in S \). The differential \( \text{id} W(z, t) \) is positive along \( \partial S \) and has \( N = 2g + m - 1 \) zeros \( \{t_r\} \) in \( S \). We assume that the points \( t_r \) are simple and they are not on \( \{C_r\}_{r=1}^{2g+m-1} \); the other cases will require only a slight modification. For simplicity, we do not distinguish between points \( z \in S \cup \partial S \) and local parameters \( z \). For an arbitrary integer \( q \) and for any positive continuous function \( \rho(z) \) on \( \partial S \), we let \( H^p_{\rho, q}(S) \) be the Banach space of analytic differentials \( f(z)(dz)^q \) of order \( q \) on \( S \) with a finite norm

\[
\left\{ \frac{1}{2\pi} \int_{\partial S} |f(z)(dz)^q|^p \rho(z) [\text{id} W(z, t)]^{1-q} \right\}^{1/p} < \infty,
\]

where \( f(z) \) means the Fatou boundary value of \( f \) at \( z \in \partial S \). Let \( K_{q, u}(z, \bar{u})(dz)^q \) be the reproducing kernel of \( H^q_{\rho, q}(S) \) which is characterized by the reproducing property

\[
f(u) = \frac{1}{2\pi} \int_{\partial S} f(z)(dz)^q K_{q, u}(z, \bar{u})(dz)^q \rho(z) [\text{id} W(z, t)]^{1-2q}
\]

for all \( f(z)(dz)^q \in H^q_{\rho, q}(S) \) (see [5]). Let \( L_{q, u}(z, u)(dz)^{1-q} \) denote the adjoint \( L \)-kernel of \( K_{q, u}(z, \bar{u})(dz)^q \). Then, \( L_{q, u}(z, u)(dz)^{1-q} \) is a meromorphic differential on \( S \) of order \( 1 - q \) with a simple pole at \( u \) having residue 1.

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Moreover:

\[ K_{q,t,\rho}(z, \bar{u})(dz)^q \rho(z)[\text{id } W(z, t)]^{1-2q} = (1/i)L_{q,t,\rho}(z, u)(dz)^{1-q} \quad \text{along } \partial S. \]  

(1.1)

We note that \( K_{q,t,\rho}(z, \bar{u}) \) and \( L_{q,t,\rho}(z, u) \) are continuous along \( \partial S \). If \( S \) is a bounded regular region in the plane, we can define these kernels for arbitrary real values of \( q \) (cf. [5, §§2 and 6]).

Next, let \( K^E(z, \bar{u}) \) and \( L^E(z, u) \) denote the exact Bergman kernel and its adjoint \( L \)-kernel on \( S \), respectively (cf. [8, p. 117]). \( L^E(z, u)(dz) \) is analytic on \( S \cup \partial S \) except for \( u \), where it has a double pole

\[ \left\{ \frac{1}{\pi} \frac{1}{(z - u)^2} + \text{regular terms} \right\} dz. \]  

(1.2)

Furthermore, it satisfies the relation

\[ -K^E(z, \bar{u})(dz) = L^E(u, z)(dz) \quad \text{along } \partial S. \]  

(1.3)

Let \( Z_\rho(z) = \int C(z, \bar{u}) dz \). Then \( \{Z_\rho(z)(dz)\}_\rho \) is a basis for the analytic differentials on \( S \) which are real along \( \partial S \). Here \( L(z, \bar{z}) \) is the adjoint \( L \)-kernel of the usual Bergman kernel \( K(z, \bar{z}) \) on \( S \) (cf. [8, §§4.3, 4.5 and 4.10]). Then from (1.1) and (1.3), we obtain

\[ K_{q,t,\rho}(z, \bar{u})K_{1-q,t,\rho}(z, \bar{u}) = \pi K^E(z, \bar{u}) + \sum_{\rho=1}^{N} \sum_{\mu=1}^{N} C_{\rho,\mu} Z_\rho(u) Z_\mu(z) \]  

(1.4)

and

\[ L_{q,t,\rho}(z, u)L_{1-q,t,\rho}(z, u) = \pi L^E(u, z) - \sum_{\rho=1}^{N} \sum_{\mu=1}^{N} C_{\rho,\mu} Z_\rho(u) Z_\mu(z), \]  

(1.5)

where the constants \( C_{\rho,\mu} \) are uniquely determined as in [5].

2. The main theorem. Let \( \{\Phi_j(z)(dz)^q\}_{j=1}^\infty \) and \( \{\Psi_j(z)(dz)^{1-q}\}_{j=1}^\infty \) be complete orthonormal systems for \( H^2_{\rho} (S) \) and \( H^{1-\rho}_\rho (S) \), respectively. Let \( H = H^2_{\rho} (S) \otimes H^{1-\rho}_\rho (S) \) denote the direct product of \( H^2_{\rho} (S) \) and \( H^{1-\rho}_\rho (S) \). The space \( H \) is composed of differentials \( f(z_1, z_2)(dz_1)^q(dz_2)^{1-q} \) on \( S \times S \) such that

\[ f(z_1, z_2) = \sum_{j=1}^\infty \sum_{k=1}^\infty A_{j,k} \Phi_j(z_1) \Psi_k(z_2), \quad \sum_{j=1}^\infty \sum_{k=1}^\infty |A_{j,k}|^2 < \infty. \]  

(2.1)

The scalar product \( (f, h)_H \) is introduced as follows:

\[ (f, h)_H = \sum_{j=1}^\infty \sum_{k=1}^\infty A_{j,k} B_{j,k}, \]

where

\[ h(z_1, z_2) = \sum_{j=1}^\infty \sum_{k=1}^\infty B_{j,k} \Phi_j(z_1) \Psi_k(z_2) \quad \text{and} \quad \sum_{j=1}^\infty \sum_{k=1}^\infty |B_{j,k}|^2 < \infty \]  

(cf. [1, §8]).
Theorem 2.1. Suppose that
\[ f(z_1, z_2)(dz_1)^q(dz_2)^{1-q} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \Phi_j(z_1) \Psi_k(z_2)(dz_1)^q(dz_2)^{1-q} \in H. \]

Then \( f(z, z) \) can be uniquely decomposed as follows:
\[ f(z, z) = h'(z) + \sum_{r=1}^{N} d_r Z_r(z) \quad \text{for} \ z \in S. \] (2.2)

It is understood that the \( d_r \) are constants, \( h(z) \) is analytic on \( S \), and
\[ \int \int_S |h'(z)|^2 \, dx \, dy < \infty \quad (z = x + iy). \]

In addition,
\[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}|^2 > \min \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi} \int_{\partial S} \psi_j(z_1)(dz_1)^q \varphi_k(z_1)(dz_1)^q \psi_k(z_2)(dz_2)^{1-q} \right\} \]
\[ \cdot \rho(z_1)[\text{id} \; W(z_1, t)]^{1-2q} \cdot \frac{1}{2\pi} \int_{\partial S} \psi_j(z_2)(dz_2)^{1-q} \psi_k(z_2)(dz_2)^{1-q} (\rho(z_2))^{-1} [\text{id} \; W(z_2, t)]^{2q-1} \]
\[ = \frac{1}{\pi} \int_{S} |h'(z)|^2 \, dx \, dy + \sum_{r=1}^{N} \sum_{\mu=1}^{N} D_{r,\mu} d_r \, d_\mu, \] (2.3)

where \( \|D_{r,\mu}\| \) is the inverse of \( \|C_{r,\mu}\| \). The minimum is taken here over all analytic functions \( \sum_{j=1}^{\infty} \psi_j(z_1) \psi_j(z_2) \) on \( S \times S \) satisfying
\[ f(z, z) = \sum_{j=1}^{\infty} \psi_j(z) \psi_j(z) \quad \text{on} \ S, \] (2.4)
\[ \psi_j(z)(dz)^q \in H_{2,q}'(S) \quad \text{and} \ \psi_j(z)(dz)^{1-q} \in H_{1,q}'(S). \]

Proof. The crucial ingredient in this proof is the observation that \( \|C_{r,\mu}\| \) is positive definite (cf. equation (1.4) and [5, p. 549]). Refer to the proof of Theorem 2.1 in [6]. The positive definiteness of \( \|C_{r,\mu}\| \) implies that
\[ k(z, \bar{u}) = \sum_{r=1}^{N} \sum_{\mu=1}^{N} C_{r,\mu} \bar{Z}_r(u) Z_\mu(z) \]
is a reproducing kernel for the finite dimensional class \( \mathcal{F}_2 \) which is generated by \( \{ Z_r(z) \}_{r=1}^{N} \) (see [1, pp. 346–347]). The scalar product is given by
\[ (f, h)_2 = \sum_{r=1}^{N} \sum_{\mu=1}^{N} D_{r,\mu} \bar{\xi}_r \eta_\mu \]
for \( f(z) = \sum_{j=1}^{\infty} \xi_j Z_j(z) \) and \( h(z) = \sum_{\eta} \eta Z_\eta(z) \). Note that 
\[ K_{q_1, q_2}(z, \bar{u})K_{1-q_1, q_2}(z, \bar{u}) \text{dz} \]
is the reproducing kernel of the space \( \mathcal{F} \) which is formed by restricting the functions in \( H \) to the diagonal set \( D = \{(z, z) | z \in S\} \) (cf. [1, p. 361, Theorem III]). For \( f \in \mathcal{F} \), the norm \( ||f||_{\mathcal{F}} \) is given by 
\[ \min \|h\|_H \]
where \( h(z_1, z_2) \) ranges over all elements of \( H \) whose restriction to \( D \) is \( f(z) \). Of course, \( \|h\|_H \) denotes the norm of \( h \) in \( H \).

On the other hand, the space \( \mathcal{F} \) must coincide with the class corresponding kernel function \( K_{q_1, q_2}(z, \bar{u})K_{1-q_1, q_2}(z, \bar{u}) \) when it is considered as the sum of the kernel functions \( \pi K_E(z, \bar{u}) \) and \( k(z, \bar{u}) \) (see [1, pp. 352–357]). We thus obtain the decomposition (2.2). The uniqueness follows from [8, pp. 104 and 108].

Finally, from the definition of the norm in \( H \) [1, pp. 357–361], we have the inequality (2.3).

Using [1] and the preceding remark about \( \mathcal{F} \), we immediately obtain

**Corollary 2.1.** Any analytic function \( f(z) \) on \( S \) with a finite Dirichlet integral can be represented as a series

\[ f'(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z) \quad \text{on } S, \quad (2.5) \]

where \( \varphi_j(z)(dz)^q \in H^q_{2\rho}(S) \) and \( \psi_j(z)(dz)^{1-q} \in H^1_{2\rho}(-S) \).

Furthermore, the equation

\[ \frac{1}{\pi} \int_S |f'(z)|^2 \, dx \, dy \]

\[ = \min \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi} \int_{\partial S} \varphi_j(z_1)(dz_1)^q \frac{\varphi_k(z_1)(dz_1)^q}{|\rho(z_1)[\text{id}W(z_1, t)]|^{1-2q}} \right\} \]

\[ \min \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi} \int_{\partial S} \psi_j(z_2)(dz_2)^{1-q} \right\} \]

is valid. The minimum is taken here over all analytic functions \( \sum_{j=1}^{\infty} \varphi_j(z_1)\psi_j(z_2) \) satisfying (2.5).

Conversely, if the \( jk \) sum in (2.6) is finite, then the exact differential \( f'(z)dz \)
defined by the series (2.5) has a finite Dirichlet integral.

**3. Some inequalities.** As an application of the main theorem, we derive some inequalities. To start with, consider the case \( f(z_1, z_2) = \varphi(z_1)\psi(z_2) \). This leads to
Theorem 3.1. For any $\varphi(z)(dz)^q \in H_{2,q}^g(S)$ and $\psi(z)(dz)^{1-q} \in H_{2,1-q}^g(S)$, we have

\[
\frac{1}{2\pi} \int_{\partial S} |\varphi(z_1)(dz_1)^q|^2 |\varphi(z_1) \cdot \psi(z_2)(dz_2)^{1-q}|^2 |\psi(z_2)\cdot \varphi(z_1)(dz_1)^q|^{-1} |\text{id } W(z_1, t) - W(z_2, t)|^{2q-1} \leq \min \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{1}{2\pi} \int_{\partial S} \varphi_j(z_1)(dz_1)^q \varphi_k(z_1)(dz_1)^q \right\}
\]

\[
= \frac{1}{2\pi} \int_{\partial S} |h'(z)|^2 \, dx \, dy + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} D_{r,\rho} \cdot \bar{d}_{\mu}, \tag{3.1}
\]

where $\varphi(z)\psi(z) = h'(z) + \sum_{r=1}^N d_r Z_r(z)$ on $S$, and where the minimum is taken over all analytic functions $\sum_{r=1}^N \varphi_j(z)\psi_j(z)$ on $S \times S$ such that

\[
\varphi(z)\psi(z) = \sum_{j=1}^{\infty} \varphi_j(z)\psi_j(z). \tag{3.2}
\]

Of course, $\varphi(z)(dz)^q \in H_{2,q}^g(S)$ and $\psi(z)(dz)^{1-q} \in H_{2,1-q}^g(S)$.

Equality holds in (3.1) if and only if $\varphi(z)\psi(z)$ is expressible in the form

\[
CK_{q,t,\rho}(z, \bar{u}) K_{1-q,1-\rho}(z, t) \text{ for some point } u \in S \text{ and for some constant } C.
\]

The equality statement in Theorem 3.1 will be proved in §5.

We can now take $q = 0$, $\varphi(z) \equiv 1$, $\psi(z) \equiv 1$, $\psi(z) \equiv f'(z)$. This yields

Corollary 3.1. For any analytic function $f(z)$ on $S \cup \partial S$, we have

\[
\int_{S} |f'(z)|^2 \, dx \, dy \leq \frac{1}{2} \int_{\partial S} |f'(z)|^2 \, |\text{id } W(z, t)|. \tag{3.3}
\]

Equality holds in (3.3) if and only if $S$ is simply connected and $f'(z)$ is expressible in the form $C\pi K_{g,1}(z, t) = CK_{1,1}(z, t)$ for some constant $C$.

Regarding the equality statement in Corollary 3.1, we note that $K_{0,1}(z, \bar{u}) \equiv 1$ if and only if $u = t \, [3]$. Furthermore, we can compare (3.3) with the inequality

\[
\left( \frac{1}{\pi} \int_{S} |f'(z)|^2 \, dx \, dy \right)^2 \leq \left( \frac{1}{2\pi i} \int_{\partial S} \overline{f(z)} f'(z) \, dz \right)^2 \leq \frac{1}{2\pi} \int_{\partial S} |f'(z)|^2 \, |\text{id } W(z, t)| \leq \frac{1}{2\pi} \int_{\partial S} |f'(z)|^2 \, |\text{id } W(z, t)|.
\]
Let \( K_{1,t,1}^E(z, \bar{u})dz \) denote the reproducing kernel of the closed subspace of \( H_{2,1}^1(S) \) composed of exact analytic differentials on \( S \) (cf. [2] and [4]). Since \( L^E(z, u) = L^E(u, z) \) if and only if \( S \) is planar \([8, pp. 114-120]\), Theorem 3.3 in [2] requires a modification when \( g > 1 \). But, this is not difficult. Using Corollary 3.1, we now obtain

**Corollary 3.2.** For all \( t \) and \( u \in S \), we have
\[
K_{1,t,1}^E(u, \bar{u}) < \pi K^E(u, \bar{u}). \tag{3.4}
\]
Equality holds in (3.4) if and only if \( S \) is simply connected and \( u = t \).

**Proof.** From (3.3) and the extremal property of \( K^E(z, \bar{u}) \) \([8, pp. 135-137]\), we have
\[
\frac{1}{K^E(u, \bar{u})} = \int_S \left| \frac{K^E(z, \bar{u})}{K^E(u, \bar{u})} \right|^2 dx \, dy < \int_S \left| \frac{K_{1,t,1}^E(z, \bar{u})}{K_{1,t,1}^E(u, \bar{u})} \right|^2 dx \, dy
\]
\[
< \frac{1}{2} \int_S \left| \frac{K_{1,t,1}^E(z, \bar{u})}{K_{1,t,1}^E(u, \bar{u})} \right|^2 \frac{|dz|^2}{id \, W(z, t)} = \frac{\pi}{K_{1,t,1}^E(u, \bar{u})}. \tag{3.5}
\]

We note that (3.3) is, in general, not valid for arbitrary analytic differentials \( f(z) \, dz \). Indeed, suppose that inequality (3.3) were valid for \( K_{1,t,1}^E(z, \bar{t}) \, dz \). Then, from the argument of Corollary 3.2, we would have
\[
K_{1,t,1}(t, \bar{t}) < \pi K(t, \bar{t}), \tag{3.6}
\]
which implies a contradiction for doubly connected regions \( S \) (cf. [6, §7]). The relationship between \( K_{1,t,1}^E(z, \bar{u}) \) and \( \pi K^E(z, \bar{u}) \) is, in general, quite complicated. (See [4, equation (2.6)].)

On the other hand, by setting \( q = 0 \) and \( \rho(z) \equiv 1 \) in Theorem 3.3, we obtain

**Corollary 3.3.** For the critical points \( \{t_\mu\}_{\mu=1}^N \) of the Green's function \( g(z, t) \) of \( G \), the matrix
\[
\left\| \sum_{\mu=1}^N \frac{Z_\nu(t_\mu)Z_\nu(t_\mu)}{W''(t_\mu, t)} - D_{\nu,\nu} \right\|_{N \times N}
\]
is positive definite. A slight modification is required here when the \( t_\mu \) are not simple.

**Proof.** We consider the decomposition \( 1 \cdot \sum_{\nu=1}^N d_\nu Z_\nu(z) \) for arbitrary constants \( d_\nu \). Then, we have
\[
\frac{1}{2\pi} \int_{\partial S} \left| \sum_{\nu=1}^N d_\nu Z_\nu(z) \, dz \right|^2 \frac{id \, W(z, t)}{id \, W(z, t)} > \sum_{\nu=1}^N \sum_{\nu'=1}^N D_{\nu,\nu'} d_\nu \overline{d_{\nu'}}. \tag{3.7}
\]
Using the residue theorem, we deduce the desired result.
4. Integral transform

By \( K_{q,1,\rho}(z, \bar{u}) K_{1-q,1,\rho}^{-1}(z, \bar{v}) \), as another application of the main theorem, we show that all the results of [7] are valid for the present \( H \).

**Theorem 4.1.** Assume that \( p > 1 \). Then, for \( \sigma \in L_p(\partial S) \),

\[
F_\sigma(z_1, z_2) = \int_{\partial S} \sigma(z) K_{q,1,\rho}(\zeta, \bar{z}_1) K_{1-q,1,\rho}^{-1}(z, \bar{z}_2) d\zeta
\]

belongs to \( H \) if and only if the projection \( h_1(z) \) of \( \sigma(z) \) onto \( H_{p,1}^0(S) \) belongs to the Bergman space of \( S \); that is,

\[
\int \int_S |h_1(z)|^2 \, dx \, dy < \infty.
\]

**Proof.** Refer to the proof of [7, Theorem 2.1]. The necessity is now apparent from Theorem 2.1. The crucial step in the sufficiency is to show that

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| -2i \int_S h_1(z) \Phi_j(z) \Psi_k(z) \, dx \, dy \right|^2
\]

converges. For any double sequence \( \{A_{j,k}\} \) satisfying

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |A_{j,k}| < \infty
\]

we consider the function \( f(z_1, z_2) \) defined by (2.1). Then, using Theorem 2.1, we have

\[
f(z, z) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \Phi_j(z) \Psi_k(z) = \tilde{f}'(z) + \sum_{r=1}^{N} \tilde{C}_r Z_r(z) \quad \text{on } S
\]

and

\[
\int \int_S |f(z, z)|^2 \, dx \, dy < \infty.
\]

Note that

\[
f_M(z, z) = \sum_{j=1}^{M} \sum_{k=1}^{M} A_{j,k} \Phi_j(z) \Psi_k(z)
\]

also can be uniquely decomposed as follows:

\[
f_M(z, z) = \tilde{f}_M'(z) + \sum_{r=1}^{N} \tilde{C}_r^{(M)} Z_r(z) \quad \text{on } S.
\]

From the main theorem, we see that the convergence of \( f_M(z, z) \, dz \) to \( f(z, z) \, dz \) in \( \mathcal{F} \) implies both the convergence of \( \tilde{f}_M(z) \) to \( \tilde{f}(z) \) in the Dirichlet norm and \( \lim_{M \to \infty} \tilde{C}_r^{(M)} = \tilde{C}_r \), for each \( \nu \). We thus obtain
\[
\int \int_S h'_1(z) \overline{f(z, z)} \, dx \, dy = \int \int_S h'_1(z) \left( \lim_{M \to \infty} f_M(z, z) \right) \, dx \, dy
\]

\[
= \int \int_S h'_1(z) \left( \lim_{M \to \infty} \tilde{f}_M(z) + \lim_{M \to \infty} \sum_{n=1}^{\infty} \tilde{C}^{(M)}_{n} Z_n(z) \right) \, dx \, dy
\]

\[
= \lim_{M \to \infty} \int \int_S h'_1(z) \left( \tilde{f}_M(z) + \sum_{n=1}^{\infty} \tilde{C}^{(M)}_{n} Z_n(z) \right) \, dx \, dy
\]

\[
= \sum_{n=1}^{\infty} \sum_{\kappa=1}^{\infty} \left( \int \int_S h'_1(z) \Phi_j(z) \Psi_k(z) \, dx \, dy \right) A_{j,k}
\] (4.9)

for any \( \{A_{j,k}\} \) satisfying (4.4). Hence, from the Landau theorem, we obtain the desired result.

The proof of this theorem shows that all the results of [7] are, in general, valid for the present \( H \).

Let \( H_{D(0)} \) denote the subspace of \( H \) formed by those functions in \( H \) which vanish along the diagonal set \( D \). Let \( (H_{D(0)})^\perp \) denote the orthocomplement of \( H_{D(0)} \) in \( H \). As in [7], we obtain

**Theorem 4.2.** Any \( f(z_1, z_2)(dz_1)^q(dz_2)^{1-q} \in (H_{D(0)})^\perp \) is expressible in the form

\[
f(z_1, z_2) = \frac{1}{2\pi} \int_{\partial S} \frac{h(\xi) \, d\xi}{id \, W(\xi, t)} K_{q, t, \rho}(\xi, \bar{z}_1) K_{1-q, t, \rho-1}(\xi, \bar{z}_2) \, d\xi
\] (4.10)

for a uniquely determined \( h(z) \) \( dz \) in \( H_{2,1}(S) \).

Furthermore:

\[
h(z) = -W'(z, t) \left\{ \int_{t}^{1} \left( \sum_{\rho=1}^{N} X_{\rho}(f) L_{q, t, \rho}(\xi, t_{\rho}) L_{1-q, t, \rho-1}(\xi, t_{\rho}) + f(\xi, \bar{z}, \bar{t}) \right) d\xi \right\},
\] (4.11)

where the constants \( X_{\rho}(f) \) are uniquely determined using the equations

\[
\sum_{\rho=1}^{N} X_{\rho}(f) \int_{C_{\rho}} L_{q, t, \rho}(\xi, t_{\rho}) L_{1-q, t, \rho-1}(\xi, t_{\rho}) \, d\xi = - \int_{C_{\rho}} f(\xi, t) \, d\xi, \quad \mu = 1, 2, \ldots, N.
\] (4.12)

5. **Proof of the equality statement in Theorem 3.1.** As an application of Theorem 4.2, we now prove the equality statement in Theorem 3.1.

If equality holds in (3.1) for \( \varphi(z_1)(dz_1)^q(\psi(z_2)(dz_2)^{1-q} \in H \), then we have \( \varphi(z_1)(dz_1)^q(\psi(z_2)(dz_2)^{1-q} \in (H_{D(0)})^\perp \) (cf. [6, equation (3.2)]). Hence, by Theorem 4.2, we have

\[
\varphi(z_1) \psi(z_2) = \frac{1}{2\pi} \int_{\partial S} \frac{h(\xi) \, d\xi}{id \, W(\xi, t)} K_{q, t, \rho}(\xi, \bar{z}_1) K_{1-q, t, \rho-1}(\xi, \bar{z}_2) \, d\xi
\] (5.1)
with \( h(\zeta) d\zeta \in H^1_S(S) \). Consider any \( \Phi^{(0)}(z)(dz)^q \) such that: (i) \( \Phi^{(0)}(z) \neq 0 \); (ii) \( \Phi^{(0)}(z) \) is analytic on \( S \cup \partial S \); (iii) \( \Phi^{(0)}(z)(dz)^q \) is orthogonal to \( \varphi(z)(dz)^q \) in \( H^q_{2,\partial}(S) \). From (5.1), we obtain

\[
\int_{\partial S} \frac{h(\zeta)d\zeta}{id W(\zeta, t)} \Phi^{(0)}(\zeta) K_{1-q,\partial^{-1}}(\zeta, z_2) d\zeta = 0 \quad \text{for all } z_2 \in S
\]

and so

\[
\int_{\partial S} \frac{h(\zeta)d\zeta}{id W(\zeta, t)} \Phi^{(0)}(\zeta) f(\zeta)d\zeta = 0,
\]

for all \( f(\zeta)(dz)^q \in H^1_{2,\partial}(S) \). Hence, from the theorem of Cauchy-Read, we obtain

\[
\frac{h(\zeta)d\zeta}{id W(\zeta, t)} \Phi^{(0)}(\zeta)(dz)^q = g(\zeta)(dz)^q \quad \text{a.e. along } \partial S,
\]

with \( g(\zeta)(dz)^q \in H^q_{2,\partial}(S) \) (cf. [5, p. 549] and [6, equation (3.9)]). From (5.1), we deduce that

\[
\varphi(z_1)\varphi(z_2) = \frac{1}{2\pi} \int_{\partial S} \frac{g(\zeta)(dz)^q K_{q,\partial}(\zeta, \bar{z}_1) K_{1-q,\partial^{-1}}(\zeta, \bar{z}_2) d\zeta}{\Phi^{(0)}(\zeta)(dz)^q}.
\]

Using (5.4) and the residue theorem, we obtain

\[
\varphi(z_1)\varphi(z_2) = \sum_{j=1}^{a} \sum_{k=0}^{b} X_{j,k} \frac{\partial^k(K_{q,\partial}(z_1, \bar{u}_j) K_{1-q,\partial^{-1}}(z_2, \bar{u}_j))}{\partial u_j^k}.
\]

and so

\[
\varphi(z_1) = \sum_{j=1}^{a} \sum_{k=0}^{b} Y_{j,k}^{(1)} \frac{\partial^k K_{q,\partial}(z_1, \bar{u}_j)}{\partial u_j^k},
\]

\[
\psi(z_2) = \sum_{j=1}^{a} \sum_{k=0}^{b} Y_{j,k}^{(2)} \frac{\partial^k K_{1-q,\partial^{-1}}(z_2, \bar{u}_j)}{\partial u_j^k},
\]

for some points \( u_j \in S \) and some constants \( \{X_{j,k}\}, \{Y_{j,k}^{(1)}\} \) and \( \{Y_{j,k}^{(2)}\} \). From (1.1), we now obtain

\[
\sum_{j=1}^{a} \sum_{k=0}^{b} X_{j,k} \frac{\partial^k(L_{q,\partial}(z_1, u_j)L_{1-q,\partial^{-1}}(z_2, u_j))}{\partial u_j^k}
\]

\[
= \sum_{j'=1}^{a} \sum_{k'=0}^{b} \sum_{j=1}^{a} \sum_{k=0}^{b} Y_{j,k}^{(1)} Y_{j',k'}^{(2)} \frac{\partial^k L_{q,\partial}(z_1, u_j)}{\partial u_j^k} \frac{\partial^k L_{1-q,\partial^{-1}}(z_2, u_j)}{\partial u_j^{k'}}
\]

for all \( z_1 \) and \( z_2 \in S \). By setting \( z_1 = z_2 = z \) and comparing the orders of the poles at each \( u_j \), we see that \( X_{j,k} = 0 \) for all \( j \) and \( k \) such that \( k \neq 0 \), and so
\[ Y_{j,k}^{(1)} = Y_{j,k}^{(2)} = 0 \text{ for all } j \text{ and } k \text{ such that } k \neq 0. \]

Thus

\[
\sum_{j=1}^{a} X_{j,0} L_{q,t,\phi}(z_1, u_j) L_{1-q,t,\phi^{-1}}(z_2, u_j) = \left( \sum_{j=1}^{a} Y_{j,0}^{(1)} L_{q,t,\phi}(z_1, u_j) \right) \left( \sum_{j=1}^{a} Y_{j,0}^{(2)} L_{1-q,t,\phi^{-1}}(z_2, u_j) \right)
\]

(5.9)

for all \( z_1 \) and \( z_2 \in S \). Without loss of generality, some \( X_{j,0} \) is nonzero. By considering (5.9) as the identity with respect to \( z_1 \),

\[
\sum_{j=1}^{a} X_{j,0} L_{1-q,t,\phi^{-1}}(z_2, u_j) = \frac{Y_{j,0}^{(1)}}{Y_{j,0}^{(2)}} \left( \sum_{j=1}^{a} Y_{j,0}^{(2)} L_{1-q,t,\phi^{-1}}(z_2, u_j) \right)
\]

(5.10)

for all \( z_2 \in S \). Therefore \( Y_{j,0}^{(2)} = 0 \) for all \( j'' \) except \( j_0 \) and so \( Y_{j,0}^{(1)} = 0 \) for all \( j'' \) except \( j_0 \). Hence \( X_{j,0} = 0 \) for all \( j \) except \( j_0 \). This yields the desired result:

\[
\psi(z_1) \psi(z_2) = X_{j,0} K_{q,t,\phi}(z_1, \bar{u}_{j_0}) K_{1-q,t,\phi^{-1}}(z_2, \bar{u}_{j_0}).
\]

(5.11)

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References

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